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Degree Sums and Path-Factors in Graphs*

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Abstract. Let G be a connected graph of order n and suppose that $n = \sum_{i=1}^{k} n_i$, where $n_i \geq 2$ are integers. In this paper we give some sufficient conditions in terms of degree sums to ensure that G contains a spanning subgraph consisting of vertex disjoint paths of orders $n_1, n_2, \ldots, n_k$.

1. Introduction

In this paper all graphs considered are finite undirected graphs without loops and multiple edges. Let G be a graph, $V(G)$ and $E(G)$ will denote the set of its vertices and edges, respectively. The neighborhood $N_G(v)$ of a vertex v is the set of vertices adjacent to v and the degree $d_G(v)$ of v is $|N_G(v)|$. For a vertex $v \in V(G)$ and a subgraph H of G, $N_H(v)$ is the set of neighbours of v contained in H, i.e., $N_H(v) = N_G(v) \cap V(H)$. We let $d_H(v) = |N_H(v)|$. We will write $N(v)$ and $d(v)$ instead of $N_G(v)$ and $d_G(v)$, respectively. A subgraph H is said to be *k-dominating* if $d_H(v) \geq k$ holds for every vertex $v \in V(G - H)$. A subgraph H is said to be *strongly dominating* if $G - H$ contains no edges. Let C be a cycle. We denote by $\overrightarrow{C}$ the cycle C with a given orientation, and by $\overleftarrow{C}$ the cycle C with the reverse orientation. If $u, v \in V(C)$ then $u\overrightarrow{C}v$ denotes the consecutive vertices of C from u to v in the direction specified by $\overrightarrow{C}$. The same vertices, in reverse order, are given by $v\overleftarrow{C}u$. If $u = v$ then $u\overrightarrow{C}v = \{u\}$. We call $u\overrightarrow{C}v$ an s-segment of C if $|u\overrightarrow{C}v| = s + 2$. We will consider $u\overrightarrow{C}v$ and $v\overleftarrow{C}u$ both as paths and vertex sets. We use u^+ to denote the successor of u and u^- to denote its predecessor. If $A \subset V(C)$ then $A^+ = \{a^+ : a \in A\}$ and $A^- = \{a^- : a \in A\}$. Similar notation is used for paths.

A path of order k is denoted by P_k. A spanning subgraph H of G is called a *path-factor* if each component of H is a path of order at least 2. Specially, H is called a *P_k-factor* if each component of H is isomorphic to P_k. The independent number, and connectivity of G are denoted by $\alpha(G)$ and $\kappa(G)$, respectively.

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Let $U \subseteq V = V(G)$. We define:
$\delta(U) = \min\{d_G(x) : x \in U\}$,
$\Delta(U) = \max\{d_G(x) : x \in U\}$,
$\sigma_3(U) = \min\{\sum_{i=1}^{3} d_G(v_i) : \{v_1, v_2, v_3\} \subseteq U$ is an independent set of G $\}$.
In particular, we write $\delta(G)$, $\Delta(G)$ and $\sigma_3(G)$ insteading of $\delta(V)$, $\Delta(V)$ and $\sigma_3(V)$, respectively. Terminology not defined here can be found in [2].

In [7] Johansson proved the following result:

Theorem 1. (*Johansson* [7]). *Let G be a connected graph of order* $n = \sum_{i=1}^{k} n_i$, *where* $n_i \geq 2$ *for all* $1 \leq i \leq k$. *If* $\delta(G) \geq \lfloor \frac{1}{2} n_1 \rfloor + \cdots + \lfloor \frac{1}{2} n_k \rfloor$, *then G contains a path-factor consisting of paths of orders* $n_1, n_2, \ldots, n_k$.

The form of the degree condition in Theorem 1 came from the following conjecture of M. El-Zahar:

Conjecture. (*El-Zahar* [4]). *If G is a graph with* $n = n_1 + \cdots + n_k$ *vertices and* $\delta(G) \geq \lceil \frac{1}{2} n_1 \rceil + \cdots + \lceil \frac{1}{2} n_k \rceil$, *then G has a spanning subgraph consisting of cycles of lengths* $n_1, \ldots, n_k$.

The case of $n_i = 3$ for all i of Theorem 1 was settled by Enomoto, Kaneko and Tuza in [6].

Theorem 2. (*Enomoto et al.* [6]). *Suppose G is a connected graph of order* $3k$ *with* $\delta(G) \geq k$. *Then G has a* P_3*-factor.*

Let $n, n_1, n_2, \ldots, n_k$ be integers. If $n = \sum_{i=1}^{k} n_i$ and $n_i \geq 2$ $(1 \leq i \leq k)$, then we call $(n_1, n_2, \ldots, n_k)$ a k-partition of n. Given a k-partition $(n_1, n_2, \ldots, n_k)$ of n, we let $\lambda = \lambda(n_1, n_2, \ldots, n_k) = |\{n_i : n_i \text{ is even}, \ 1 \leq i \leq k\}|$. It is easy to see that Theorem 1 is equivalent to the following:

Theorem 1′. *Let G be a connected graph of order n and* $(n_1, n_2, \ldots, n_k)$ *a k-partition of n. If* $\delta(G) \geq (n - k + \lambda)/2$ *then G contains a path-factor consisting of paths of orders* $n_1, n_2, \ldots, n_k$.

We note that if G contains a hamiltonian path then Theorem 1 holds trivially. So we may always assume that G has no hamiltonian path. Thus we can get that $\alpha(G) \geq 3$ by a result of Chvátal and Erdös [3] which says that if $\alpha(G) \leq \kappa(G) + 1$ then G has a hamiltonian path. Hence we may consider the existences of path-factors in connected graphs with the assumption that $\alpha(G) \geq 3$.

In this paper we give some sufficient conditions in terms of degree sums for the existence of path-factors in a connected graph. The following are main results of this paper.

Theorem A. *Let G be a connected graph of order n and* $(n_1, n_2, \ldots, n_k)$ *a k-partition of n. If* $n \geq 3(k - \lambda) + 4$ *and* $\sigma_3(G) \geq 3(n - k + \lambda)/2 - 2$ *then G contains a path-factor consisting of paths of orders* $n_1, n_2, \ldots, n_k$.

For the case $n < 3(k - \lambda) + 4$, we first note the following:

Proposition 1. *Let $(n_1, n_2, \ldots, n_k)$ be a k-partition of n. Then $n < 3(k-\lambda)+4$ if and only if one of the following three cases holds*:

(a) $n = 3k+2$, $n_1 = n_2 = \cdots = n_{k-1} = 3$ *and* $n_k = 5$;
(b) $n = 3k$, $n_1 = n_2 = \cdots = n_k = 3$;
(c) $n = 3k-1$, $n_1 = n_2 = \cdots = n_{k-1} = 3$ *and* $n_k = 2$.

Proof. It is enough to show the part "only if". By the definition of the k-partition, we have $n \geq 2\lambda + 3(k-\lambda)$ i.e. $n \geq 3k - \lambda$. Thus the assumption $n < 3(k-\lambda)+4$ implies $3k - \lambda \leq n \leq 3k - 3\lambda + 3$ and hence $\lambda \leq 1$. It is easy to see that the cases (a) and (b) occur if $\lambda = 0$, and the case (c) occurs if $\lambda = 1$.

Let G be a connected graph of order $n = \sum_{i=1}^{k} n_i$. A path-factor of G consisting of paths of orders $n_1, n_2, \ldots, n_k$ is called a $P_{s,t}$-factor if $n_1 = \cdots = n_{k-1} = s$ and $n_k = t$.
For the cases (a) and (b), we will show the following:

Theorem B. *Let G be a connected graph of order n. If $\sigma_3(G) \geq n$, then*

(a) *G contains a P_3-factor if $n = 3k$;*
(b) *G contains a $P_{3,5}$-factor if $n = 3k+2$.*
For the case (c), we have

Theorem C. *Let G be a connected graph of order $n = 3k-1$. If $\sigma_3(G) \geq n-1$, then G contains a $P_{3,2}$-factor.*

It is easy to see that $\frac{3}{2}(n-k+\lambda) \geq n$. Hence combining Theorems A, B and C we can get the following:

Corollary. *Let G be a connected graph of order n and $(n_1, n_2, \ldots, n_k)$ a k-partition of n. If $\sigma_3(G) \geq \frac{3}{2}(n-k+\lambda)$ then G contains a path-factor consisting of paths of orders $n_1, n_2, \ldots, n_k$.*

Since $\delta(G) \geq \frac{1}{2}(n-k+\lambda)$ implies that $\sigma_3(G) \geq \frac{3}{2}(n-k+\lambda)$, the corollary above generalizes Theorem 1.

Remark. *Theorems A, B and C are best possible in the following sense*:

The bound of Theorem A is sharp.
Take the complete bipartite graph $K_{a,b}$ with bipartition (A, B), where $|A| = a \leq b = |B|$. Suppose that $n = \sum_{i=1}^{k} n_i$, $a = \frac{1}{2}(n-k+\lambda)-1$ and $b = n-a$. Clearly, $K_{a,b}$ is connected and $\sigma_3(K_{a,b}) = \frac{3}{2}(n-k+\lambda)-3$. Since $a = \lfloor \frac{1}{2}n_1 \rfloor + \cdots + \lfloor \frac{1}{2}n_k \rfloor - 1$ and each path contributes at least $\lfloor \frac{1}{2}n_i \rfloor$ vertices to A, $K_{a,b}$ can not have vertex disjoint paths of order $n_1, \ldots, n_k$.
The bound of Theorem B is sharp.
(a) Let $G_{p,q,r}$ be a graph of three complete graphs $K_{p+1}, K_{q+1}, K_{r+1}$ with one vertex in common, where $|G_{p,q,r}| = n = p+q+r+1$ and $p \equiv q \equiv 2 \pmod 3$, $r \equiv 1 \pmod 3$. It is easy to see that $\sigma_3(G) = n-1$ and G has no P_3-factor.
(b) Let G be a graph of three complete graphs K_3, K_5, K_{p+1} with one vertex in common, where $|G| = n = p+7$ and $p \equiv 1 \pmod 3$. It is easy to see $\sigma_3(G) = n-1$ and G does not contain a $P_{3,5}$-factor.

The bound in Theorem C is also sharp.

Consider complete bipartite graph $K_{k-1,2k}$. It's easy to see $\sigma_3(K_{k-1,2k}) = 3k-3 = n-2$ and $K_{k-1,2k}$ does not contain a $P_{3,2}$-factor.

2. Lemmas

To prove our results, we need some lemmas.

Lemma 1. (*Enomoto et al.* [6]). *Suppose G is a connected graph of order n with $\sigma_3(G) \geq n$ or $\alpha(G) \leq 2$. Then either G contains a hamiltonian path or every longest cycle of G is strongly dominating.*

Lemma 2. (*Erdös and Gallai* [5]). *Let $C = x_1x_2 \cdots x_m x_1$ be a cycle of G. and $x_i, x_j \in V(C)$ with $i \neq j$. If $d_C(x_i) + d_C(x_j) \geq m+1$, then G has a path P from x_i^+ to x_j^+ such that $V(P) = V(C)$.*

Lemma 3. *Let G be a connected graph and C a maximal cycle of G. Suppose that $v \in V(G-C)$ and $d_C(v) \geq 2$. Then for any two distinct vertices y and z in $N_C^+(v)$ or $N_C^-(v)$, $yz \notin E(G)$ and $N(y) \cap N(z) \cap V(G-C) = \emptyset$.*

Lemma 4. *Let G be a connected graph of order n and k an integer with $k \leq \frac{1}{3}(n-4)$. Suppose $\sigma_3(G) \geq 3k_0 - 2$, where $k_0 = \frac{1}{2}(n-k)$. Then either G has a hamiltonian path or G contains a k_0-dominating path.*

Proof. Suppose G has no hamiltonian path. Let C be a longest cycle of G. Since $\sigma_3(G) \geq 3k_0 - 2 = \frac{3}{2}(n-k) - 2$ and $k \leq \frac{1}{3}(n-4)$, we have $\sigma_3(G) \geq \frac{3}{2}(n-k) -2 \geq n$. Hence C is a strongly dominating cycle of G by Lemma 1. Let $Y = V(G) - V(C)$ and $|Y| = l$. Because G has no hamiltonian path, we have $l \geq 2$. If Y contains at most one vertex of degree less than k_0, the conclusion holds trivially. On the other hand, because of $\sigma_3(G) \geq 3k_0 - 2$, Y contains at most two vertices of degree less than k_0. Hence we may assume Y contains exactly two vertices, say y_1, y_2, such that $d(y_1) \leq d(y_2) < k_0$ in the following proof.

The following proposition is obvious.

Proposition 2. *If $v \in V(G)$ such that $\{v, y_1, y_2\}$ is an independent set, then $d(v) \geq 3k_0 - 2 - d(y_1) - d(y_2)$.*

Let $N(y_1) = \{u_1, \ldots, u_s\}$ and $N(y_2) = \{v_1, v_2, \ldots, v_t\}$. Assume that $v_1, v_2, \ldots, v_t$ occur on $\overrightarrow{C}$ in the order of their indices. The vertices of $N(y_2)$ divides C into t segments. Let t_1, t_2 and t_3 be the numbers of 1-segments, 2-segments and m-segments for all $m \geq 3$, respectively, among these t segments. Thus $t_1 + t_2 + t_3 = t$. Let V_i be the union of the vertex sets of all i-segment and $X_i = V_i - N(y_2)$, where $i = 1, 2$. Let $X = X_1 \cup X_2$. If $N(y_1) \cap N^+(y_2) \neq \emptyset$ or $N(y_1) \cap N^-(y_2) \neq \emptyset$, then G has a path P containing $V(C)$ from y_1 to y_2. Obviously P is a k_0-dominating path of G, and hence Lemma 4 holds. Therefore we may assume that

$$N(y_1) \cap N^+(y_2) = N(y_1) \cap N^-(y_2) = \emptyset \qquad (*)$$

in the following proof.

Claim 2.1. *For any $x_1, x_1' \in X_1$, and for any $x_2, x_2' \in X_2$, we have:*

(a) $N_Y(x_1) \cap N_Y(x_1') = N_Y(x_1) \cap N_Y(x_2) = \emptyset$;
(b) $N_Y(x_2) \cap N_Y(x_2') = \emptyset$.

Proof. Noting that $x_1, x_1' \in N^+(y_2) \cap N^-(y_2)$ and $x_2, x_2' \in N^+(y_2) \cup N^-(y_2)$, by Lemma 3, we get (a).

For (b), we may assume that $x_2 \in N^-(y_2)$ and $x_2' \in N^+(y_2)$ by (a). Assume that $z \in N_Y(x_2) \cap N_Y(x_2'), x_2 = v_i^-$ and $x_2' = v_j^+$. It is easy to see that $i \neq j-1$ by the maximality of C. Let $C' = x_2 \overrightarrow{C} v_j y_2 v_{i-1} \overleftarrow{C} x_2' z x_2$. Obviously, $V(C') = (V(C) - \{v_{i-1}^+\}) \cup \{y_2, z\}$, which contradicts the maximality of C. The proof of Claim 2.1 is complete. □

We consider the following four cases separately.

Case 1. $l = 2$ and $d(y_2) = t = k_0 - 1$.

Since $k_0 = \frac{1}{2}(n-k)$ and $k \leq \frac{1}{3}(n-4)$, we have $t = k_0 - 1 \geq \frac{1}{3}(n-1)$. Hence $t_1 \geq 1$. Otherwise we have $3t \leq |C| = n-2$, that is $t \leq \frac{1}{3}(n-2)$. Assume, without loss of generality, that $v_1 \overrightarrow{C} v_2 = v_1 x_1 v_2$ be a 1-segment. Let $P = y_1 u_1 \overrightarrow{C} v_1 y_2 v_2 \overrightarrow{C} u_1^-$. Since $\{x_1, y_1, y_2\}$ is an independent set, we have $d_P(x_1) = d_C(x_1) = d(x_1) \geq k_0$ by Proposition 2. Therefore P is a path as required.

Case 2. $l = 2$ and $d(y_2) = t \leq k_0 - 2$.

Since $\sigma_3(G) \geq 3k_0 - 2 \geq n$, we have $t \geq 2$. Hence we can choose two vertices, say u_1, v_1, such that $u_1 \neq v_1$. By the maximality of C and $(*)$, we have that both $\{u_1^-, y_1, y_2\}$ and $\{v_1^-, y_1, y_2\}$ are independent sets.

Subcase 2.1. There is an m-segment with $m \leq 3$ among these t segments.

Without loss of generality, we may assume $v_1 \overrightarrow{C} v_2 = v_1 x_1 x_2 \cdots x_m v_2$ is an m-segment with $m \leq 3$. Since both $\{x_1, y_1, y_2\}$ and $\{x_m, y_1, y_2\}$ are independent sets, we have $d_C(x_1) \geq k_0 + 2$ and $d_C(x_m) \geq k_0 + 2$ by Proposition 2. Let $P = y_1 u_1 \overrightarrow{C} v_1 y_2 v_2 \overrightarrow{C} u_1^-$. If $m = 1$, then P is a path as required since $d_P(x_1) = d_C(x_1) \geq k_0 + 2$. If $m = 2$, since $d_P(x_i) = d_C(x_i) - 1 \geq k_0 + 1$ for $i = 1, 2$, P is a path as required. Now we assume $m = 3$. If $x_2 \in N(y_1)$, then $y_1 x_2 \overrightarrow{C} v_1 y_2$ is a path as required. If $x_2 \notin N(y_1)$, then $\{x_2, y_1, y_2\}$ is an independent set. Hence we have $d_C(x_2) \geq k_0 + 2$ by Proposition 2. Thus $d_P(x_i) \geq d_C(x_i) - 2 \geq k_0$ for $i = 1, 2, 3$. Therefore P is a path as required.

Subcase 2.2. There is no m-segment with $m \leq 3$ among these t segments.

In this case, $d(y_1) \leq d(y_2) \leq \frac{1}{5}(n-2)$. Hence we have $d_C(u_1^-) = d(u_1^-) \geq \frac{1}{5}(3n+4)$ and $d_C(v_1^-) = d(v_1^-) \geq \frac{1}{5}(3n+4)$. This implies $d_C(u_1^-) + d_C(v_1^-) \geq \frac{1}{5}(6n+8) > |C|$. By Lemma 2, G has a path P from u_1 to v_1 such that

$V(P) = V(C)$. Therefore G has a hamiltionian path from y_1 to y_2. This is a contradiction.

Case 3. $l \geq 3$ and $d(y_2) = t \leq k_0 - 2$.

To prove Case 3, we need some claims.

Claim 2.2. $t \geq \frac{1}{4}(n + l) = \frac{1}{4}(|C| + 2l)$.

Proof. Since C is a longest cycle of G, we have that for each $y \in Y$, $d(y) \leq \frac{1}{2}(n - l)$. Since $l \geq 3$, we may take $y_3 \in Y - \{y_1, y_2\}$. Noting that $\sigma_3(G) \geq 3k_0 - 2 \geq n$ and $d(y_2) \geq d(y_1)$, we have $2t \geq d(y_1) + d(y_2) \geq \sigma_3(G) - d(y_3) \geq n - \frac{1}{2}(n - l)$, and hence $t \geq \frac{1}{4}(n + l)$. □

Claim 2.3. $t_1 + t_2 \geq l$ *and if* $t_1 = 0$, *then* $t_2 \geq 2l$.

Proof. It is easy to see that $t + t_1 + 2t_2 + 3t_3 \leq |C|$. This implies that $4t \leq |C| + 2t_1 + t_2$ as $t = t_1 + t_2 + t_3$. If $t_1 + t_2 < l$, we have $4t \leq |C| + 2t_1 + t_2 \leq |C| + 2(t_1 + t_2) < |C| + 2l$ which contradicts Claim 2.2. Similarly, we have $t_2 \geq 2l$ if $t_1 = 0$. □

Claim 2.4. *If G has no* k_0-*dominating path*, *then both (a) and (b) hold.*

(a) Let $v_i \overrightarrow{C} v_{i+1} = v_i x_1 v_{i+1}$ be any 1-segment. Then $|N_Y(x_1)| \geq 3$;
(b) Let $v_i \overrightarrow{C} v_{i+1} = v_i x_1 x_2 v_{i+1}$ be any 2-segment. Then $|N_Y(x_1) \cup N_Y(x_2)| \geq 2$.

Proof. Let $P = y_1 u_1 \overrightarrow{C} v_i y_2 v_{i+1} \overrightarrow{C} u_1^-$. Since $\{x_1, y_1, y_2\}$ is an independent set, by Proposition 2, we have $d(x_1) \geq k_0 + 2$. By Claim 2.1 we get that $|N(y) \cap X| \leq 1$ holds for any $y \in Y$. Hence we have that $d_P(y) \geq k_0 + 1$ holds for any $y \in Y - \{y_1, y_2\}$. If $|N_Y(x_1)| \leq 2$, then P is a path as required since $d_P(x_1) \geq k_0$, and therefore (a) holds. We can prove (b) similarly. □

By Claim 2.1 and Claim 2.4, we have the following

Claim 2.5. $|N_Y(X_1)| \geq 3t_1$, $|N_Y(X_2)| \geq 2t_2$. □

Now we begin to prove Case 3.

Suppose to the contrary that G contains no k_0-dominating path. By Claim 2.1, $N_Y(X_1) \cap N_Y(X_2) = \emptyset$. Hence $|N_Y(X)| \geq 3t_1 + 2t_2$. If $t_1 \geq 1$, we have $|N_Y(X)| \geq 3t_1 + 2t_2 > 2(t_1 + t_2) \geq 2l > l = |Y|$, a contradiction. If $t_1 = 0$, by Claim 2.3 we have $t_2 \geq 2l$. Hence $|N_Y(X)| \geq 3t_1 + 2t_2 = 2t_2 \geq 4l > l = |Y|$, also a contradiction.

Case 4. $l \geq 3$ and $d(y_2) = t = k_0 - 1$.

Suppose that G has no path as required.

Claim 2.6. *For any* $x_1 \in X_1$, $N(x_1) \cap Y \neq \emptyset$.

Proof. Let $P = y_1 u_1 \overrightarrow{C} x_1^- y_2 x_1^+ \overrightarrow{C} u_1^-$. If there is some vertex $x_1 \in X_1$ such that $N(x_1) \cap Y = \emptyset$, then P is a path as required since $d_P(x_1) \geq k_0$ and $d_P(y) \geq k_0$ for any vertex $y \in Y - \{y_1, y_2\}$. □

Claim 2.7. $|N_Y(X_1)| \geq |X_1| \geq l-1 \geq 2$.

Proof. First we show that $t_1 \geq l-1$. Otherwise, we have $n-l=|C| \geq t+t_1+2t_2+3t_3=3t-t_1+t_3 \geq 3t-t_1 \geq 3t-l+2$. This implies that $t \leq \frac{1}{3}(n-2)$ which contradicts that $d(y_2)=t=k_0-1 \geq \frac{1}{3}(n-1)$.

By Claim 2.1 and Claim 2.6, we can get that $|N_Y(X_1)| \geq |X_1|$. □

We now begin to prove Case 4.

By $|Y|=l$ and Claim 2.7, we have $l-1 \leq |X_1| \leq |N_Y(X_1)|$. By $(*)$, we have $y_1 \notin N(x_1)$ for any vertex x_1 in X_1. This implies that $N(x_1) \subseteq Y-\{y_1,y_2\}$ for any vertex $x_1 \in X_1$. Hence we have $|N(X_1)| \leq |Y-\{y_1,y_2\}|=l-2$, a contradiction.

The proof of Lemma 4 is complete. □

Lemma 5. (*Johansson* [7]). *Let G be a graph of order n and* $(n_1,n_2,\ldots,n_k)$ *a k-partition of n, where all* n_i *are odd positive integers. Suppose furthermore that G contains a path P such that every vertex* $v \in V(G)-V(P)$ *has no two consecutive neighbours on P and satisfies* $d_P(v) \geq (n-k)/2$. *Then G contains a path-factor consisting of paths of orders* $n_1,n_2,\ldots,n_k$. □

Bondy [1] showed that if G is a 2-connected graph of order n with $\sigma_3(G) > \frac{3}{2}(n-1)$, then G contains a hamiltonian cycle. From this result we can get the following

Lemma 6. *Let G be a connected graph of order n. If* $\sigma_3(G) \geq (3n-5)/2$, *then G has a hamiltionian path.* □

3. Proofs of theorems

Proof of Theorem A. If all n_i's are even then $\lambda=k$, and hence $\sigma_3(G) \geq (3n-4)/2$. Thus, Theorem A holds by Lemma 6. Hence we may assume at least one of the n_i's is odd. Without loss of generality, we can assume $n_1,n_2,\ldots,n_{p-1}$ are even and $n_p,\ldots,n_k$ are odd. Hence $\lambda=p-1$. Set $n'_1=n_1+n_2+\cdots+n_p$ and $n'_{i+1}=n_{i+p}$ for all $1 \leq i \leq k-p$. Clearly, $(n'_1,n'_2,\ldots,n'_{k-p+1})$ is a k'-partition of n, where $k'=k-p+1$. Since each $n'_i(1 \leq i \leq k')$ is odd, we have $\lambda'=\lambda(n'_1, n'_2,\ldots,n'_{k-p+1})=0$. By the assumptions of Theorem A we get $n \geq 3(k-\lambda)+4=3(k-p+1)+4=3k'+4$ and $\sigma_3(G) \geq \frac{3}{2}(n-k+\lambda)-2=\frac{3}{2}(n-k+p-1)-2=3k'_0-2$, where $k'_0=\frac{1}{2}(n-k')$. Then the assumptions of Lemma 4 are satisfied. If G contains a hamiltonian path, then Theorem A holds. Hence we may assume that G contains a $\frac{1}{2}(n-k')$-dominating path. Let P be a longest $\frac{1}{2}(n-k')$-dominating path, then no vertex in $V(G)-V(P)$ has two consecutive neighbours on P. Hence G and P satisfy the hypothesis of Lemma 5. Therefore G contains a path-factor consisting of paths of orders $n'_1,n'_2,\ldots,n'_{k'}$, and hence G contains a path-factor consisting of paths of orders $n_1,n_2,\ldots,n_k$. The proof of Theorem A is complete. □

In order to prove Theorems B and C, we first prove a result which is slightly stronger than Theorems B and C.

Theorem D. *Suppose G is a connected graph of order n and G contains a maximal strongly dominating cycle C such that* $\sigma_3(Y) \geq n-1$ *or* $|Y| \leq 2$, *where* $Y = V - V(C)$. *Then the following three conclusions hold.*

(1) *If* $n = 3k$, *then G has a* P_3*-factor.*
(2) *If* $n = 3k+2 \geq 11$, *then G has a* $P_{3,5}$*-factor.*
(3) *If* $n = 3k-1$, *then G contains a* $P_{3,2}$*-factor.*

Proof. We shall prove this result by induction on n.

Obviously, if $n = 3$, then the conclusion (1) holds, and if $n = 5$, the conclusion (3) holds. As the bases of induction, we now need show that if $n = 11$ the conclusion (2) holds.

If $n = 11$ and $|Y| \leq 2$, then the conclusion (2) holds. Hence we may assume that $|Y| \geq 3$. Since $\sigma_3(Y) \geq n-1 = 10$, we have $\Delta(Y) \geq 4$. By the maximality of C and $\Delta(Y) \geq 4$, we have $|C| \geq 8$. Because of $n = 11$, we get that $\Delta(Y) = 4$, $|C| = 8$ and $|Y| = 3$. Let $Y = \{y_1, y_2, y_3\}$ with $d(y_1) \leq d(y_2) \leq d(y_3)$. Clearly $d(y_1) \geq 2$ and $d(y_2) \geq 3$. Suppose that $\overrightarrow{C} = v_1 v_2 \cdots v_8$. Without loss of generality, we can assume that $N(y_3) = \{v_1, v_3, v_5, v_7\}$ and $N_C^+(y_3) = N_C^-(y_3) = \{v_2, v_4, v_6, v_8\}$. By Lemma 3, we have that $|N(y_i) \cap N_C^+(y_3)| \leq 1$, where $i = 1, 2$. This implies that $|N(y_2) \cap N(y_3)| \geq 2$ and $|N(y_1) \cap N(y_3)| \geq 1$. If $N(y_1) \cap N(y_2) \cap N(y_3) \neq \emptyset$, say $v_1 \in N(y_1) \cap N(y_2) \cap N(y_3)$, then $y_1 v_1 y_2$, $v_2 v_3 y_3$ and $v_4 \overrightarrow{C} v_8$ is a $P_{3,5}$-factor as required. We now assume that $N(y_1) \cap N(y_2) \cap N(y_3) = \emptyset$. By symmetry, we may assume, without loss of generality, that $\{v_1, v_3\} \subseteq N(y_2) \cap N(y_3)$ and $v_5 \in N(y_1) \cap N(y_3)$ or $\{v_1, v_5\} \subseteq N(y_2) \cap N(y_3)$ and $v_3 \in N(y_1) \cap N(y_3)$. In the former case $y_2 v_3 y_3$, $v_4 v_5 y_1$ and $v_6 \overrightarrow{C} v_2$ is a $P_{3,5}$-factor as required. In the latter case $y_2 v_1 y_3, v_2 v_3 y_1$ and $v_4 \overrightarrow{C} v_8$ is a $P_{3,5}$-factor as required. Hence the conclusion (2) holds when $n = 11$.

We now assume that the conclusions hold for small values of n.

It is easy to see the conclusions hold when $|Y| \leq 2$. If $\delta(Y) = 1$, then we must have $|Y| \leq 2$. Since if $|Y| \geq 3$, we can get a vertex $y \in Y$ such that $d_C(y) = d(y) \geq \frac{1}{2}(n-2) > \frac{1}{2}(n-3) \geq \frac{1}{2}|C|$. This contradicts that C is maximal. So we may assume that $|Y| \geq 3$ and $\delta(Y) \geq 2$ in the rest of the proof.

Let $y \in Y$ such that $d(y) = \Delta(Y) = t$. Since $\sigma_3(Y) \geq n-1$ and $n \not\equiv 1 \pmod 3$, we have $t \geq \frac{1}{3}n$. Set $N(y) = \{v_1, \ldots, v_t\}$ which divides C into t segments $v_1 \overrightarrow{C} v_2, v_2 \overrightarrow{C} v_3, \ldots, v_t \overrightarrow{C} v_1$. Here we assume that $v_1, v_2, \ldots, v_t$ occur on $\overrightarrow{C}$ in the order of their indices. Suppose that there exists t_1 1-segments, t_2 2-segments and t_3 m-segments with $m \geq 3$ among these t segments. We first claim that $t_1 \geq 3$. Otherwise we have $n-3 \geq |C| \geq t + t_1 + 2t_2 + 3t_3 \geq 3t - t_1$. This implies that $t \leq \frac{1}{3}(n-1) < \frac{1}{3}n$, a contradiction.

Case 1. There is some $j (1 \leq j \leq t)$ such that both $v_{j-1} \overrightarrow{C} v_j$ and $v_j \overrightarrow{C} v_{j+1}$ are 1-segments.

Without loss of generality we may assume the two 1-segments are $u_1 u_2 u_3$ and $u_3 u_4 u_5$. Let now $P = u_2 u_3 u_4$ and $G' = G - P$. Obviously $n = n' \pmod 3$, where

$n' = |G'|$. It is easy to see that the cycle $C' = u_1 y u_5 \overrightarrow{C} u_1$ is a strongly dominating cycle of G'. Let $Y' = Y - \{y\}$, then for any vertex $y' \in Y'$, we have $|N(y') \cap \{u_2, u_3, u_4\}| \leq 1$. Otherwise, if $\{u_2, u_3\}$ or $\{u_3, u_4\} \subseteq N(y')$, then C is not a maximal cycle in G. And if $\{u_2, u_4\} \subseteq N(y')$, then $C^* = u_4 \overrightarrow{C} u_1 y u_3 u_2 y' u_4$ is a cycle such that $V(C) \subset V(C^*)$, a contradiction. Thus we have $|N(y') \cap \{u_2, u_3, u_4\}| \leq 1$ and G' is connected since $\delta(Y) \geq 2$. Moreover, we have $\sigma_3(Y') \geq n' - 1$ or $|Y'| \leq 2$. If C' is maximal in G', then by induction hypothesis, G' has a P_3-factor ($P_{3,5}$-factor or $P_{3,2}$-factor, respectively) $\mathscr{P}'$. Let $\mathscr{P} = \mathscr{P}' \cup \{P\}$. Clearly $\mathscr{P}$ is a P_3-factor ($P_{3,5}$-factor or $P_{3,2}$-factor, respectively) of G. If C' is not maximal in G', then we must be able to find a cycle C'' such that C'' is maximal strongly dominating cycle of G' and $V(C') \subset V(C'')$. Let $Y'' = V(G') - V(C'')$, it is not difficult to see $|Y''| \leq 2$ or $\sigma_3(Y'') \geq n' - 1$. Similar to the discussion above, we get the conclusions.

Case 2. For any $j (1 \leq j \leq t)$, there is at most one 1-segment in $v_{j-1} \overrightarrow{C} v_j$ and $v_j \overrightarrow{C} v_{j+1}$.

To prove Case 2, we need the following claims.

Let $u_1 \overrightarrow{C} u_k = u_1 u_2 u_3 \cdots u_{k-2} u_{k-1} u_k$ be a segment in C, $k \equiv 2 \pmod 3$ and $k \geq 8$. If $N(y) \cap u_1 \overrightarrow{C} u_k = \{u_1, u_k\} \cup \{u_i : i \equiv 0 \pmod 3 \text{ and } 3 \leq i \leq k-2\}$, we call $u_1 \overrightarrow{C} u_k$ a $(1,2)$-segment.

Claim 3.1. *There exists at least three* $(1,2)$*-segments in* C.

Proof. If there are at most two $(1,2)$-segments in C, then $t_3 \geq (t_1 - 2)$. Hence $n - 3 \geq |C| \geq t + t_1 + 2t_2 + 3t_3 \geq 3t - t_1 + t_3 \geq 3t - 2$. This implies that $t \leq \frac{1}{3}(n-1)$, contrary to that $t \geq \frac{1}{3}n$. □

We now choose two 1-segments $v_j \overrightarrow{C} v_{j+1}$ and $v_s \overrightarrow{C} v_{s+1}$ such that $v_j \overrightarrow{C} v_{s+1}$ is a $(1,2)$-segment and the number of vertices between v_{j+1} and v_s along $\overrightarrow{C}$ is as small as possible. Without loss of generality, we may assume $u_1 u_2 u_3$ and $u_i u_{i+1} u_{i+2}$ are the two 1-segments. Set $u_4 \overrightarrow{C} u_{i-1} = x_1 x_2 \cdots x_m$. By Claim 3.1, we have $m \leq \frac{1}{3}(n - 12) = \frac{1}{3}n - 4$ and $m \equiv 2 \pmod 3$.

Claim 3.2. $\delta(Y) \geq \frac{1}{5}n + \frac{7}{5}$

Proof. Since for any $j (1 \leq j \leq t)$, there is at most one 1-segment in $v_{j-1} \overrightarrow{C} v_j$ and $v_j \overrightarrow{C} v_{j+1}$, hence we have $\Delta(Y) = t \leq (n-3) - t_1 - 2t_2 - 3t_3 \leq (n-3) - \frac{3}{2}t$. That is $t \leq \frac{2}{5}(n-3)$. Therefore $\delta(Y) \geq \sigma_3(Y) - 2\Delta(Y) \geq \frac{1}{5}n + \frac{7}{5}$. □

Claim 3.3. *Let* $y_0 \in Y$, *then*:

$$N(y_0) \cap V(C) - \{u_2, u_3, x_1, \ldots, x_m, u_i, u_{i+1}\} \neq \emptyset$$

Proof. If $N(y_0) \cap V(C) - \{u_2, u_3, x_1, \ldots, x_m, u_i, u_{i+1}\} = \emptyset$, then we must have $\delta(Y) \leq d(y_0) \leq \frac{1}{2}(m+4) \leq \frac{1}{6}n < \frac{1}{5}n + \frac{7}{5} \leq \delta(Y)$, a contradiction. □

Now we begin to prove Case 2.

Case 2.1. There is some x_j with $j \not\equiv 0 \pmod 3$ such that $N(x_j) \cap Y \neq \emptyset$, say $x_1y_1 \in E(G)$, where $y_1 \in Y$.

Let $P = y_1x_1x_2$, Set $G' = G - P$ and $Y' = Y - \{y, y_1\}$. Then $C' = u_3yx_3\overrightarrow{C}u_3$ is a strongly dominating cycle of G' and $|Y'| \leq 2$ or $\sigma_3(Y') \geq n' - 1$, where $n' \equiv n \pmod 3$ and $n' = |G|$. Similar to the discussion in Case 1, we get the conclusions.

Case 2.2. For any x_j with $j \not\equiv 0 \pmod 3$, $N(x_j) \cap Y = \emptyset$.

Let $m = 3m_1 + 2$. The segment $u_2\overrightarrow{C}u_{i+1}$ can be partitioned into $(m_1 + 2)$ P_3's: $u_2u_3x_1, x_2x_3x_4, \ldots, x_{m-3}x_{m-2}x_{m-1}, x_mu_iu_{i+1}$. We denote by $\mathscr{R}$ the set of these $m_1 + 2$ P_3's and set $G^* = G - \{u_2, u_3, x_1, \ldots, x_m, u_i, u_{i+1}\}$ and $n^* = n - m - 4$. Then G^* is a graph of order $n^* \equiv n \pmod 3$. Let $C^* = u_1yu_{i+2}\overrightarrow{C}u_1$. Then C^* is a strongly dominating cycle of G^*. Set $Y^* = V(G^*) - V(C^*)$. We claim that for any vertex $y' \in Y - \{y\}$, both $|N(y') \cap \{u_2, u_3, x_1\}| \leq 1$ and $|N(y') \cap \{x_m, u_i, u_{i+1}\}| \leq 1$ hold by the maximality of C and Lemma 3. Therefore we have $\sigma_3(Y^*) \geq n^* - 1$ if $|Y^*| \geq 3$. By Claim 3.3, G^* is connected. If C^* is maximal in G^*, then by induction hypothesis, G^* has a P_3-factor ($P_{3,5}$-factor or $P_{3,2}$-factor, respectively) $\mathscr{P}^*$. Let $\mathscr{P} = \mathscr{P}^* \cup \mathscr{R}$. then $\mathscr{P}$ is a P_3-factor ($P_{3,5}$-factor or $P_{3,2}$-factor, respectively) of G. If C^* is not maximal in G^*, then similar to the discussion in Case 1, we can get the conclusions.

Hence we complete the proof of Theorem D. □

Proof of Theorem B. (a) It is a direct consequence of Lemma 1 and Theorem D(1).

(b) When $n = 5$ or $n = 8$, we can see the conclusion holds by checking it directly. When $n \geq 11$, it is a direct consequence of Lemma 1 and Theorem D(2).

Therefore the proof of Theorem B is completed. □

Proof of Theorem C. If G contains a hamiltonian path, then the conclusion holds trivially. If G contains a strongly dominating cycle, then the conclusion holds by Theorem D(3). Hence we may assume that G has neither hamiltonian path nor strongly dominating cycle in the following proof.

Let $P = x_1x_2 \cdots x_m$ be a longest path of G and y any vertex such that $y \in V - V(P)$. Set $A = N_P^-(x_1)$, $B = N_P^+(x_m)$, $D = N_P(y)$. By the maximality of P we get that $A \cap D = B \cap D = \emptyset$ and $\{x_1, y, x_m\}$ is independent.

Claim 3.4. $A \cap B = \emptyset$.

Proof. Suppose that $A \cap B \neq \emptyset$, say $x_i \in A \cap B$. Then $C = x_1\overrightarrow{P}x_{i-1}x_m\overleftarrow{P}x_{i+1}x_1$ is a cycle of length $|P| - 1$. Let $U = V - V(C)$. Obviously, $N(x_i) \cap U = \emptyset$. If $G[U]$ contains at least one edge, then since G is connected, we can get a path P' of length at least $|P| + 1$, a contradiction. If $G[U]$ contains no edges, then $G[V - V(C)]$ contains no edges, and hence C is a strongly dominating cycle of G, also a contradiction. □

Claim 3.5. *$G[V - V(P)]$ is a complete graph.*

Proof. Otherwise, there exists a vertex $y \in V - V(P)$ such that $d_{G-P}(y) \leq n - |P| - 2$. Thus, we have $n - 1 \leq \sigma_3 \leq d(x_1) + d(x_m) + d(y) \leq |P| + n - |P| - 2 = n - 2$ since x_1 and x_m has no neighbours in $G - P$ and are not connected to each other, furthermore $A \cap B = B \cap D = A \cap D = \emptyset$. This contradiction proves Claim 3.5. □

By Claim 3.5, we can assume that the only component of $G[V - V(P)]$ is H_y and $H_y = K_{n-m}$. If $n - m \equiv 0, 2 \pmod 3$, then we can get the conclusion easily. Let now $n - m \equiv 1 \pmod 3$. Without loss of generality, we can assume $N(y) \cap V(P) \neq \emptyset$. Clearly $H_y - \{y\}$ has a P_3-factor. Let $x_{i_0} \in N_P(y)$. If $i_0 \equiv 1 \pmod 3$, then $P' = x_1x_2 \cdots x_{i_0}y$ has a $P_{3,2}$-factor and $P'' = x_{i_0+1}x_{i_0+2} \cdots x_m$ has a P_3-factor. Hence we get a path-factor as required. If $i_0 \equiv 2 \pmod 3$, then $P' = x_1x_2 \cdots x_{i_0}y$ has a P_3-factor and $P'' = x_{i_0+1} \cdots x_m$ has a $P_{3,2}$-factor. Hence G has a $P_{3,2}$-factor. If $i_0 \equiv 0 \pmod 3$, then $P' = x_1x_2 \cdots x_{i_0-1}$ has a $P_{3,2}$-factor and $P'' = yx_{i_0}x_{i_0+1} \cdots x_m$ has a P_3-factor. Hence we can also get a path-factor as required.

The proof of Theorem C is complete. □

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