Probab. Theory Relat. Fields 120, 445–496 (2001)
Digital Object Identifier (DOI) 10.1007/s004400100129

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# Elliptic equations for measures on infinite dimensional spaces and applications

Received: 23 January 2000 / Revised version: 4 October 2000 /
Published online: 5 June 2001 – 

**Abstract.** We introduce and study a new concept of a weak elliptic equation for measures on infinite dimensional spaces. This concept allows one to consider equations whose coefficients are not globally integrable. By using a suitably extended Lyapunov function technique, we derive a priori estimates for the solutions of such equations and prove new existence results. As an application, we consider stochastic Burgers, reaction-diffusion, and Navier-Stokes equations and investigate the elliptic equations for the corresponding invariant measures. Our general theorems yield a priori estimates and existence results for such elliptic equations. We also obtain moment estimates for Gibbs distributions and prove an existence result applicable to a wide class of models.

## 1. Introduction

In this work we consider weak elliptic equations for measures on infinite dimensional spaces that can be formally written as $L^*_{A,B}\mu = 0$ in the sense that

$$\int L_{A,B}\psi \, d\mu = 0, \qquad \forall\, \psi \in \mathscr{K}, \tag{1.1}$$

where $\mathscr{K}$ is a certain class of test functions on $X$ and $L_{A,B}$ is formally given by

$$L_{A,B}\psi = \sum_{i,j=1}^{\infty} A_{ij}\partial_{e_i}\partial_{e_j}\psi + \sum_{j=1}^{\infty} B_j\partial_{e_j}\psi \tag{1.2}$$

with some $\mu$-measurable functions $A_{ij}$ and $B_j$ and vectors $e_j \in X$. In fact, $A$ and $B$ are regarded merely as collections $A := \{A_{ij}\}_{i,j\in\mathbb{N}}$ and $\{B_j\}_{j\in\mathbb{N}}$, respectively. In many cases, however, $A$ is an operator-valued function and $B$ is a vector field. Typical examples of such a situation are elliptic equations for invariant measures

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*Mathematics Subject Classification (2000):* Primary 46G12, 35J15, 28C20; Secondary: 60H15, 82B20, 60J60, 60K35

*Key words or phrases:* Invariant measures of diffusions – Elliptic equation for measures – Lyapunov function – Gibbs distribution – Logarithmic gradient – Stochastic Burgers equation – Stochastic Navier–Stokes equation – Stochastic reaction-diffusion equation

of diffusion processes and integration by parts formulas for measures. In the finite dimensional case, a natural choice of $\mathscr{K}$ is the class $C_0^\infty(\mathbb{R}^n)$ of smooth compactly supported functions (in this case the series becomes a finite sum). In infinite dimensions, there are many natural possibilities to choose $\mathscr{K}$; for example, one can take the class of smooth cylindrical functions or smooth functions $\psi$ of bounded support possessing bounded partial derivatives $\partial_{e_j}\psi$ and $\partial_{e_j}\partial_{e_j}\psi$. Moreover, the interpretations of (1.1) and (1.2) may be different. Our definition is this: if, say $A_{ij} = \delta_{ij}$, then (1.1) and (1.2) are interpreted as

$$\sum_{n=1}^{\infty} \int_X \big[\partial_{e_n}^2 \psi + B_n \partial_{e_n} \psi\big]\, d\mu = 0, \qquad \forall\, \psi \in \mathscr{K}. \tag{1.3}$$

We show that under certain technical conditions, it is possible to obtain a priori estimates for solutions of such equations and prove existence results for them. Particular emphasis is given to applications. More precisely, we consider elliptic operators of type (1.2) corresponding to stochastic partial differential equations such as Burgers, Navier–Stokes, and reaction-diffusion equations. We obtain new existence results and also improve results obtained in [38], [39], [5], [6]. As compared to [5], [6], we prove existence results (in particular, for Gibbs measure) under partly weaker assumptions. An advantage of the method employed in this paper is that it is universal enough to apply to all these cases. Furthermore, from a more technical point of view it does not require constructing scales of Hilbert spaces, and the verification of the various conditions in concrete examples becomes more direct and elementary. In addition, our method extends to the manifold case (see [18]).

A typical feature of the above mentioned examples of applications is that the drift term $B$ is only defined on a $\mu$-measure zero set. For example, in the case of the stochastic Burgers equation, the measure $\mu$ is defined on the Sobolev space $H_0^{2,1}$ of functions $u$ such that $u(0) = u(1) = 0$ and $u, u' \in L^2(0,1)$, whereas $B$ is heuristically given by the expression $B(u) = u'' - \psi(u)u' + f$. However, there are well-defined “coordinate functions” $B_n$ of this non-existing drift $B$: $B_n(u) = -(u', \eta_n')_2 - \big(\psi(u)u', \eta_n\big)_2 + (f, \eta_n)_2$, where $\{\eta_n\}$ is an orthormal basis in $L^2(0,1)$ such that $\eta_n \in H_0^{2,1}$. The situation is similar in the cases of the stochastic Navier–Stokes equation, stochastic reaction-diffusion equation, and Gibbs measures. Nevertheless, even in the case where the functions $B_n$ are really components of a well-defined mapping $B$, we obtain new results. In many cases, these results enable us to find a mapping $B$ defined $\mu$-a.e. and taking values in an appropriate enlargement of the original space $X$ such that the $B_n$’s become indeed the coordinates of $B$. For example, in the above considered case of the stochastic Burgers equation, $B$ can be regarded as a mapping to a suitable negative Sobolev space.

We want to emphasize that although one of our motivations is the study of invariant measures of infinite dimensional diffusion processes, we do not discuss the processes themselves in this paper. We even never assume the existence of the processes associated with the elliptic operators in question. However, it is known (see, e.g., [48]) that, under very broad assumptions, once we have a probability measure $\mu$ that solves (1.1), one can construct a diffusion process with generator $L_{A,B}$ having $\mu$ as an invariant (or sub-invariant) measure.

The organization of this paper is as follows: after introducing some notation and definitions in Sections 2 and 3, we prove some a priori estimates (in Section 4) which are of their own interest and which are necessary for the subsequent existence proofs given in Section 5. Section 6 is devoted to the special symmetric case. In particular, Gibbs measures are considered here. In Section 7 we present other applications. The question of regularity of invariant measures (discussed previously in [1], [12], [15]) is addressed in Section 8. More precisely, we give conditions ensuring that an invariant measure has partial logarithmic derivatives. Some results in this paper have been announced in [17].

## 2. Notation

Throughout, $X$ is a locally convex space with Borel $\sigma$-algebra $\mathscr{B}(X)$ and topological dual $X^*$. Let $\mathscr{M}(X)$ denote the set of all signed measures on $\mathscr{B}(X)$ with finite total variation. Given a family of linear functionals $\Lambda \subset X^*$, we denote by $\mathscr{F}\mathscr{C}_b^\infty(X, \Lambda)$ the class of all functions $f$ on $X$ of the form

$$f(x) = \psi\big(l_1(x), \dots, l_n(x)\big), \quad \text{where} \quad \psi \in C_b^\infty(\mathbb{R}^n), l_i \in \Lambda.$$

If $\Lambda = X^*$, then we write $\mathscr{F}\mathscr{C}_b^\infty(X)$ instead of $\mathscr{F}\mathscr{C}_b^\infty(X, X^*)$. In particular, we shall deal below with the classes $\mathscr{F}\mathscr{C}_b^\infty(X, \{l_n\})$ corresponding to countable sets $\{l_n\} \subset X^*$. Replacing $C_b^\infty(\mathbb{R}^n)$ by $C_0^\infty(\mathbb{R}^n)$ we obtain the classes $\mathscr{F}\mathscr{C}_0^\infty(X, \{l_n\})$ and $\mathscr{F}\mathscr{C}_0^\infty(X)$ (these classes are not linear spaces).

Given a Banach space $X$, we denote by $C_0^k(X)$, $k \in \mathbb{N}$, the class of all functions $f$ with bounded supports such that $f$ has $k$ bounded and continuous Fréchet derivatives.

**Definition 2.1.** *We say that a measure $\mu \in \mathscr{M}(X)$ is differentiable (in the sense of Fomin) along a vector $h \in X$ with respect to a certain class $\mathscr{K}$ of bounded Borel functions if every $f \in \mathscr{K}$ has a bounded (or just $\mu$-integrable) partial derivative $\partial_h f$ and there exists a $\mu$-measurable function $\beta_h^\mu$ such that, for all $f \in \mathscr{K}$, one has $f\beta_h^\mu \in L^1(\mu)$ and*

$$\int_X \partial_h f \, d\mu = -\int_X f \, \beta_h^\mu \, d\mu. \tag{2.1}$$

*The function $\beta_h^\mu$ is called the partial logarithmic derivative of $\mu$ along $h$ with respect to $\mathscr{K}$.*

The above definition gives a kind of "local" partial logarithmic derivatives. If $\mathscr{K} = \mathscr{F}\mathscr{C}_b^\infty(X)$ and $\beta_h^\mu \in L^1(\mu)$, then we arrive at the usual Fomin differentiability (see, e.g., [9]). An advantage of our more general definition is that it enables us to consider logarithmic derivatives with very weak integrability properties, which is convenient, e.g., in the study of Gibbs measures. Below we consider concrete examples; we only observe here that if $\mathscr{K} = \mathscr{K}_c^1(X)$ is the class of all bounded Borel functions $f$ on $X$ with compact supports such that the partial derivatives $\partial_h f$

exist and are bounded, then one can consider logarithmic derivatives $\beta_h^\mu$ that are $\mu$-integrable only on compact sets.

Let $H \subset X$ be a separable Hilbert space (fixed for the rest of this section) continuously and densely embedded into $X$. This embedding generates a standard embedding $j_H : X^* \to H$ defined by means of the Riesz representation as follows:

$$(j_H(l), h)_H = \langle l, h\rangle, \quad \forall\, l \in X^*,\ h \in H.$$

A typical example is $X = \mathbb{R}^\infty$ (the space of all sequences with the topology of pointwise convergence) and $H = l^2$. Then $X^* = \mathbb{R}_0^\infty$ is the space of finite sequences and $j_H(l)$ is represented by $l$ itself.

**Definition 2.2.** *A $\mu$-measurable mapping $\beta^\mu : X \to X$ is called logarithmic gradient of $\mu$ associated to $H$ with respect to a fixed class $\mathscr{K}$ if, for every $l \in X^*$, the measure $\mu$ is differentiable along $h = j_H(l)$ with respect to $\mathscr{K}$ and $\langle l, \beta^\mu\rangle = \beta_h^\mu$ $\mu$-a.e.*

Logarithmic gradients were introduced in [2], where the case of globally integrable $\beta_h^\mu$ was considered. Gibbs distributions and invariant measures of diffusion processes provide important examples which motivate the study of local logarithmic gradients. The logarithmic gradient may not exist even if $\mu$ is differentiable along all directions in $H$ (see examples in [15], [9]). This is one of the reasons why it is useful to consider "generalized logarithmic gradients" of the form $(B_n)_{n=1}^\infty$, where $B_n = \beta_{e_n}^\mu$, for a given sequence $\{e_n\}$ in $H$. This was already done in [15, Section 6] and was essential for the main result obtained there. We shall see below that under broad assumptions, it is possible to enlarge the space $X$ with a measure $\mu$ differentiable along a dense linear subspace such that the logarithmic derivative exists on the enlargement. Henceforth, however, we shall not necessarily assume that we are additionally given such an embedded Hilbert space $H$.

## 3. Weak elliptic equations for measures

Let $B = (B_n)$ be a sequence of Borel functions on $X$ and let $\{e_n\}$ be a sequence in $X$. Suppose we are given a certain class $\mathscr{K}$ of Borel functions on $X$ such that the partial derivatives $\partial_{e_i}\psi$ and $\partial_{e_j}\partial_{e_i}\psi$ exist for all $\psi \in \mathscr{K}$ and $i,\ j \in \mathbb{N}$ and are bounded.

**Definition 3.1.** *We shall say that $\mu \in \mathscr{M}(X)$ satisfies the weak elliptic equation*

$$L_B^*\mu = 0 \tag{3.1}$$

*with respect to the class $\mathscr{K}$ if, for every $\psi \in \mathscr{K}$, one has $B_n\partial_{e_n}\psi \in L^1(\mu)$ for all $n$ and*

$$\sum_{n=1}^\infty \int_X \left[\partial_{e_n}^2\psi + B_n\partial_{e_n}\psi\right] d\mu = 0, \quad \forall\, \psi \in \mathscr{K}. \tag{3.2}$$

In this case, we say that $L_{A,B} = L_B$ is heuristically given by

$$L_B\psi = \sum_{j=1}^{\infty}\big[\partial_{e_j}^2\psi + B_j\partial_{e_j}\psi\big]. \tag{3.3}$$

However, no assumptions about convergence of the series in (3.3) are made.

*Remark 3.2.* The above definition depends on the choice of $\{e_n\}$ in an essential way (e.g., $\mathscr{K}$ already depends on $\{e_n\}$). For simplicity, we nevertheless only use the terminology "with respect to $\mathscr{K}$" without mentioning $\{e_n\}$ explicitly.

As in the *symmetric case* (i.e., $B_n = \beta_{e_n}^{\mu}$ and the latter exists) already noticed in [5], [6], this definition enables one to consider equation (3.1) without global integrability assumptions on $B_n$ made in our previous work [14], [15], [12]. Note that also here the functions $B_n$ may not correspond to any "drift" $B$ on $X$.

Let $\{l_n\}$ be a sequence in $X^*$ such that $l_n(e_n) = 1$ and $l_n(e_m) = 0$ for all $n$ and $m \neq n$. Given a Borel mapping $B\colon\ X \to X$, we set $B_n = \langle l_n, B\rangle$. Then we can consider the operator $L_B = \sum\limits_{n=1}^{\infty}\big[\partial_{e_n}^2 + B_n\partial_{e_n}\big]$ and (3.1) becomes the characteristic equation for the invariant measures of the diffusion with the drift $B/2$ (and Wiener process associated with $\{e_n\}$) provided it exists. In order to make sense of $L_B$ it suffices to assume that $B_n \in L^1(\mu)$ and take for $\mathscr{K}$ the class $\mathscr{F}\mathscr{C}_b^{\infty}(X, \{l_n\})$, on which $L_B$ is defined by its natural expression.

However, there is a lot of examples (some of which are discussed below) where $L_B$ is not defined on $\mathscr{K}$ (hence (3.3) has no sense) and (3.1) in itself really only is a symbolic expression for (3.2).

Let us observe that equation (3.1) is trivially satisfied if the $\beta_{e_n}^{\mu}$'s exist with respect to the class $\mathscr{K}$, $\partial_{e_n}\psi \in \mathscr{K}$ for all $\psi \in \mathscr{K}$ and $n \in \mathbb{N}$, and $B_n = \beta_{e_n}^{\mu}$ $\mu$-a.e. Indeed, then every term in the series in (3.2) vanishes separately. In particular, large classes of Gibbs measures satisfy weak elliptic equations with respect to suitably chosen classes of test functions (see [3], [4], [5], [6]).

## 4. Some a priori estimates

Our next goal is to establish some a priori estimates for the solutions of (3.1). In the subsequent theorem we extend the Lyapunov functions technique to our situation. Our reasoning uses a modification of standard arguments going back to Hasminskii (see, e.g., [37]) and used also by many other authors studying stochastic differential equations (see, e.g., [8], [14], [15], [16], [22], [26], [38], [39], [40], [41], [42], [43], [52], [53]). Let $B := (B_n)$, $\mathscr{K}$, and $\{e_n\}$ be as defined at the beginning of Section 3.

**Theorem 4.1.** *Let $\mu$ be a probability measure on $X$ satisfying equation* (3.1) *with respect to $\mathscr{K}$. Suppose that $V$ is a nonnegative Borel function on $X$ such that $\partial_{e_n}^2 V$, $n \in \mathbb{N}$, exist and $\varphi \circ V \in \mathscr{K}$ for every $\varphi \in C_0^{\infty}(\mathbb{R}^1)$. Let $\Theta$ be a nonnegative Borel function on $X$ that is $\mu$-integrable on the sets $\{V \le c\}$, $c \in [0, \infty)$ (e.g., let $\Theta = \chi \circ V$, where $\chi$ is a nonnegative locally bounded Borel function on $\mathbb{R}^1$). Assume, in addition, that $L_B V \le C - \Theta$ $\mu$-a.e. in the following sense: there exist*

*$\mu$-measurable functions $\lambda_i$ such that on the sets $\{V \le c\}$, $c \in [0, +\infty)$, the series $\sum_{n=1}^{\infty} \lambda_n$ converges in $L^1(\mu)$ and one has*

$$\partial_{e_n}^2 V(x) + B_n(x)\partial_{e_n} V(x) \le \lambda_n(x) \ \ \text{and} \ \ \sum_{n=1}^{\infty} \lambda_n(x) \le C - \Theta(x) \quad \mu\text{-a.e.}, \tag{4.1}$$

*where $C > 0$. Then*

$$\int_X \Theta \, d\mu \le C. \tag{4.2}$$

*Finally, the hypothesis that $\Theta \ge 0$ can be replaced by the following one: $\Theta = \Theta_1 + \Theta_2$, where $\Theta_1 \ge 0$ and $\Theta_2 \in L^1(\mu)$.*

*Proof.* Certainly, (4.2) follows trivially by integrating the estimate $LV_B \le C - \Theta$ and making use of the equality $\int LV_B \, d\mu = 0$. However, due to the above interpretation of both relations, some justification is needed. By our hypothesis, we have (3.2) with $\psi = \varphi \circ V$ for every $\varphi \in C_0^\infty(\mathbb{R}^1)$. Then the same is true for every $\varphi \in C^\infty(\mathbb{R}^1)$ such that $\varphi = const$ outside some interval, since $\varphi - const \in C_0^\infty(\mathbb{R}^1)$ and (3.2) is trivially true for $\psi = const$. Now let us fix an even function $\zeta \in C^\infty(\mathbb{R}^1)$ such that $\zeta(t) = t$ if $|t| \le 1$, $\zeta(t) = 2$ if $t \ge 3$, $0 \le \zeta'(t) \le 1$ and $\zeta''(t) \le 0$ if $t \ge 0$. For $j \in \mathbb{N}$ set $\zeta_j(t) = j\zeta(t/j)$ if $t \ge 0$ and $\zeta_j(t) = \zeta_j(-t)$ if $t \le 0$. Clearly, $0 \le \zeta_j'(t) \le 1$ and $\zeta_j''(t) \le 0$ if $t \ge 0$. In addition, $\zeta_j(t) = t$ if $t \in [0, j]$ and $\zeta_j(t) = 2j$ if $t \ge 3j$. Hence, (3.2) is satisfied for $\psi = \zeta_j \circ V$. We observe that

$$\begin{aligned}
\partial_{e_n}^2(\zeta_j \circ V) + B_n\partial_{e_n}(\zeta_j \circ V) &= \zeta_j' \circ V\big[\partial_{e_n}^2 V + B_n\partial_{e_n} V\big] + \zeta_j'' \circ V(\partial_{e_n} V)^2 \\
&\le \zeta_j' \circ V\big[\partial_{e_n}^2 V + B_n\partial_{e_n} V\big] \le (\zeta_j' \circ V)\lambda_n.
\end{aligned}$$

Integrating with respect to $\mu$ and making use of (3.2) we arrive at the estimate

$$\int_X (\zeta_j' \circ V)\Theta \, d\mu \le C \int_X \zeta_j' \circ V \, d\mu \le C.$$

Now the desired estimate follows by Fatou's lemma, since $\zeta_j' \circ V \ge 0$ and $\lim_{j\to\infty} \zeta_j' \circ V = 1$ $\mu$-a.e. The case where $\Theta = \Theta_1 + \Theta_2$, where $\Theta_1 \ge 0$ and $\Theta_2$ is $\mu$-integrable, is proved similarly. □

*Remark 4.2.* Suppose that the functions $\lambda_n$ in the above theorem can be written as $\lambda_n = u_n - w_n$, where $u_n$ and $w_n$ are nonnegative functions, integrable on the sets $\{V \le r\}$. Then the $L^1(\mu)$-convergence of the series $\sum_{n=1}^{\infty} \lambda_n$ on the sets $\{V \le r\}$

follows from the integrability of the series $\sum_{n=1}^{\infty} u_n$ on the sets $\{V \le r\}$. Indeed, let $\zeta_r$ be the function introduced in the proof of Theorem 4.1. Then, as we have seen,

$$\partial_{e_n}^2(\zeta_r \circ V) + B_n \partial_{e_n}(\zeta_r \circ V) \le (\zeta_r' \circ V)\lambda_n = (\zeta_r' \circ V)u_n - (\zeta_r' \circ V)w_n.$$

By the above estimate and (3.2) it follows that

$$\begin{aligned}\sum_{n=1}^{\infty} \int_{\{V\le r\}} w_n \,\mu(dx) &\le \sum_{n=1}^{\infty} \int_X (\zeta_r' \circ V)w_n \,\mu(dx) \le \sum_{n=1}^{\infty} \int_X (\zeta_r' \circ V)u_n \,\mu(dx) \\ &\le \sum_{n=1}^{\infty} \int_{\{V\le 3r\}} u_n \,\mu(dx) < \infty,\end{aligned}$$

since $0 \le \zeta_r' \le 1$, $\zeta_r' \circ V = 1$ on $\{V \le r\}$, and $\zeta_r' \circ V = 0$ on $\{V > 3r\}$ .

*Remark 4.3.* The assumption that the functions in $\mathscr{K}$ and the functions $V$ and $\Theta$ in Theorem 4.1 are Borel can be replaced (as is obvious from the proof) by the assumption that those functions are $\mu$-measurable. We required the Borel measurability just in order to make the initial setting independent of $\mu$. The condition that $V$ and functions from $\mathscr{K}$ have partial derivatives along $e_n$ everywhere serves the same purpose. It is obvious from the above proof that we only need those derivatives $\mu$-a.e. Finally, one can assume that the functions $V$ and $\Theta$ take values in $[0, +\infty]$, but are finite on a linear subspace of full $\mu$-measure containing the sequence $\{e_n\}$. The same concerns all the results below. In typical applications, functions $\varphi \in \mathscr{K}$, $V$, and $\Theta$ are defined on a proper linear subspace $X_1$ of the initial space $X$ with $\mu(X_1) = 1$ such that the hypotheses of Theorem 4.1 are fulfilled on this smaller subspace (in particular, $\{e_n\} \subset X_1$ and the functions in question are differentiable along $e_n$). We shall use this simple observation below.

In the *symmetric case*, i.e., when $B_n$ is the logarithmic derivative of $\mu$ along $e_n$ with respect to $\mathscr{K}$, our hypotheses on $V$ can be modified under assumptions which are weaker in many cases.

**Theorem 4.4.** *Let $\mu$ be a probability measure on $X$ and let $B_n$ be the logarithmic derivative of $\mu$ along $e_n$ with respect to $\mathscr{K}$ and let $U$ and $V$ be $\mu$-measurable nonnegative functions such that* (i) *$\partial_{e_n}^2 V$ and $\partial_{e_n} U$ exist $\mu$-a.e. and $\partial_{e_n} U \partial_{e_n} V \ge 0$ $\mu$-a.e.,* (ii) *for every $\varphi \in C_0^\infty(\mathbb{R}^1)$, one has $\varphi(U)\partial_{e_n} V \in \mathscr{K}$, $\varphi(U)\partial_{e_n}^2 V \in L^1(\mu)$. Finally, suppose that* (4.1) *holds, where the series $\sum_{n=1}^{\infty} \lambda_n$ converges in $L^1(\mu)$ on the sets $\{U \le c\}$, $c \ge 0$. Then one has* (4.2).

*Proof.* Let $\varphi \in C_0^\infty(\mathbb{R}^1)$. Since $\varphi(U)\partial_{e_n} V \in \mathscr{K}$, one has

$$\partial_{e_n}\Big[\varphi(U)\partial_{e_n} V\Big] = \varphi(U)\partial_{e_n}^2 V + \partial_{e_n} U \varphi'(U)\partial_{e_n} V,$$

where both terms on the right are $\mu$-integrable, since their sum is bounded and $\partial^2_{e_n} V\varphi(U) \in L^1(\mu)$. If $\varphi$ is such that $\varphi(U) \geq 0$ and $\varphi'(U) \leq 0$, we obtain

$$\int \Big[\partial^2_{e_n} V + B_n \partial_{e_n} V\Big]\varphi(U)\, d\mu = -\int \varphi'(U)\partial_{e_n} V \partial_{e_n} U\, d\mu \geq 0.$$

Hence $\int \lambda_n \varphi(U)\, d\mu \geq 0$, consequently

$$C\int \varphi(U)\, d\mu \geq \int \varphi(U)\Theta\, d\mu.$$

Taking $\varphi_n \in C_0^\infty(\mathbb{R}^1)$ such that $0 \leq \varphi_n \leq 1$, $\varphi_n' \leq 0$ on $[0, +\infty)$, $\varphi_n = 1$ on $[-n, n]$, we arrive at (4.2) by Fatou's theorem. □

**Example 4.5.** *Let $\mu$ be a probability measure on $X$ satisfying equation* (3.1) *with respect to $\mathscr{K}$. Suppose that $V$ is a nonnegative Borel function on $X$ such that $\partial_{e_n} V$, $\partial^2_{e_n} V$ exist for all $n \in \mathbb{N}$ and $\varphi \circ V \in \mathscr{K}$ for every $\varphi \in C_0^\infty(\mathbb{R}^1)$. Assume, in addition, that $L_B V \leq C - kV^\alpha$ $\mu$-a.e. in the same sense as in the above theorem for some $\alpha \geq 0$, $k > 0$. Then*

$$\int_X V^\alpha\, d\mu \leq \frac{C}{k}. \tag{4.3}$$

Let us consider an example which extends a result from [5], [6], where the special symmetric case was considered and more restrictive assumptions on $B_i$ were used. Relations of this example to the results in [38], [39], [15] will be commented below.

If $J = (J_{s,t})_{s,\,t\in S}$ is an infinite symmetric matrix with nonnegative entries $J_{s,t}$ indexed by a countable set $S$ and $q = (q_s)$, $s \in S$, are positive numbers such that $\sum_{s\in S} q_s < \infty$, then we write $Jq \leq Cq$ with $C \in [0, \infty)$ if $\sum_{s\in S} q_s J_{s,t} \leq Cq_t$ for every $t \in S$. We denote by $l^1(q)$ the weighted $l^1$-space of all families $x = (x_s)_{s\in S}$ such that

$$\|x\|_{l^1(q)} = \sum_{s\in S} q_s |x_s| < \infty.$$

We observe that the condition $Jq \leq Cq$ is satisfied if $J_{t,s} = b_{t,s}c_{t,s}$, where $b_{t,u}q_u \leq C_1 q_t$ and $\sum_{u\in S} c_{t,u} \leq C_2$ for all $t, u \in S$. In particular, the latter condition is fulfilled if $S = \mathbb{Z}^d$ and $J_{n,j} = a(n - j)$, where $a$ is an even nonnegative function such that $a(n) \leq \text{const.}\, q_n^2$ and $q_j q_{n-j} \leq \text{const.}\, q_n$ for all $n, j \in \mathbb{Z}^d$. For example, the latter holds if $q_n \sim |n|^{-r}$, $r > d$, and $a(n) \leq \text{const.}|n|^{-2r}$.

We shall assume throughout that $J$ induces a bounded operator on $l^1(q)$, i.e., one has $\sum_{s,t\in S} q_t J_{t,s}|x_s| \leq \lambda \sum_{t\in S} q_t |x_t|$ for all $x \in l^1(q)$ and some $\lambda \geq 0$. The minimal possible $\lambda$ is the operator norm $\|J\|_{\mathscr{L}(l^1(q))}$. Clearly, the condition $Jq \leq Cq$ implies $\|J\|_{\mathscr{L}(l^1(q))} \leq C$.

As an example of suitable $J$ and $q$ let us take the integer lattice $S = \mathbb{Z}^d$ in $\mathbb{R}^d$ and for $n \in \mathbb{Z}^d$ set $q_n = (|n| + 1)^{-r}$, where $r > d$, so that $\sum_{n\in S} q_n < \infty$. Let the following condition employed in [6] be satisfied:

$$\|J\|_p = \sup_{n\in\mathbb{Z}^d} \sum_{j\in\mathbb{Z}^d} (1 + |j|)^{2p} J^2_{n,j+n} < \infty, \qquad \forall\, p \in \mathbb{N}. \tag{4.4}$$

Then $\|J\|_{\mathscr{L}(l^1(q))} < \infty$. Indeed, even

$$\begin{aligned}\sum_{j\in S} J_{n,j} q_j &\le C_1\big(|n/2| + 1\big)^{-r} + \sum_{|j|<|n|/2} J_{n,j} q_j \\ &\le C_1\big(|n/2| + 1\big)^{-r} + C_2\big(|n/2| + 1\big)^{-r} \\ &\le C_3 q_n,\end{aligned}$$

since $J_{n,j} \le \|J\|_r (1 + |n - j|)^{-r}$.

**Example 4.6.** Let $X = \mathbb{R}^S$ and let $X_0$ be the weighted Banach space of sequences such that

$$|x|_0 = \Big(\sum_{s\in S} q_s |x_s|^\alpha\Big)^{1/\alpha} < \infty,$$

where $\alpha \ge 2$ and $q = (q_s)_{s\in S} \in l_1$. Suppose that for $J = (J_{s,t})_{s,t\in S}$ as above

$$\|J\|_{\mathscr{L}(l^1(q))} \le \lambda, \qquad \sum_{t\in S} J_{s,t} \le \lambda.$$

Assume that $B = (B_s)_{s\in S}$ is a collection of Borel functions on $X_0$ with

$$x_s B_s(x) \le c - (\lambda + \varepsilon)|x_s|^\alpha + \sum_{t\in S} J_{s,t} |x_t|^\alpha \tag{4.5}$$

for some positive $c$, $\varepsilon$. Suppose that $\mu$ is a probability measure on $\mathbb{R}^S$ such that $\mu(X_0) = 1$ and that the $B_i$'s are $\mu$-integrable on all balls in $X_0$. If (3.1) is satisfied with respect to $C_0^2(X_0)$, where $e_s$, $s \in S$, are the standard unit vectors so that the $s$-th coordinate of $e_s$ is 1 and all other coordinates are zero, then one has

$$\int_X |x_s|^\alpha \, d\mu \le \frac{c+1}{\varepsilon}, \qquad \forall\, s \in S, \tag{4.6}$$

$$\int |x|_0^\alpha \, \mu(dx) \le \frac{c+1}{\varepsilon} \sum_{s\in S} q_s. \tag{4.7}$$

*Proof.* We observe that $\sum_{t\in S} J_{s,t}|x_t|^\alpha < \infty$ if $x \in X_0$, since $\|J\|_{\mathscr{L}(l^1(q))} < \infty$. We shall apply Theorem 4.1 to the function $V(x) = \sum_{s\in S} q_s x_s^2$, which is possible by our hypothesis that $\alpha \geq 2$. It should be noted that in order to apply the theorem cited, we can either refer to Remark 4.3 or simply restrict everything to the space $X_0$ (equipped with the topology from $\mathbb{R}^S$ in order to make $|\cdot|_0$-balls compact). We have

$$\partial_{e_s} V(x) = 2q_s x_s, \qquad \partial_{e_s}^2 V(x) = 2q_s.$$

Let

$$\lambda_s(x) := 2q_s + 2q_s\Big(c - (\lambda+\varepsilon)|x_s|^\alpha + \sum_{t\in S} J_{s,t}|x_t|^\alpha\Big).$$

By our hypothesis, $\partial_{e_s}^2 V + B_s \partial_{e_s} V \leq \lambda_s$. For every $x \in X_0$, because $\big(|x_s|^\alpha\big)_{s\in S} \in l^1(q)$, we have that

$$\sum_{s,t\in S} q_s J_{s,t}|x_t|^\alpha \leq \lambda \sum_{s\in S} q_s |x_s|^\alpha.$$

Therefore,

$$\sum_{s\in S} \lambda_s(x) \leq 2(c+1)\sum_{s\in S} q_s - 2\varepsilon \sum_{s\in S} q_s|x_s|^\alpha.$$

By Theorem 4.1, we obtain the estimate

$$\int_X \sum_{s\in S} q_s |x_s|^\alpha \,\mu(dx) \leq \frac{c+1}{\varepsilon}\sum_{s\in S} q_s.$$

In particular, letting

$$\xi_s = \int_X |x_s|^\alpha \,\mu(dx), \quad s\in S$$

we have $\xi := (\xi_s) \in l^1(q)$. Let us fix $s \in S$, $\delta > 0$ and consider the function

$$V_\delta(x) = x_s^2 + \delta\sum_{t\neq s} q_t x_t^2, \qquad x \in X_0.$$

Then

$$\partial_{e_s}^2 V_\delta + B_s \partial_{e_s} V_\delta \leq 2q_s\Big(1 + c - (\lambda+\varepsilon)|x_s|^\alpha + \sum_{t\in S} J_{s,t}|x_t|^\alpha\Big) = \lambda_s,$$

and for all $t \in S\setminus\{s\}$

$$\partial_{e_t}^2 V_\delta + B_t \partial_{e_t} V_\delta \leq 2\delta q_t\Big(1 + c - (\lambda+\varepsilon)|x_t|^\alpha + \sum_{u\in S} J_{t,u}|x_u|^\alpha\Big) = \lambda_t.$$

By Theorem 4.1, we arrive at the estimate

$$(\lambda+\varepsilon)\xi_s \leq 1 + \delta\sum_{t\neq s} q_t + c + \sum_{t\in S} J_{s,t}\xi_t + \delta\sum_{t\neq s} q_t\Big(c - (\lambda+\varepsilon)\xi_t + \sum_{u\in S} J_{t,u}\xi_u\Big).$$

Letting $\delta \to 0$ we have

$$\xi_s \leq \frac{1+c}{\lambda+\varepsilon} + \frac{1}{\lambda+\varepsilon}\sum_{t\in S} J_{s,t}\xi_t,$$

Denoting the element $(k, k, \dots) \in l^1(q)$ by $k$, the above estimate means that

$$\xi \leq (1+c)(\lambda+\varepsilon)^{-1} + (\lambda+\varepsilon)^{-1}J\xi.$$

Therefore, for every $m \in \mathbb{N}$, one has

$$\xi \leq \frac{1+c}{\lambda+\varepsilon}\sum_{k=0}^{m-1}(\lambda+\varepsilon)^{-k}J^k(1) + (\lambda+\varepsilon)^{-m}J^m\xi.$$

Noting that $\|J\|_{l^1(q)} \leq \lambda$ and that $(\lambda+\varepsilon)^{-k}J^k(1) \leq \lambda^k(\lambda+\varepsilon)^{-k}$, since $\sum\limits_{t\in S} J_{s,t} \leq \lambda$,

we arrive at the estimate $\xi \leq \dfrac{1+c}{\lambda+\varepsilon}\sum\limits_{k=0}^{\infty}\lambda^k(\lambda+\varepsilon)^{-k} = (1+c)/\varepsilon.$ □

*Remark 4.7.* By considering the functions $V_m(x) = \Big(\sum\limits_{s=1}^{\infty} q_s x_s^2\Big)^m$ in the previous example, one obtains by induction that

$$\int \Big(\sum_{s=1}^{\infty} q_s x_s^2\Big)^m \mu(dx) < \infty, \qquad \forall\, m \in \mathbb{N}. \tag{4.8}$$

A typical situation to which Example 4.6 applies is the case of a Gibbs measure $\mu$ on $\mathbb{R}^S$ with the conditional distributions $\mu(\,\cdot\,|x_s^c)$ given by continuously differentiable densities $p(x_s|x_s^c)$ such that

$$B_s(x) = \frac{\partial_{x_s} p(x_s|x_s^c)}{p(x_s|x_s^c)}, \qquad x = (x_s, x_s^c),$$

provided that the functions $B_s$ satisfy (4.5) and are locally bounded on $X_0$ and $\mu(X_0) = 1$. Indeed, in this case $\mu$ satisfies (3.1) with respect to $C_0^2(X_0)$. We refer to [5], [6] for specific examples.

Assume that $\{l_n\}$ is a sequence of continuous linear functionals on $X$ separating the points in $X$ and that $\{e_n\} \subset X$ is such that $l_i(e_j) = \delta_{ij}$ for all $i, j$.

**Example 4.8.** Suppose that $\{q_n\} \in l^1$ is a sequence of positive numbers such that $X_0 = \Big\{x\colon\ |x|_0^2 = \sum\limits_{n=1}^{\infty} q_n l_n(x)^2 < \infty\Big\}$ is a separable Hilbert space continuously embedded into $X$. Let $\mu$ be a probability measure on $X_0$ such that (3.1) is satisfied with respect to $C_0^2(X_0)$ and let $B_n$ be $\mu$-measurable functions that are $\mu$-integrable on balls in $X_0$. Assume that

$$\sum_{n=1}^{\infty} q_n l_n(x) B_n(x) \leq C - \varrho(|x|_0) \tag{4.9}$$

in the same sense as in Theorem 4.1, i.e., there exist functions $\zeta_n$ which are $\mu$-integrable on balls in $X_0$ such that the series $\sum_{n=1}^{\infty} \zeta_n$ converges in $L^1(\mu)$ on balls in $X_0$ and one has

$$q_n l_n(x) B_n(x) \le \zeta_n(x) \ \text{ and } \ \sum_{n=1}^{\infty} \zeta_n(x) \le C - \varrho(|x|_0) \ \ \mu\text{-a.e.}, \tag{4.10}$$

where $\varrho$ is a nonnegative locally bounded function on $\mathbb{R}^1_+$ with

$$\liminf_{t\to+\infty} \varrho(t) > C + \sum_{n=1}^{\infty} q_n + \varepsilon$$

for some $\varepsilon > 0$. Then

$$\int_X |x|_0^p \, \mu(dx) \le \frac{K\Big(p, \varrho, C, \varepsilon, \sum_{n=1}^{\infty} q_n\Big)}{\varepsilon - p \sup_n q_n} \tag{4.11}$$

for all $p \in [0, \varepsilon/\sup_n q_n)$. In particular, if, in addition, $\lim\limits_{t\to\infty} \varrho(t) = +\infty$, then

$$\int_X |x|_0^m \, \mu(dx) < \infty, \quad \forall m \in \mathbb{N}. \tag{4.12}$$

*Proof.* Let $V_m(x) = |x|_0^{2m}$, $m \ge 1$, and $Q = \sum_{n=1}^{\infty} q_n$. Then

$$\begin{aligned}\partial_{e_n}^2 V_m + B_n \partial_{e_n} V_m &= 2mq_n V_{m-1} + 4m(m-1)q_n^2 l_n^2 V_{m-2} + 2mq_n B_n l_n V_{m-1} \\ &\le \lambda_n := 2mq_n V_{m-1} + 4m(m-1)q_n^2 l_n^2 V_{m-2} + 2m\zeta_n V_{m-1}.\end{aligned}$$

The series $\sum_{n=1}^{\infty} \lambda_n$ converges in $L^1(\mu)$ on every ball in $X_0$ and

$$\begin{aligned}\sum_{n=1}^{\infty} \lambda_n &\le 2mQV_{m-1} + 4m(m-1)\sup_n q_n V_{m-1} + 2mV_{m-1}\Big(C - \varrho(|x|_0)\Big) \\ &= 2mV_{m-1}\Big(Q + (2m-2)\sup_n q_n + C - \varrho(|x|_0)\Big).\end{aligned}$$

Let us take $m := 1 + p/2$. There exists $R > 0$ such that $\varrho(t) \ge C + \sum_{n=1}^{\infty} q_n + \varepsilon$ for all $t \ge R$. Then for $\varepsilon' := \varepsilon - p \sup_n q_n$ one has

$$Q + (2m-2)\sup_n q_n + C - \varrho(|x|_0) = Q + p \sup_n q_n + C - \varrho(|x|_0) \le -\varepsilon'$$

if $|x|_0 \geq R$. Therefore,

$$\sum_{n=1}^{\infty} \lambda_n \leq 2mR^{2m-2}\big(Q+2(m-1)\sup_n q_n + C\big) - 2m\varepsilon' V_{m-1} = 2mM - 2m\varepsilon' V_{m-1},$$

where $M = M(m, Q, C, \sup_n q_n, R) > 0$. Hence $\int V_{m-1}\, d\mu \leq M/\varepsilon'$. It remains to note that $2m - 2 = p$. □

In a standard way, one also gets exponential moment estimates in the situation of Example 4.8 when making an appropriate choice of a Lyapunov function:

**Example 4.9.** Consider the situation of Example 4.8 and assume that $\lim_{t\to+\infty} \varrho(t) = +\infty$. Let $V(x) = u(|x|_0^2)$, where $u \in C^2(\mathbb{R}^1)$ is an increasing function such that for some $\delta > 0$ one has

$$(2\sup_n q_n)t^2 u''(t^2) \leq (1-\delta)u'(t^2)\varrho(t).$$

Then

$$\int u'(|x|_0^2)\varrho(|x|_0)\, d\mu < \infty.$$

For example, in order to obtain the integrability of $\exp\big(|x|_0^m\big)$, it suffices to take $u(t) = \exp(t^{m/2})$ and to require the estimate $\varrho(t) \geq Kt^m$ with $K$ sufficiently large. If one needs the integrability of $\exp\Big(\exp\big(|x|_0^2\big)\Big)$, then a suitable function is $u(t) = \exp(\exp t)$ and a sufficient estimate is $\varrho(t) \geq Kt^2\exp(t^2)$ with $K$ large enough.

*Proof.* We have $\partial_{e_n} V = 2q_n l_n u'(|x|_0^2)$, $\partial_{e_n}^2 V = 2q_n u'(|x|_0^2) + 4q_n^2(l_n)^2 u''(|x|_0^2)$, and

$$\begin{aligned}\partial_{e_n}^2 V + B_n(x)\partial_{e_n} V(x) &= \big(2q_n + 2q_n l_n(x)B_n(x)\big)u'(|x|_0^2) + 4q_n^2 l_n(x)^2 u''(|x|_0^2)\\ &\leq \big(2q_n + 2\zeta_n(x)\big)u'(|x|_0^2) + 4q_n^2 l_n(x)^2 u''(|x|_0^2),\end{aligned}$$

where $\zeta_n$ is as in (4.10). By our hypothesis,

$$\begin{aligned}&\sum_{n=1}^{\infty}\Big(\big(2q_n + 2\zeta_n(x)\big)u'(|x|_0^2) + 4q_n^2 l_n^2 u''(|x|_0^2)\Big)\\ &\quad\leq \Big(2Q + 2C - 2\varrho(|x|_0)\Big)u'(|x|_0^2) + 4\sup_n q_n u''(|x|_0^2)|x|_0^2\\ &\quad\leq \Big(2Q + 2C - 2\delta\varrho(|x|_0)\Big)u'(|x|_0^2) \leq \widehat{C} - \delta\varrho(|x|_0)u'(|x|_0^2),\end{aligned}$$

for some $\widehat{C} > 0$, where we used that $\lim_{t\to+\infty}\varrho(t) = +\infty$ and that $x \mapsto \varrho(|x|_0)$ $u'(|x|_0^2)$ is bounded on balls in $X_0$. □

**Example 4.10.** Suppose that in Example 4.8 one has $\varrho(t) = kt^2$ with $k > 2\lambda \sup_n q_n$. Then

$$\int_X \exp\bigl(\lambda |x|_0^2\bigr)\, \mu(dx) < \infty.$$

In particular, this is the case in Example 4.6, provided $\alpha > 2$.

*Remark 4.11.* We recall that in the above results the functions $B_n$ need not be globally $\mu$-integrable. However, if $B_n$ in Example 4.6 or Example 4.8 satisfies the estimate $|B_n(x)| \le C_n + K_n |x|_0^{d_n}$, the integrability of all powers of $|\cdot|_0$ yields that $B_n$ is in all $L^p(\mu)$. Therefore, in the situation of Example 4.6, the mapping $B = (B_n)$ $\mu$-a.e. takes values in the weighted Hilbert space

$$Y = \Bigl\{x \in \mathbb{R}^S\colon\ \sum_{n\in S} c_n x_n^2 < \infty\Bigr\}, \qquad \text{where } c_n > 0 \text{ and } \sum_{n\in S} c_n \|B_n\|^2_{L^2(\mu)} < \infty.$$

In a similar manner, one can construct a suitable Hilbert space $Y$ in Example 4.8 such that $X_0$ is contained in $Y$ and the functions $B_n$ coincide $\nu$-a.e. with the coordinates of a mapping $B\colon\ X_0 \to Y$.

The above results extend to the case of a non-constant diffusion term. Let us give the precise formulations.

Let $A_{ij}, B_j\colon\ X \to \mathbb{R}^1$, $i, j \in \mathbb{N}$, be Borel functions. We shall now define solutions to the elliptic equation

$$L^*_{A,B}\mu = 0, \tag{4.13}$$

where $L_{A,B}$ is heuristically given by $L_{A,B}\psi = \sum_{i,j=1}^\infty A_{ij}\partial_{e_i}\partial_{e_j}\psi + \sum\limits_{n=1}^\infty B_n \partial_{e_n}\psi$. We say that a Radon measure $\mu$ satisfies equation (4.13) with respect to the class $\mathscr{K}$ and the sequence $\{e_n\} \subset X$ if

$$\sum_{j=1}^\infty \int_X \Bigl(\sum_{i=1}^\infty A_{ij}\partial_{e_i}\partial_{e_j}\psi + B_j \partial_{e_j}\psi\Bigr)\, d\mu = 0, \quad \forall\, \psi \in \mathscr{K}, \tag{4.14}$$

where, for every $j$, the series $\sum\limits_{i=1}^\infty A_{ij}\partial_{e_i}\partial_{e_j}\psi$ converges $\mu$-a.e. and the existence of the above integrals is assumed in advance. We think, of course, of cases where the matrix $(A_{ij})$ is positive definite. However, this is only used in the following section. Let us give an analogue of Theorem 4.1 in this more general setting; the proof is the same as above, and we do not repeat it.

**Theorem 4.12.** *Let $\mu$ be a probability measure on $X$ satisfying equation* (4.13) *with respect to $\mathscr{K}$. Suppose that $V$ is a nonnegative Borel function on $X$ such that $\partial_{e_n}V$, $\partial_{e_n}\partial_{e_j}V$, $n, j \in \mathbb{N}$, exist and $\varphi \circ V \in \mathscr{K}$ for every $\varphi \in C_0^\infty(\mathbb{R}^1)$. Assume that, for every $c \in [0,\infty)$, the series $\sum\limits_{i=1}^\infty A_{ij}\partial_{e_i} V I_{\{V\le c\}}$ converges $\mu$-a.e. to* *a function from* $L^1(\mu)$ *for every* $j$ *and the series* $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} A_{ij}\partial_{e_i} V I_{\{V\le c\}}$ *converges in* $L^1(\mu)$ *to a nonnegative function. Let* $\Theta$ *be a nonnegative Borel function on* $X$ *that is* $\mu$*-integrable on the sets* $\{V \le c\}$, $c \in [0,\infty)$ (*e.g., let* $\Theta = \chi \circ V$, *where* $\chi$ *is a nonnegative locally bounded Borel function on* $\mathbb{R}^1$). *Finally, assume that* $L_{A,B}V \le C - \Theta$ $\mu$*-a.e. in the following sense: there exist* $\mu$*-measurable functions* $\lambda_n$ *such that the series* $\sum_{n=1}^{\infty}\lambda_n$ *converges in* $L^1(\mu)$ *on the sets* $\{V \le c\}$, $c \in [0,\infty)$, *and one has*

$$\sum_{j=1}^{\infty} A_{nj}(x)\partial_{e_n}\partial_{e_j} V(x) + B_n(x)\partial_{e_n} V(x) \le \lambda_n(x) \quad \textit{and}$$

$$\sum_{n=1}^{\infty}\lambda_n(x) \le C - \Theta(x) \qquad \mu\textit{-a.e.},$$

*where* $C \in [0,\infty)$. *Then*

$$\int_X \Theta\, d\mu \le C.$$

*Remark 4.13.* As a consequence of Theorem 4.12 it is clear that Examples 4.5 to 4.10 above have their obvious generalizations to the case of non-constant diffusion coefficients.

Here is e.g. an analogue of Example 4.8.

**Example 4.14.** Let $\{q_n\}$ and $X_0$ be the same as in Example 4.8. Let $\mu$ be a probability measure on $X_0$ and let $B_n$, $A_{ij}$ be $\mu$-measurable functions that are $\mu$-integrable on balls in $X_0$. Suppose that

$$\sup_x \sum_{i,j=1}^{\infty} A_{ij}^2(x) q_i q_j < \infty \quad \text{and} \quad \sup_x \sum_{n=1}^{\infty} q_n |A_{nn}(x)| < \infty. \tag{4.15}$$

Assume furthermore that $\mu$ satisfies (4.13) with respect to $C_0^2(X_0)$ including all the integrability assumptions. Suppose that $\sum_{n=1}^{\infty} q_n l_n(x) B_n(x) \le C - \varrho(|x|_0)$ in the same sense as in Example 4.8, where $\varrho$ is a nonnegative function on $[0,+\infty)$ such that $\lim_{t\to\infty}\varrho(t) = +\infty$. Then

$$\int_X |x|_0^m\, \mu(dx) < \infty, \qquad \forall\, m \in \mathbb{N}. \tag{4.16}$$

Clearly, (4.15) is fulfilled if $\sup_{x,i,j} |A_{ij}(x)| < \infty$.

Note that the first condition in (4.15) is used in the proof to obtain the following estimate:

$$\sum_{i,j=1}^{\infty} |A_{ij} q_i q_j l_i l_j| \le \Big(\sum_{i=1}^{\infty} q_i l_i^2\Big)^{1/2} \sum_{j=1}^{\infty}\Big(\sum_{i=1}^{\infty} A_{ij}^2 q_i\Big)^{1/2} q_j |l_j|$$
$$\le \Big(\sum_{i,j=1}^{\infty} A_{ij}^2 q_i q_j\Big)^{1/2} \sum_{i=1}^{\infty} q_i l_i^2 \le const. \sum_{i=1}^{\infty} q_i l_i^2 .$$

## 5. Existence results

We now turn to the existence results. The next two theorems are proved by the same method as in [15, Theorem 5.2]. However, global integrability assumptions on the coefficients of the drift are replaced by local ones. In order to make this paper self-contained we give complete proofs.

In this section, $\{l_n\}$ is a sequence of continuous linear functionals separating the points in $X$ and $\{e_n\} \subset X$ is such that $l_n(e_k) = \delta_{nk}$. We shall start with the existence results for equation (4.13) in the special case where $A_{ij} = 0$ if $i \neq j$ and $A_{nn} = A_n$, i.e., we are concerned with the operator $L_{A,B}$ heuristically given by $L_{A,B} = \sum_{n=1}^{\infty} (A_n \partial_{e_n}^2 + B_n \partial_{e_n})$. We note, however, that in the theorems below we deal with classes of cylindrical functions, on which $L_{A,B}$ makes sense as a finite sum.

We recall that a function $G\colon X \to [0, +\infty]$ on a topological space $X$ is called compact if the sets $\{G \le c\}$, $c \in \mathbb{R}^1$, are compact.

**Theorem 5.1.** *Suppose that $\Theta\colon X \to [0, +\infty]$ is compact and is finite on the finite dimensional spaces $E_n$ spanned by $e_1, \dots, e_n$. Let $A_n \ge 0$ and $B_n$ be functions on $X$ which are continuous on the sets $\{\Theta \le c\}$, $c \in \mathbb{R}^1$, as well as on the subspaces $E_j$. Assume that there exists $C \in (0, +\infty)$ and a nonnegative function $V$ on $X$ such that, for every $n$, the restriction of $V$ to $E_n$ is compact and twice continuously differentiable and one has*

$$\sum_{j=1}^{n}\Big[A_j(x)\partial_{e_j}^2 V(x) + \partial_{e_j} V(x) B_j(x)\Big] \le C - \Theta(x), \quad x \in E_n. \tag{5.1}$$

*Finally, let us assume that*

$$A_n(x) + |B_n(x)| \le C_n + \delta_n\big(\Theta(x)\big)\Theta(x), \quad x \in \{\Theta < +\infty\}, \tag{5.2}$$

*where $\delta_n$ is a nonnegative bounded Borel function on $[0, +\infty)$ with $\lim_{r\to+\infty} \delta_n(r) = 0$ and $C_n \in (0, +\infty)$. Then there exists a probability measure $\mu$ on $X$ such that $L_{A,B}^*\mu = 0$ with respect to the class $\mathscr{F}\mathscr{C}_b^\infty(X, \{l_n\})$. In addition,*

$$\int_X \Theta(x)\,\mu(dx) \le C. \tag{5.3}$$

*Proof.* Let $E_n$ be equipped with the inner product making $e_1, \dots, e_n$ an orthonormal basis. Then the sets $\{x \in E_n : \ \Theta(x) \le c\}$ are compact in $E_n$. Hence $\Theta(x) \to +\infty$ and $V(x) \to +\infty$ as $|x|_{E_n} \to +\infty$. According to [16, Corollary 1.3], there exists a probability measure $\mu_n$ on $E_n$ such that $L_n^*\mu_n = 0$ with respect to $C_0^\infty(E_n)$, where

$$L_n\psi(x) = \sum_{j=1}^n \big[A_j(x)\partial_{e_j}^2\psi(x) + \partial_{e_j}\psi(x)B_j(x)\big], \quad \psi \in C_0^\infty(E_n).$$

Clearly, we also have $L_n^*\mu_n = 0$ with respect to the class $C_0^2(E_n)$. The functions $A_j\partial_{e_j}^2 V$ and $B_j\partial_{e_j}V$ are bounded on the compact sets $\{\Theta \le c\} \cap E_n$. According to (5.1) the function $\Theta$ is bounded on balls in $E_n$. Since $\varphi \circ V \in C_0^2(E_n)$ for every $\varphi \in C_0^\infty(\mathbb{R}^1)$, it follows from (5.1) and Theorem 4.12 that

$$\int_{E_n} \Theta(x)\,\mu_n(dx) \le C. \tag{5.4}$$

Let $K_n = \sup_r \delta_n(r)$. By (5.2) we obtain

$$\int_{E_n} \big[A_j(x) + |B_j(x)|\big]\,\mu_n(dx) \le C_j + CK_j, \qquad \forall n, j \in \mathbb{N}. \tag{5.5}$$

In particular, we conclude that $L_n^*\mu_n = 0$ with respect to $C_b^\infty(E_n) = \mathscr{F}\mathscr{C}_b^\infty(E_n, \{l_j\}_{j=1}^n)$. We shall consider $\mu_n$ as a measure on $X$ (i.e., we extend $\mu_n$ to $X$ setting $\mu_n(X\backslash E_n) = 0$). Since the sets $\{\Theta \le c\}$ are compact, the sequence $\{\mu_n\}$ is uniformly tight. This yields (since compact sets in $X$ are metrizable because the sequence $\{l_n\}$ is separating) that there is a subsequence $\mu_{n_i}$ which converges weakly to some Radon probability measure $\mu$ on $X$. We may assume that the whole sequence $\{\mu_n\}$ converges weakly to $\mu$. It is readily seen that the measure $\mu$ is concentrated on the union of the compact sets $\{\Theta \le m\}$, $m \in \mathbb{N}$. We observe that we have not used so far that $\delta_n$ in (5.2) tends to zero at infinity. Note that, for every function $\psi$ of the form $\psi(x) = \psi_0\big(l_1(x), \dots, l_m(x)\big)$, $\psi_0 \in C_b^\infty(\mathbb{R}^m)$, and every $n \ge m$, we have

$$\begin{aligned}\int_X L_{A,B}\psi\,d\mu_n &= \int_X \sum_{j=1}^m \Big[A_j(x)\partial_{e_j}^2\psi(x) + \partial_{e_j}\psi(x)B_j(x)\Big]\,\mu_n(dx)\\ &= \int_X \sum_{j=1}^n \Big[A_j(x)\partial_{e_j}^2\psi(x) + \partial_{e_j}\psi(x)B_j(x)\Big]\,\mu_n(dx) = 0.\end{aligned}$$

Let us show that (5.3) holds and $L_{A,B}^*\mu = 0$ with respect to $\mathscr{F}\mathscr{C}_b^\infty(X, \{l_n\})$. Clearly, it suffices to show that, for every $\psi \in \mathscr{F}\mathscr{C}_b^\infty(X, \{l_n\})$ and every fixed $j$, one has $A_j, B_j \in L^1(\mu)$ and

$$\lim_{n\to\infty} \int_{E_n} A_j(x)\psi(x)\,\mu_n(dx) = \int_X A_j(x)\psi(x)\,\mu(dx),$$

$$\lim_{n\to\infty}\int\limits_{E_n} B_j(x)\psi(x)\,\mu_n(dx) = \int\limits_X B_j(x)\psi(x)\,\mu(dx).$$

We verify the second equality, with the first one can proceed completely analogously. Let $R > 0$ and $\Omega_R = \{\Theta \le R\}$. Define

$$\varepsilon_j(R) := R^{-1} \sup_{t\in[0,R]} t\delta_j(t). \tag{5.6}$$

Then $\varepsilon_j$ is a nonnegative function on $[0, +\infty)$ such that

$$\lim_{R\to+\infty} \varepsilon_j(R) = 0 \quad \text{and} \quad \sup_{t\in[0,R]} \delta_j(t)t \le \varepsilon_j(R)R.$$

Hence $\sup_{\Omega_R} |B_j| \le C_j + \varepsilon_j(R)R$. By (5.4) we have

$$\mu_n(X\backslash\Omega_R) \le CR^{-1}.$$

By the weak convergence and compactness of $\Omega_R$ we have

$$\mu(X\backslash\Omega_R) \le \liminf_{n\to\infty} \mu_n(X\backslash\Omega_R), \qquad \forall\, R \ge 0,$$

and since $\Theta$ is lower semicontinuous

$$\int\limits_X \Theta(x)\,\mu(dx) \le \liminf_{n\to\infty} \int\limits_X \Theta(x)\,\mu_n(dx) \le C, \quad \mu(X\backslash\Omega_R) \le CR^{-1}.$$

By (5.2) we obtain that $A_j, B_j \in L^1(\mu)$. Since $B_j$ is continuous on the compact set $\Omega_R$, there is a continuous function $G_R$ on $X$ such that $G_R = B_j$ on $\Omega_R$ and $|G_R| \le C_j + \varepsilon_j(R)R$. By the weak convergence we have

$$\lim_{n\to\infty}\int\limits_{E_n} G_R(x)\psi(x)\,\mu_n(dx) = \int\limits_X G_R(x)\psi(x)\,\mu(dx).$$

By the above estimates and the equality $G_R = B_j$ on $\Omega_R$ we obtain

$$\begin{aligned}
&\int\limits_{E_n} |G_R(x)\psi(x) - B_j(x)\psi(x)|\,\mu_n(dx)\\
&\quad\le \big[C_j + \varepsilon_j(R)R\big] \sup|\psi|\mu_n(X\backslash\Omega_R)\\
&\qquad + \sup|\psi| \int\limits_{X\backslash\Omega_R} \Big[C_j + \delta_j\big(\Theta(x)\big)\Theta(x)\Big]\,\mu_n(dx)\\
&\quad\le C\Big[C_jR^{-1} + \varepsilon_j(R) + C_jR^{-1} + \sup_{t\ge R}\delta_j(t)\Big]\sup|\psi|.
\end{aligned}$$

The right-hand side of this estimate goes to zero as $R \to +\infty$. The same is true for $\mu$ in place of $\mu_n$, which completes the proof. □

We shall also employ the following modification of Theorem 5.1 proved by a similar method.

**Theorem 5.2.** *Suppose that in the situation of Theorem* 5.1 *condition* (5.2) *is replaced by the following conditions:*

$$A_n(x) + |B_n(x)| \le C_n + K_n V(x)^{d_n}\Big(1 + \delta_n\big(\Theta(x)\big)\Theta(x)\Big),\ x \in \{\Theta < +\infty\}, \tag{5.7}$$

$$\sum_{j=1}^{n} A_j(x)|\partial_{e_j} V(x)|^2 \le C + \delta\big(\Theta(x)\big)\Theta(x)V(x), \qquad x \in E_n, \tag{5.8}$$

*where* $C_n, K_n, d_n \ge 0$, $\delta_n$ *and* $\delta$ *are nonnegative bounded Borel functions with* $\lim\limits_{r\to+\infty} \delta_n(r) = \lim\limits_{r\to+\infty} \delta(r) = 0$. *Assume, in addition, that* $V$ *is bounded on the sets* $\{\Theta \le c\}$, $c \in [0, +\infty)$. *Then there exists a probability measure* $\mu$ *on* $X$ *such that* $A_n$, $B_n \in L^1(\mu)$, $L^*_{A,B}\mu = 0$ *with respect to the class* $\mathscr{F}\mathscr{C}_b^\infty(X, \{l_n\})$, *and* (5.3) *holds. Moreover, if* $V$ *is continuous on the set* $\{\Theta < \infty\}$ (*or, more generally, the functions* $V^m\Theta$ *are lower semicontinuous*), *then*

$$\int_X V^m[1+\Theta]\,d\mu < \infty, \qquad \forall m \in \mathbb{N}. \tag{5.9}$$

*Proof.* The same reasoning as in Theorem 5.1 applies except for the justification of the equality $L^*_{A,B}\mu = 0$, which is deduced from the estimates

$$\sup_n \int_{E_n} V(x)^m\Theta(x)\,\mu_n(dx) = M_m < \infty, \qquad m \in \mathbb{N}. \tag{5.10}$$

In order to prove these estimates, we consider the functions $V_m := V^m$, find that

$$\begin{aligned} L_nV_m &= \sum_{j=1}^{n}\Big[m(m-1)A_jV^{m-2}|\partial_{e_j}V|^2 + mV^{m-1}A_j(x)\partial_{e_j}^2 V + mV^{m-1}B_j\partial_{e_j}V\Big] \\ &\le mV^{m-1}L_nV + Cm(m-1) + m(m-1)(\delta\circ\Theta)\Theta V^{m-1} \le \widehat{C}_m - V^{m-1}\Theta \end{aligned}$$

with some constants $\widehat{C}_m$ and apply Theorem 4.12. The rest of the proof is the same as above. Namely, we may assume that $V \ge 1$ (otherwise we replace $V$ by $V+1$). Let $\widetilde{\delta}_j(s) = \sup\limits_{1\le t\le s} \delta_j(s/t)t^{-1/2}, s \ge 1$. Then $\widetilde{\delta}_j(s) \le \max\Big(s^{-1/4}\sup\limits_t \delta_j(t),\ \sup\limits_{z\ge\sqrt{s}} \delta_j(z)\Big)$ $\to 0$ as $s \to +\infty$. In addition, $\delta_j(\Theta) \le \widetilde{\delta}_j(V^{2d_j}\Theta)V^{d_j}$ if $\Theta \ge 1$. Hence there exist constants $C'_j$ such that

$$A_j + |B_j| \le C'_j + K_jV^{d_j} + K_j\widetilde{\delta}_j(V^{2d_j}\Theta)V^{2d_j}\Theta. \tag{5.11}$$

Let $\widetilde{\varepsilon}_j$ be defined analogously to $\varepsilon_j$ in (5.6) with $\widetilde{\delta}_j$ in place of $\delta_j$. Let $R \in \mathbb{N}$ be fixed. The sets $\{V^{2d_j}\Theta \le R\}$ are contained in the sets $\{\Theta \le R\}$, hence have compact closures $\widetilde{\Omega}_R$. The functions $A_j$ and $B_j$ are continuous on the set $\{\Theta \le R\}$, hence also on $\widetilde{\Omega}_R$. Therefore, by our choice of $\widetilde{\varepsilon}_j$, we obtain $\sup_{\widetilde{\Omega}_R}\bigl[A_j + |B_j|\bigr] \le C''_j + K_j\widetilde{\varepsilon}_j(R)R$. Together with (5.10) this yields that $A_j$ and $B_j$ are $\mu$-integrable. Indeed, for a fixed $j$, we take a bounded continuous function $G_R$ which agrees with $B_j$ on $\widetilde{\Omega}_R$ and is majorized by $C''_j + K_j\widetilde{\varepsilon}_j(R)R$ outside $\widetilde{\Omega}_R$. By (5.10) and Chebyshev's inequality, $R\mu_n(X\backslash\widetilde{\Omega}_R) \le M_{2d_j}$ for all $n$. Hence $R\mu(X\backslash\widetilde{\Omega}_R) \le M_{2d_j}$. By (5.11), for some constants $N_j$ and $K'_j$, we have

$$A_j + |B_j| \le N_j + K'_j V^{2d_j}\Theta.$$

Therefore,

$$\int_{E_n} G_R\, d\mu_n \le N_j + K'_j M_{2d_j} + [C''_j + K_j\widetilde{\varepsilon}_j(R)R]\mu_n(X\backslash\widetilde{\Omega}_R) \le N'_j,$$

where $N'_j$ is independent of $n$ and $R$. Hence, by the weak convergence of $\mu_n$ to $\mu$, the integral of $G_R$ with respect to $\mu$ is estimated by $N'_j$. Letting $R \to \infty$, we obtain the integrability of $B_j$ with respect to $\mu$. The same is true for $A_j$. Finally, the equality $L^*_{A,B}\mu = 0$ is justified as in the previous theorem. If the functions $V^m\Theta$ are lower semicontinuous on the space $\{\Theta < \infty\}$ (say, $V$ is continuous on this space), then (5.9) follows from (5.10) by the weak convergence of $\mu_n$ to $\mu$ and the boundedness of $V$ on the set $\{\Theta \le 1\}$. □

**Theorem 5.3.** *Let $X$, $\{q_n\}$, and $X_0$ be the same as in Example* 4.8. *Assume, in addition, that the embedding $X_0 \subset X$ is compact. Let $B_n\colon\ X_0 \to \mathbb{R}^1$ be continuous on all balls in $X_0$ with respect to the topology of $X$ and satisfy the estimates*

$$|B_n(x)| \le C_n + K_n|x|_0^{d_n}, \qquad \forall\, x \in X_0. \tag{5.12}$$

*Assume that*

$$\sum_{n=1}^{\infty} q_n l_n(x) B_n(x) \le C - \varrho(|x|_0) \tag{5.13}$$

*on the linear span of the $e_n$'s, where $\varrho$ is a nonnegative bounded Borel function on* $[0,+\infty)$ *such that* $\lim\limits_{t\to\infty}\varrho(t) = +\infty$. *Then there exists a probability measure $\mu$ on $X_0$ satisfying equation* (3.1) *with respect to $\mathscr{F}\mathscr{C}_b^\infty(X,\{l_n\})$ such that*

$$\int_X \varrho(|x|_0)\,\mu(dx) \le C + \sum_{n=1}^{\infty} q_n, \qquad \int_X |x|_0^m\bigl[1+\varrho(|x|_0)\bigr]\,\mu(dx) < \infty, \quad \forall\, m \in \mathbb{N}.$$

*Proof.* Let $V(x) = \sum\limits_{n=1}^{\infty} q_n l_n(x)^2 = |x|_0^2$. We have

$$\sum_{j=1}^{n}\bigl[\partial_{e_j}^2 V(x) + B_j(x)\partial_{e_j}V(x)\bigr] \le 2\sum_{j=1}^{n} q_j + 2C - 2\varrho(|x|_0), \quad x \in E_n.$$

Let $\Theta(x) = 2\varrho(|x|_0)$. The functions $B_n$ and $V$ are bounded and continuous on compact sets $\{\Theta \le c\}$, which are balls in $X_0$. In addition, $A_n = 1$ and $\sum_{j=1}^{\infty} |\partial_{e_j} V|^2 \le 4(\sup_j q_j)V$. Now we can use Theorem 5.2 with $\delta(t) = (|t|+1)^{-1}$ and $\delta_n = 0$. □

*Remark 5.4.* Let $\kappa_n = \|B_n\|_{L^1(\mu)}$. Let $\Lambda$ be the collection of continuous linear functionals $l$ on $X_0$ such that $\sum_{n=1}^{\infty} \kappa_n |l(e_n)| < \infty$. Then it follows that $\mu$ satisfies (3.1) also with respect to $\mathscr{F}\mathscr{C}_b^\infty(X_0, \Lambda)$. In particular, $\mu$ satisfies (3.1) also with respect to $\mathscr{F}\mathscr{C}_b^\infty(X)$ if $X^* \subset \Lambda$. This is the case, e.g., if $\sup_n[C_n + K_n + d_n] < \infty$. Then $\mu$ satisfies (3.1) even with respect to $\mathscr{F}\mathscr{C}_b^\infty(X_0)$. Indeed, $\sup_n \kappa_n < \infty$ and $\sum_{n=1}^{\infty} |l(e_n)|^2/q_n < \infty$ for every $l \in X_0^*$. Hence $\sum_{n=1}^{\infty} \kappa_n |l(e_n)| \le \Bigl(\sum_{n=1}^{\infty} \kappa_n^2 q_n\Bigr)^{1/2} \Bigl(\sum_{n=1}^{\infty} |l(e_n)|^2/q_n\Bigr)^{1/2} < \infty$. It is worth noting that if $X = \mathbb{R}^S$, where $S$ is a countable set, then $X^*$ is the space of all finite sequences, hence $\mathscr{F}\mathscr{C}_b^\infty(X) = \mathscr{F}\mathscr{C}_b^\infty(X, \{l_n\})$, where $l_n$ are the natural coordinate functions.

**Corollary 5.5.** *Assume that in Theorem 5.3 one has $\varrho(t) = kt^2$ and that $k > 2\lambda \sup_n q_n$. Then*

$$\int_X \exp\bigl(\lambda |x|_0^2\bigr)\, d\mu < \infty. \tag{5.14}$$

*Moreover, for existence of $\mu$ it suffices to replace the power estimates on $B_n$ by $|B_n(x)| \le C_n + K_n \exp\bigl(d_n|x|_0\bigr)$. If $\varrho(t) = kt^{2+\delta}$, where $k,\ \delta > 0$, then it is enough to have the estimates $|B_n(x)| \le C_n + K_n \exp\bigl(d_n|x|_0^2\bigr)$.*

*Remark 5.6.* The global polynomial bound on the $B_n$'s can be dropped if there exist a continuously embedded Hilbert space $X_1$ such that $X_0 \subset X_1$ and the embedding is compact and a mapping $B\colon\ X_1 \to X_1$ which is continuous (with respect to the norm of $X_1$) and bounded on balls in the Hilbert space $X_1$ such that $B_n = \langle l_n, B\rangle$. In this case, there exists a probability measure $\mu$ on $X_0$ such that equation (3.1) is satisfied with respect to the class $C_0^2(X_1)$. The proof is the same as above taking into account that, for any $\varphi \in C_0^2(X_1)$, the function $\sum_{n=1}^{\infty} \partial_{e_n}^2 \varphi$ is bounded continuous on $X_1$, since $|\partial_{e_n}^2 \varphi| \le \text{const}\, |e_n|_0^2 = \text{const}\, q_n^2$.

We observe that the result in Theorem 5.3 could be equivalently reformulated in terms of a single Hilbert space $Z = X_0$ as follows.

**Theorem 5.7.** *Let $Z$ be a separable Hilbert space with an orthonormal basis $\{\eta_n\}$ and let $B_n\colon\ Z \to \mathbb{R}^1$ be continuous on balls with respect to the weak topology. Let $P_n$ be the orthogonal projection onto the linear span $E_n$ of $\eta_1, \dots, \eta_n$. Suppose that there exist constants $C$, $C_n$, $K_n$, $d_n$ and a locally bounded nonnegative Borel function $\varrho$ on $[0, +\infty)$ such that $\lim_{R\to+\infty} \varrho(R) = +\infty$ and for all $n \in \mathbb{N}$*

$$\sup_{|P_n x|_Z \ge R} \Bigl(B_1(P_n x)\eta_1 + \ldots + B_n(P_n x)\eta_n,\, P_n x\Bigr)_Z \le C - \varrho(R), \tag{5.15}$$

$$|B_n(x)| \le C_n + K_n |x|_Z^{d_n}. \tag{5.16}$$

*If* $t_n > 0$ *and* $\sum\limits_{n=1}^{\infty} t_n^2 < \infty$, *then there exists a probability measure* $\mu$ *on* $Z$ *such that*

$$\sum_{n=1}^{\infty} \int_Z \left[ t_n^2 \partial_{\eta_n}^2 f + B_n \partial_{\eta_n} f \right] d\mu = 0, \qquad \forall\, f \in \mathscr{F}\mathscr{C}_b^\infty(Z, \{\eta_n\}). \tag{5.17}$$

*If* $\sum_{n=1}^{\infty} \|B_n\|^2_{L^2(\mu)} < \infty$, *then* (5.17) *is true for all* $f \in \mathscr{F}\mathscr{C}_b^\infty(Z)$.

*Proof.* This theorem follows from Theorem 5.3 by considering the natural embedding of $Z$ into $Z$ with the weak topology and setting $q_n = t_n^2$, $e_n = t_n \eta_n$, $l_n(x) = t_n^{-1}(x, \eta_n)_Z$, $\widehat{B}_n = t_n^{-1} B_n$. Then $l_n(e_k) = \delta_{nk}$ and

$$\sum_{i=1}^{n} q_i l_i(x) \widehat{B}_i(x) = \sum_{i=1}^{n} (x, \eta_i)_Z B_i(x), \qquad \forall\, x \in E_n.$$

In addition, $\partial_{e_n}^2 f + \widehat{B}_n \partial_{e_n} = t_n^2 \partial_{\eta_n}^2 f + B_n \partial_{\eta_n}$, so that we are in the situation of Theorem 5.3. □

This result is an extension of [15, Remark 5.4], where it was assumed that $B_n = (B, \eta_n)_Z$ for some Borel mapping $B$ on $Z$, and the estimate $\lim\limits_{R\to+\infty} \sup\limits_{|x|>R} \big(B(x), x\big)_Z = -\infty$ was required instead of (5.15). However, that estimate was used in the form of (5.15) and the existence of $B$ was never used. Moreover, the reasoning in [15] was exactly the same as the one above. We have not been able to prove the statement of [15, Theorem 5.2] for $B$ merely continuous.

Let us apply our existence result to the situation in Example 4.6.

**Theorem 5.8.** *Let* $X = \mathbb{R}^S$, *let* $S$ *be a countable set, and let* $X_0$ *be the weighted Banach space of sequences such that*

$$|x|_0 = \Big( \sum_{s\in S} q_s |x_s|^\alpha \Big)^{1/\alpha},$$

*where* $\alpha \ge 2$. *Suppose that* $J$ *and* $q$ *satisfy the same conditions as in Example* 4.6. *Let* $B = (B_s)_{s\in S}$ *be a collection of continuous functions on* $(X_0, |\cdot|_0)$ *satisfying* (4.5) *and* (5.12). *Then there exists a probability measure* $\mu$ *on* $X_0$ *such that* (3.1) *is satisfied with respect to* $\mathscr{F}\mathscr{C}_b^\infty(X)$.

*Proof.* We may assume that $S = \mathbb{N}$. As in the proof of Theorem 5.3, we find probability measures $\mu_n$ on the $n$-dimensional linear subspaces $\mathbb{R}^n \subset \mathbb{R}^{\mathbb{N}}$ satisfying the elliptic equations $L_n^* \mu_n = 0$, where

$$L_n \psi(x_1, \dots, x_n) = \Delta\psi(x_1, \dots, x_n) + \sum_{i=1}^{n} B_i(x_1, \dots, x_n, 0, 0, \dots) \partial_{x_i} \psi(x_1, \dots, x_n).$$

It follows from Example 4.6 that

$$\sup_{n,j} \int |x_j|^p \, \mu_n(dx) < \infty, \qquad \forall\, p \in \mathbb{N}.$$

We can find $\widetilde{q}_n > 0$ such that $\sum_{n=1}^{\infty} \widetilde{q}_n < \infty$ and $\lim\limits_{n\to\infty} q_n/\widetilde{q}_n = 0$. Therefore, by considering the function $\Psi := \sum\limits_{n=1}^{\infty} \widetilde{q}_n |x_n|^{\alpha}$ and using that $\sup\limits_{n} \int \Psi \, d\mu_n < \infty$, we see that the sequence $\{\mu_n\}$ is uniformly tight on $X_0$. Let us take for $l_n$ the natural coordinate functions on $\mathbb{R}^S$. Now the same reasoning as in Theorem 5.3 completes the proof. $\square$

It is clear from the above proofs that by [16] our results extend to the case of a non-constant diffusion term. Let us give the exact formulation.

**Theorem 5.4.** *Let $X$, $\{q_n\}$ and $X_0$ be the same as in Theorem* 5.3. *Let $B_n$ and $A_{ij}$ be functions on $X_0$ which are continuous on balls in $X_0$ with respect to the topology of $X$ and satisfy the estimates*

$$|B_n(x)| \le C_n + K_n |x|_0^{d_n}, \quad \forall x \in X_0, \, \forall\, n \in \mathbb{N}, \qquad \sup_{x,n,j} |A_{nj}(x)| < \infty.$$

*Assume that $(A_{ij}|_{E_n})_{i,j\le n}$ is nonnegative definite for all $n \in \mathbb{N}$, where $E_n$ is the linear span of $e_1, \ldots, e_n$. Suppose, furthermore, that*

$$\sum_{n=1}^{\infty} q_n l_n(x) B_n(x) \le C - \varrho(|x|_0)$$

*on the linear span of the $e_n$'s, where $\varrho$ is a nonnegative function on* $[0, +\infty)$ *such that $\lim\limits_{t\to\infty} \varrho(t) = +\infty$. Then there exists a probability measure $\mu$ on $X_0$ satisfying equation* (4.13) *with respect to* $\mathcal{F}\mathcal{C}_b^{\infty}(X_0, \{l_n\})$.

The proof is the same as in Theorem 5.3 taking into account Example 4.14.

## 6. The symmetric case

In this section, we discuss the so-called symmetric (Gibbsian) case, i.e., the situation where the functions $B_n$ are logarithmic derivatives of the measure $\mu$ that satisfies (3.1), so that every term in (3.2) vanishes separately. If $\mathcal{K} = \mathcal{F}\mathcal{C}_b^{\infty}(X)$ and $B_n \in L^2(\mu)$, then this is equivalent to the symmetry of $L_B$ on $\mathcal{F}\mathcal{C}_b^{\infty}(X)$ (see [15] and Proposition 8.6 below).

Let $(E, \mathcal{E})$ and $(Y, \mathcal{F})$ be two measurable spaces and let $\mu$ be a measure on $\mathcal{B} =: \mathcal{E} \otimes \mathcal{F}$ such that the projection of $|\mu|$ to $Y$ is $\nu$. We recall that measures $\mu^y$ on the sets $E \times \{y\}$, $y \in Y$, equipped with the trace $\sigma$-fields generated by $\mathcal{B}$, are called regular conditional measures if the sets $E \times \{y\}$ belong to $\mathcal{E} \otimes \mathcal{F}$ (i.e., $\mathcal{F}$ contains all single point sets in $Y$), for every $B \in \mathcal{E} \otimes \mathcal{F}$ the function $\mu^y(B) = \mu^y(B \cap E \times \{y\})$ is $\nu$-measurable and

$$\mu(B) = \int_Y \mu^y(B) \, \nu(dy).$$

It is well known that regular conditional measures exist under very broad assumptions (e.g., if $E$ and $Y$ are Souslin spaces with their Borel $\sigma$-fields).

The relation of Gibbs measures to elliptic equations is seen from the following simple example. Suppose that $\mu$ is a probability measure on $\mathbb{R}^2$ with a smooth positive density $f$. It is easily seen that the projections of $\mu$ on the first and second coordinate axis have densities $f_1(x) = \int f(x, y)\,dy$ and $f_2(y) = \int f(x, y)\,dx$, respectively. Hence the conditional measures $\mu^x$ on the lines $\{x\}\times\mathbb{R}^1$ have densities $f^x(y) = f(x, y)/f_1(x)$ and similarly for the conditional measures $\mu^y$ on the lines $\mathbb{R}^1 \times \{y\}$. Suppose we want to reconstruct $\mu$ from $\mu^x$ and $\mu^y$. Of course, in our trivial example one can find $f_1$ knowing $f^x$ and $f^y$, but we shall discuss an approach which works also in infinite dimensions. Namely, we can find the partial logarithmic derivatives $\beta_1(x, y) = \partial_x f^y(x)/f^y(x)$ and $\beta_2(x, y) = \partial_y f^x(y)/f^x(y)$. From the above expressions we find that $(\beta_1, \beta_2) = \nabla f/f$. Therefore, we have to find a probability measure $\mu$ with the given logarithmic gradient $\beta$. One can show that this is equivalent to finding a probability measure $\mu$ such that it satisfies the equation $L^*_{I,\beta}\mu = 0$ (the equation is verified through integration by parts) and, in addition, $L_{I,\beta}$ with domain $C_0^\infty$ is symmetric on $L^2(\mu)$. Thus, the initial problem is replaced by the following two problems: solving an elliptic equation and distinguishing its symmetric solutions. The situation is similar in infinite dimensions. This is why the method of Lyapunov functions comes naturally into play. In some examples Lyapunov functions can be used directly without involving the elliptic equation, but the equation is helpful in order to find appropriate Lyapunov functions. We shall see this in the examples below.

We shall first discuss relations between the integration by parts formula and existence of differentiable conditional measures. The next lemma is a straightforward modification of a result in [50], where it was proved for globally integrable logarithmic derivatives. Later special cases of that result were derived in the context of “the integration by parts characterization of Gibbs measures” (cf. [46]). For the reader’s convenience and due to some additional technicalities we include a complete proof.

**Lemma 6.1.** *Let $X = \mathbb{R}^n \times Y$, where $(Y, \mathscr{F})$ is a measurable space, let $\mu$ be a signed measure of finite total variation on $\mathscr{B} = \mathscr{B}(\mathbb{R}^n) \otimes \mathscr{F}$ with regular conditional measures $\mu^y$ on $\mathbb{R}^n \times \{y\}$, and let $\nu$ be the projection of $|\mu|$ to $Y$. Suppose that $\mathscr{K}$ is a class of bounded $\mathscr{B}$-measurable functions that satisfies the following conditions:*

(i) *for every $\psi \in \mathscr{K}$ and $y \in Y$, the function $x \mapsto \psi(x, y)$ is continuously differentiable and $\nabla_x\psi$ is bounded;*

(ii) *$(x, y) \mapsto \psi(x + v, y) \in \mathscr{K}$ and $\varphi \circ \psi \in \mathscr{K}$ whenever $\psi \in \mathscr{K}$, $v \in \mathbb{R}^n$, $\varphi \in C_0^\infty(\mathbb{R}^1)$, $\varphi(0) = 0$, and $\psi_1\psi_2 \in \mathscr{K}$ if $\psi_1$ and $\psi_2$ are in $\mathscr{K}$;*

(iii) *the class $\mathscr{K}$ separates the measures on $\mathscr{B}$.*

*Let $\beta\colon X \to \mathbb{R}^n$ be a $\mu$-measurable mapping such that, for every $\psi \in \mathscr{K}$ and $v \in \mathbb{R}^n$, one has $\psi|\beta| \in L^1(\mu)$ and*

$$\int_X (\nabla_x\psi, v)\,d\mu = -\int_X \psi(\beta, v)\,d\mu. \tag{6.1}$$

*Then, for $\nu$-a.e. $y$, $\mu^y$ admits a density $f^y$ on the fibre $\mathbb{R}^n \times \{y\}$ such that*

$$f^y \in W^{1,1}_{loc}(\mathbb{R}^n) \quad \textit{and} \quad \beta(x,y) = \nabla_x f^y(x)/f^y(x) \ \ \mu^y\text{-}a.e. \tag{6.2}$$

*Proof.* We can find a sequence of measurable sets $A_j \subset X$ such that $\bigcup_{j=1}^{\infty} A_j$ has full measure and there exist functions $\varphi_j \in \mathcal{K}$ with $\varphi_j > 0$ on $A_j$. Indeed, let $\mathcal{K}_0 = \{\psi \in \mathcal{K} : 0 \le \psi \le 1\}$. By [30, Theorem IV.11.6], there is a sequence $\varphi_j \in \mathcal{K}_0$ such that, for every $\psi \in \mathcal{K}_0$, one has $\psi \le \sup_j \varphi_j$ $\mu$-a.e. Then the union of the sets $A_j = \{\varphi_j > 0\}$ has full measure. Indeed, if $\sup_j \varphi_j = 0$ on a positive measure set $A$, then for every $\varphi \in \mathcal{K}_0$, one has $\varphi = 0$ $\mu$-a.e. on $A$, hence the same is true for every $\varphi \in \mathcal{K}$, which easily follows by taking compositions with smooth compactly supported functions vanishing at the origin. Thus, the measure $\mu|_A$ and the zero measure are not separated by $\mathcal{K}$, which is a contradiction. Moreover, we may assume that $\varphi_j = 1$ on $A_j$. To this end, one can replace every function $\varphi_j$ be the sequence of functions $\theta_k \circ \varphi_j$, where $\theta_k \in C_0^\infty(\mathbb{R}^1)$, $0 \le \theta_k \le 1$, $\theta_k(t) = 0$ if $t \le 0$ or $t \ge k+1$ and $\theta_k(t) = 1$ if $k^{-1} \le t \le k$. Then the sets $\{\theta_k \circ \varphi_j = 1\}$ cover the set $\{\varphi_j > 0\}$. Let us consider the measure

$$\mu_j = \varphi_j\, \mu.$$

Let $\beta_j = \beta + \nabla_x \varphi_j/\varphi_j$. Note that (6.1) holds for $\psi = \psi_1\psi_2$ if $\psi_1, \psi_2 \in \mathcal{K}$. Hence we obtain from (6.1) that

$$\int_X (\nabla_x \psi, v)\, d\mu_j = -\int_X \psi(\beta_j, v)\, d\mu_j$$

for every $\psi \in \mathcal{K}$ and every $v \in \mathbb{R}^n$. In addition, $|\beta_j| \in L^1(\mu_j)$, since $|\beta|\varphi_j \in L^1(\mu)$. Let $v \in \mathbb{R}^n$ be a fixed vector and let $t \in \mathbb{R}^1$. Then we have

$$\begin{aligned}\int_X \big[\psi(x+tv,y) - \psi(x,y)\big]\, d\mu_j \\ = -\int_0^t \int_X \psi(x+sv,y)\big(\beta_j(x,y), v(x)\big)\, d\mu_j\, ds\end{aligned} \tag{6.3}$$

for all bounded $\psi \in \mathcal{K}$, which is proved as follows. Both sides of (6.3) are continuously differentiable in $t$ and vanish at $t = 0$. It follows by (6.1) that their derivatives coincide, since $(x,y) \mapsto \psi(x+tv,y) \in \mathcal{K}$ by our hypothesis. The left-hand side of (6.3) equals the integral of $\psi$ with respect to the measure $(\mu_j)_t - \mu_j$, where $(\mu_j)_t$ is the image of $\mu_j$ under the shift $(x,y) \mapsto (x+tv,y)$. The right-hand side of (6.3) is the integral of $\psi$ against the measure

$$\sigma_j^t := \int_0^t \Big((\beta_j, v)\mu_j\Big)_s\, ds.$$

Hence, by our assumption on $\mathcal{K}$, we have

$$(\mu_j)_t - \mu_j = \sigma_j^t. \tag{6.4}$$

This implies that (6.3) holds for all bounded $\mathscr{B}$-measurable functions $\psi$. We set

$$\mu_j^y = \varphi_j \mu^y, \quad \text{i.e.} \quad \mu_j(B) = \int_Y \mu_j^y(B)\,\nu(dy).$$

Now (6.4) yields the absolute continuity of the measures $\mu_j^y$ for $\nu$-a.e. $y$. Indeed, let $p$ be a probability density on $\mathbb{R}^n$ with support in the unit ball $U$, $p_\varepsilon(t) = \varepsilon^{-n} p(t/\varepsilon)$, $\gamma_\varepsilon = p_\varepsilon\,dx$, $\varepsilon \in (0, 1)$, and let

$$\pi_\varepsilon(B) = \int_Y \mu_j^y * \gamma_\varepsilon(B)\,\nu(dy).$$

Then, for every bounded Borel function $g$, one has

$$\begin{aligned}\int_X g(x, y)\,d\pi_\varepsilon &= \int_Y \int_{\mathbb{R}^n\times\{y\}} \int_{\mathbb{R}^n} g(x + \varepsilon z, y) p(z)\,dz\,\mu_j^y(dx)\,\nu(dy)\\ &= \int_{\mathbb{R}^n} \int_X g(x + \varepsilon z, y) p(z)\,d\mu_j\,dz.\end{aligned} \tag{6.5}$$

It follows from (6.4) and (6.5) that

$$\begin{aligned}&\Big|\int_X g\,d\mu_j - \int_X g\,d\pi_\varepsilon\Big|\\ &\quad= \Big|\int_U \int_X g\left[d(\mu_j) - d(\mu_j)_{\varepsilon z}\right] p(z)\,dz\Big|\\ &\quad= \Big|\int_U \int_0^\varepsilon \int_X g(x + sz, y)(\beta_j, z)\,d\mu_j\,ds\;p(z)\,dz\Big| \le \varepsilon \sup|g|\;\|\beta_j\|_{L^1(\mu_j,\mathbb{R}^n)},\end{aligned}$$

since $|(\beta_j, z)| \le |\beta_j|$ on the support of $p$. Therefore,

$$\|\mu_j - \pi_\varepsilon\| \le 2\varepsilon\|\beta_j\|_{L^1(\mu_j,\mathbb{R}^n)}.$$

Clearly, every measure $\pi_\varepsilon$ with $\varepsilon > 0$ has absolutely continuous conditional measures on $\mathbb{R}^n \times \{y\}$. Hence, for $\nu$-a.e. $y$, the conditional measure $\mu_j^y$ admits a density $q_j^y(x)$ with respect to Lebesgue measure. Thus, we obtain from (6.4) that there exists a measurable set $Y_0$ of full $\nu$-measure such that, for every $i = 1, \ldots, n$, every rational $t$, and every $y \in Y_0$, one has for a.e. $x$

$$q_j^y(x + te_i) - q_j^y(x) = \int_0^t \Big[(\beta_j, e_i)\,q_j^y\Big](x + se_i)\,ds.$$

Therefore, for every $y \in Y_0$, we obtain $q_j^y \in W^{1,1}_{loc}(\mathbb{R}^n)$ and $\nabla_x q_j^y(x)/q_j^y(x) = \beta_j(x, y)$.

Recall that $\varphi_j\,\mu^y = \mu_j^y$ for $\nu$-a.e. $y$. Since the union of the sets $A_j$ has full $\mu$-measure, the set $(\mathbb{R}^n \times \{y\}) \cap (\bigcup_{j=1}^\infty A_j)$ has full $\mu^y$-measure for $\nu$-a.e. $y$. Therefore, for $\nu$-a.e. $y$, the measure $\mu^y$ admits a density $f^y$ such that $\varphi_j(x,y) f^y(x) = q_j^y(x)$ for every $j$ and a.e. $x$. In addition, $\nabla_x \varphi_j = 0$ $\mu$-a.e. on $A_j$, since the derivative of any differentiable function $F$ on $\mathbb{R}^n$ vanishes almost everywhere on the set $\{F = 1\}$. Hence we obtain a set $Y_1$ of full $\nu$-measure such that, for every $i = 1, \dots, n$, every rational $t$, and every $y \in Y_1$, one has for a.e. $x$

$$f^y(x + te_i) - f^y(x) = \int_0^t \Big[ (\beta, e_i)\, f^y \Big](x + se_i)\, ds,$$

which implies that $f^y \in W^{1,1}_{loc}(\mathbb{R}^n)$ and $\nabla_x f^y(x)/f^y(x) = \beta(x,y)$ $\mu^y$-a.e. $\square$

**Corollary 6.2.** *Let $\mu$ be as in Lemma* 6.1 *and let $f$ be a $\mu$-measurable function such that, for every $y \in Y$, the function $x \mapsto f(x,y)$ is in $W^{1,1}_{loc}(\mathbb{R}^n)$. Suppose that $|\nabla_x f|,\ f|\beta| \in L^1(\mu)$. Then*

$$\int_X (\nabla_x f, v)\, d\mu = -\int_X f(v, \beta)\, d\mu, \qquad \forall\, v \in \mathbb{R}^n. \tag{6.6}$$

*The same is true if instead of $x \mapsto f(x,y) \in W^{1,1}_{loc}(\mathbb{R}^n)$, one has that the usual partial derivatives $\partial_{x_i} f(x,y)$ exist for every $x$.*

*Proof.* It suffices to prove (6.6) for every $v = e_i$, where $\{e_i\}$ is the standard basis in $\mathbb{R}^n$. By Lemma 6.1 and our integrability assumptions, it is enough to show that, for all $p,\ g \in W^{1,1}_{loc}(\mathbb{R}^n)$ such that $p, p\partial_{e_i} g, g\partial_{e_i} p \in L^1(\mathbb{R}^n)$, one has

$$\int_X \partial_{e_i} g\, p\, dx = -\int_X g \partial_{x_i} p\, dx. \tag{6.7}$$

We may assume that $i = 1$. It is known that the functions $p$ and $g$ admit versions, denoted by the same letters, such that $t \mapsto p(t, x_2, \dots, x_n)$ and $t \mapsto g(t, x_2, \dots, x_n)$ are locally absolutely continuous and their partial derivatives represent the generalized partial derivatives $\partial_{e_1} p$ and $\partial_{e_1} g$. Therefore, by Fubini's theorem, (6.7) reduces to the one dimensional case. If $g$ is bounded, then the desired relation follows by the integration by parts formula, since there exist $a_j \to -\infty$ and $b_j \to +\infty$ such that $pg(b_j) \to 0$, $pg(b_j) \to 0$. The case of unbounded $g$ follows by considering the compositions $\theta_j(g)$, where $\theta_j \in C_b^\infty(\mathbb{R}^1)$, $\theta_j(t) = t$ if $|t| \le j$, $\theta_j(t) = j \operatorname{sign} t$ if $|t| \ge j + 1$, and $\sup |\theta_j'| \le 2$. A justification in the case where $g$ is differentiable everywhere, but is not locally absolutely continuous can be found in [19, Theorem 2.6]. $\square$

*Remark 6.3.* (i) It is clear from the above proof that the separation assumption (iii) on $\mathscr{K}$ can be weakened; e.g., it would be enough to replace it by the following condition:

(iii$'$) there exists a measurable set $\Omega \subset X$ of full measure with respect to all shifts $(\mu)_t$ generated by the vectors $v$ as above such that $\mathscr{K}$ separates the measures on the set $\Omega$.

In particular, this is the case when $\Omega$ has full $\mu$-measure and is mapped by the shifts $(x, y) \mapsto (x + v, y)$ into itself.

(ii) The requirement $\varphi \circ \psi \in \mathscr{K}$ for all $\psi \in \mathscr{K}$ and $\varphi \in C_0^\infty(\mathbb{R}^1)$ with $\varphi(0) = 0$ in condition (ii) can be replaced by the following assumption:

there exist functions $\psi_j \in \mathscr{K}$ such that the sets $\{\psi_j = 1\}$ cover $\mathbb{R}^n \times Y$ up to a $\mu$-measure zero set.

Finally, note that if $\mathscr{K}$ is a linear space of bounded functions such that it is stable under compositions with $C_0^\infty$-functions vanishing at 0, then $\psi_1\psi_2 \in \mathscr{K}$ for all $\psi_1,\ \psi_2 \in \mathscr{K}$.

Note that the class of Lipschitzian functions with bounded supports on a separable Banach space $X_0$ separates the Borel measures on $X_0$. If $X_0$ is reflexive, then $X_0$ has nontrivial Lipschitzian continuously Fréchet differentiable functions with bounded supports (see, e.g., [29, Ch. I, 2.1 and 3.1]). It is readily seen that in this case, the class $C_0^1(X_0)$ separates the Borel measures on $X_0$. Therefore, we get the following application of Lemma 6.1.

**Example 6.4.** Let $\mu$ be a signed measure on a locally convex space $X$ of finite total variation. Let $(X_0, \|\cdot\|_0)$ be a continuously embedded separable Banach space of full $\mu$-measure. Let $h \in X_0$. Suppose that there exists a $\mu$-measurable function $\beta$ such that $\beta$ is $\mu$-integrable on all $\|\cdot\|_0$-bounded sets and

$$\int\limits_{X_0} \partial_h \psi \, d\mu = -\int\limits_{X_0} \psi\beta \, d\mu \tag{6.8}$$

for every function $\psi$ from the class $Lip_0(X_0)$ of all Lipschitzian functions with bounded supports on $X_0$. Let $X = \mathbb{R}^1 h \otimes Y$, where $Y$ is a closed linear subspace. Then $\mu$ has regular conditional measures $\mu^y$ on the line $y + \mathbb{R}^1 h$, $y \in Y$, that have locally absolutely continuous densities $f^y$ with $\big(f^y(t)\big)'/f^y(t) = \beta(y + th)$. If $X_0$ is reflexive, then the same is true for $C_0^1(X_0)$ in place of $Lip_0(X_0)$. Finally, if $X_0$ is Hilbert, then $C_0^1(X_0)$ can be replaced by $C_0^\infty(X_0)$.

Let us observe that one can take even smaller classes of test functions in the above example. Namely, there is a compactly embedded separable reflexive Banach space $X_1 \subset X_0$ of full $\mu$-measure. Then the class $C_0^1(X_1)$ separates the Borel measures on $X_1$ and can be used in the above example in place of $Lip_0(X_0)$. If $X_0$ is Hilbert, then one can choose for $X_1$ also a Hilbert space. This enables us to weaken the integrability assumption on $\beta$ by requiring the integrability of $\beta$ only on compact sets in $X_0$.

*Remark 6.5.* It is often of interest to know that the conditional distributions have strictly positive continuous densities. Sufficient conditions for this can be expressed in terms of logarithmic derivatives. Let $\mu$ be a locally finite nonnegative measure on $\mathbb{R}^1$ with a locally absolutely continuous density $\varrho$. Then $\varrho > 0$ if and only if

$\beta^\mu := \varrho'/\varrho$ is locally integrable with respect to Lebesgue measure. The necessity is obvious, and the sufficiency is readily verified by showing the continuity of $\log \varrho$. If $\mu$ is a locally finite measure on $\mathbb{R}^n$ with a density $\varrho \in W^{1,1}_{loc}(\mathbb{R}^n)$, then a sufficient condition for the existence of a positive continuous modification of $\varrho$ is the local integrability of $|\beta^\mu|^p = |\nabla\varrho/\varrho|^p$, where $p > n$, with respect to Lebesgue measure. Another sufficient condition is the following: every point $x$ has a neighborhood $U$ such that $\exp\bigl(\varepsilon|\beta^\mu|\bigr)$ is $\mu$-integrable on $U$ for some $\varepsilon > 0$ (see [13, Proposition 2.18], where the proof is given in the global case but works locally as well). Although, the latter condition is stronger than the previous one, its advantage is that it is expressed entirely in terms of $\mu$ without reference to Lebesgue measure.

We now proceed to analogues of the existence results from the previous section in the symmetric case. The method of proof is exactly the same. The idea to construct measures with given logarithmic derivatives by the method of Lyapunov functions employing the dissipativity condition is already in [38], [39]. Later the same method was used and further developed in [14], [15], [5], [6]. The first application of this method to construct Gibbs measures has been given in [5], [6]. We shall see below how the reasoning in [15] enables one to obtain even stronger results on the existence of Gibbs measures.

We recall that $\{l_n\} \subset X^*$ is a separating sequence, $\{e_n\} \subset X$, and $l_n(e_k) = \delta_{nk}$.

**Theorem 6.6.** *Let $X, q = \{q_n\} \in l^1$ and $X_0$ be the same as in Theorem* 5.3 *and let $B_n\colon\ X_0 \to \mathbb{R}^1$ be continuous on balls in $X_0$ with respect to the topology of $X$ such that* (5.13) *is satisfied. Assume, in addition, that there exist continuously differentiable functions $G_n$ on $E_n :=$ linear span of $\{e_1, \dots e_n\}$ such that $B_i = \partial_{e_i} G_n$ on $E_n$ for every $i \le n$. Then there exists a probability measure $\mu$ on $X_0$ such that, for every $n$, the function $B_n$ is the logarithmic derivative of $\mu$ along $e_n$ with respect to $Lip_0(X_0)$. If* (5.12) *is fulfilled, then $\mu$ is differentiable along every $e_n$ with respect to $\mathscr{F}\mathscr{C}_b^\infty(X)$.*

*Proof.* Let

$$H := \Bigl\{x \in X_0\colon\ |x|_H^2 = \sum_{n=1}^{\infty} l_n(x)^2 < \infty\Bigr\}.$$

For any differentiable function $\psi$ on $E_n$, we set

$$D_H\psi = \partial_{e_1}\psi\, e_i + \dots + \partial_{e_n}\psi\, e_n.$$

In the gradient case, the measures $\mu_n$ on $E_n$ constructed in the proof of Theorem 5.3 are given by the following explicit densities $f_n$ with respect to the Lebesgue measures on $E_n$ associated with the norm of $H$: $f_n = z_n \exp(G_n)$, where $z_n$ is the normalization constant. The integrability of $f_n$ is obvious, since by (5.13) there exists $t_0 > 0$ such that for every $x$ in the unit sphere of $E_n$, one has

$$\frac{d}{dt}G_n(tx) = \frac{1}{t}\bigl(D_H G_n(tx), tx\bigr)_H \le \frac{C}{t} - \frac{1}{t}\varrho(t|x|_0) \le \frac{-2n}{t} \quad \text{for all } t \ge t_0$$

by the equivalence of the norms $|\cdot|_0$ and $|\cdot|_H$, whence $f_n(tx) \le const.|t|^{-2n}$, $t \ge t_0$. Looking at $f_n$ in polar coordinates, it is clear that it is integrable. Let us observe that we may assume that $X = \mathbb{R}^\infty$. Indeed, $X$ can be injected into $\mathbb{R}^\infty$ by means of the sequence $\{l_n\}$ separating the points in $X$. The balls from $X_0$ are compact in $\mathbb{R}^\infty$ and the topology of $X$ on them coincides with the one from $\mathbb{R}^\infty$ (i.e., the one generated by the functionals $l_n$). Assuming that $X = \mathbb{R}^\infty$, we can choose a bigger reflexive separable Banach space $X_1 \supset X_0$ such that both embeddings $X_0 \to X_1$ and $X_1 \to X$ are compact (see, e.g., the proof in [10, Theorem 3.6.5]). In fact, every completely metrizable locally convex space $X$ has such a property (but not an arbitrary locally convex space; this is the point to consider the injection into $\mathbb{R}^\infty$). Then $\mu_n \to \mu$ weakly also on $X_1$. Since $D_H f_n/f_n = (B_1, \dots, B_n)$, we obtain by the integration by parts formula for every $\psi \in C_0^1(X_1)$ and $i \le n$ that

$$\int_{X_1} \partial_{e_i}\psi \, d\mu_n = -\int_{X_1} \psi \, B_i \, d\mu_n.$$

The same is true for $\mu$ in place of $\mu_n$, since $|B_i|$ is bounded and continuous on the support of $\psi$ (which is compact in $X$) and $\mu_n \to \mu$ weakly on $X_1$. It now follows by Example 6.4 and Corollary 6.2 that

$$\int_X \partial_{e_i}\psi \, d\mu = -\int_X \psi \, B_i \, d\mu \tag{6.9}$$

for every Borel function $\psi$ such that $\partial_{e_i}\psi$ exists and $\partial_{e_i}\psi$, $\psi B_i \in L^1(\mu)$. In particular, this is true if $\psi \in Lip_0(X_0)$. As shown above, $|x|_0 \in L^p(\mu)$ for all $p \in [1, \infty)$. Therefore, in the case where $|B_n|$ is majorized by $C_n + K_n|x|_0^{d_n}$, one has $B_n \in L^1(\mu)$, hence $B_n$ is the logarithmic derivative also with respect to $\mathscr{F}\mathscr{C}_b^\infty(X)$ (and with respect to $\mathscr{F}\mathscr{C}_b^\infty(X_0)$). □

As in the non symmetric case, the above result can be reformulated in terms of a single Hilbert space. We only give the formulation, since its relation to Theorem 6.6 is the same as the relation of Theorem 5.7 to Theorem 5.3.

**Theorem 6.7.** *Let $Z$, $B_n$, $E_n$ and $t_n$ be the same as in Theorem* 5.7, *but instead of condition* (5.16) *we shall assume that there exist continuously differentiable functions $G_n$ on $E_n$ such that, letting $e_n = t_n u_n$, one has for all $n$*

$$B_i = \partial_{e_i} G_n, \qquad \forall\, i \le n, \ \forall\, x \in E_n. \tag{6.10}$$

*Then there exists a probability measure $\mu$ on $Z$ such that $B_n$ is the logarithmic derivative of $\mu$ along $t_n e_n = t_n^2 u_n$ with respect to $Lip_0(Z)$. If, in addition,* (5.16) *holds true, then $B_n$ is the logarithmic derivative of $\mu$ along $t_n e_n = t_n^2 u_n$ with respect to $\mathscr{F}\mathscr{C}_b^\infty(Z)$.*

*Remark 6.8.* Condition $\lim\limits_{R\to+\infty} \varrho(R) = +\infty$ required in Theorem 6.6 and Theorem 6.7 to ensure the differentiability of $\mu$ with respect to $Lip_0(X_0)$ and $Lip_0(Z)$, respectively, can be replaced by the weaker condition

$$\liminf_{R\to+\infty} \varrho(R) > C + \sum_{n=1}^\infty q_n, \tag{6.11}$$

where $q_n = t_n^2$ in the case of Theorem 6.7, provided that, for every $n$, the mapping $(B_1, \dots, B_n)$ on $E_n$ is the logarithmic gradient of some probability measure $\mu_n$ on $E_n$ with respect to the inner product $(u, v)_H = \sum_{i=1}^{n} t_i^{-2}(u, u_i)_E (v, u_i)_E$ on $E_n$. Indeed, only for this purpose the existence of the functions $G_n$ was used, whereas the argument showing tightness remains valid by Example 4.8.

*Remark 6.9.* (i) We observe that the above results improve [6, Theorem 2.3], where more restrictive assumptions were made about $B_n$. First, we do not require any bound on the growth of $B_n$. Secondly, in [6, Theorem 2.3], it is required that there exist continuous logarithmic gradients $\beta_n = \sum_{m=1}^{n} b_m^n e_m$ on $E_n$ such that for each $m$ and $\varepsilon > 0$ there exists $N$ with $|B_m - b_m^n \circ P_n| \le \varepsilon(1 + |x|_0^M)$ for all $x \in X_0$ and $n \ge N$. Clearly, this implies the weak continuity on balls in $X_0$, since the functions $b_m^n \circ P_n$ are continuous cylindrical and converge uniformly on balls in $X_0$. In addition, this yields that the restriction of $(B_1, \dots, B_n)$ to $E_n$ is a continuous logarithmic gradient (with respect to the $H$-norm as in Remark 6.8). To see this, we observe that if $\{F_j\}$ is a sequence of continuously differentiable functions on $\mathbb{R}^n$ such that the gradients $\nabla F_j$ converge uniformly on every ball to a continuous mapping $\Psi$, then there exists $G \in C^1(\mathbb{R}^n)$ such that $\Psi = \nabla G$. Indeed, in this case the sequence $\widehat{F}_j(x) = F_j(x) - F_j(0)$ converges uniformly on every ball (by the uniform convergence of $\nabla\widehat{F}_j$ and the convergence at the origin), hence we can take $G = \lim \widehat{F}_j$.

(ii) It should be noted that Theorem 6.7 as well as its predecessor [6, Theorem 2.3] can be deduced also from the constructions in [38], [39], although they are not formal corollaries of the corresponding results for two reasons. First there are some extra technical assumptions used in [38], [39]. But these were only used for the finite dimensional estimates, and were removed in [15], [16]. Second, in [38], [39] global assumptions on $B$ were imposed a priori. As mentioned in the above introduction they were only removed in the recent work [6] by considering test functions with bounded support (as $C_0^2(X_0)$ above). This step was essential to include applications to Gibbs measures. Nevertheless, both here and in [6, Theorem 2.3], the method of proof of existence is in spirit of that in [38], [39].

*Remark 6.10.* It is clear from Remark 4.11 that if in the situation of Theorem 6.6 condition (5.12) is satisfied, then one can choose a separable Hilbert space $H \subset X_0$ and a separable Hilbert space $Y \supset X_0$ such that the natural embeddings $H \to X_0$, $H \to Y$, $X_0 \to Y$ are continuous and dense, and there exists a mapping $\beta\colon Y \to Y$ such that $\beta$ is the logarithmic gradient of $\mu$ with respect to $H$ and $\mathscr{F}\mathscr{C}_b^\infty(Y)$, $|\beta|_Y \in L^2(\mu)$, and $\langle l_n, B\rangle = B_n$. This is well-known, but we repeat the construction for the convenience of the reader. Let

$$H = \Big\{x\colon\ |x|_H^2 = \sum_{n=1}^{\infty} l_n(x)^2 < \infty\Big\},$$

$$Y = \Big\{x\colon\ |x|_Y^2 = \sum_{n=1}^{\infty} c_n l_n(x)^2 < \infty\Big\},$$

where $c_n = 2^{-n}\min\Bigl(q_n, \|B_n\|^{-2}_{L^2(\mu)}\Bigr)$, and $B(x) = \sum_{n=1}^{\infty} B_n(x)e_n$. Then $\{e_n\}$ is an orthonormal basis in $H$ and $j_H(l_n) = e_n$, since $l_n(e_k) = \delta_{nk}$. Hence $\langle l_n, B\rangle = B_n = \beta^{\mu}_{e_n}$.

Here is an analogue of Theorem 5.8 for the symmetric case. Its proof is analogous to the proof of Theorem 6.6.

**Theorem 6.11.** *Let $X$, $X_0$, $J$, and $q$ be the same as in Example* 4.6 *and let $B = (B_n)_{n\in S}$ be a collection of functions on $X_0$ that are continuous on balls in $X_0$ with respect to the topology from $X$ and satisfy* (4.5). *Suppose that $S$ is a union of increasing finite sets $S_k$, $k \in \mathbb{N}$, and that there exist continuously differentiable functions $G_k$ on $\mathbb{R}^{S_k}$ such that $B_i = \partial_{x_i} G_k$ on $\mathbb{R}^{S_k}$ for all $i \in S_k$. Then there exists a probability measure $\mu$ on $X_0$ such that, for every $n$, the function $B_n$ is the logarithmic derivative of $\mu$ along $e_n$ with respect to $Lip_0(X_0)$.*

Let us consider a class of Gibbs distributions that fits the above framework and which has been analyzed in detail in [5], [6]. However, as we shall see in some cases we can relax the assumptions made there. Let us consider a classical spin system on the lattice $\mathbb{Z}^d$ with the configuration space $\mathbb{R}^S$, $S = \mathbb{Z}^d$, having the formal energy functional

$$E(x) = \sum_{n\in S} V_n(x_n) + \sum_{n,j\in S} W_{n,j}(x_n, x_j),$$

where $W_{n,j}(x_n, x_j) = W_{n,j}(x_j, x_n)$ and $W_{n,n} = 0$. We shall assume that the functions $V_n$ and $W_{n,j}$ are continuously differentiable and satisfy the following estimates:

$$|W_{n,j}(x_n, x_j)| \le J_{n,j}(1 + |x_n|^{\alpha} + |x_j|^{\alpha}), \tag{6.12}$$

$$|\partial_{x_n} W_{n,j}(x_n, x_j)| \le J_{n,j}(1 + |x_n|^{\alpha-1} + |x_j|^{\alpha-1}), \tag{6.13}$$

$$x_n \partial_{x_n} V_n(x_n) \le C - M|x_n|^{\alpha}, \tag{6.14}$$

where $J_{n,j} \ge 0$, $C, M > 0$.

**Example 6.12.** Let (6.12), (6.13), and (6.14) be satisfied, where $\alpha \ge 2$ and $J = (J_{n,j})_{n,j\in S}$ is a symmetric matrix such that there exists a family of positive numbers $q = (q_s)_{s\in S}$ such that $\sum_{s\in S} q_s < \infty$ and $\|J\|_{\mathscr{L}(l^1(q))} \le \lambda < M/3$, $\sum_{n\in S} J_{n,j} \le \lambda$. Then there exists a probability measure $\mu$ concentrated on the space

$$X_0 = \Bigl\{x\colon\ |x|_0 = \Bigl(\sum_{n\in S} q_n|x_n|^{\alpha} < \infty\Bigr)^{1/\alpha}\Bigr\}$$

such that $\mu$ is differentiable along the standard unit vectors $e_n \in \mathbb{R}^S$ with respect to the class $Lip_0(X_0)$ and

$$\beta^{\mu}_{e_n}(x) = \partial_{x_n} V_n(x_n) + \sum_{j\in S} \partial_{x_n} W_{n,j}(x_n, x_j).$$

In particular, the regular conditional measures of $\mu$ on the lines $\mathbb{R}^1 e_n + y$, $y \in \Pi_n := \{x\colon\ x_n = 0\}$, have continuously differentiable densities $p(x_n|x_n^c)$ with

$$\partial_{x_n} p(x_n|x_n^c)/p(x_n|x_n^c) = \beta^{\mu}_{e_n}(x), \qquad x = (x_n, x_n^c),$$

i.e.,

$$p(x_n|x_n^c) = c(x_n^c)\exp\Big[V_n(x_n) + \sum_{j\in S} W_{n,j}(x_n, x_j)\Big], \tag{6.15}$$

where $c(x_n^c)$ is a normalization constant.

*Proof.* The functions

$$B_n(x) = \partial_{x_n} V_n(x_n) + \sum_{j\in S} \partial_{x_n} W_{n,j}(x_n, x_j)$$

are continuous on balls in $X_0$, since the corresponding series converges uniformly on every ball in $X_0$ by (6.13) and our assumption on $J$. There exists $\varepsilon > 0$ such that $M = 3\lambda + \varepsilon$. Note that by the inequality $z \le 1 + z^\alpha/\alpha$ for $z \ge 0$ one has $|x_n|\,|x_j|^{\alpha-1} \le |x_n|^\alpha/\alpha + |x_j|^\alpha$, and consequently $|x_n| + |x_n|^\alpha + |x_n|\,|x_j|^{\alpha-1} \le 1 + 2|x_n|^\alpha + |x_j|^\alpha$. Hence, by the estimate $\sum_{j\in S} J_{n,j} \le \lambda$, we obtain

$$\begin{aligned} x_n B_n(x) &\le C - (3\lambda + \varepsilon)|x_n|^\alpha + \sum_{j\in S} J_{n,j}(1 + 2|x_n|^\alpha + |x_j|^\alpha) \\ &\le C + \lambda - (\lambda + \varepsilon)|x_n|^\alpha + \sum_{j\in S} J_{n,j}|x_j|^\alpha, \end{aligned}$$

i.e., (4.5) holds. Let $S_k = \{s \in S\colon\ |s| \le k\}$, $k \in \mathbb{N}$. It follows from what was said in Remark 6.9 that the restrictions of $(B_i)_{i\in S_k}$ to $\mathbb{R}^{S_k}$ are gradients of continuously differentiable functions. Therefore, Theorem 6.11 applies. Note that the series in (6.15) converges uniformly on balls in $X_0$ by (6.13). □

We observe that Example 6.12 improves Theorem 3.1 from [6], where stronger assumptions were made on $V_n$ and the matrix $J$.

## 7. Applications to SPDEs

Let us apply the above results to the elliptic equations associated with invariant measures for diffusion processes generated by certain stochastic partial differential equations. We shall consider some generalizations or modifications of stochastic Burgers and Navier–Stokes equations. The same techniques apply to reaction-diffusion equations. We shall show how to get the existence of a probability measure solving our elliptic equation by the above method of Lyapunov functions. In order to construct suitable finite dimensional subspaces, we employ usual Galerkin approximations used by many authors in related problems (see, e.g., [49]).

We shall first consider the elliptic equation for invariant measures of the diffusion governed by the following SPDE of the Burgers type:

$$du(t,x)=\sqrt{2}dW^Q(t,x)+\Big[\mathscr{K}u(t,x)-\psi\big(u(t,x)\big)\partial_x u(t,x)+f(x)\Big]dt \qquad (7.1)$$

with zero boundary conditions on [0, 1], where $\mathscr{K}$ is a self-adjoint operator on $X = L^2(0,1)$ with domain $D(\mathscr{K}) \subset H_0^{2,1}(0,1)$ such that its eigenfunctions $\eta_n$ (with eigenvalues $\lambda_n$) are in $H_0^{2,1}(0,1)$ and form an orthonormal basis in $L^2(0,1)$. Suppose that there is $\lambda > 0$ such that

$$\int u\,\mathscr{K}u\,dx \le -\lambda \int (u')^2\,dx, \qquad \forall\, u \in \operatorname{span}\{\eta_n\}. \qquad (7.2)$$

Here (and below) $u'$ denotes derivative with respect to $x \in (0,1)$. We assume that $f \in L^\infty(0,1)$, that $\psi$ is a locally bounded Borel function and that $W_t^Q$ is a "Wiener process with covariance $Q$" in $L^2(0,1)$ or also a cylindrical Wiener process. It is well known that in the case $\mathscr{K} = \Delta$, $\psi(x) = x$, there exists a process $u$ in $L^2(0,1)$ satisfying (7.1) (in the sense of "mild solutions") and having an invariant probability measure $\mu$ (see [24], [25], [27]). However, we make no assumptions concerning the solvability of (7.1). All our assumptions will be specified later. We emphasize that we consider only the elliptic equations for measures and hence deal with the so called infinitesimal invariance of measures, which enables us to weaken the assumptions on the coefficients. By using the results from [48] one can construct Markov processes which satisfy in a certain sense the corresponding stochastic equations. This as well as the exact connection of measures satisfying our elliptic equations and invariant measures of SPDEs will be a subject of a forthcoming paper.

Let us take for $X_0$ the Sobolev space $H_0^{2,1}(0,1)$ of functions $u$ with $u' \in L^2(0,1)$ and $u(0) = u(1) = 0$. This is a Hilbert space with the norm $\|u\|_{X_0} := \|u'\|_2$, compactly embedded into $L^2(0,1)$. Let us set $u_n = (u, \eta_n)_2$, where $(\,\cdot\,,\,\cdot\,)_2$ is the inner product in $L^2(0,1)$, and

$$\begin{aligned} B_n(u) &= \lambda_n u_n - \big(\psi(u)u', \eta_n\big)_2 + (f,\eta_n)_2 \\ &= \lambda_n u_n + \big(\Psi(u), \eta_n'\big)_2 + (f,\eta_n)_2, \qquad \forall\, u \in X_0, \end{aligned} \qquad (7.3)$$

where $f \in L^\infty(0,1)$ and

$$\Psi(y) = \int_0^y \psi(s)\,ds$$

with some locally bounded Borel function $\psi$.

Finally, suppose that $\sum\limits_{n=1}^{\infty} \alpha_n^2 < \infty$. We are going to apply Theorem 5.2 to the operator

$$L_{A,B}\varphi = \sum_{n=1}^{\infty} \alpha_n^2 \partial_{\eta_n}^2 \varphi + \sum_{n=1}^{\infty} B_n \partial_{\eta_n}\varphi$$

on $\mathscr{F}\mathscr{C}_b^\infty(X, \{l_n\})$, where $l_n(u) = u_n$.

This operator arises if we consider the process $W^Q(t) = \sum_{n=1}^{\infty} \alpha_n w_n(t)\eta_n$, where $\{w_n(t)\}$ is a sequence of independent standard real Wiener processes, i.e., $Q\eta_n = \alpha_n^2\eta_n$. This is the so called time white noise case, i.e., $W^Q(t)$ is a Wiener process in $L^2(0,1)$ unlike the case of a space–time white noise discussed below, where $\alpha_n = 1$.

**Proposition 7.1.** *Let $|\Psi(y)| \le c_1 + c_2|y|^d$, where $d < 6$ and let $\eta_n' \in L^\infty(0,1)$ for all $n$. Then there exists a probability measure $\mu$ on $X_0$ which satisfies the equation $L_{A,B}^*\mu = 0$ with respect to $\mathscr{F}\mathscr{C}_b^\infty(X, \{l_n\})$, where $l_n(u) = u_n$. In addition,*

$$\int \|x\|_{X_0}^2\, d\mu < \infty, \qquad \int \|x\|_2^m\, d\mu < \infty,\ \forall\, m \in \mathbb{N}. \tag{7.4}$$

*Proof.* We shall apply Theorem 5.2. Let $A_n(x) = \alpha_n^2/2$, $e_n = \eta_n$, $V(u) = \|u\|_2^2$, and $\Theta(u) = \|u'\|_2^2$. We observe that $B_n$ is continuous on any ball $S$ in $X_0$ with respect to the topology from $X = L^2(0,1)$. Indeed, $u \mapsto u_n$ is continuous. Now let $u^k \to u$ in $X$ be such that $u^k \in S$. Then the functions $u^k$ converge to $u$ uniformly, whence the claim follows. It remains to note that for all $u \in \operatorname{span}\{\eta_n\}$ and any $\varepsilon \in (0,\lambda)$, one has

$$\begin{aligned}\sum_{n=1}^{\infty} B_n(u)u_n &= \sum_{n=1}^{\infty} \lambda_n u_n^2 + \int_0^1 f u\, dx \\ &\le -\lambda\int_0^1 (u')^2\, dx + \int_0^1 f u\, dx \le \frac{1}{4\varepsilon}\|f\|_2^2 - (\lambda-\varepsilon)\|u\|_{X_0}^2.\end{aligned}$$

Indeed, letting $G$ be any primitive of $x\psi(x)$, we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\big(\psi(u)u', \eta_n\big)_2 (u,\eta_n)_2 &= \int_0^1 \psi(u)u'u\, dx \\ &= \int_0^1 [G(u)]'\, dx = G\big(u(1)\big) - G\big(u(0)\big) = 0\end{aligned}$$

for each $u \in X_0$ and, by (7.2), we have for all $u \in \operatorname{span}\{\eta_n\}$

$$\sum_{n=1}^{\infty} \lambda_n u_n^2 \le -\lambda\int_0^1 (u')^2\, dx.$$

For every $n \in \mathbb{N}$, we obtain

$$\begin{aligned}L_n V(u) &= \sum_{j=1}^{n} \alpha_j^2 \partial_{\eta_j}^2 V(u) + \sum_{j=1}^{n} B_j \partial_{\eta_j} V(u) \\ &= 2\sum_{j=1}^{n} \alpha_j^2 + 2\sum_{j=1}^{n} \lambda_j u_j^2 + 2\big(\psi(u)u', u\big)_2 + 2(f,u)_2 \\ &\le 2\sum_{j=1}^{\infty} \alpha_j^2 - 2\lambda(u',u')_2 + 2(f,u)_2 \le C(\varepsilon) - (\lambda-\varepsilon)\|u\|_{X_0}^2,\end{aligned}$$

hence (5.1) holds. Since

$$\int u^6\,dx = \int u^2(x)\biggl(\int_0^x 2uu'\,dt\biggr)^2 dx \le 4\|u\|_2^4\|u'\|_2^2, \qquad u \in X_0,$$

it follows that

$$\int_0^1 |\Psi(u)|\,dx \le c_1+c_2\int_0^1 |u|^d\,dx \le c_1+c_2\|u\|_6^d \le c_1+c_2 4^{d/6}\|u\|_2^{4d/6}\|u'\|_2^{2d/6}.$$

Hence (5.7) holds with $d_n = \max(\frac{d}{3}, 1)$ and $\delta_n(r) = r^{-s}$, where $s = (1-d/6) > 0$. Clearly, (5.8) also holds with $\delta(r) = r^{-1}$. Now the claim follows by Theorem 5.2. □

It is clear that the same reasoning applies also to the elliptic operator associated with the equation

$$du(t,x) = \sqrt{2}dW^Q(t,x) + \Bigl[\partial_x^2 u(t,x) - \psi\bigl(u(t,x)\bigr)\partial_x u(t,x) + P\bigl(u(t,\,\cdot\,),x\bigr)\Bigr]\,dt, \tag{7.5}$$

where the additional nonlinear term $P(u,x)$ has the following properties: $(u,x) \mapsto P(u,x)$ is continuous on $C[0,1]\times[0,1]$ and

$$P(u,x)u(x) \le c_1 + c_2|u(x)|^2, \qquad \int_0^1 |P(u,x)|\,dx \le c_1 + c_3|u|_{L^p}^p$$

for some $c_1, c_2, c_3 \in \mathbb{R}^1$, $c_2 < 1$, and $p \in [1,6)$. Now the functions $B_n$ take the form

$$B_n(u) = -n^2 u_n + \bigl(\Psi(u), \eta_n'\bigr)_2 + \bigl(P(u,\,\cdot\,), \eta_n\bigr)_2, \quad u \in X_0. \tag{7.6}$$

Clearly, these functions are still continuous on the balls in $X_0$ with the topology induced by $X$. We consider the same Lyapunov function $V(u) = \|u\|_2^2$; a minor change in the proof concerns the term $P(u,x)$. As in Proposition 7.1 we apply Theorem 5.2 to obtain a probability measure $\mu$ which satisfies the equation $L_{A,B}^*\mu = 0$ with respect to $\mathscr{F}\mathscr{C}_b^\infty(X, \{l_n\})$.

Let us now consider the elliptic equation associated with the space-time white noise $W^I(t)$, i.e.,

$$W^I(t) = \sum_{n=1}^\infty w_n(t)\eta_n,$$

where $\eta_n(x) = \sqrt{2}\sin\pi nx$ (we recall that $\{\sqrt{2}\sin\pi nx\}$ is a complete orthonormal system in $L^2(0,1)$). Thus, we consider the operator

$$L_{I,B}\varphi = \sum_{j=1}^\infty \bigl[\partial_{\eta_j}^2\varphi + B_j\partial_{\eta_j}\varphi\bigr]$$

on $\mathscr{F}\mathscr{C}_b^\infty(X, \{l_n\})$, where the $B_n$'s are given by (7.3). However, in this case we assume that $Lu = u''$, i.e., $\lambda_n = -n^2$, and that

$$|\Psi(y)| \le c_1 + c_2|y|,$$

where $c_1, c_2 \in \mathbb{R}^1$, and $c_2 < 1$. We have to modify the arguments in the previous example in order to obtain a convergent series of $\partial^2_{\eta_n} V$. To this end, we shall consider the following Lyapunov function: $V(u) = \sum_{n=1}^{\infty} n^{-3/2} u_n^2$. Letting

$$L_n\varphi = \sum_{j=1}^{n} \left[\partial^2_{\eta_j}\varphi + B_j \partial_{\eta_j}\varphi\right]$$

on the linear span $E_n$ of the vectors $\eta_1, \ldots, \eta_n$, we obtain

$$L_n V(u) = 2\sum_{j=1}^{n} j^{-3/2} - 2\sum_{j=1}^{n} j^2 j^{-3/2} u_j^2$$
$$+2\sum_{j=1}^{n} j^{-3/2}\Big[\big(\Psi(u), \eta_j'\big)_2 u_j + (f, \eta_j)_2 u_j\Big].$$

Since $\{\sqrt{2}\cos \pi n x\} = \{n^{-1}\eta_n'\}$ is an orthonormal system, we obtain

$$L_n V(u) \le 2\sum_{j=1}^{\infty} j^{-3/2} - 2\sum_{j=1}^{n} j^{1/2} u_j^2 + 2\big(c_1 + c_2\|u\|_2\big)\|u\|_2 + 2\|f\|_2\|u\|_2.$$

Applying Theorem 5.1 (or Theorem 5.2) with $\Theta(u) = \sum_{j=1}^{\infty} j^{1/2} u_j^2$, we arrive at the following assertion.

**Proposition 7.2.** *There exists a probability measure $\mu$ on the space*

$$X_0 = H_0^{2,1/4}[0,1] = \left\{u\colon\ \sum_{j=1}^{\infty} j^{1/2} u_j^2 < \infty\right\}$$

*which satisfies the equation $L^*_{I,B}\mu = 0$ with respect to the class $\mathscr{F}\mathscr{C}_b^\infty(X_0, \{l_n\})$.*

Clearly, the same is true for the fractional Sobolev class $H_0^{2,r}[0,1]$ with $r < 1/2$ in place of $H_0^{2,1/4}[0,1]$.

Suppose now that functions $A_{ij}$ on $X_0$ satisfy the following conditions:

1) $A_{ij}$ is continuous on balls in $X_0$ with respect to the topology of $X$ and, for every $n$, the restrictions of the matrix-valued mapping $(A_{ij})_{i,j}^n$ to the linear span of $\eta_1, \ldots, \eta_n$ is nonnegative symmetric,

2) there exist constants $C_1$, $C_2$, $C_{ij}$ such that

$$|A_{ij}(u)| \le C_{ij}(1+\|u\|_{X_0}^{\gamma}), \qquad \sum_{j=1}^{\infty} j^{-3/2} A_{jj}(u) \le C_1 + C_2\|u\|_{X_0}^{\gamma},$$

where $\gamma < 2$ (or, if $\gamma = 2$, the same is true for every $\varepsilon > 0$ and some $C_2(\varepsilon)$ in place of $C_1$ and $C_2$).

Suppose that $\Psi_0$ is a continuous function on $\mathbb{R}^1$ such that $|\Psi_0(s)| \le C_3 + C_4|s|^{\alpha}$, where $\alpha < 1$. Let

$$B_n(u) = -n^2 u_n - \big(\psi(u)u', \eta_n\big)_2 + \big(\Psi_0(u), \eta_n\big)_2 + (f, \eta_n)_2,$$

Then there exists a probability measure $\mu$ on $X_0$ such that $L^*_{A,B}\mu = 0$ with respect to the class $\mathscr{F}\mathscr{C}_b^{\infty}(X, \{l_n\})$, where

$$L_{A,B}\varphi := \sum_{i,j=1}^{\infty} A_{ij}\partial_{\eta_i}\partial_{\eta_j}\varphi + \sum_{n=1}^{\infty} B_n \partial_{\eta_n}\varphi.$$

In fact, the mapping $u \mapsto \Psi_0 \circ u$ can be replaced by any mapping $G$ on $X_0$ which is continuous on balls in $X_0$ with respect to the topology of $L^2(0, 1)$ and satisfies the estimate $|G(u)(s)| \le C_3 + C_4|u(s)|^{\alpha}$.

The functions $A_{ij}$ can be given, e.g., by the expression

$$A_{ij}(u) = \int_0^1 \sigma(x, u)^2 \eta_i(x)\eta_j(x)\, dx$$

with some function $\sigma$ on $[0, 1] \times X_0$, which corresponds to a non constant diffusion coefficient if we deal with an SPDE.

Let $D$ be a bounded region with smooth boundary $\partial D$ in $\mathbb{R}^d$ and let $f\colon\ \mathbb{R}^1 \times D \to \mathbb{R}^d$. We shall now consider the elliptic equation associated with the following SPDE of the Navier–Stokes type:

$$\begin{aligned} du(t, x) = \sqrt{2}dW^{Q}(t, x) + \Big[\, &\mathscr{H}u(t, x) - \big(u(t, x)\cdot\nabla\big)u(t, x) \\ &+F\big(x, u(t, x)\big) + \nabla p(t, x)\Big]\, dt \end{aligned} \tag{7.7}$$

with the incompressibility condition

$$\operatorname{div} u = 0$$

and the boundary condition $u(t, x) = 0$, $(t, x) \in [0, T] \times \partial D$. We assume that $W_t^Q$ is a Wiener process in $L^2(D, \mathbb{R}^d)$ (the exact conditions on the corresponding elliptic operator are given below). In case of the classical Navier–Stokes equation, one has $\mathscr{H} = \Delta$ and $F = 0$. We shall actually deal with the projection of (7.7) to the space of divergence free fields, hence we do not take the pressure $p$ into account. It is well known that in the case $\mathscr{H} = \Delta$, at least for $d = 2$, under reasonable assumptions on $F$ and $Q$, there exists a process $u$ in $L^2(D, \mathbb{R}^d)$ satisfying (7.7) and having an invariant probability measure $\mu$ (see [23], [27], [31], [33], [49]). In the case $d = 3$ a stationary solution to the classical stochastic Navier–Stokes equation has been constructed in [32]. The corresponding marginals should be related (or even coincide) with the (infinitesimally) invariant measures constructed by us. This will be the subject of further study. It should also be noted that in the classical case $\mathscr{K} = \Delta$ and $F = 0$ the existence of invariant measures was first proved in [49] (see Appendix II due to M.I. Vishik and A.I. Komech) in any dimension by arguments very close in the spirit to the ones employed below. In this particular case, the solutions to our elliptic equation constructed below coincide with the invariant measures constructed in [49, Appendix II, §9].

Let

$$X_0 = \{u = (u^1, \dots, u^d) \in H_0^{2,1}(D, \mathbb{R}^d) : \operatorname{div} u = 0\}$$

with norm

$$\|u\|_{X_0}^2 = \sum_{j=1}^d \|\nabla u^j\|^2_{L^2(D,\mathbb{R}^d)},$$

and let $X$ be the closure of $X_0$ in $L^2(D, \mathbb{R}^d)$ equipped with the inner product from $L^2(D, \mathbb{R}^d)$.

In fact, we shall consider the following more general equation:

$$du(t,x) = \sqrt{2}dW^Q(t,x) + \Big[\mathscr{K}u(t,x) - \sum_{j=1}^d u^j \partial_j u(t,x) + F\big(x, u(t,\,\cdot)\big)\Big]\,dt, \tag{7.8}$$

where $F\colon D \times X_0 \to \mathbb{R}^d$ is a uniformly bounded mapping and $\mathscr{K}$ is a linear operator from $X$ to $L^2(D, \mathbb{R}^d)$ with a domain $D(\mathscr{K})$ dense in $X$. In addition, we shall assume that the domains of $\mathscr{K}$ and $\mathscr{K}^*$ contain an orthonormal basis $\{\eta_n\}$ of $X$ such that $\eta_n \in X_0 \cap L^\infty(D, \mathbb{R}^d)$. Suppose that for some $\lambda > 0$ one has

$$(\mathscr{K}u, u)_2 \le -\lambda\|u\|_{X_0}^2, \qquad \forall\, u \in \operatorname{span}\{\eta_n\}.$$

where $(\,\cdot\,,\,\cdot\,)_2$ is the inner product in $L^2(D, \mathbb{R}^d)$. Finally, we assume that for all $n$ the functions $u \mapsto \big(F(\,\cdot\,,u), \eta_n\big)_2$ are continuous on balls in $X_0$ with respect to the topology induced by $L^2(D, \mathbb{R}^d)$.

From now on, we take for $F$ the mapping $F(x,u) = F_0\big(x, u(x)\big)$, where $F_0\colon D \times \mathbb{R}^d \to \mathbb{R}^d$ is a bounded Borel mapping continuous in the second argument.

Let

$$B_n(u) = (u, \mathscr{K}^*\eta_n)_2 + \sum_{j=1}^d (\partial_j u, u^j\eta_n)_2 + \big(F(\,\cdot\,,u), \eta_n\big)_2, \qquad u \in X_0.$$

We observe that $B_n$ is continuous on any ball $S$ in $X_0$ with respect to the topology from $X$ (i.e., from $L^2(D, \mathbb{R}^d)$). Indeed, if $u_k \to u$ in $X$ and $u_k \in S$, then, for each $i = 1, \dots, d$, the sequence $\{u_k^i\}$ is bounded in $H_0^{2,1}(D)$, hence $u \in H_0^{2,1}(D, \mathbb{R}^d)$

and $\lim_{k\to\infty} \partial_j u_k^i = \partial_j u^i$ in the weak topology of $L^2(D)$. Then it follows by the embedding theorem that $\lim_{k\to\infty} u_k^j = u^j$ in $L^{2d/(d-2)}(D)$ (resp. in all $L^p(D)$, $p \geq 1$, if $d = 2$). Hence $\lim_{k\to\infty} u_k^j \eta_n = u^j \eta_n$ in $L^2(D, \mathbb{R}^d)$ for every $j = 1, \ldots, d$. This yields that $\lim_{k\to\infty} (\partial_j u_k, u_k^j \eta_n)_2 = (\partial_j u, u^j \eta_n)_2$. For every $u \in \operatorname{span}\{\eta_n\}$, we have

$$\sum_{n=1}^{\infty}\sum_{j=1}^{d}(u^j\partial_j u, \eta_n)_2(u, \eta_n)_2 = \sum_{j=1}^{d}(u^j\partial_j u, u)_2 = -\frac{1}{2}\int_D |u|^2 \operatorname{div} u \, dx = 0.$$

In addition, for every $u \in \operatorname{span}\{\eta_n\}$, one has

$$\sum_{n=1}^{\infty}(u, \mathscr{H}^*\eta_n)_2(u, \eta_n)_2 = (\mathscr{H}u, u)_2 \leq -\lambda\|u\|_{X_0}^2. \tag{7.9}$$

Therefore, (5.1) obviously holds. Clearly, (5.7) is fulfilled with $\Theta(u) = \|u\|_{X_0}^2$ and $V(u) = \|u\|_{X_0}^2$. (5.8) holds with $\delta(r) = r^{-1}$.

Now, Theorem 5.2 applies to the corresponding elliptic equation: Let

$$L_{A,B}\varphi = \sum_{n=1}^{\infty}\alpha_n^2\partial_{\eta_n}^2\varphi + \sum_{n=1}^{\infty} B_n\partial_{\eta_n}\varphi$$

be defined on $\mathscr{F}\mathscr{C}_b^\infty(X, \{l_n\})$, where $l_n(u) = (u, \eta_n)_2$, and $\sum_{n=1}^\infty \alpha_n^2 < \infty$.

**Proposition 7.3.** *Under the above conditions, there exists a probability measure $\mu$ on $X_0$ which satisfies the equation $L_{A,B}^*\mu = 0$ with respect to $\mathscr{F}\mathscr{C}_b^\infty(X, \{l_n\})$.*

This result corresponds to the process $W^Q(t) = \sum_{n=1}^{\infty}\alpha_n w_n(t)\eta_n$, where $\sum_{n=1}^{\infty}\alpha_n^2 < \infty$ and $\{w_n(t)\}$ is a sequence of independent standard real Wiener processes, i.e., to the time white noise in the corresponding SPDE.

As in the case of the Burgers equation, the same reasoning applies to the following more general situation. Suppose that functions $A_{ij}$ on $X_0$ are continuous on balls in $X_0$ with respect to the topology of $X$ and, for every $n$, the restriction of the matrix-valued mapping $(A_{ij})_{i,j}^n$ to the linear span of $\eta_1, \ldots, \eta_n$ is nonnegative symmetric. Assume that there exist constants $C_1$, $C_2$, $C_{ij}$ such that

$$|A_{ij}(u)| \leq C_{ij}(1 + \|u\|_{X_0}^\gamma), \qquad \sum_{j=1}^{\infty} A_{jj}(u) \leq C_1 + C_2\|u\|_{X_0}^\gamma,$$

where $\gamma < 2$ (or, if $\gamma = 2$, the same is true for every $\varepsilon > 0$ and some $C_2(\varepsilon)$ in place of $C_1$ and $C_2$). Let

$$L_{A,B}\varphi = \sum_{i,j=1}^{\infty} A_{ij}\partial_{\eta_i}\partial_{\eta_j}\varphi + \sum_{n=1}^{\infty} B_n\partial_{\eta_n}\varphi$$

with

$$B_n(u) = (u, \mathscr{H}^*\eta_n)_2 - \sum_{j=1}^{d}(\partial_j u, u^j \eta_n)_2 + \big(\Psi_0(u), \eta_n\big)_2 + \big(F(u), \eta_n\big)_2,$$

where $\Psi_0\colon \mathbb{R}^d \to \mathbb{R}^d$ is a continuous mapping such that $|\Psi_0(x)| \le C_3 + C_4|x|^\alpha$ with $\alpha < 2d/(d-2)$ and $\big(\Psi_0(x), x\big) \le C_3 + C_4|x|^\kappa$ with $\kappa < 2$.

For concrete examples of $\mathscr{H}$ one can take $\mathscr{H} = \Delta$ or a more general nondegenerate second order elliptic operator with smooth coefficients.

In a similar manner one can study the reaction-diffusion equation

$$du(t,x) = \Big[\partial_x^2 u(t,x) + F\big(u(t,x)\big)\Big]\,dt + \sqrt{2}dW^I(t),$$

where $W^I(t)$ is the space-time white noise process, i.e.,

$$W^I(t) = \sum_{n=1}^{\infty} w_n(t)\eta_n,$$

where $w_n$ are independent standard Wiener processes and $\eta_n(x) = \sqrt{2}\sin \pi n x$ as in the example considered in Proposition 7.2. Invariant measures for this equation are considered, e.g., in [20], [27], [34], [35], [44], [47], [51]. The corresponding elliptic operator is given by

$$L_{I,B}\varphi = \sum_{n=1}^{\infty}\big[\partial_{\eta_n}^2\varphi + B_n\partial_{\eta_n}\varphi\big],$$

where the functions $B_n(u) = -n^2 u_n + (F\circ u, \eta_n)_2$ are defined on $C[0,1]$ or on a suitable $L^p$. In order to apply our results, it suffices to assume that $F$ is continuous, has at most polynomial growth at infinity and that $F(x)x \le C + \varepsilon|x|^2$ with a sufficiently small $\varepsilon > 0$. One can explicitly find invariant measures for the above reaction-diffusion equation. For example, let $F(x) = \Pi'(x)$, where $\Pi$ is a continuously differentiable function on $\mathbb{R}^1$ such that $\Pi(x) \le C + \varepsilon|x|^2$ and $\varepsilon > 0$ is sufficiently small. Indeed, let $\nu$ be the centered Gaussian measure on $L^2(0,1)$ corresponding to $F = 0$, i.e., to the functions $B_n(u) = \lambda_n u_n$. It is readily seen that $\nu$ is the distribution of the Gaussian random vector $Y(\omega) = \sum_{n=1}^{\infty} |\lambda_n|^{-1/2}\xi_n(\omega)\eta_n$, where $\xi_n$ are independent standard Gaussian random variables. The measure $\nu$ is in fact concentrated on the space $E = C[0,1]$ of continuous functions (or on a smaller subspace of Hölder continuous functions). Hence the function

$$\varrho(u) = \exp \int_0^1 \Pi\big(u(x)\big)\,dx$$

is continuous on the space $E$ with its natural norm and

$$\frac{\partial_{\eta_n}\varrho(u)}{\varrho(u)} = \int_0^1 \Pi'\big(u(x)\big)\eta_n(x)\,dx = \big(F(u), \eta_n\big)_2.$$

Due to the estimate

$$\int_0^1 \Pi\bigl(u(x)\bigr)\,dx \le C + \varepsilon \int_0^1 |u(x)|^2\,dx,$$

the function $\varrho$ is $\nu$-integrable, provided that $\varepsilon$ is sufficiently small. Since $\partial_{\eta_n}\varrho$ is bounded on balls in $E$, we obtain by the integration by parts formula that the measure $\mu = \varrho\,\nu$ is differentiable along $\eta_n$ with respect to $Lip_0(E)$ and

$$\beta^{\mu}_{\eta_n}(u) = \beta^{\nu}_{\eta_n}(u) + \frac{\partial_{\eta_n}\varrho(u)}{\varrho(u)} = B_n(u).$$

Moreover, if we have the estimate $|\Pi'(x)| \le C + \exp\bigl(\varepsilon|x|^2\bigr)$ and $\varepsilon > 0$ is sufficiently small, then the function $\partial_{\eta_n}\varrho/\varrho$ is $\nu$-integrable, hence $B_n$ is the logarithmic derivative of $\mu$ along $\eta_n$ with respect to the classes $C_b^1(X)$, $C_b^1(E)$, in particular, with respect to $\mathscr{F}\mathscr{C}_b^\infty(X)$. Of course, we can take a more general mapping for $F$. The same proof as above applies to the mapping $F\colon\ E \to L^2(0,1)$, where $\bigl(F(u), \eta_n\bigr)_2 = \partial_{\eta_n}\Pi(u)$, provided that $\Pi$ is a Borel function on $E$ differentiable along all vectors $\eta_n$ and

$$\Pi(u) \le C + \varepsilon\|u\|_E^2, \qquad \|F(u)\|_2 \le C + \exp\bigl(\varepsilon\|u\|_E^2\bigr)$$

for a sufficiently small $\varepsilon > 0$. The smallness of $\varepsilon$ in these examples is determined by the $\nu$-integrability of the function $\exp\bigl(\varepsilon\|u\|_E^2\bigr)$ (the existence of such $\varepsilon$ is ensured by Fernique's theorem). Certainly, we could assume just as well the $\nu$-integrability of $\exp\Pi$. The above explicit expression for $\mu$ was obtained in [35], [34], [51] under stronger assumptions on $\Pi$. It should be noted that, according to [11], every probability measure $\mu_0$ on $X = L^2(0,1)$ such that $u_n \in L^2(\mu_0)$, $\|F\circ u\|_{L^2(0,1)} \in L^2(\mu_0)$ and $L^*_{I,B}\mu_0 = 0$ with respect to the class $\mathscr{F}\mathscr{C}_b^\infty(X,\{l_n\})$, is absolutely continuous with respect to the Gaussian measure $\nu$.

In the case of the reaction-diffusion equation

$$du(t,x) = \Bigl[\partial_x^2 u(t,x) + F\bigl(x, u(t,\,\cdot\,)\bigr)\Bigr]\,dt + \sqrt{2}\sigma\bigl(x, u(t,\,\cdot\,)\bigr)\,dW^I(t)$$

with a non-constant diffusion coefficient $\sigma$, invariant measures are not absolutely continuous with respect to Gaussian measures and do not admit explicit expressions. If $\sigma$ does not depend on $x$, then the corresponding operator is given by

$$L_{A,B}\varphi = \sum_{n=1}^\infty \bigl[\sigma^2\partial_{\eta_n}^2\varphi + B_n\partial_{\eta_n}\varphi\bigr],$$

where $B_n(u) = -n^2 u_n + \bigl(F(\,\cdot\,,u), \eta_n\bigr)_2$. If $\sigma$ depends on $x$, then

$$L_{A,B}\varphi = \sum_{i,j=1}^\infty A_{ij}\partial_{\eta_i}\partial_{\eta_j}\varphi + \sum_{n=1}^\infty B_n\partial_{\eta_n}\varphi,$$

where $A_{ij}(u) = \displaystyle\int_0^1 \sigma(x,u)^2\eta_i(x)\eta_j(x)\,dx$.

The existence of solutions to the associated elliptic equation is obtained by our method under the following conditions. Let $E = L^p[0,1]$ with some $p \in [2,\infty)$ be equipped with the norm $\|u\|_E = \|u\|_{L^p}$. Assume that $F$ and $\sigma$ are Borel real-valued functions on $[0,1] \times E$ such that

$$\|F(\,\cdot\,,u)\|_2 \|u\|_2 \le c_1 + \varepsilon_1 \|u\|_E^2, \qquad \int_0^1 \sigma(x,u)^2\,dx \le c_2 + \varepsilon_2 \|u\|_E^2,$$

and $\varepsilon_1, \varepsilon_2 > 0$ are such that $\varepsilon_1 + 2\varepsilon_2 \sum_{j=1}^{\infty} j^{\delta-2} < 2$, where $\delta \in (0,1)$ is such that we have the embedding

$$X_0 = \Big\{ u\colon\ |u|_0^2 = \sum_{j=1}^{\infty} j^{\delta} u_j^2 < \infty \Big\} \subset L^p[0,1].$$

By the Sobolev embedding theorem, it suffices to take $\delta$ sufficiently close to 1.

Finally, suppose that, for all $n$ and $j$, the functions $u \mapsto \int_0^1 F(x,u)\eta_n(x)\,dx$ and $u \mapsto \int_0^1 \sigma(x,u)^2 \eta_n(x)\eta_j(x)\,dx$ are continuous on $E$.

**Proposition 7.4.** *Under the above conditions, there exists a probability measure $\mu$ on $E$ such that $L_{A,B}^{*}\mu = 0$ with respect to $\mathscr{F}\mathscr{C}_b^{\infty}(E,\{l_n\})$.*

*Proof.* Let us consider the Lyapunov function similar to the one considered in the case of the space-time white noise Burgers equation (see Proposition 7.2): $V(u) = \sum_{j=1}^{\infty} j^{\delta-2} u_j^2$. In the case when $\sigma$ does not depend on $x$, the elliptic operators $L_n$ on $E_n$ are given by

$$L_n\varphi(u) = \sum_{j=1}^{n} \sigma^2(u)\partial_{\eta_j}^2 \varphi(u) + \sum_{j=1}^{n} B_j(u)\partial_{\eta_j}\varphi(u).$$

In order to apply the same techniques as above, it suffices to observe that

$$\sum_{j=1}^{n} j^{\delta-2}\big(F(u),\eta_j\big)_2 (u,\eta_j)_2 \le \|F(u)\|_2 \|u\|_2$$

and $\sum_{j=1}^{\infty} j^2 u_j \partial_{\eta_j} V(u) = 2|u|_0^2$. In the case when $\sigma$ may depend on $x$, the reasoning is similar, although the first term in the expression for $L_n$ becomes $\sum_{i,j\le n} A_{ij}(u)\partial_{\eta_i}\partial_{\eta_j}\varphi$. Now it remains to use the estimate $|A_{ij}(u)| \le \displaystyle\int_0^1 \sigma(x,u)^2\,dx$. $\square$

Clearly, the above assumptions on $F$ and $\sigma$ are fulfilled if $F(x,u) = F_0\big(x,u(x)\big)$ and $\sigma(x,u) = \sigma_0\big(x,u(x)\big)$, where $F_0$ and $\sigma_0$ are uniformly bounded Borel functions continuous in the second argument. We remark that our assumptions on $F$ and $\sigma$ are weaker than those in [47] where both functions were uniformly bounded and uniformly Lipschitzian. The condition of the linear growth of $F$ can be replaced by a polynomial bound provided that some extra coercivity condition is imposed. In the case of a bounded $\sigma$, results in this direction (concerning existence of the processes and their invariant measures) have been obtained in [20]. Finally, note that in the above results the functions $B_n$ do not correspond to any $X_0$-valued drift $B$, i.e., they are typical for application of the technique developed in this paper.

## 8. Regularity

A difficult problem is to prove the existence of logarithmic derivatives for solutions of the elliptic equation (3.1). Considerable progress has been achieved in the finite dimensional case, but in infinite dimensions only a few special resuls are known (cf. [1], [12], [15]). Yet another special result will be proved below, but now we give a sufficient condition which ensures that the measure $\mu$ constructed in Theorem 5.3 has a logarithmic gradient (cf. [38], [39], [14], [15]).

In this section, we assume as above that $\{l_n\}$ is a point separating sequence of continuous linear functionals on $X$ and $\{e_n\} \subset X$ is such that $l_n(e_k) = \delta_{nk}$.

**Theorem 8.1.** *Let $\mu$ be a probability measure that satisfies* (3.1) *with respect to the class $\mathscr{F}\mathscr{C}_0^\infty(X, \{l_n\})$. Suppose that $(B_1, \dots, B_n) = \nabla G_n(l_1, \dots, l_n) + D_n$, where $G_n$ is a continuously differentiable function on $\mathbb{R}^n$, $D_n$ is a Borel mapping with values in $\mathbb{R}^n$, and*

$$\sup_k \|D_k\|_{L^2(\mu,\mathbb{R}^k)} < \infty. \tag{8.1}$$

*Suppose that $B_n \in L^2(\mu)$ for every $n$. Then $\mu$ is differentiable along each $e_n$ and*

$$\|\beta^\mu_{e_n}\|_{L^2(\mu)} \le \|B_n\|_{L^2(\mu)} + \sup_k \|D_k\|_{L^2(\mu,\mathbb{R}^k)}. \tag{8.2}$$

*Proof.* Let $P_n x = l_1(x)e_1 + \cdots + l_n(x)e_n$ and let $\mathbb{E}_n$ stand for the conditional expectation with respect to the measure $\mu$ and the $\sigma$-field $\sigma_n$ generated by $P_n$. Set $S_n = B_1 e_1 + \cdots + B_n e_n$. We denote by $\mu_n$ the image of $\mu$ under the projection $P_n$. It is readily seen (see the proof of [15, Proposition 3.3]), that the measure $\mu_n$ solves the elliptic equation $L_n^*\mu_n = 0$ with

$$L_n f = \sum_{j=1}^n \bigl[\partial^2_{e_j} f + \partial_{e_j} f\, \mathbb{E}_n B_j\bigr]$$

and has the logarithmic gradient $\beta^n$ on $E_n$, $E_n$ being equipped with the $H$-norm. We shall identify $(E_n, |\cdot|_H)$ with $\mathbb{R}^n$. Let us show that

$$\beta^n = \mathbb{E}_n S_n + \Lambda_n,$$

where

$$\int_{E_n} |\Lambda_n|_H^2 \, d\mu_n \le \|\mathbb{E}_n D_n\|^2_{L^2(\mu,\mathbb{R}^n)} \le \|D_n\|^2_{L^2(\mu,\mathbb{R}^n)}. \tag{8.3}$$

By our hypothesis,

$$\mathbb{E}_n S_n = \sum_{j=1}^{n} \mathbb{E}_j B_j e_i = \nabla G_n(l_1, \dots, l_n) + \mathbb{E}_n D_n.$$

As shown in [15, Theorem 3.1], $\beta^n$ is the orthogonal projection of $\mathbb{E}_n S_n$ to the closure $\Gamma$ of $\{D_H \varphi,\ \varphi \in C_0^\infty(E_n)\}$ in the Hilbert space $L^2(\mu_n, E_n)$. We observe that $\nabla G_n(l_1, \dots, l_n) \in \Gamma$ (see [45] or [21]). Therefore, we get (8.3). Let now $\psi \in \mathscr{F}\mathscr{C}_0^\infty(X, \{l_i\})$ and let $n \in \mathbb{N}$ be fixed. For any $m \geq n$ such that $\psi$ depends only on $l_j$ with $j \leq m$, one has

$$\begin{aligned}
\int_{E_m} \partial_{e_n} \psi \, d\mu_m &= -\int_{E_m} \psi (\beta^m, e_n)_H \, d\mu_m \\
&= -\int_{E_m} \psi (\mathbb{E}_m S_m, e_n)_H \, d\mu_m - \int_{E_m} \psi (\Lambda_m, e_n)_H \, d\mu_m \\
&\leq \|\psi\|_{L^2(\mu_m)} \sqrt{\int_{E_m} |\mathbb{E}_m B_n|^2 \, d\mu_m} + \|\psi\|_{L^2(\mu_m)} \sqrt{\int_{E_m} |\Lambda_m|_H^2 \, d\mu_m} \\
&\leq \|\psi\|_{L^2(\mu_m)} \|B_n\|_{L^2(\mu)} + \|\psi\|_{L^2(\mu_m)} \sup_k \|D_k\|_{L^2(\mu, \mathbb{R}^k)}.
\end{aligned}$$

It follows that $\beta_{e_n}^\mu$ exists and its $L^2(\mu)$-norm is majorized by the right-hand side of (8.2) (see, e.g., [15, Lemma 1.4] or [9, Proposition 2.6.1]). □

*Remark 8.2.* It follows from (8.2) that the measure $\mu$ is differentiable along all directions $h$ in the Hilbert space

$$H_0 = \Big\{x\colon\ |x|_{H_0}^2 := \sum_{j=1}^{\infty} c_j l_j(x)^2\Big\},$$

where $c_n = \|B_n\|_{L^2(\mu)}^2 + 1$. Moreover, $\|\beta_h^\mu\|_{L^2(\mu)}^2 \leq |h|_{H_0}^2$.

Theorem 8.1 applies to the situation of Example 4.6 provided the additional condition (8.1) is satisfied. However, in that specific case, more can be shown: namely, any solution $\mu$ of (3.1) is its symmetric solution, i.e., the $\beta_{e_n}^\nu$'s exist and $\beta_{e_n}^\mu = B_n$. In the probabilistic interpretation, this means that any invariant probability for the diffusion generated by the Gibbs measure is Gibbsian with respect to the same specifications.

**Theorem 8.3.** *Suppose that in the situation of Theorem* 8.1 *one has the following stronger condition:*

$$\lim_{n\to\infty} \|\mathbb{E}_n D_n\|_{L^2(\mu, \mathbb{R}^n)} = 0. \tag{8.4}$$

*Then* $\beta_{e_n}^\mu = B_n$ $\mu$*-a.e.*

*Proof.* Let $\varepsilon > 0$ be fixed and let us choose $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ one has

$$\int |\mathbb{E}_k P_k B - \nabla G_k|_H^2 \, d\mu = \int \sum_{n=1}^{k} |\mathbb{E}_k D_n|^2 \, d\mu \leq \varepsilon^2, \tag{8.5}$$

which is possible by (8.4), and set

$$R_k := (\beta_{e_1}^{\mu}, \ldots, \beta_{e_k}^{\mu}).$$

Keeping $k \geq k_0$ fixed, for every $\psi \in C_0^\infty(\mathbb{R}^k)$, one has from the elliptic equation that

$$\int (\mathbb{E}_k P_k B - \mathbb{E}_k R_k, \nabla\psi)_H \, d\mu = \int (P_k B - R_k, \nabla\psi)_H \, d\mu = 0. \tag{8.6}$$

We observe that the projection $\mu_k$ of $\mu$ under $P_k$ has the logarithmic gradient $\beta^{\mu_k} = \mathbb{E}_k R_k$ on $E_k$ (where as before $E_k$ is equipped with the $H$-norm and we identify $E_k$ with $\mathbb{R}^k$). There exist two functions $\psi_1,\ \psi_2 \in C_0^\infty(\mathbb{R}^k)$ such that

$$\int |\beta^{\mu_k} - \nabla\psi_1|_H^2 \, d\mu_k + \int |\nabla G_k - \nabla\psi_2|_H^2 \, d\mu_k < \frac{\varepsilon^2}{2}.$$

Then

$$\int |\beta^{\mu_k} - \nabla G_k - (\nabla\psi_1 - \nabla\psi_2)|_H^2 \, d\mu_k \leq \varepsilon^2. \tag{8.7}$$

Taking into account (8.6) with $\psi = \psi_1 - \psi_2$, (8.5), and (8.7), we obtain

$$\begin{aligned}
&\|\mathbb{E}_k P_k B - \beta^{\mu_k}\|_{L^2(\mu,H)}^2 \\
&\quad = (\mathbb{E}_k P_k B - \beta^{\mu_k}, \nabla\psi)_{L^2(\mu,H)} \\
&\qquad + (\mathbb{E}_k P_k B - \beta^{\mu_k}, \mathbb{E}_k P_k B - \beta^{\mu_k} - \nabla\psi)_{L^2(\mu,H)} \\
&\quad = (\mathbb{E}_k P_k B - \beta^{\mu_k}, \mathbb{E}_k P_k B - \beta^{\mu_k} - \nabla\psi)_{L^2(\mu,H)} \\
&\quad = (\mathbb{E}_k P_k B - \beta^{\mu_k}, \nabla G_k - \beta^{\mu_k} - \nabla\psi)_{L^2(\mu,H)} \\
&\qquad + (\mathbb{E}_k P_k B - \beta^{\mu_k}, \mathbb{E}_k P_k B - \nabla G_k)_{L^2(\mu,H)} \\
&\quad \leq \|\mathbb{E}_k P_k B - \beta^{\mu_k}\|_{L^2(\mu,H)} \|\nabla G_k - \beta^{\mu_k} - \nabla\psi\|_{L^2(\mu,H)} \\
&\qquad + \|\mathbb{E}_k P_k B - \beta^{\mu_k}\|_{L^2(\mu,H)} \|\mathbb{E}_k P_k B - \nabla G_k\|_{L^2(\mu,H)} \\
&\quad \leq 2\varepsilon \|\mathbb{E}_k P_k B - \beta^{\mu_k}\|_{L^2(\mu,H)}.
\end{aligned}$$

Therefore,

$$\int |\mathbb{E}_k B_i - \mathbb{E}_k \beta_{e_i}^{\mu}|^2 \, d\mu = \int |\mathbb{E}_k B_i - \beta_{e_i}^{\mu_k}|^2 \, d\mu < 4\varepsilon^2.$$

By the martingale convergence theorem, letting $k$ tend to infinity we obtain

$$\int |B_i - \beta_{e_i}^{\mu}|^2 \, d\mu < 4\varepsilon^2,$$

whence $B_i = \beta_{e_i}^{\mu} = 0$ $\mu$-a.e., since $\varepsilon > 0$ was arbitrary. □

**Corollary 8.4.** *Suppose that in the situation of Theorem* 8.1*, there exists a non-negative function* $\Theta$ *as in Theorem* 4.1 *such that*

$$|D_m|^2 \le \varepsilon_m \Theta, \tag{8.8}$$

*where* $\lim\limits_{m\to\infty} \varepsilon_m = 0$. *Then* $\beta^{\mu}_{e_n} = B_n$ $\mu$*-a.e.*

**Example 8.5.** Condition (8.8) is fulfilled in the situation of Example 4.6 with

$$B_n(x) = \partial_{x_n} V_n(x_n) + \sum_{j\neq n} \partial_{x_n} W_{n,j}(x_n, x_j)$$

provided that $W_{n,j} = W_{j,n}$, $\partial_{x_n} V_n(x_n)x_n \le c_1 - k|x_n|^{\alpha}$, and

$$|\partial_{x_n} V_n(x_n)| \le c_3(1+|x_n|^{\alpha}), \quad |\partial_{x_n} W_{n,j}(x_j, x_n)| \le c_{n,j}(1+|x_n|^{\alpha-1}+|x_j|^{\alpha-1}),$$

where the numbers $c_{n,j}$ satisfy the following additional condition: $c_{n,j} \le q_n q_j$.

The following result on regularity of solutions can be informally interpreted in terms of the time-reversal of the corresponding diffusions. Its finite dimensional version is exactly this: if $\mu$ solves elliptic equation (3.1) on $\mathbb{R}^n$ with $B$ $\mu$-square-integrable, then there exists a Markovian semigroup $(T_t)_{t\ge 0}$ with invariant measure $\mu$ and generator $L_B f = \Delta f + (B, \nabla f)$; moreover, there is a diffusion $\xi$ with transition semigroup $(T_t)_{t\ge 0}$. The drift term can be written as $\beta + \delta$, where $\beta = \nabla p/p$ and $\delta$ is orthogonal in $L^2(\mu, \mathbb{R}^n)$ to the gradients of smooth compactly supported functions. Clearly, $\mu$ also solves equation (3.1) with the "dual" drift $\widehat{B} = \beta - \delta$. The diffusion with generator $L_{\widehat{B}} f = \Delta f + (\widehat{B}, \nabla f)$ is the time-reversal of $\xi$; in analytic terms, $L_{\widehat{B}}$ is the generator of the dual semigroup. It is now obvious that $\beta$ can be found from the equality $B + \widehat{B} = 2\beta$. In the next proposition, we do not assume in advance that the adjoint operator has the structure $L_{\widehat{B}} f = \Delta f + (\widehat{B}, \nabla f)$ with some $\widehat{B}$, but this anticipated formula is, of course, implicitly behind our calculations.

**Proposition 8.6.** *Let* $\mu$ *be a probability measure on* $X$ *and let* $B_n \in L^2(\mu)$, $n \in \mathbb{N}$. *Suppose that* $\mu$ *satisfies equation* (3.1) *with respect to* $\mathscr{K} = \mathscr{F}\mathscr{C}^{\infty}_b(X, \{l_n\})$. *Assume also that* $l_j \in L^2(\mu)$ *for some* $j$. *Then* $\mu$ *is differentiable along* $e_j$ *with* $\beta_{e_j} \in L^2(\mu)$ *precisely when* $l_j$ *belongs to the domain of the adjoint operator for*

$$L_B f = \sum_{n=1}^{\infty} (\partial^2_{e_n} f + B_n \partial_{e_n} f) \qquad \text{with } D(L_B) = \mathscr{F}\mathscr{C}^{\infty}_b(X, \{l_n\}).$$

*In addition,*

$$\beta^{\mu}_{e_j} = \frac{L^*_B l_j + B_j}{2}. \tag{8.9}$$

*Proof.* We first look at the case $X = \mathbb{R}^n$ in more detail. Let $L_B^*\mu = 0$ with $|B| \in L^2(\mu)$ and $L_B f = \Delta f + (B, \nabla f)$, $f \in C_0^\infty(\mathbb{R}^n)$; note that in the case of a globally $\mu$-integrable drift $B$, there is no difference between the classes $C_0^\infty(\mathbb{R}^n)$ and $C_b^\infty(\mathbb{R}^n)$ for the interpretation of (3.1). We recall that, as shown in [15], $\mu$ has a density $p \in W^{1,1}(\mathbb{R}^n)$ and that $\beta := \nabla p/p \in L^2(\mu, \mathbb{R}^n)$. Let us verify that the operator $L_{\widehat{B}} f = \Delta f + (\widehat{B}, \nabla f)$, where $\widehat{B} = 2\beta - B$, coincides on $C_b^\infty(\mathbb{R}^n)$ with the adjoint to $L_B$ on $L^2(\mu)$. Indeed, let $f,\ g \in C_0^\infty(\mathbb{R}^n)$. We know that $B - \beta = \delta$, where $\delta$ is orthogonal to every $\nabla\zeta$, $\zeta \in C_0^\infty(\mathbb{R}^n)$, in $L^2(\mu, \mathbb{R}^n)$. Then $\widehat{B} = \beta - \delta$. Integrating by parts, we obtain

$$\begin{aligned}\int L_B f\, g\, d\mu &= \int L_B f\, gp\, dx \\ &= -\int (\nabla f, \nabla g)\, d\mu - \int (\nabla f, \nabla p) g\, dx + \int (B, \nabla f) g\, d\mu \\ &= -\int (\nabla f, \nabla g)\, d\mu + \int (\delta, \nabla f) g\, d\mu.\end{aligned}$$

In a similar manner,

$$\int L_{\widehat{B}} g\, f\, d\mu = -\int (\nabla f, \nabla g)\, d\mu - \int (\delta, \nabla g) f\, d\mu.$$

In order to conclude that

$$\int L_B f\, g\, d\mu = \int L_{\widehat{B}} g\, f\, d\mu, \tag{8.10}$$

it remains to note that $\int \big(\delta, \nabla(fg)\big)\, d\mu = 0$. Clearly, (8.10) holds true also for all $f,\ g \in C_b^2(\mathbb{R}^n)$. Assume now that $x_j \in L^2(\mu)$ (which is not always the case, of course). Then we set $L_B^* x_j := L_{\widehat{B}} x_j = \widehat{B}_j$ and observe that (8.10) holds true also for $x_j$ in place of $g$. Actually, this follows from the above calculations, but is also a direct consequence of (8.10) with $g(x) = \zeta(x_j)$, where $\zeta \in C_0^\infty(\mathbb{R}^1)$ is such that $\zeta(t) = t$ on $[-r, r]$ with $r$ so large that the support of $f$ is contained in the centered ball of radius $r$. Since $\widehat{B}_j = 2(\beta, e_j) - B_j \in L^2(\mu)$, (8.10) is valid for all $f \in C_b^\infty(\mathbb{R}^n)$ and $g = x_j$, and the functional $f \mapsto \int L_B f\, x_j\, d\mu$ is continuous with respect to the $L^2(\mu)$ norm on $C_b^\infty(\mathbb{R}^n)$, which is equivalent to the inclusion $x_j \in D(L_B^*)$.

The above calculation enables us to reduce the general case to that of $\mathbb{R}^n$. Suppose first that $\beta_{e_j}$ exists and is in $L^2(\mu)$. Let $f \in \mathscr{F}\mathscr{C}_b^\infty(X, \{l_n\})$; we observe that equation (3.1) is satisfied also with respect to $\mathscr{F}\mathscr{C}_b^\infty(X, \{l_n\})$ due to the integrability of the $B_n$'s. We may assume that $f$ depends only on $l_1, \dots, l_n$ with $n \geq j$. In fact, everything reduces to the case $X = \mathbb{R}^\infty$, since we can take the embedding $(l_n)_{n=1}^\infty\colon\ X \to \mathbb{R}^\infty$. Therefore, we assume further on that the $l_n$'s are the coordinate functions on $\mathbb{R}^\infty$ and $e_n$ is the standard $n$-th "unit" vector in $\mathbb{R}^\infty$. The necessity part could be proved directly by the same calculations as above if we knew the differentiability of $\mu$ along $e_1, \dots, e_n$, and not only along $e_j$. However, it is easy to overcome this difficulty by considering the image $\mu_n$ of $\mu$ under the projection

$P_n x = (x_1, \dots, x_n)$ to $\mathbb{R}^n$. Let $\mathbb{E}_n g$ denote the conditional expectation of $g$ with respect to the measure $\mu$ and the $\sigma$-field generated by $P_n$. We know that $\mu_n$ satisfies the elliptic equation on $\mathbb{R}^n$ with drift coefficient $D^n := (\mathbb{E}_n B_1, \dots, \mathbb{E}_n B_n)$ and that it has $\beta_{e_j}^{\mu_n}$ as $\mu_n$-square integrable partial logarithmic derivatives. Moreover, as is easily verified (see [15]), one has

$$\mathbb{E}\beta_{e_j}^{\mu}(x) = \beta_{e_j}^{\mu_n}(P_n x) \qquad \mu\text{-a.e.} \tag{8.11}$$

According to the finite dimensional case and (8.11), we have by the definition of the conditional expectation

$$\begin{aligned}\int_X L_B f\, l_j\, d\mu &= \int_{\mathbb{R}^n} \Big[\sum_{i=1}^n \partial_{e_i}^2 f\, l_j + \partial_{e_i} f\, \mathbb{E}_n B_i l_j\Big]\, d\mu_n \\ &= \int_{\mathbb{R}^n} f\,(2\beta_{e_j}^{\mu_n} - \mathbb{E}_n B_j)\, d\mu_n = \int_X f(2\beta_{e_j}^{\mu} - B_j)\, d\mu.\end{aligned}$$

Therefore, $L_B^* l_j = 2\beta_{e_j}^{\mu} - B_j$, where $L_B$ is considered on $\mathscr{F}\mathscr{C}_b^\infty(X, \{l_n\})$ (or on the smaller domain $\mathscr{F}\mathscr{C}_0^\infty(X, \{l_n\})$, which makes no difference in the present situation). Conversely, assume that $L_B^* l_j$ exists. We have to verify the equality

$$\int_X \partial_{e_j} f\, d\mu = -\frac{1}{2}\int_X f(L_B^* l_j + B_j)\, d\mu = -\frac{1}{2}\int_X (L_B f l_j + f B_j)\, d\mu \tag{8.12}$$

for smooth cylindrical $f$. We may assume again that $f(x) = \psi(x_1, \dots, x_n)$, where $\psi \in C_b^\infty(\mathbb{R}^n)$ and $n \geq j$. Employing the same notation as above and making use of (8.11), we rewrite (8.12) as

$$\int_{\mathbb{R}^n} \partial_{e_j} f\, d\mu_n = -\frac{1}{2}\int_{\mathbb{R}^n} \Big(l_j \sum_{i=1}^n [\partial_{e_i}^2 f + \partial_{e_i} f\, \mathbb{E}_n B_i] + f\,\mathbb{E}_n B_j\Big)\, d\mu_n. \tag{8.13}$$

Note that the right-hand side of (8.13) equals

$$-\frac{1}{2}\int_{\mathbb{R}^n} \Big(f(2\beta_{e_j}^{\mu_n} - \mathbb{E}_n B_j) + f\,\mathbb{E}_n B_j\Big)\, d\mu_n = -\int_{\mathbb{R}^n} f\beta_{e_j}^{\mu_n}\, d\mu_n,$$

which is exactly the left-hand side of (8.13). □

Many results in this paper admit extensions to the manifold case. This is done in [18].

*Acknowledgements.* This work has been supported in part by the RFBR project 00–15–99267, the INTAS Grant 99–01317, the DFG Grant 436 RUS 113/343/0(R), the EU–TMR–Project ERB–FMRX–CT96–0075, and the SFB–343. We thank G. Da Prato and T. Tsykalenko for useful discussions.

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