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Colouring Arcwise Connected Sets in the Plane I

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Abstract. Let $\mathcal{G}$ be the family of finite collections $\mathcal{S}$ where $\mathcal{S}$ is a collection of bounded, arcwise connected sets in $\mathbb{R}^2$ which for any $S, T \in \mathcal{S}$ where $S \cap T \neq \emptyset$, it holds that $S \cap T$ is arcwise connected. We investigate the problem of bounding the chromatic number of the intersection graph G of a collection $\mathcal{S} \in \mathcal{G}$.

Assuming G is triangle-free, suppose there exists a closed Jordan curve $C \subset \mathbb{R}^2$ such that C intersects all sets of $\mathcal{S}$ and for all $S \in \mathcal{S}$, the following holds:

(i) $S \cap (C \cup int(C))$ is arcwise connected or $S \cap int(C) = \emptyset$.
(ii) $S \cap (C \cup ext(C))$ is arcwise connected or $S \cap ext(C) = \emptyset$.

Here $int(C)$ and $ext(C)$ denote the regions in the interior, resp. exterior, of C. Such being the case, we shall show that $\chi(\mathcal{S})$ is bounded by a constant independent of $\mathcal{S}$.

1. Introduction

For any graph G in this paper $\chi(G)$ will denote the least number of colours necessary to colour the vertices of G so that any pair of adjacent vertices recieve different colours; that is, the **chromatic number** of G. The **clique number** of G is the order of the largest clique of G and we shall denote it by $\omega(G)$. For a collection $\mathcal{F}$ of subsets of $\mathbb{R}^n$ we define the **intersection graph** of $\mathcal{F}$, $G(\mathcal{F})$ to be the graph whose vertices correspond to sets in $\mathcal{F}$ where two vertices are adjacent if and only if their corresponding sets have nonempty intersection.

Let $\mathcal{G}$ be the family of finite collections $\mathcal{S}$ of bounded, arcwise connected sets in $\mathbb{R}^2$ with the property that for each $S, T \in \mathcal{S}$ where $S \cap T \neq \emptyset$, it holds that $S \cap T$ is arcwise connected. Technically speaking, a set $X \subseteq \mathbb{R}^2$ is **arcwise connected** if for any 2 points $x, y \in X$ there is a continuous injection $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$. The function f and its range are referred to as an **arc** where x and y are its endpoints. To avoid pathological varieties of arcwise connected sets (ie. space-filling curves etc.), we shall assume in our definition of arcwise connectedness that arcs have the additional property that they can be closely approximated by polygonal curve; a curve which is a finite union of line segments. That is, for an arc, we assume that one can find an arbitrarily good approximation to it via polygonal curves.

Throughout, we shall implicitly use the Jordan curve Theorem (see [8]).

Following [5] we call a family of graphs $\mathscr{H}$ χ**-bound with binding function f** if for each $H \in \mathscr{H}$, it holds that

$$\chi(H) \leq f(\omega(H)).$$

It is known (see [2]) that if G is an interval graph, that is, if $G = G(\mathscr{F})$ where $\mathscr{F}$ is a finite collection of closed intervals of $\mathbb{R}$, then $\chi(G) = \omega(G)$ (ie. G is perfect). Other families which are χ-bound include circular-arc graphs, where $\mathscr{F}$ is the collection of closed, circular arcs of a circle. In this case, it is not difficult to show that $\chi(G) \leq 2\omega(G) - 1$. In [6], Gyárfás showed that if $\mathscr{F}$ is a collection where each member of $\mathscr{F}$ is a union of at most t closed intervals of $\mathbb{R}$, then $G(\mathscr{F})$ satisfies $\chi \leq 2t(\omega - 1)$ for $\omega \geq 2$. Gyárfás (see [7]) also showed that for the intersection graph of chords of a circle, it holds that $\chi \leq 2^{2\omega}\omega^2$. Recently, Kostochka and Kratochvil [9] showed that this bound can be improved to $50(2^{\omega})$. They also showed that, among other things, for the class of intersection graphs representing the intersection of convex polygons inscribed in a circle, has a binding function f where $f(\omega) = 2^{\omega}$. In [10], Kostochka and Nešetřil obtained estimates for the chromatic numbers of intersection graphs of intervals, rays, and strings in the plane. Their estimates are valid for such graphs having girth at least 5, and they mention that even bounds for the chromatic number of the intersection graph of rays in the plane is unknown when such graphs have girth 4. The main result in this paper shows that such bounds do exist. In fact, it can be shown (see Corollary 1.2) that a much more general result holds for collections of unbounded, arcwise connected sets. In [1], Asplund and Grunbaum showed that for the intersection graph of rectangles in the plane (so-called box intersection graphs) it holds that $\chi \leq 4\omega^2 - 3\omega$. Burling [4] subsequently showed via a non-trivial example that χ cannot be bounded by any function of ω for the intersection graph of 3-dimensional boxes; that is, this is not a χ-bound family.

For some intersection graphs, it is difficult to determine whether they are χ-bound or not. Erdős (see [5]) posed as a problem to determine whether the family of intersection graphs of line segments in the plane is χ-bound. It was shown recently in [11], that the family of infinite-L-graphs is χ-bound. We shall use some of the ideas in that paper to prove more general results for the intersection graph of sets in the plane. More precisely, we focus on the intersection graphs formed by the intersection of bounded, arcwise connected sets in $\mathbb{R}^2$.

For convenience, we shall drop the induced graph notation $G(\mathscr{S})$ and instead refer directly to $\mathscr{S}$. Thus we will speak of the chromatic number of $\mathscr{S}$, $\chi(\mathscr{S})$ as being $\chi(G(\mathscr{S}))$, and we shall speak of paths and cycles of sets of $\mathscr{S}$, whose counterparts in $G(\mathscr{S})$ are paths and cycles. We may also use χ-bounded in referring to collections of sets (a family $\mathscr{F}$ of collections of sets of $\mathscr{S}$ being χ-bounded iff the family of graphs $\{G(\mathscr{S}) : \mathscr{S} \in \mathscr{F}\}$ is). We say that a collection $\mathscr{S} \in \mathscr{G}$ is **triangle-free** if $G(\mathscr{S})$ is triangle-free. We define $\mathscr{G}_3 = \{\mathscr{S} \in \mathscr{G} : \mathscr{S} \textit{ triangle-free}\}$.

For a subset $S \subseteq \mathbb{R}^2$, we let $Int(S)$ denote the set of interior points of S. Let $x \in S$ and let C_x be a maximal arcwise connected set in S containing x. The set C_x is unique for each x and if for some x and y we have $C_x \neq C_y$, then $C_x \cap C_y = \emptyset$. The sets C_x, $x \in S$ are called the **components** of S. We say that S is **bounded** if it is contained in some disk of finite radius.

In light of the above-mentioned results and problems, we pose the following conjecture:

Conjecture. The family $\mathscr{G}$ is χ-bound.

For a collection of sets $\mathscr{S}$ from $\mathbb{R}^2$ and $\mathscr{S}' \subset \mathscr{S}$, we let $\mathscr{L}_{\mathscr{S}}(\mathscr{S}')$ denote the sets of $\mathscr{S}$ contained in the bounded components of $\mathbb{R}^2 - \bigcup_{S \in \mathscr{S}'} S$. When $\mathscr{S}$ is implicit, we shall just write $\mathscr{L}(\mathscr{S}')$. We say that a set of $S \in \mathscr{S}$ is **surrounded** by $\mathscr{S}'$ if $S \in \mathscr{L}(\mathscr{S}')$.

In extending the results on infinite-L-graphs [11], we shall prove the following which is the main result of this paper.

Theorem 1.1. *Let $\mathscr{S} \in \mathscr{G}_3$. Suppose there exists a closed Jordan curve $C \subset \mathbb{R}^2$ such that C intersects all sets of $\mathscr{S}$ and for all $S \in \mathscr{S}$ the following hold*:

(i) *$S \cap (C \cup int(C))$ is arcwise connected or $S \cap int(C) = \varnothing$.*
(ii) *$S \cap (C \cup ext(C))$ is arcwise connected or $S \cap ext(C) = \varnothing$.*

Then $\chi(\mathscr{S})$ is bounded by a constant independent of $\mathscr{S}$.

In [10] it was posed as a problem to determine whether the triangle-free intersection of rays in the plane have bounded chromatic number. The above theorem gives an affirmative answer to this problem. Moreover, one can prove the following result for intersections of unbounded arcwise connected sets.

Corollary 1.2. *Let $\mathscr{S}$ be a finite, triangle-free collection of unbounded, arcwise connected sets in the plane where for any two intersecting sets S and T of $\mathscr{S}$ it holds that $S \cap T$ is arcwise connected. For such a collection, $\chi(\mathscr{S})$ is bounded by a constant which is independent of $\mathscr{S}$.*

The principle idea in the proof of Theorem 1.1 is to surround a group of sets $\mathscr{S}' \subset \mathscr{S}$ for which $\mathscr{S}'$ has large chromatic number when $\mathscr{S}$ does. Let $\alpha \geq 1$. For sake of convenience, we define a relation $\leftrightharpoons_\alpha$ on 2 collections of sets $\mathscr{T} \subseteq \mathscr{S}$ $\mathscr{S}, \mathscr{T} \in \mathscr{G}_3$, in the following way:

$$\mathscr{T} \leftrightharpoons_\alpha \mathscr{S} \quad \Leftrightarrow \quad \chi(\mathscr{T}) > \frac{1}{\alpha}\chi(\mathscr{S}).$$

As a matter of notation, we shall write $\mathscr{S}_1 \leftrightharpoons_{\alpha_1} \mathscr{S}_2 \leftrightharpoons_{\alpha_2} \cdots \leftrightharpoons_{\alpha_n} \mathscr{S}_n$ to mean $\mathscr{S}_1 \leftrightharpoons_{\alpha_1} \mathscr{S}_2$, $\mathscr{S}_2 \leftrightharpoons_{\alpha_2} \mathscr{S}_3, \ldots, \mathscr{S}_{n-1} \leftrightharpoons_{\alpha_n} \mathscr{S}_n$. Note that if $\mathscr{L} \leftrightharpoons_\alpha \mathscr{T}$ and $\mathscr{T} \leftrightharpoons_\beta \mathscr{S}$, then $\mathscr{L} \leftrightharpoons_{\alpha\beta} \mathscr{S}$.

For each vertex v in an intersection graph $G = G(\mathscr{S})$ we let $S(v)$ denote the corresponding set in $\mathscr{S}$, and for $V \subseteq V(G)$ we let $S(V) = \{S(v) : v \in V\}$. If we are given an intersection graph $G(\mathscr{S})$, for each $S \in \mathscr{S}$ we let $v(S)$ denote the vertex corresponding to S. For any subset $\mathscr{S}' \subseteq \mathscr{S}$ we let $v(\mathscr{S}') = \{v(S) : S \in \mathscr{S}'\}$. Here we shall always assume that there is some implicit one-to-one correspondence between vertices of $G(\mathscr{S})$ and the sets of $\mathscr{S}$.

For a graph H and $V \subseteq V(G)$ we say that V **induces the subgraph** H, if $V = V(H)$ and for all $u, v \in V$, $uv \in E(H)$ if and only if $uv \in E(G)$. For $G = G(\mathscr{S})$ and $\mathscr{S}' \subseteq \mathscr{S}$, the collection $\mathscr{S}'$ is said to **induce a graph** being its intersection graph $G(\mathscr{S}')$.

For a graph G and $u, v \in V(G)$ we shall write $u \sim v$ to mean that u is adjacent to v. For two sets S and T we shall write $S \sim T$ to mean that S intersects T and $S \nsim T$ to mean that S does not intersect T.

2. Colouring Lemmas

We shall make use of the following lemma:

Lemma 2.1. *Let G be a graph whose vertices are partitioned into vertex disjoint subgraphs $G_1, G_2, \ldots, G_k$ where $E(G) = \bigcup_i E(G_i)$. Suppose also that $V_1, V_2, \ldots, V_l$ are subsets of vertices which partition $V(G)$ such that for $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, l |V_j \cap V(G_i)| \leq 1$. Let $t = \max_i |V_i|$ and let H be a graph on k vertices $g_1, g_2, \ldots, g_k$ where $g_i \sim g_j$ if and only if for some $1 \leq s \leq l$, $V_s \sim V(G_i)$ and $V_s \sim V(G_j)$. Let G^* be the graph with vertices $v_1, v_2, \ldots, v_l$ where $v_i \sim v_j$ in G^* if there is an edge from V_i to V_j in G. Then*

$$\chi(G^*) \leq \binom{\chi(H)}{t} \max_i \chi(G_i)^t.$$

Proof. We suppose that H has a proper colouring c_H with $\chi(H)$ colours $1, 2, \ldots, \chi(H)$. We can partition G into $\chi(H)$ subgraphs $\mathscr{G}_1, \mathscr{G}_2, \ldots, \mathscr{G}_{\chi(H)}$ where each subgraph $\mathscr{G}_i$ is the union of all subraphs G_j for which $c_H(g_j) = i$. Each $\mathscr{G}_i$ has a proper colouring c_i with colours $0, 1, 2, \ldots, \max_j \chi(G_j) - 1$. Assign to each $v_i \in V(G^*)$ a $\chi(H)$-tuple $c_{G^*}(v_i) = (x_1^i, x_2^i, \ldots, x_{\chi(H)}^i)$ where

$$x_j^i = \left\{ \begin{array}{rl} c_j(u) & \text{if} \quad V_i \cap V(\mathscr{G}_j) = \{u\} \\ 0 & \text{if} \quad V_i \cap V(\mathscr{G}_j) = \varnothing. \end{array} \right\}$$

The function c_{G^*} is easily seen to be a proper colouring of G^* with at most $\binom{\chi(H)}{t} \max_i \chi(G_i)^t$ colours. □

We mention here a well-known result (see [6]), which will be used extensively.

Lemma 2.2. *Let G be a graph and let $v \in V(G)$. Suppose $G_0, G_1, G_2, \ldots$ are the subgraphs induced by vertices at distance $0, 1, 2, \ldots$ respectively from v. Then for some d, $\chi(G_d) \geq \dfrac{\chi(G)}{2}$.* □

3. Proof of the Main Theorem

We define a **dendrite** to be an arcwise connected set which is a finite union of arcs, no sub-collection of which contains a closed Jordan curve.

Let $\mathscr{S}$ be a collection of arcwise connected sets where for any pair $S, S' \in \mathscr{S}$ having nonempty intersection, the set $S \cap S'$ has finitely many arcwise connected components. We denote the set of such components by $K(S \cap S')$. We shall associate to each set $S \in \mathscr{S}$ a dendrite $T_S \subset S$ in the following way: For each set $S \in \mathscr{S}$, we link all the components in $\bigcup_{S' \sim S} K(S \cap S')$ together with a finite number of disjoint dendrites $W_S \subset S$ where for each component $K \in \bigcup_{S' \sim S} K(S \cap S')$ and each dendrite T of W_S we may assume $T \cap K$ is either empty or a single point. So if one were to contract each component K into a single point, then W_S would become a dendrite containing all these points. Now in each component $K \in \bigcup_{S' \sim S} K(S \cap S')$ we may find a dendrite T_K containing $(W_S \cup W_{S'}) \cap K$. For each $S \in \mathscr{S}$, we let $T_S = W_S \cup \bigcup_{\substack{S \cap S' \neq \varnothing \\ K \in K(S \cap S')}} T_K$. We see that for all $S, S' \in \mathscr{S}$ it holds that $S \sim S'$ iff $T_S \sim T'_S$. By construction, if $S \cap S'$ is arcwise connected, then so is $T_S \cap T_{S'}$. We summarize the above in the following proposition:

Proposition 3.1. *For each $S \in \mathscr{S}$ we may associate a dendrite T_S so that $S \sim S'$ iff $T_S \sim T_{S'}$. Moreover, if $S \cap S'$ is arcwise connected, then $T_S \cap T_{S'}$ is also arcwise connected.* □

For any dendrite D and points $x, y \in D$ there is a unique arc in D having endpoints x and y. We shall let $D(xy)$ denote this arc.

Let C and $\mathscr{S}$ be as stated in Theorem 1.1. Let $\mathscr{S}_1$ be the collection of intersections $S \cap (C \cup int(C))$ where $S \in \mathscr{S}$ and $S \cap int(C) \neq \varnothing$. Similarly, let $\mathscr{S}_2$ be the collection of all intersections $S \cap (C \cup ext(C))$ where $S \in \mathscr{S}$ and $S \cap ext(C) \neq \varnothing$. It suffices to prove that $\mathscr{S}_1$ and $\mathscr{S}_2$ have chromatic number bounded by a constant independent of $\mathscr{S}$.

To see this, we first find proper colourings for $\mathscr{S}_1$ and $\mathscr{S}_2$, using colours $1, 2, \ldots, k$ where k does not depend on $\mathscr{S}$. We then associate a pair (c_i, c_j) of integers to each set $S \in \mathscr{S}$ in the following way: if $S \cap (C \cup int(C)) \in \mathscr{S}_1$, then let c_1 be the colour it recieves in $\mathscr{S}_1$; otherwise let $c_1 = 0$. If $S \cap (C \cup ext(C)) \in \mathscr{S}_2$, then let c_2 be the colour it recieves in $\mathscr{S}_2$; otherwise let $c_2 = 0$. It is easy to see that this gives a proper colouring of $\mathscr{S}$ with fewer than $(k+1)^2$ colours.

For convenience we shall only prove that $\mathscr{S}_2$ has bounded chromatic number. The sets of $\mathscr{S}_2$ are arcwise connected (by assumption). However, for 2 sets $S, S' \in \mathscr{S}_2$ which intersect, the intersection need not be arcise connected, but is a finite union of components. In the case where $S \cap S' \cap C = \varnothing$, we observe that $S \cap S'$ is arcwise connected. For convenience, we shall let $\mathscr{S}$ be the collection $\mathscr{S}_2$. According to Proposition 3.1, we may replace each set $S \in \mathscr{S}$ by a dendrite T_S, where the collection of dendrites $T_S, S \in \mathscr{S}$ preserves the same intersection properties as $\mathscr{S}$. Moreover, we may assume each T_S intersects C at a finite number of points, and for any $K \in K(S \cap S')$, C intersects T_K in at most one point (by preturbing C if necessary). For convenience, we shall assume $S = T_S$ for all $S \in \mathscr{S}$. Since each component K of a nonempty intersection $S \cap S'$ is a dendrite, we may contract K to a single point without changing the intersections of $\mathscr{S}$, and each set S remains a dendrite after all contractions. Note that contracting components which intersect C is allowed since they intersect C in exactly one point. Thus if $S \sim S'$ and $S \cap S' \cap C = \varnothing$, then we may assume $S \cap S'$ is a single point in

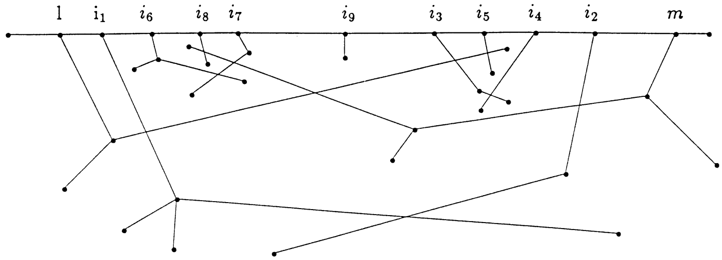

Fig. 1

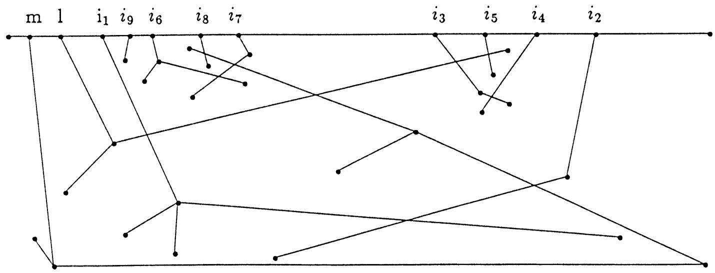

Fig. 2

$ext(C)$. If on the other hand $S \sim S'$ and $S \cap S' \cap C \neq \emptyset$, then we may assume that $S \cap S'$ is a finite collection of points, all of which lie on C.

The sets of $\mathcal{S}$ shall be enumerated as $S_0, S_1, S_2, \ldots, S_n$ in order of appearance as we move counterclockwise around C, ie. in their "chronological order". For $i = 0, 1, 2, \ldots$ we let x_i be the first point of S_i we encounter while moving along C. If two sets appear coincidentally along C, we shall enumerate one before the other in an arbitrary way.

For a subset $I \subseteq [0, n]$ and any subset $\mathcal{S}' \subseteq \mathcal{S}$, we let $\mathcal{S}'(I) = \{S_i \in \mathcal{S}' : i \in I\}$ Given that $\mathcal{S}$ has high chromatic number, Lemma 2.2 implies that for some d, the set of dendrites $\mathcal{S}_d$ at distance d from S_0 will also have high chromatic number. The basic idea we pursue here is to show that given $\mathcal{S}_d$ has high chromatic number (for example $\chi \geq 2^{100}$), we can find dendrites $S_m, S_l, S_{i_1}, S_{i_2}, \ldots, S_{i_9}$ which intersect in one of 2 ways similar to those illustrated by the configurations in Figs. 1 and 2. In either case, the dendrite S_{i_9} is "surrounded"; that is, any dendrite $S \in \mathcal{S}$ which chronologically lies to the left or right of all the dendrites in the configuration can not intersect S_{i_9} without first creating a triangle or crossing a dendrite twice. However, since $S_{i_9} \in \mathcal{S}_d$, there is a path of length d from S_0 to S_{i_9}, and the $(d-1)'$th dendrite in the path will play the role of S, as it must pierce through the configuration and intersect S_{i_9}, either creating a triangle or crossing a dendrite twice.

Suppose for some $i < j$ we have $S_i \sim S_j$. Let $y \in S_i \cap S_j$ and let $A_i = S_i(x_i y)$ and $A_j = S_j(x_j y)$.

Let

$$(i)_1 = \{k \in (i,j) : S_k \sim A_i\}$$
$$(i)_0 = \{k \in (i,j) : S_k \nsim A_i\}$$

We can define $(j)_1$ and $(j)_0$ similarly.

For $\delta_i, \delta_j \in \{0,1\}$ let

$$(i,j)_{\delta_i\delta_j} = (i)_{\delta_i} \cap (j)_{\delta_j}.$$

Note that $(i,j)_{11} = \varnothing$ since $\mathscr{S}$ is triangle-free.

For $\delta_i, \delta_j \in \{0,1\}$, $(\delta_i, \delta_j) \neq (0,0)$ let

$$(i,j)_{00}^{\delta_i\delta_j} = \{k \in (i,j)_{00} : \exists k' \in (i,j)_{\delta_i\delta_j}\ s.t.\ S_k \sim S_{k'}\}.$$

We let

$$(i,j)_{00}^{00} = (i,j)_{00} - (i,j)_{00}^{01} - (i,j)_{00}^{10}.$$

Lemma 3.2. *The collection of sets* $\{S_k : k \in (i,j)_{00}^{10} \cup (i,j)_{00}^{01}\}$ *has chromatic number bounded by a constant c which is independent of* $\mathscr{S}$.

Proof. It suffices to show that both $\mathscr{S}((i,j)_{00}^{10})$ and $\mathscr{S}((i,j)_{00}^{01})$ have chromatic number bounded by constants which are independent of $\mathscr{S}$. We shall show this is true for the first collection, a similar proof applying to the second as well.

For simplicity, let $\mathscr{S}_{ij} = \mathscr{S}((i,j)_{00}^{10})$. We let $\mathscr{S}((i,j)_{10}) = \{S_{r_1}, S_{r_2}, \dots, S_{r_\alpha}\}$ where $i < r_1 < r_2 \cdots < r_\alpha < j$. For $k = 1,2,\dots,\alpha$ let

$$\{y_{r_k}\} = S_{r_k} \cap A_i, \quad A_{r_k} = S_{r_k}(x_{r_k} y_{r_k}).$$

It is easily seen that the subset of dendrites $S \in S_{ij}$ which intersect at most one arc A_{r_k}, $k \in \{1,2,\dots,\alpha\}$ has chromatic number at most 2. This being the case, we assume for convenience that each $S \in S_{ij}$, S intersects at least 2 different arcs A_{r_k}.

For each $S \in \mathscr{S}_{ij}$ let

$$m(S) = \min_{S \sim A_{r_l}} l \quad \text{and} \quad n(S) = \max_{S \sim A_{r_l}} l.$$

The arcs $A_{r_1}, A_{r_2}, \dots, A_{r_\alpha}$ are disjoint and divide the region R bounded by C, A_i, and A_j into regions $R_1, R_2, \dots, R_{\alpha+1}$ where R_1 is the region between and including A_i and A_{r_1}, $R_{\alpha+1}$ is the region between and including A_{r_α} and A_j, and for $2 \le k \le \alpha$, R_k is the region between and including$A_{r_{k-1}}$ and A_{r_k}. For each $S \in \mathscr{S}_{ij}$, let A_S be the arc in S joining $S \cap A_{r_{m(S)}}$ to $S \cap A_{r_{n(S)}}$. The arcs A_S, $S \in \mathscr{S}_{ij}$ are disjoint for if $S \sim T$ for some $S, T \in \mathscr{S}_{ij}$, then either $n(S) = m(T) - 1$ or $n(T) = m(S) - 1$. The arcs A_S, $S \in \mathscr{S}_{ij}$ divide each region R_k, $k = 1,2,\dots,\alpha$ into subregions $R_{k1}, R_{k2}, \dots, R_{k\alpha_k}$. For $S \in \mathscr{S}_{ij}$, we let $V_S = \{S \cap R_{m(S)}, S \cap R_{n(S)+1}\}$ and we call the members of V_S the **ends** of S. Each end is seen to belong to a subregion R_{kl}. For $k = 1,2,\dots,\alpha$ and $l = 1,2,\dots,\alpha_k$, let G_{kl} be the intersection graph of the ends of dendrites of $\mathscr{S}_{ij}$ contained in R_{kl}. Let H be a graph having vertices g_{kl}, where $k = 1,2,\dots,\alpha$, $l = 1,2,\dots,\alpha_k$, and $g_{kl} \sim g_{k'l'}$ iff for some $S \in S_{ij}$, S has 2 ends, one in R_{kl} and another in $R_{k'l'}$. The graph H is seen to be planar, for if $g_{k_1l_1} \sim g_{k_1'l_1'}$, $k_1 < k_1'$ and $g_{k_2l_2} \sim g_{k_2'l_2'}$, $k_2 < k_2'$, then there exist $S, T \in \mathscr{S}_{ij}$ for which

$S \sim R_{k_1l_1}$, $R_{k'_1l'_1}$, and $T \sim R_{k_2l_2}$, $R_{k'_2l'_2}$. Now $S \sim T$ iff either $(k'_1, l'_1) = (k_2, l_2)$ or $(k'_2, l'_2) = (k_1, l_1)$. We conclude that H has a planar representation, and hence $\chi(H) \leq 5$ by the 5-colour Theorem [3, p. 156].

Let G^* be the intersection graph of $\mathcal{S}_{ij}$. We note that 2 dendrites $S, T \in \mathcal{S}_{ij}$ intersect iff they have ends which intersect in some R_{kl}; that is, there are ends in V_S and V_T whose corresponding vertices are adjacent in the intersection graph G_{kl}. Applying Lemma 2.1, we obtain

$$\chi(G^*) \leq \binom{\chi(H)}{2} \max_{k,l} \chi(G_{kl})^2.$$

Clearly $\chi(G_{kl}) \leq 2$, for all k and l as each dendrite of $\mathcal{S}_{ij}$ having an end in R_{kl} must intersect exactly one of A_{r_k} or $A_{r_{k+1}}$. We obtain from the above that $\chi(G^*) \leq 40$. □

Remark. The proof above indicates that the constant c in the statement of Lemma 3.2 is at most 100.

For $\lambda \in \mathbb{Z}^+$, a finite sequence $\{r_i\}_{i=0}^q$ is called a λ-**sequence** if $r_0 = -1$, $r_q = n$ and for $i = 1, 2, \ldots, q$, $\chi(\mathcal{S}(r_{i-1}, r_i]) \leq \lambda$ subject to r_i, $i = 1, \ldots, q-1$ being as large as possible. We note that for $i = 1, 2, \ldots, q-1$, $\chi(\mathcal{S}(r_{i-1}, r_i]) = \lambda$.

Lemma 3.3. *Let $\xi \in \mathbb{Z}^+$ and suppose $\chi(\mathcal{S}) \geq 16\xi$. Then there exists a subcollection $\mathcal{S}' \subset \mathcal{S}$ where*

(i) $\chi(\mathcal{S}') \geq 8$.
(ii) *for all $S_i, S_j \in \mathcal{S}'$ where $S_i \sim S_j$ it holds that $\chi(\mathcal{S}(i, j)) \geq \xi$.*
(iii) *there exists $S_{i_1}, S_{i_2} \in \mathcal{S}'$ such that $S_{i_1} \sim S_{i_2}$ and $\chi(\mathcal{S}[1, i_1)) \geq \xi$ and $\chi(\mathcal{S}(i_2, n])$ $\geq \xi$.*

Proof. Let $\{r_i\}_{i=0}^q$ be a ξ-sequence. Colour each of $\mathcal{S}(r_0, r_1], \mathcal{S}(r_1, r_2], \ldots,$ $\mathcal{S}(r_{q-1}, r_q]$ with ξ colours. Since $\chi(\mathcal{S}) \geq 16\xi$ at least one of the ξ colour classes forms a collection $\mathcal{T}$ with $\chi(\mathcal{T}) \geq 16$. The collection $\mathcal{T}$ can be partitioned into two subcollections $\mathcal{T}_1$ and $\mathcal{T}_2$ where $\mathcal{T}_1$ is the subcollection $\mathcal{T}$ intersected with $\mathcal{S}(r_0, r_1] \cup \mathcal{S}(r_2, r_3] \cup \cdots$ and $\mathcal{T}_2$ is the subcollection $\mathcal{T}$ intersected with $\mathcal{S}(r_1, r_2]$ $\cup \mathcal{S}(r_3, r_4] \cup \cdots$. We have that either $\chi(\mathcal{T}_1) \geq \dfrac{\chi(\mathcal{T})}{2} \geq 8$ or $\chi(\mathcal{T}_2) \geq \dfrac{\chi(\mathcal{T})}{2} \geq 8$. Assume, without loss of generality that the former holds. Suppose for some $i < j$ that $S_i, S_j \in \mathcal{T}_1$, and $S_i \sim S_j$. Then for some $0 \leq s < t$ we have $S_i \in \mathcal{S}(r_{2s}, r_{2s+1}]$ and $S_j \in \mathcal{S}(r_{2t}, r_{2t+1}]$. Thus $\mathcal{S}(r_{2s+1}, r_{2s+2}] \subseteq \mathcal{S}(i, j)$ and hence $\chi(\mathcal{S}(i, j)) \geq$ $\chi(\mathcal{S}(r_{2s+1}, r_{2s+2}]) = \xi$. We now see that $\mathcal{S}' = \mathcal{T}_1$ fullfills (i) and (ii). To see that it satisfies (iii), we note that since $\chi(\mathcal{T}_1) \geq 8$, we can pick $S_{i_1} \in \mathcal{S}(r_{2k}, r_{2k+1})$, $k > 0$, and $S_{i_2} \in \mathcal{S}(r_{2l}, r_{2l+1})$, $2k < 2l \leq q-3$, such that $S_{i_1} \sim S_{i_2}$. This being the case, we have $\chi(\mathcal{S}(0, i_1)) \geq \chi((0, r_1]) \geq \xi$ and $\chi(\mathcal{S}(i_2, n]) \geq \chi(\mathcal{S}(r_{q-2}, r_{q-1}]) \geq \xi$. □

Proof of Theorem 1.1. We shall assume that $\chi(\mathcal{S})$ is large (for example $\chi \geq 2^{100}$). Let $\mathcal{S}_0, \mathcal{S}_1, \ldots$ be the subcollections of dendrites of $\mathcal{S}$ at distance $0, 1, 2, \ldots$ from S_0 in $\mathcal{S}$. We aim to find a subcollection of dendrites in one of the distance classes which essentially corresponds to one of the configurations in either Fig. 1 or 2. To do this we shall use a repeated application of Lemma 1.2. By Lemma 1.2, there exists $a > 0$ such that $\chi(\mathscr{S}_a) \geq \frac{\chi(\mathscr{S})}{2}$. We have that $\mathscr{S}_a \leftrightarrows_2 \mathscr{S}$. Let S_{n_a} be the dendrite of smallest index which belongs to $\mathscr{S}_a$. Let $\mathscr{S}_{a0}, \mathscr{S}_{a1}, \mathscr{S}_{a2}, \ldots$ be the subcollections of dendrites of $\mathscr{S}_a$ at distance $0, 1, 2, \ldots$ resp. from S_{n_a}. As in the above, there is a $b > 0$ such that $\mathscr{S}_{ab} \leftrightarrows_2 \mathscr{S}_a \leftrightarrows_2 \mathscr{S}$. Now let S_{n_b} be the dendrite of smallest index which belongs to $\mathscr{S}_{ab}$, and let $\mathscr{S}_{ab0}, \mathscr{S}_{ab1}, \mathscr{S}_{ab2}, \ldots$ be the subcollections of dendrites of $\mathscr{S}_{ab}$ at distance $0, 1, 2, \ldots$ resp. from S_{n_b}. Again, there is a $c > 0$ such that $\mathscr{S}_{abc} \leftrightarrows_2 \mathscr{S}_{ab} \leftrightarrows_2 \mathscr{S}_a \leftrightarrows_2 \mathscr{S}$.

Pick $S_{i_1}, S_{i_2} \in \mathscr{S}_{abc}$ where $S_{i_1} \sim S_{i_2}$ and $\mathscr{S}_{abc}(i_1, i_2) \leftrightarrows_{2^4} \mathscr{S}_{abc}$. Such a pair exists by Lemma 3.3. Since the dendrites of $\mathscr{S}$ intersecting S_{i_1} and S_{i_2} have chromatic number at most 2, we have that $\chi(\mathscr{S}_{abc}((i_1, i_2)_{00})) \geq \chi(\mathscr{S}_{abc}(i_1, i_2)) - 2$. Applying Lemma 3.3 once again, we see that we may choose $S_{i_3}, S_{i_4} \in \mathscr{S}_{abc}((i_1, i_2)_{00})$ such that $i_1 < i_3 < i_4 < i_2$, and

(i) $S_{i_3} \sim S_{i_4}$
(ii) $\mathscr{S}_{abc}(i_1, i_3) \leftrightarrows_{2^5} \mathscr{S}_{abc}$, $\mathscr{S}_{abc}(i_3, i_4) \leftrightarrows_{2^5} \mathscr{S}_{abc}$, and $\mathscr{S}_{abc}(i_4, i_2) \leftrightarrows_{2^5} \mathscr{S}_{abc}$

Since $\chi(\mathscr{S})$ is large, $\mathscr{S}_{abc}$ is also large and consequently, Lemma 3.2 implies that we may pick

$$S_{i_5} \in \mathscr{S}((i_1, i_2)^{00}_{00}) \cap \mathscr{S}_{abc}((i_3, i_4)_{00}).$$

In addition, if S_{i_5} lies between 2 points of $S_{i_3} \cap C$ where x and y are the points nearest S_{i_5} coming respectively, before and after S_{i_5}, then we may choose S_{i_5} so that it intersects no other dendrite (besides S_{i_3}) containing x and y.

Since $S_{i_5} \in \mathscr{S}_{abc}$, there is a shortest path from S_{n_b} to S_{i_5} in $\mathscr{S}_{ab}$ of length c, say $S_{u_0} S_{u_1} \cdots S_{u_c}$. Since $\chi(\mathscr{S})$ is large (and hence $\chi(\mathscr{S}_{ab})$ is large) we may assume that $c \geq 2$. Since there is no such path which is shorter, it follows that $S_{u_i} \nsim S_{i_5}$ for $i = 0, 1, \ldots, c-2$. Moreover, S_{u_i} does not intersect any dendrite of $\mathscr{S}_{abc}$ for $i = 0, 1, \ldots, c-2$, but $S_{u_{c-1}} \sim S_{i_5}$. Since $n_b < i_1 < i_2$, it follows that $S_{u_{c-1}}$ intersects either S_{i_1} or S_{i_2}. Since $n_b < i_1 < i_2$, we conclude from the choice of S_{i_5}, $S_{u_{c-1}} \notin \mathscr{S}(i_1, i_2)$. Let $u_{c-1} = l$, and assume that (without loss of generality) $l < i_1$. Since $\mathscr{S}_{abc}(i_1, i_3) \leftrightarrows_{2^5} \mathscr{S}_{abc}$, $\chi(\mathscr{S}_{abc}(i_1, i_3))$ is large. Lemma 3.3 asserts that we may pick $S_{i_6}, S_{i_7} \in \mathscr{S}_{abc}(i_1, i_3)$ where $i_1 < i_6 < i_7 < i_3$ and

(iii) $S_{i_6} \sim S_{i_7}$
(iv) $\mathscr{S}_{abc}(i_1, i_6) \leftrightarrows_{2^5} \mathscr{S}_{abc}(i_1, i_3), \mathscr{S}_{abc}(i_6, i_7) \leftrightarrows_{2^5} \mathscr{S}_{abc}(i_1, i_3)$
$\mathscr{S}_{abc}(i_7, i_3) \leftrightarrows_{2^5} \mathscr{S}_{abc}(i_1, i_3)$
(v) $S_{i_6}, \mathscr{S}_{i_7} \nsim S_{i_1}, S_{i_2}, S_{i_3}, S_{i_4}, S_{i_5}, S_{i_6}, S_l$.

Since $\mathscr{S}_{abc}(i_6, i_7) \leftrightarrows_{2^5} \mathscr{S}_{abc}(i_1, i_3)$, we may pick $S_{i_8} \in \mathscr{S}_{abc}(i_6, i_7)$ such that $S_{i_8} \in \mathscr{S}((i_1, i_2)^{00}_{00} \cap (l, i_5)^{00}_{00}) \cap \mathscr{S}_{abc}((i_6, i_7)_{00})$. In addition, if S_{i_8} lies between 2 points of $S_{i_6} \cap C$ where x and y are the points nearest S_{i_8} coming respectively, before and after S_{i_8}, then we may choose S_{i_8} so that it intersects no other dendrite (besides S_{i_6}) containing x and y.

We have that $S_{i_8} \in \mathscr{S}_{ab}$ and thus there is a shortest path from $\mathscr{S}_{n_a}$ to S_{i_8} in $\mathscr{S}_{ab}$ of length b. Reasoning in a similar way as before, we conclude that there exists $S_m \in \mathscr{S}_a$, $m \notin [l, i_2]$ such that $S_m \sim S_{i_8}$.

Suppose $m > i_2$. Since $\mathscr{S}_{abc}(i_7, i_3) \leftrightarrows_{2^5} \mathscr{S}_{abc}(i_1, i_3)$, $\mathscr{S}_{abc}(i_7, i_3)$ has large chromatic number, and we may pick $S_{i_9} \in \mathscr{S}((l,m)_{00}^{00}) \cap \mathscr{S}_{abc}(i_7, i_3)$ such that $S_{i_9} \nsim S_m, S_l, S_{i_7}, S_{i_3}$ (see Figure 1). Since $S_{i_9} \in \mathscr{S}_a$, there is a shortest path from S_0 to S_{i_9} in $\mathscr{S}$ of length a. Reasoning in a similar way as was done for S_{i_5} and S_{i_8}, there is an $S_p \in \mathscr{S}_{a-1}$ where $p \notin [l,m]$. and $S_p \sim S_{i_9}$. Since $S_p \sim S_{i_9}$, there is an arc $A \subset S_p$ from the first point of S_p occuring along C to a point of $S_p \cap S_{i_9}$. To avoid creating a triangle, A must either intersect S_{i_8} or S_{i_5} before intersecting S_{i_9}.

We suppose $A \sim S_{i_5}$. Then A either intersects S_{i_3} before and after intersecting S_{i_5}, or it intersects S_{i_4} before before and after intersecting S_{i_5}. Clearly A can not intersect S_{i_4} before and after having intersected S_{i_3}, for then such intersection points would belong to C, and consequently, S_{i_4} would have been enumerated before S_{i_3} along C. We suppose therefore that A intersects S_{i_3} before and after intersecting S_{i_5}. Such intersection points must belong to C (since all points of $S_p \cap S_{i_3}$ must lie on C), and we let x and y be the nearest intersection points to S_{i_3} coming respectively, before and after S_{i_3} along C. Then x and y are the nearest points of $S_{i_3} \cap C$ occuring respectively before and after S_{i_5} along C. By the choice of S_{i_5}, no dendrite containing x and y intersects S_{i_5}. This yields a contradiction since $x, y \in S_p$ and $S_p \sim S_{i_5}$.

From the above we conclude that $A \nsim S_{i_5}$ and we deduce in a similar fashion that $A \nsim S_{i_8}$.

Suppose now that $m < l < i_2$ (see Fig. 2). We may now choose $S_{i_9} \in \mathscr{S}((m, i_2)_{00}^{00}) \cap \mathscr{S}_{abc}(i_1, i_6)$ such that $S_{i_9} \nsim S_{i_1}, S_l, S_{i_6}, S_{i_7}$. Reasoning in a similar way as before, there is an $S_p \in \mathscr{S}_{a-1}$ where $p \notin [m, i_2]$, and $S_p \sim S_{i_9}$. There is an arc $A \subset S_p$ from the first point of S_p along C to a point of $S_p \cap S_{i_9}$. To avoid creating a triangle, A must intersect S_{i_8} before intersecting S_{i_9}. One can show in a manner similar to the previous case that this can not happen. This concludes the proof of Theorem 1.1. □

Lastly, we include a proof of Corollary 1.2.

Proof of Corollary 1.2. By Proposition 3.1 we may replace each set $S \in \mathscr{S}$ by a dendrite T_S where for any pair $S, S' \in \mathscr{S}$ we have $S \sim S'$ iff $T_S \sim T_{S'}$. Furthermore, if $S \sim S'$, then $T_S \cap T_{S'}$ is arcwise connected (and bounded). Since there are only finitely many such dendrites T_S, we may choose a circle C in the plane so that each dendrite T_S lies inside C. We may extend each dendrite T_S to a dendrite T'_S in the following way: for a pair $S, S' \in \mathscr{S}$ where $S \sim S'$ and $S \cap S' \cap C \neq \varnothing$ let $A_{SS'}$ be an arc in $S \cap S'$ from $T_S \cap T_{S'}$ to C. Extend both T_S and $T_{S'}$ by adding $A_{SS'}$ to them. Repeat this operation for every such pair S and S'.

Suppose that for some T_S, it holds for all sets $S' \in \mathscr{S} \setminus \{S\}$ where $S \sim S'$ that $S \cap S' \cap C \neq \varnothing$. Since S is unbounded, it intersects C and there is an arc $A_S \subset S$ from C to T_S which we may assume intersects at most one other set S', and if such happens, then $A_S \cap S'$ is an arc in $S \cap S'$ which terminates at $T_S \cap T_{S'}$ (this we can assume since $S \cap S'$ is arcwise connected). Extend T_S to a larger dendrite T'_S which intersects C by adding A_S. If A_S intersects some other set S', then extend $T_{S'}$ to a larger dendrite $T'_{S'}$ by adding the arc $A_S \cap S'$. Now $T'_S \cap T'_{S'}$ is still a dendrite, and hence is arcwise connected. Repeat this proceedure for every such dendrite T_S.

When we have finished, we will have obtained a collection of dendrites T'_S, $S \in \mathscr{S}$ which preserves the same intersections as $\mathscr{S}$ where nonempty intersections are still arcwise connected and each T'_S is such that it intersects C but is contained in $C \cup int(C)$. It follows by Theorem 1.1 that the dendrites T'_S, $S \in \mathscr{S}$ have chromatic number bounded by a constant which is independent of $\mathscr{S}$. The proof of the corollary now follows. □

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