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# Self-adjointness of some infinite-dimensional elliptic operators and application to stochastic quantization

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**Abstract.** We consider an operator $\mathring{K}\varphi = L\varphi - \langle CDU(x), D\varphi\rangle$ in a Hilbert space $H$, where $L$ is an Ornstein–Uhlenbeck operator, $U \in W^{1,4}(H,\mu)$ and $\mu$ is the invariant measure associated with $L$. We show that $\mathring{K}$ is essentially self-adjoint in the space $L^2(H,\nu)$ where $\nu$ is the "Gibbs" measure $\nu(dx) = Z^{-1}e^{-2U(x)}dx$. An application to Stochastic quantization is given.

## 1. Introduction

We are concerned with the following operator in a separable Hilbert space (norm $|\cdot|$, inner product $\langle\cdot,\cdot\rangle$):

$$\mathring{K}\varphi = L\varphi - \langle C^{1/2}DU(x), C^{1/2}D\varphi\rangle, \tag{1.1}$$

defined on $D(\mathring{K}) = D(L) \cap C_b^2(H)$ and where the operator $L$, defined by

$$L\varphi = \frac{1}{2}\,\mathrm{Tr}\,[CD^2\varphi] + \langle Ax, D\varphi\rangle, \quad \varphi \in D(L), \tag{1.2}$$

is the Ornstein–Uhlenbeck operator, see Section 2 for precise definitions.

Here $A$ and $C$ are linear operators in $H$, and $U$ a is mapping from $H$ into $\mathbb{R}$. Moreover $D\varphi$ represents the Fréchet derivative of $\varphi$.

Such operators arise in several applications as the Landau–Ginzburg equations [3], and in stochastic quantization. An application will be given at the end of this paper.

Let us formulate our assumptions.

**Hypothesis 1.1.**
*(i)* $A : D(A) \subset H \to H$ *is self-adjoint strictly negative.*
*(ii)* $C = (-A)^{-\varepsilon}$ *for some* $\varepsilon \in (0,1)$.

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*(iii) The operator* $Q = \frac{1}{2}(-A)^{-(1+\varepsilon)}$ *is of trace class.*

Hence we shall denote by $\{-\alpha_k\}$ the sequence of (negative) eigenvalues of $A$ and by $\{e_k\}$ the corresponding complete orthonormal system. We have also, by denoting $\lambda_k = \frac{1}{2}\alpha_k^{-(1+\varepsilon)}$

$$Qe_k = \lambda e_k, \qquad k \in \mathbb{N}.$$

Moreover we shall denote by $\mu$ the gaussian measure $\mathcal{N}(0, Q)$ of mean 0 and covariance operator $Q$.

Now we consider the probability measure

$$\nu(dx) = Z^{-1}\, e^{-2U(x)}\, \mu(dx),$$

where

$$Z = \int_H e^{-2U(x)}\, \mu(dx),$$

under the following assumptions

**Hypothesis 1.2.**

*(i)* $e^{-U}$ *belongs to* $L^p(H, \mu)$ *for any* $p \geq 1$.
*(ii)* $U : H \to \mathbb{R}$ *belongs to* $W_C^{1,4}(H, \nu)$. ([1])

Our goal is to show that, under Hypotheses 1.1 and 1.2, $\mathring{K}$ is essentially self–adjoint in $L^2(H, \nu)$. In this way we are able to construct a semigroup $e^{tK},\ t \geq 0$ were $K$ is the closure of $\mathring{K}$. We notice that formally $e^{tK},\ t \geq 0$ is the transition semigroup corresponding to the differential stochastic equation

$$dX = (AX - CDU)dt + C^{1/2}dW(t), \quad X(0) = x, \tag{1.3}$$

where $W$ is a cylindrical Wiener process taking values in $H$. We notice that under assumptions above we are not able to find even a weak solution of equation (1.3).

The problem of self-adjointness of the operator $\mathring{K}$ has been studied by several authors under different assumptions using the Dirichlet forms theory, see e.g [17], [1], [16], and references therein. The existence of a self-adjoint extension $K$ of $\mathring{K}$ follows from the closability of the Dirichlet form naturally associated with $\mathring{K}$. Then the main problem consists in proving that $K$ is the closure of $\mathring{K}$ (Uniqueness problem).

Our approach is different. We show that the operator $\mathring{K}$ is symmetric, and that the image of $\lambda - \mathring{K}$ is dense on $L^2(H, \nu)$ for $\lambda > 0$. This will imply, by the Lumer-Phillips theorem, see [18], that $\mathring{K}$ is closable and its closure $K$ is self-adjoint (We will denote by $D(K)$ its domain).

[1] Spaces $W_C^{1,4}(H, \nu)$ are introduced in the next section. Moreover if $H$ and $K$ are Hilbert spaces we denote by $C_b(H; K)$ the Banach space of all uniformly continuous and bounded mappings from $H$ into $K$, endowed with the sup norm $\|\cdot\|_0$. Moreover, for any $k \in \mathbb{N}$, $C_b^k(H; K)$ will represent the Banach space of all mappings from $H$ into $K$, that are uniformly continuous and bounded together with their Fréchet derivatives of order less or equal to $k$ endowed with their natural norm $\|\cdot\|$. If $K = \mathbb{R}$ we set $C_b(H; K) = C_b(H)$ and $C_b^k(H; K) = C_b^k(H)$

The main tools in order to prove density of $(\lambda - \mathring{K})(L^2(H,\nu))$ are an approximation of $U$ by smooth functions and an *a priori* estimate on $W^{1,4}_C(H,\nu)$. This estimate is similar to one proved in the papers [17] and [16].

We can also show, see Remark 3.4 below, that the set $\mathscr{E}_A(H)$ of all functions $\varphi$ of the form

$$\varphi(x) = \mathrm{Re}\,\sum_{k=1}^n a_k e^{i\langle x,h_k\rangle},$$

where $n \in \mathbb{N}$, $h_1, ..., h_n \in D(A)$ and $a_1, a_2, \ldots, a_n \in \mathbb{C}$, is a core for $K$.

We notice that our assumptions are close to that of [16], but our method seems to be simpler and can be applied to non gradient Dirichlet operator, by replacing symmetry with dissipativity, see Remark 3.5.

As in [16] we give finally an application to Stochastic Quantization in dimensions 2.

Stochastic Quantization has been studied by several authors see [15], [5], [2], [14], [20]. In particular in [15], a transition semigroup for equation (1.3) was built by giving a meaning to Girsanov formula for $\varepsilon < 1/10$.

## 2. Preliminary results

We first introduce the Ornstein–Uhlenbeck semigroup $R_t,\ t \ge 0$, as a family of bounded operators in $C_b(H)$:

$$R_t\varphi(x) = \int_H \varphi(y)\mathscr{N}(e^{tA}x, Q_t)(dy),\ \ \varphi \in C_b(H), \tag{2.1}$$

where

$$Q_t = Q\left(1 - e^{2tA}\right),\ \ t \ge 0, \tag{2.2}$$

and $\mathscr{N}(e^{tA}x, Q_t)$ is the gaussian measure on $H$ of mean $e^{tA}x$ and covariance operator $Q_t$.

**Proposition 2.1.** *For any $\varphi \in C_b(H)$ and any $t > 0,\ h \in H$, $R_t\varphi$ is differentiable in the direction $C^{1/2}h$ and we have*

$$\langle DR_t\varphi, C^{1/2}h\rangle = \int_H \langle \Lambda(t)h, Q_t^{-1/2}y\rangle\varphi(e^{tA}x + y)\mathscr{N}(0, Q_t)(dy), \tag{2.3}$$

*where*

$$\Lambda(t) = \sqrt{2}\,(-A)^{1/2}(1 - e^{2tA})^{-1/2}e^{tA},\ \ t \ge 0. \tag{2.4}$$

*Moreover*

$$|C^{1/2}DR_t\varphi(x)| \le t^{-\frac{1}{2}}\|\varphi\|_0,\ \ t \ge 0. \tag{2.5}$$

*Proof*. (2.3) follows easily from the Cameron-Martin formula, see e.g. [11]. Let us check (2.5). Since

$$\Lambda(t)e_k = \sqrt{2}\ \frac{\alpha_k^{1/2}e^{-\alpha_k t}}{(1 - e^{-2\alpha_k t})^{1/2}}\ e_k,$$

we have

$$\|\Lambda(t)\| = \sup_{k\in N} \sqrt{2}\ \frac{\alpha_k^{1/2} e^{-\alpha_k t}}{(1-e^{-2\alpha_k t})^{1/2}} \le t^{-1/2}.$$

Now the conclusion follows integrating with respect to $\mu$. □

The semigroup $R_t$ is not strongly continuous in $C_b(H)$. However one can define its infinitesimal generator $L$, see [6], as the unique linear operator $L : D(L) \subset C_b(H) \to C_b(H)$ whose resolvent is given by

$$R(\lambda, L)\varphi(x) = \int_0^{+\infty} e^{-\lambda t} R_t\varphi(x)dt, \quad x \in H,\ \varphi \in C_b(H).$$

The following result is a consequence of (2.5).

**Proposition 2.2.** *For any $\varphi \in D(L)$ we have that $C^{1/2}D\varphi \in C_b(H; H)$ and*

$$|C^{1/2}D\varphi(x)| \le \sqrt{\pi}\ \ (\|\varphi\|_0 + \|L\varphi\|_0), \quad x \in H. \tag{2.6}$$

*Proof*. Let $\varphi \in D(L)$ and set $f = \varphi - L\varphi$. Then from (2.5) it follows that

$$|C^{1/2}DR(\lambda, L)f(x)| \le \sqrt{\pi}\ \|f\|_0,$$

that yields the conclusion. □

Let us also recall the following identity, see e.g. [13]

$$\int_H L\varphi\, \varphi\, d\nu = -\frac{1}{2} \int_H |C^{1/2}D\varphi|^2 d\nu, \quad \varphi \in D(L). \tag{2.7}$$

We now define the Sobolev space $W_C^{1,2}(H, \nu)$.

For any $k \in \mathbb{N}$, and for any $\varphi \in C_b^1(H)$, we denote by $D_k\varphi$ the derivative of $\varphi$ on the direction $e_k$, and we set $x_k = \langle x, e_k\rangle$. The following identity is well known, see e.g. [13],

$$\int_H D_k\varphi\psi d\mu = -\int_H \varphi D_k\psi d\mu + \frac{1}{\lambda_k} \int_H x_k\varphi\psi d\mu,\ \ \varphi, \psi \in C_b^1(H), \tag{2.8}$$

and so we obtain the result,

**Lemma 2.3.** *Let $\varphi, \psi \in C_b^1(H)$. Then for any $k \in \mathbb{N}$ the following identity holds*

$$\int_H D_k\varphi\psi d\nu = -\int_H \varphi D_k\psi d\nu + 2\int_H D_kU\varphi\psi d\nu + \frac{1}{\lambda_k} \int_H x_k\varphi\psi d\nu. \tag{2.9}$$

*Moreover $D_k$ is closable in $L^2(H, \nu)$; we shall still denote by $D_k$ its closure.*

We can finally define the Sobolev space $W_C^{1,2}(H,\nu)$ by setting

$$W_C^{1,2}(H,\nu) = \Big\{\varphi \in L^2(H,\nu) :\ D_i\varphi \in L^2(H,\nu) \quad \forall\, i \in \mathbb{N},$$
$$\sum_{i=1}^{\infty} \alpha_i^{-\varepsilon} \int_H |D_i\varphi(x)|^2 \nu(dx) < +\infty\Big\}.$$

We conclude this section with a result needed later.

**Lemma 2.4.** *Let $\beta : \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous* ([2]), *$\varphi \in C_b^1(H)$ and $p > 1$. Then $\beta \circ \varphi \in W_C^{1,p}(H,\nu)$.*

*Proof*. Let $\{\beta_n\} \subset C^1(\mathbb{R})$ be uniformly convergent to $\beta$ and such that $\sup_{n\in\mathbb{N}} \|\beta_n\|_{\mathrm{Lip}} \le \|\beta\|_{\mathrm{Lip}}$. Then $\beta_n \circ \varphi \to \beta \circ \varphi$ uniformly and

$$\int_H |C^{1/2} D\beta_n \circ \varphi|^2 d\nu = \int_H |D\beta_n(\varphi(x))|^2 |C^{1/2} D\varphi(x)|^2 d\nu \le \|\beta\|_{\mathrm{Lip}}^2 \|\varphi\|_1^2.$$

Thus, by a standard argument, we have $\beta \circ \varphi \in W_C^{1,p}(H,\nu)$. □

**Proposition 2.5.** *Let $p \ge 1$, and let $\varphi \in W_C^{1,p}(H,\nu)$. Then for any constant $\kappa > 0$ we have* $\min\{\varphi,\kappa\} \in W_C^{1,p}(H,\nu)$.

*Proof.* Let $\{\varphi_n\}$ be a sequence in $C_b^1(H)$ convergent to $\varphi$ in $W_C^{1,p}(H,\nu)$. Then $\psi_n = \min\{\varphi_n,\kappa\},\ n \in \mathbb{N}$ belongs to $W_C^{1,p}(H,\nu)$ by Lemma 2.4. Therefore we have $\psi_n \to \min\{\varphi,\kappa\}$ in $L^p(H,\nu)$, and moreover

$$\|C^{1/2} D\psi_n\|_{L^p(H,\nu)} \le \|C^{1/2} D\varphi_n\|_{L^p(H,\nu)} \le \sup_{n\in\mathbb{N}} \|C^{1/2} D\varphi_n\|_{L^p(H,\nu)} < +\infty.$$

The same argument as before implies the conclusion. □

## 3. The main result

Here we assume that Hypotheses 1.1 and 1.2 hold. We are concerned with the operator $\mathring{K}$ defined by (1.1). By (2.7) it follows immediately that for any $\varphi, \psi \in D(\mathring{K})$ we have

$$\int_H \mathring{K}\varphi\psi\, d\nu = -\frac{1}{2} \int_H \langle CD\varphi, D\psi\rangle d\nu. \tag{3.1}$$

Therefore $\mathring{K}$ is symmetric and consequently closable in $L^2(H,\nu)$.

We now introduce an approximating problem. Let $\{U_n\}$ be a sequence in $C_b^\infty(H)$ convergent to $U$ in $W_C^{1,4}(H,\nu)$. Such a sequence can be easily constructed by setting

$$U_n(x) = S_{1/n}[V_n(x)],$$

[2] $\|\beta\|_{\mathrm{Lip}} = \sup_{x,y\in\mathbb{R}} \frac{|\beta(x)-\beta(y)|}{|x-y|}$

where $S_t$ is an auxiliary strong Feller Ornstein–Uhlenbeck semigroup, see [11, (9.50)], ([3]) and

$$V_n(x) = \begin{cases} U(x) & \text{if } |U(x)| \le n, \\ \frac{U(x)}{|U(x)|} n & \text{if } |U(x)| > n, \end{cases}$$

and recalling Proposition 2.5.

We define

$$K_n\varphi = L\varphi - \langle CDU_n, D\varphi\rangle,\ \ \varphi \in D(L). \tag{3.2}$$

Then it is easy to check, by a simple fixed point argument taking into account Proposition 2.2, that the resolvent set of $K_n$ contains the half line $(0, +\infty)$, and (since $DU_n$ is regular) that $K_n$ is the infinitesimal generator of the transition semigroup corresponding to the differential stochastic equation

$$\begin{cases} dX_n(t,x) = (AX_n(t,x) - DU_n(X_n(t,x)))dt + \sqrt{C}\, dW(t), \\ X_n(0,x) = x. \end{cases} \tag{3.3}$$

Consequently

$$R(\lambda, K_n)f(x) = \int_0^{+\infty} e^{-\lambda t}\,\mathbb{E}(f(X_n(t,x))dt,\ \ f \in C_b(H), \tag{3.4}$$

Let now $f \in C_b^2(H)$ and let $\varphi_n$ be the solution to

$$\lambda\varphi_n - K_n\varphi_n = f. \tag{3.5}$$

It is easy to see that $\varphi_n \in D(L) \cap C_b^2(H)$, and

$$\lambda\varphi_n - \mathring{K}\varphi_n + \langle C^{1/2}(DU - DU_n), C^{1/2}D\varphi_n\rangle = f. \tag{3.6}$$

In order to prove that $\mathring{K}$ is essentially self–adjoint we will show that the image of $\lambda - \mathring{K}$ is dense in $L^2(H,\nu)$. For this we need an estimate of $|C^{1/2}D\varphi_n|$ in $L^4(H,\nu)$. This is provided by the following result.

**Proposition 3.1.** *Let $f \in C_b^2(H)$. Then the solution $\varphi_n$ to* (3.5) *belongs to $D(\mathring{K})$ and there is a positive constant $\kappa$ depending only on $\lambda$, $\|f\|_\infty$, and $\|C^{1/2}DU\|_{L^4(H,\nu)}$ such that*

$$\int_H \langle CD\varphi_n, D\varphi_n\rangle^2 d\nu \le \kappa. \tag{3.7}$$

*Proof.* We proceed in several steps.

*Step 1*. For any $\varphi, \psi \in D(\mathring{K})$ we have.

$$\int_H K_n\varphi\ \psi d\nu = -\frac{1}{2}\int_H \langle CD\varphi, D\psi\rangle d\nu + \int_H \langle CD\varphi, D(U - U_n)\rangle\psi d\nu. \tag{3.8}$$

[3] We can choose $S_t\varphi(x) = \int_H \varphi(e^{\frac{t}{2}Q^{-1}}x + y)\mathcal{N}(0, Q(1 - e^{tQ^{-1}}))(dy)$

The simple verification is left to the reader.

*Step 2*. For any $\varphi \in D(\mathring{K})$ we have

$$\int_H \langle K_n D\varphi, CD\varphi\rangle d\nu = -\frac{1}{2}\int_H \mathrm{Tr}\,[(CD^2\varphi)^2]d\nu + \int_H \langle D^2\varphi\ CD\varphi, C(DU - DU_n)\rangle d\nu. \tag{3.9}$$

We have in fact, taking into account (3.8),

$$\begin{aligned}\int_H \langle K_n D\varphi, CD\varphi\rangle d\nu &= \sum_{i=1}^\infty \alpha_i^{-\varepsilon}\int_H K_n D_i\varphi\ D_i\varphi d\nu\\ &= -\frac{1}{2}\sum_{i=1}^\infty \alpha_i^{-\varepsilon}\int_H \langle CDD_i\varphi, DD_i\varphi\rangle d\nu\\ &\quad + \sum_{i=1}^\infty \alpha_i^{-\varepsilon}\int_H \langle CDD_i\varphi, D(U-U_n)\rangle D_i\varphi d\nu,\end{aligned}$$

and the conclusion follows.

*Step 3*. For any $\varphi \in D(\mathring{K})$ we have

$$\begin{aligned}&2\int_H (K_n\varphi)^2 d\nu + \int_H \langle D^2\varphi\cdot CD\varphi, CDU_n\rangle d\nu + 2\int_H K_n\varphi\langle D\varphi, CDU_n\rangle d\nu\\ &\quad + \int_H \langle D^2\varphi\cdot CD\varphi, C(DU-DU_n)\rangle d\nu - 2\int_H K_n\varphi\langle D\varphi, C(DU-DU_n)\rangle d\nu\\ &\quad -2\int_H \langle CD\varphi, D(U-U_n)\rangle\ \langle CD\varphi, DU_n\rangle d\nu\\ &\qquad = \frac{1}{2}\int_H \mathrm{Tr}\,[(CD^2\varphi)^2]d\nu - \int_H \langle ACD\varphi, D\varphi\rangle d\nu.\end{aligned} \tag{3.10}$$

In fact, setting in (3.8) $\psi = K_n\varphi$ we find

$$\begin{aligned}\int_H (K_n\varphi)^2 d\nu &= -\frac{1}{2}\int_H \langle CD\varphi, DK_n\varphi\rangle d\nu + \int_H \langle CD\varphi, D(U-U_n)\rangle K_n\varphi d\nu\\ &= -\frac{1}{2}\int_H \langle CD\varphi, [D, K_n]\varphi\rangle d\nu - \frac{1}{2}\int_H \langle CD\varphi, CK_n\varphi\rangle d\nu\\ &\quad + \int_H \langle CD\varphi, D(U-U_n)\rangle K_n\varphi d\nu\end{aligned}$$

Now, taking into account (3.9) and (3.10), we find

$$\begin{aligned}\int_H (K_n\varphi)^2 d\nu &= -\frac{1}{2}\int_H \langle CD\varphi, AD\varphi\rangle d\nu + \frac{1}{2}\int_H \langle CD\varphi, D\{\langle CDU, D\varphi\rangle\}\rangle d\nu\\ &= -\frac{1}{2}\int_H \langle D^2\varphi\ CDU, D\varphi\rangle d\nu + \frac{1}{4}\int_H \mathrm{Tr}\,[(CD^2\varphi)^2]d\nu\end{aligned}$$

$$= -\frac{1}{2} \int_H \langle D^2\varphi\, CD\varphi, C(DU - DU_n)\rangle d\nu$$
$$+ \int_H \langle CD\varphi, DU - DU_n\rangle K_n\varphi d\nu \tag{3.11}$$

Setting in (3.8) $\psi = \langle CDU, D\varphi\rangle$ we find

$$\int_H \langle CD\varphi, D\{\langle CDU, D\varphi\rangle\}\rangle d\nu$$
$$-2\int_H K_n\varphi\langle CDU, D\varphi\rangle d\nu + 2\int_H \langle CD\varphi, D(U - U_n)\rangle\langle CDU_n, D\varphi\rangle d\nu.$$

By substituting this in (3.11) the conclusion follows.

*Step 4*. For any $\varphi \in D(\mathring{K})$ we have

$$\begin{aligned}\int_H \langle CD\varphi, D\varphi\rangle^2 d\nu = &-2\int_H K_n\varphi\ \varphi\ \langle CD\varphi, D\varphi\rangle d\nu\\ &-2\int_H \varphi\ \langle D^2\varphi \cdot CD\varphi, CD\varphi\rangle d\nu\\ &+2\int_H \langle CD\varphi, DU - DU_n\rangle\langle CD\varphi, D\varphi\rangle\varphi d\nu.\end{aligned} \tag{3.12}$$

Setting in (3.8) $\psi = \varphi\langle CD\varphi, D\varphi\rangle$ we find

$$\begin{aligned}\int_H K_n\varphi\ \varphi\ \langle CD\varphi, D\varphi\rangle d\nu = &-\frac{1}{2} \int_H \langle CD\varphi, D(\varphi\langle CD\varphi, D\varphi\rangle)\rangle d\nu\\ &+ \langle CD\varphi, DU - DU_n\rangle\varphi\langle CD\varphi, D\varphi\rangle,\end{aligned}$$

that yields (3.12).

Now it is easy to see that

$$\|\varphi_n\|_\infty \le \frac{1}{\lambda}\ \|f\|_\infty, \tag{3.13}$$

and

$$\|K_n\varphi_n\|_\infty \le 2\|f\|_\infty. \tag{3.14}$$

*Step 5*. We have

$$\|C^{1/2}D\varphi_n\|_{L^2(H,\nu)} \le \frac{2}{\lambda}\ \|f\|_\infty\left[\sqrt{2\lambda}\ + \|C^{1/2}D(U - U_n)\|_{L^2(H,\nu)}\right]. \tag{3.15}$$

We first note that, multiplying (3.5) by $\varphi_n$ and integrating, we have

$$\lambda\int_H \varphi_n^2 d\nu - \int_H K_n\varphi_n\ \varphi_n d\nu = \int_H f\varphi_n d\nu.$$

Taking into account (3.8), (3.13), and (3.14), it follows

$$\begin{aligned}\frac{1}{2}\int_H \langle CD\varphi_n, CD\varphi_n\rangle d\nu &\le \int_H f\varphi_n d\nu + \int_H \langle CD\varphi_n, D(U-U_n)\rangle \varphi_n d\nu\\ &\le \frac{1}{\lambda}\|f\|_\infty^2 + \frac{1}{\lambda}\|f\|_\infty \int_H |C^{1/2}D\varphi_n||C^{1/2}D(U-U_n)|d\nu\\ &\le \frac{1}{\lambda}\|f\|_\infty^2 + \frac{1}{\lambda}\|f\|_\infty \left(\int_H |C^{1/2}D\varphi_n|^2 d\nu\right)^{1/2}\\ &\quad\times\left(\int_H |C^{1/2}D(U-U_n)|^2 d\nu\right)^{1/2},\end{aligned}$$

that yields the conclusion.

*Step 6*. We have

$$\left|\int_H \langle D^2\varphi\cdot Cu, Cv\rangle d\nu\right| \le \left(\int_H \mathrm{Tr}\,[(CD^2\varphi)^2]d\nu\right)^{1/2}\left(\int_H \langle Cu,u\rangle\ \langle Cv,v\rangle d\nu\right)^{1/2}.$$

*Step 7*. Conclusion.

We now estimate $\int_H \langle CD\varphi_n, D\varphi_n\rangle^2\, d\nu$ starting from (3.12), that we write in the form

$$\int_H \langle CD\varphi_n, D\varphi_n\rangle^2\, d\nu = \Lambda_1 + \Lambda_2 + \Lambda_3. \tag{3.16}$$

By (3.15) it follows

$$\begin{aligned}\Lambda_1 &\le \frac{2}{\lambda}\|f\|_\infty^2 \int_H |C^{1/2}D\varphi_n|^2 d\nu\\ &\le \frac{4}{\lambda^3}\|f\|_\infty^4\left[\sqrt{2\lambda} + \|C^{1/2}D(U-U_n)\|_{L^2(H,\nu)}\right]\end{aligned} \tag{3.17}$$

Moreover, taking into account (3.15) we find

$$\Lambda_2 \le \frac{1}{\lambda}\|f\|_\infty^2 \left(\int_H \mathrm{Tr}\,[(CD^2\varphi_n)^2]d\nu\right)^{1/2}\left(\int_H |C^{1/2}D\varphi_n|^4 d\nu\right)^{1/2}, \tag{3.18}$$

and

$$\begin{aligned}\Lambda_3 &\le \frac{2}{\lambda}\|f\|_\infty \int_H |C^{1/2}D\varphi_n|^3|DU-DU_n|d\nu\\ &\le \frac{2}{\lambda}\|f\|_\infty \left(\int_H |D(U-U_n)|^4 d\nu\right)^{1/4}\left(\int_H |C^{1/2}D\varphi_n|^4 d\nu\right)^{3/4}.\end{aligned} \tag{3.19}$$

Substituting (3.17), (3.18) and (3.19) in (3.16), we find

$$\int_H |C^{1/2}D\varphi_n|^4 d\nu \le \frac{4}{\lambda^3}\ \|f\|_\infty^4 \left[\sqrt{2\lambda}\ +\|C^{1/2}D(U-U_n))\|_{L^2(H,\nu)}\right]$$

$$+\frac{1}{\lambda}\ \|f\|_\infty^2 \left(\int_H \mathrm{Tr}\,[(CD^2\varphi_n)^2]d\nu\right)^{1/2}\left(\int_H |C^{1/2}D\varphi_n|^4 d\nu\right)^{1/2} \tag{3.20}$$

$$+\frac{2}{\lambda}\ \|f\|_\infty \left(\int_H |D(U-U_n)|^4 d\nu\right)^{1/4}\left(\int_H |C^{1/2}D\varphi_n|^4 d\nu\right)^{3/4}.$$

Consequently, there exists a constant $\kappa_1 > 0$ depending on $\lambda$, $\|f\|_\infty$, and $\|C^{1/2}DU\|_{L^4(H,\nu)}$, but not on $n$, such that

$$\int_H |C^{1/2}D\varphi_n|^4 d\nu \le \kappa_1\left(1+\int_H \mathrm{Tr}\,[(CD^2\varphi_n)^2]d\nu\right). \tag{3.21}$$

Now we estimate the second hand side.

By (3.10), using the Hölder inequality, we find

$$\begin{aligned}
\frac{1}{2}\ \int_H \mathrm{Tr}\,[(CD^2\varphi_n)^2]d\nu &\le 4\|f\|_\infty^2\\
&+\left(\int_H \mathrm{Tr}\,[(CD^2\varphi_n)^2]d\nu\right)^{1/2}\left(\int_H |C^{1/2}D\varphi_n|^2|C^{1/2}DU_n|^2 d\nu\right)^{1/2}\\
&+2\|f\|_\infty\int_H |C^{1/2}D\varphi_n||C^{1/2}DU_n|d\nu\\
&+\left(\int_H \mathrm{Tr}\,[(CD^2\varphi_n)^2]d\nu\right)^{1/2}\left(\int_H |C^{1/2}D\varphi_n|^2|C^{1/2}D(U-U_n)|^2 d\nu\right)^{1/2}\\
&+4\|f\|_\infty\int_H |C^{1/2}D\varphi_n||C^{1/2}D(U-U_n)|d\nu\\
&+2\int_H |C^{1/2}D\varphi_n|^2|C^{1/2}DU_n||C^{1/2}D(U-U_n)|d\nu.
\end{aligned}$$

Using the Hölder inequality we find

$$\begin{aligned}
\frac{1}{2}\ \int_H \mathrm{Tr}\,[(CD^2\varphi_n)^2]d\nu &\le 4\|f\|_\infty^2\\
&+\left(\int_H \mathrm{Tr}\,[(CD^2\varphi_n)^2]d\nu\right)^{\frac12}\left(\int_H |C^{1/2}D\varphi_n|^4 d\nu\right)^{\frac14}\left(\int_H |C^{1/2}DU_n|^4 d\nu\right)^{\frac14}\\
&+2\|f\|_\infty\left(\int_H |C^{1/2}D\varphi_n|^2 d\nu\right)^{\frac12}\left(\int_H |C^{1/2}DU_n|^2 d\nu\right)^{\frac12}\\
&+\left(\int_H \mathrm{Tr}\,[(CD^2\varphi_n)^2]d\nu\right)^{\frac12}\left(\int_H |C^{1/2}D\varphi_n|^4 d\nu\ \int_H |C^{1/2}D(U-U_n)|^4 d\nu\right)^{\frac14}
\end{aligned}$$

$$+4\|f\|_\infty \left(\int_H |C^{1/2}D\varphi_n|^2 d\nu\right)^{\frac{1}{2}} \left(\int_H |C^{1/2}D(U-U_n)|^2 d\nu\right)^{\frac{1}{2}}$$
$$+2\left(\int_H |C^{1/2}D\varphi_n|^4 d\nu\right)^{\frac{1}{2}}\left(\int_H |C^{1/2}DU_n|^4 d\nu \cdot \int_H |C^{1/2}D(U-U_n)|^4 d\nu\right)^{\frac{1}{4}}.$$

Therefore there exists $\kappa_2 > 0$ depending only on $\lambda$ and $\|f\|_\infty$ such that

$$\int_H \mathrm{Tr}\,[(CD^2\varphi_n)^2]d\nu$$
$$\leq \kappa_2\left[1+\left(\int_H \mathrm{Tr}\,[(CD^2\varphi_n)^2]d\nu\right)^{1/2}\left(\int_H |C^{1/2}D\varphi_n|^4\,d\nu\right)^{1/4}\right.$$
$$\left. +\left(\int_H |C^{1/2}D\varphi_n|^4\,d\nu\right)^{1/2}\right]. \tag{3.22}$$

Setting

$$x=\int_H |C^{1/2}D\varphi_n|^4\,d\nu, \quad y=\int_H |C^{1/2}D\varphi_n|^4\,d\nu,$$

we find by (3.21) and (3.22)

$$y \leq \kappa_1(1+x), \quad x \leq \kappa_2(1+y^{1/2}+x^{1/2}y^{1/4}).$$

This gives the conclusion. □

Now we are ready to prove

**Theorem 3.2.** *Assume that Hypotheses 1.1 and 1.2 holds. Then $\mathring{K}$ is essentially self-adjoint in $L^2(H,\nu)$. Moreover, denoting by $K$ its closure, we have*

$$\int_H K\varphi\psi d\nu = -\frac{1}{2}\int_H \langle CD\varphi, D\psi\rangle\,d\nu. \qquad \varphi,\psi \in D(K). \tag{3.23}$$

*Proof.* Since $\mathring{K}$ is symmetric, it is sufficient to show that the image of $\lambda-\mathring{K}$ is dense in $L^2(H,\nu)$, see [18].

Let $f\in C_b^2(H)$ and let $\varphi_n$ be the solution of (3.5). By the Hölder inequality, we have

$$\left[\int_H |\langle C^{1/2}(DU-DU_n), C^{1/2}D\varphi_n\rangle|^2 d\nu\right]^2$$
$$\leq \int_H |C^{1/2}(DU-DU_n)|^4 d\nu \int_H |C^{1/2}D\varphi_n|^4 d\nu$$

Taking into account Proposition 3.1 we obtain

$$\lim_{n\to\infty}\int_H |\langle C^{1/2}(DU-DU_n), C^{1/2}D\varphi_n\rangle|^2 d\nu = 0,$$

so that

$$\lambda\varphi_n - \mathring{K}\varphi_n \to f$$

in $L^2(H,\nu)$ and, since $C_b^2(H)$ is dense in $L^2(H,\nu)$, it follows that $\mathring{K}$ is essentially self-adjoint. The last statement follows from (3.1). □

**Remark 3.3.** By (3.23) for any $\varphi \in L^2(H, \nu)$ we have

$$\int_H |e^{Kt}\varphi|^2 \, d\nu + \int_0^t ds \int_H |C^{1/2} D e^{Kt}\varphi|^2 \, d\nu = \int_H |\varphi|^2 \, d\nu. \tag{3.24}$$

Using (3.24) it is not difficult to show

$$\lim_{t\to\infty} e^{Kt}\varphi = \int_H \varphi \, d\nu \qquad \text{in } L^2(H, \nu).$$

Thus the measure $\nu$ is strongly mixing.

**Remark 3.4.** Let $K$ be the closure of $\mathring{K}$. Then obviously $D(L) \cap C_b^2(H)$ is a core for $K$. Since $\mathscr{E}_A(H) \subset D(L_2) \cap C_b^2(H)$([4]) and is dense in $D(L_2)$, endowed with the graph norm, then it is not difficult to show that $\mathscr{E}_A(H)$ is a core for $K$.

**Remark 3.5.** Let $F \in L^4(H, \nu; H)$ and consider the operator

$$\mathring{N}\varphi = L\varphi - \langle F(x), D\varphi\rangle, \;\; \varphi \in C_b^1(H) \cap D(L). \tag{3.25}$$

Assume that there exists a Borel measure $\nu$ on $H$ such that

$$\int_H \mathring{N}\varphi\varphi d\nu = -\frac{1}{2} \int_H |C^{1/2} D\varphi|^2 d\nu. \tag{3.26}$$

Then all previous considerations can be repeated and we can conclude that $\mathring{N}$ is closable on $L^2(H, \nu)$ and its closure is $m$–dissipative.

## 4. Application to stochastic quantization

Let $\mathbb{S}$ be the square $[0, 2\pi]^2$. Let us consider the differential stochastic equation

$$\begin{cases} dX = \frac{1}{2} \; ((\Delta - 1)X + \sigma X^n)\, dt + dW(t), \\ \\ X(0) = x, \end{cases} \tag{4.1}$$

where $n$ is odd, $\sigma > 0$, and $W$ is the cylindrical Wiener process on $L^2(\mathbb{S})$, $W(t)$ is defined as

$$W(t)(\xi) = \sum_{h\in\mathbb{Z}^2} e_h(\xi)\eta_h,$$

where $\{\eta_h\}$ is a sequence of gaussian random variables $\mathscr{N}(0, 1)$ mutually independent and $\{e_h\}$ is the complete orthonormal system in $L^2(\mathbb{S})$ defined by

$$e_h(\xi) = \frac{1}{2\pi} \, e^{i\langle h,\xi\rangle}, \qquad h = (h_1, h_2) \in \mathbb{Z}^2, \; \xi \in \mathbb{S}.$$

Moreover $\Delta$ is the realization of the Laplace operator with $2\pi$–periodic boundary conditions.

[4] Here we denote by $D(L_2)$ the domain of Ornstein-Uhlenbeck operator in $L^2(H, \mu)$

Equation (4.1) describes a gradient system having formally the Gibbs invariant probability measure:

$$\nu(dx) = Z^{-1} e^{\frac{1}{n+1}\int_{\mathbb{S}} x^{n+1}(\xi)d\xi} \mathscr{N}(0, C)(dx),$$

where $C = (1 - \Delta)^{-1}$. Notice that this definition is not meaningful since

$$\text{Tr } C = \sum_{h\in\mathbb{Z}^2} \frac{1}{1 + |h|^2} = +\infty.$$

Equation (4.1) takes in account an interacting field. The corresponding free field is described by

$$\begin{cases} dZ = \frac{1}{2}\ (\Delta - 1)Zdt + dW(t), \\ \\ Z(0, x) = x, \end{cases} \tag{4.2}$$

so that

$$Z(t, x) = e^{\frac{1}{2}(\Delta-1)t}x + \int_0^t e^{\frac{1}{2}(\Delta-1)(t-s)}\, dW_s.$$

Since

$$\mathbb{E}|Z(t, 0)|^2 = \text{ Tr } [C(1 - e^{(\Delta-1)t})] = +\infty,$$

then $Z(t)$ does not live in $L^2(\mathbb{S})$. However it lives in $H^{-1}(\mathbb{S})$ defined as the completion of $L^2(\mathbb{S})$ with respect to the inner product

$$\langle x, y\rangle_{-1} = \sum_{h\in\mathbb{Z}^2} (1 + |h|^2)^{-1} \langle x, e_h\rangle \overline{\langle y, e_h\rangle}.$$

We have in fact

$$\mathbb{E}|Z(t, 0)|^2_{-1} = \text{ Tr } [C^2(1 - e^{(\Delta-1)t})] < +\infty,$$

as easily checked.

Now we interpret equation (4.1) as an equation in $H^{-1}(\mathbb{S})$. For this we set $W_1(t) = C^{-1/2}W(t)$ so that $W_1$ is a cylindrical Wiener process in $H^{-1}(\mathbb{S})$ and replace the Gibbs measure by

$$\nu(dx) = Z^{-1} \exp\left(\frac{1}{n+1} \int_{\mathbb{S}} : x^{n+1}(\xi) : d\xi\right) \mathscr{N}(0, C)(dx), \tag{4.3}$$

where $: x^{n+1}(\xi) :$ is the Wick product that we recall below and $Z$ is the normalization constant. In this way (4.1) is replaced by

$$\begin{cases} dX = \frac{1}{2}\ ((\Delta - 1)X + \sigma C : X^n :)\, dt + C^{1/2}dW_1(t), \\ \\ X(0) = x. \end{cases} \tag{4.4}$$

We notice that the factor $C$ in front to $: X^n :$ is necessary to make (4.4) a gradient system (having (4.3) as invariant measure).

We fix now $n \in \mathbb{N}$ and recall the definition of the Wick monomial of order $n$:

$$:x^n:\,=\,G_n(x) : H^{-1}(\mathbb{S}) \to H^{-1}(\mathbb{S}).$$

where $G_n(x)$ is the limit in $L^2(H^{-1}(\mathbb{S}), \mu, H^{-1}(\mathbb{S}))$ of the functions $G_{n,N}(x)$ defined as

$$G_{n,N}(x)(\xi) = \sqrt{n!}\rho_N^n\ \mathbb{H}_n\left(\frac{1}{\rho_N}\sum_{|h|\le N}\langle x, e_h\rangle e_h(\xi)\right),$$

where $\mathbb{H}_n$ denotes the Hermite polynomial of order $n$ (see [10]) and

$$\rho_N^2 = \frac{1}{(2\pi)^2}\sum_{|h|\le N}\frac{1}{1+|h|^2}.$$

Moreover, setting

$$U_n(x) = \frac{1}{n+1}\int_{\mathbb{S}} G_{n+1}(x)(\xi)d\xi,$$

we have that $U_n \in W_C^{1,p}(H^{-1}(\mathbb{S}), \mu)$ for all $p \ge 1$, and $DU_n(x) = G_n(x)$. Also if $n$ is a positive odd integer

$$e^{-pU_n(x)} \text{ is in } L^1(H^{-1}(\mathbb{S}), \mu)$$

for all $p \ge 1$. For a proof of these results we refer to Simon [22]; see also [10].

Finally fix $\sigma > 0$ and a positive odd integer $n$. Then consider the following linear operator in $H = H^{-1}(\mathbb{S})$:

$$\mathring{K}\varphi = \tfrac{1}{2}\ \mathrm{Tr}\,[CD^2\varphi] - \tfrac{1}{2}\ \langle C^{-1}x, D\varphi\rangle_{-1} - \tfrac{\sigma}{2}\ \langle CG_n(x), D\varphi\rangle_{-1}.$$

Setting $A = -\frac{1}{2}\ C^{-1}$, $U = -\frac{\sigma}{2}\ U_n$, we can apply Theorem 3.2 and conclude that $\mathring{K}$ is essentially selfadjoint on $L^2(H, \nu)$.

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