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RESEARCH ARTICLE

# Regular Semigroups with Inverse Transversals

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If S is a regular semigroup with set of idempotents $E(S)$ then an inverse subsemigroup of $S$ is called an *inverse transversal* of $S$ if $S^\circ$ contains a unique inverse $x^\circ$ of each element $x$ of $S$. The class of regular semigroups with inverse transversals was introduced by Blyth and McFadden [1] in 1982. This large class contains, for example, elementary rectangular bands of inverse semigroups [10], naturally ordered regular semigroups with a biggest idempotent [7], regular 4-spiral semigroups [2], and split orthodox semigroups [6]. Several authors have investigated regular semigroups with inverse transversals; see, for example, [1], [4], [11], [12]. A general structure theorem for regular semigroups with inverse transversals was obtained by Saito [13]. In a regular semigroup with an inverse transversal $S^\circ$ , the subsets

$$\begin{aligned} I &= \{e \in E(S) : e = ee^\circ\}, \\ \Lambda &= \{f \in E(S) : f = f^\circ f\}, \end{aligned}$$

are of considerable importance. In this paper we show that both $I$ and $\Lambda$ are subsemigroups of $S$. This means that, in the terminology of [13], every inverse transversal of $S$ is an *S-inverse transversal*, so that this latter concept becomes superfluous. Consequently, the construction in [13] can be replaced by the simpler form given in [12]. We also obtain necessary and sufficient conditions for an inverse subsemigroup $T$ of $S$ to be an inverse transversal, in particular when S is *E-solid(quasi-orthodox)* or *locally inverse*.

We recall that for idempotents $e$, $f$ of a regular semigroup $S$ the sandwich set $S(e,f)$ [8] is defined by

$$S(e,f) = \{g \in E(S) : ge = g = fg, egf = ef\}.$$

It is well-known that $S(e,f) = fV(ef)e$. Moreover, if $a' \in V(a)$ and $b' \in V(b)$, then, writing $S(a,b)$ for $S(a'a, bb')$, we have

$$(\forall g \in S(a,b)) \qquad b'ga' \in V(ab).$$

In particular, if $e\mathcal{L}g\mathcal{R}f$ with $e,f,g \in E(S)$ then $fge = g$ and $ef \in V(g)$.

In a regular semigroup S, we list the following basic facts which will be used in the sequel:

1. if $aa' = a$ or $a'a = a$ for $a' \in V(a)$ then $a, a' \in E(S)$ and $a\mathcal{L}a'$ or $a\mathcal{R}a'$;

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2. if $e, f \in E(S)$ with $e\mathcal{L}f$ or $e\mathcal{R}f$ then $e$ and $f$ are mutually inverse.

Let $S$ be a regular semigroup with an inverse transversal $S^\circ$. It was shown in [1] that

$$\begin{aligned} I &= \{aa^\circ : a \in S\}, \\ \Lambda &= \{a^\circ a : a \in S\}. \end{aligned}$$

We give the following lemma which will be needed in Theorem 2.

**Lemma 1.** *Let $S$ be a regular semigroups with an inverse transversal $S^\circ$. Then*

(i) $I = \{e \in E(S) : e\mathcal{L}e^\circ\}$;

(ii) $\Lambda = \{f \in E(S) : f\mathcal{R}f^\circ\}$;

(iii) *if $x \in I$, or $x \in \Lambda$, then $x^\circ \in E(S)$.* ■

At first, for $I$ and $\Lambda$ we begin with the following foundamental theorem.

**Theorem 2.** *Both $I$ and $\Lambda$ are subsemigroups of $S$.*

**Proof.** Suppose that $e, f \in I$ and consider the sandwich element $g = f(ef)^\circ e$. Using the fact, established in [5], that

$$(xy)^\circ = (x^\circ xy)^\circ x^\circ = y^\circ (xyy^\circ)^\circ$$

for any $x, y \in S$, and the observation that if $e \in I$ then $e^\circ e = e^\circ$ so that $e^\circ \in E(S)$ and $e\mathcal{L}e^\circ$, we have

$$g = f(e^\circ ef)^\circ e^\circ e = f(e^\circ ef)^\circ e^\circ = f(ef)^\circ.$$

Therefore, $eg = ef(ef)^\circ \in I$. We then have $(eg)^\circ \in E(S)$ and $g\mathcal{L}eg\mathcal{L}(eg)^\circ$ so that $(eg)^\circ \in V(g) \cap S^\circ$. Consequently, $g^\circ = (eg)^\circ \mathcal{L} g$ and $g^\circ \in E(S)$. Also

$$g^\circ = [f^\circ f(ef)^\circ e]^\circ f^\circ = [f^\circ f(ef)^\circ e]^\circ f^\circ f = g^\circ f.$$

from which we obtain that $g = gg^\circ = gg^\circ f = gf$. By the proof above, we have $eg \in I$. Therefore $ef = egf \in I$. Hence $I$ is a subsemigroup; and similarly so is $\Lambda$. ■

It follows immediately from Theorem 2 that all inverse transversals of a regular semigroup are necessarily S-inverse transversals in the terminology of [13]. From [13] we therefore deduce for all $x, y \in S$ that we have in general the identity

$$(x^{\circ\circ} y)^\circ = (xy^{\circ\circ})^\circ = y^\circ x^\circ.$$

For a subset $X$ of a regular semigroup $S$ we define $V(X) = \cup\{V(x) : x \in X\}$.

**Theorem 3.** *Let $S$ be a regular semigroup and let $T$ be an inverse subsemigroup of $S$ such that $V(T) = S$. Let*

$$P = \{e \in S : \exists e' \in V(e) \cap T \text{ such that } ee' = e\},$$

$$Q = \{e \in S : \exists e' \in V(e) \cap T \text{ such that } e'e = e\}.$$

*Then the following statements are equivalent:*

(1) *$T$ is an inverse transversal of $S$;*

(2) *$P$ and $Q$ are subsemigroups of $S$;*

(3) *$V(P) \cap T \subseteq E(T)$ and $V(Q) \cap T \subseteq E(T)$;*

(4) *$\mid V(e) \cap T \mid = 1$ for $e \in P$ or $e \in Q$.*

**Proof.** Note that

$$P = \{aa' : a' \in V(a) \cap T\}$$

and

$$Q = \{a'a : a' \in V(a) \cap T\}.$$

In fact, $P \subseteq \{aa' : a' \in V(a) \cap T\}$ is trivial. For $a \in S$ let $a' \in V(a) \cap T$ and $a'' \in V(a') \cap T$. Then $a''a' \in V(aa') \cap T$ and $aa' = (aa')(a''a')$. So $\{aa' : a' \in V(a) \cap T\} \subseteq P$.

(1) $\Longrightarrow$ (2): It follows immediately from Theorem 2.

(2) $\Longrightarrow$ (3): Suppose that (2) holds. Then $P$ is a left regular band. To see this, let $e, f \in P$. Then there exist $e', f' \in T$ such that $ee' = e$, $e'e = e'$, $ff' = f$ and $f'f = f'$. Since $e', f' \in E(T)$ it is clear that $e', f' \in P$. Since $P$ is a subsemigroup we have $e'f \in P$ and so, since the idempotents of $T$ commute,

$$e'f = e'fe'f = e'ff'e'f = e'fe'f' = e'ff'e' = e'fe'.$$

Consequently,

$$efe = ee'fe = ee'fe'e = ee'fe' = ee'f = ef$$

and therefore the band $P$ is left regular. Now let $e \in P$ and let $e'' \in V(e) \cap T$. Since $P$ is left regular and $ee'' \in P$ we have

$$ee'' = ee''eee'' = ee''e = e,$$

so that $e'' = e''e \in E(T)$. Thus $V(P) \cap T \subseteq E(T)$. Similarly, $Q$ is a right regular band and $V(Q) \cap T \subseteq E(T)$.

(3) $\Longrightarrow$ (4): Given $e \in P$ let $e'$, $e'' \in V(e) \cap T$ with $ee' = e$. By (3) we have $e', e'' \in E(T)$ and so

$$e'' = e''ee'' = e''ee'e'' = e''ee''e' = e''e' = e''e'e = e'e''e = e'ee''e = e'e = e'.$$

Hence $\mid V(e) \cap T \mid = 1$ for every $e \in P$. Similarly, we have $\mid V(f) \cap T \mid = 1$ for every $f \in Q$.

(4)$\Longrightarrow$ (1): Suppose first that $e, f \in P$ with $e\mathcal{R}f$. Then there exist $e' \in V(e) \cap T$ and $f' \in V(f) \cap T$ such that $ee' = e$ and $ff' = f$. Since $ef = f$ and $fe = e$ it follows readily that $e'f \in V(f'e) \cap V(e)$. Now let $(f'e)' \in V(f'e) \cap T$. Then we have

$$e'f = e'ff'ee'f = e'ff'e(f'e)'f'ee'f = e'(f'e)'f' \in T.$$

By (4) it follows that $e'f = e'$ and therefore $f = ef = e$. Similarly, we can show that if $e, f \in Q$ with $e\mathcal{L}f$ then $e = f$. Suppose now that $a \in S$ and let $a'$, $a" \in V(a) \cap T$. Since $aa'$, $aa" \in P$ and $aa'\mathcal{R}aa"$ we have $aa' = aa"$, and similarly, $a'a = a"a$. Consequently, $a' = a'aa' = a"aa" = a"$. Thus $T$ is an inverse transversal of $S$. ■

For any $a \in S$, the $\mathcal{L}-$(resp. $\mathcal{R}-$)class containing $a$ will be written by $L_a$ (resp. $R_a$). Now let $R_X = \cup\{R_x : x \in X\}$ and $L_X = \cup\{L_x : x \in X\}$ for any subset $X$ of $S$. For a regular semigroup $S$, the subsemigroup generated by the idempotents of $S$ is called the *core* of $S$ and is denoted by $C$.

**Theorem 4.** *A regular semigroup $S$ has an inverse transversal $S^\circ$ if and only if there exists an inverse transversal $C^\circ$ of $C$ such that $R_{C^\circ} \cap L_{C^\circ}$ forms a subsemigroup of $S$.*

**Proof.** Suppose that $S$ has an inverse transversal $S^\circ$. Let $C^\circ = C \cap S^\circ$. Then $C^\circ$ is an inverse subsemigroup of $C$. For any $x \in C$, we have $x = e_1 \dots e_n \in C$ with $e_1, \dots, e_n \in E$. By the well-known fact that:

if $E$ is the set of idempotents in a regular semigroup $S$ then

$$V(E^n) = E^{n+1} \quad (\forall n \in \mathbb{N})$$

we then have $x^\circ \in E^{n+1}$ and so $x^\circ \in C \cap S^\circ$. Therefore $x^\circ \in V(x) \cap C^\circ$. Since $V(x) \cap C^\circ \subseteq V(x) \cap S^\circ$, we obtain that $\mid V(x) \cap C^\circ \mid = 1$ for each $x \in C$. Consequently, $C^\circ$ is an inverse transversal of $C$.

Let $a \in S^\circ$. Then $aa^\circ$, $a^\circ a \in C^\circ$, which gives that $a \in R_{C^\circ} \cap L_{C^\circ}$. Conversely, if $a \in R_{C^\circ} \cap L_{C^\circ}$ then there exist $x, y \in C^\circ$ such that $x\mathcal{R}a\mathcal{L}y$. Thus $xx^\circ\mathcal{R}a\mathcal{L}y^\circ y$. Denote $xx^\circ$ and $y^\circ y$ by $e$ and $f$ respectively. Then $e$, $f \in E(C^\circ)$. Let $a^\circ \in V(a) \cap S^\circ$. Then $fa^\circ e \in V(a) \cap S^\circ$, which yields that $a^\circ = fa^\circ e$. Furthermore, $aa^\circ = a^{\circ\circ}a^\circ$ and $a^\circ a = a^{\circ\circ}a^\circ$. It is a routine matter to show that $a = aa^\circ a^{\circ\circ}a^\circ a = a^{\circ\circ} \in S^\circ$. Therefore $S^\circ = R_{C^\circ} \cap L_{C^\circ}$, as required.

Now suppose that there exists an inverse transversal $C^\circ$ of $C$ such that $R_{C^\circ} \cap L_{C^\circ}$ is a subsemigroup.

If $a \in R_{C^\circ} \cap L_{C^\circ}$ then there exist $x, y \in C^\circ$ such that $x\mathcal{R}a\mathcal{L}y$, which yields that $xx^\circ\mathcal{R}a\mathcal{L}y^\circ y$. Thus there exists $b \in V(a)$ such that $ab = xx^\circ$ and $ba = y^\circ y$. This gives that $b \in R_{C^\circ} \cap L_{C^\circ}$ and so $R_{C^\circ} \cap L_{C^\circ}$ is a regular semigroup.

Let $e \in E(R_{C^\circ} \cap L_{C^\circ})$. Then $g\mathcal{R}e\mathcal{L}f$ for some $f, g \in E(C^\circ)$. Therefore

$$g = eg = efg = egf = gf = gfe = fge = fe = f.$$

Thus $e = f = g \in E(C^\circ)$ and so $R_{C^\circ} \cap L_{C^\circ}$ is an inverse subsemigroup.

Let $a \in S$ and $a' \in V(a)$. Then $aa', a'a \in C$. Without difficulty, we obtain that

$$(a'a)^{\circ}(a'a)\mathcal{L}a\mathcal{R}(aa')(aa')^{\circ}.$$

Thus there exists $x \in V(a)$ such that $ax = (aa')(aa')^{\circ}$ and $xa = (a'a)^{\circ}(a'a)$, which means that $x \in R_{C^{\circ}} \cap L_{C^{\circ}}$. This element $x$ is denoted by $a^{\circ}$. Thus for each $a \in S$ there exists $a^{\circ} \in V(a) \cap R_{C^{\circ}} \cap L_{C^{\circ}}$. So $S = V(R_{C^{\circ}} \cap L_{C^{\circ}})$. Let

$$\begin{aligned} I &= \{aa^{\circ} : a \in S\}; \\ \Lambda &= \{a^{\circ}a : a \in S\}; \\ I_C &= \{aa^{\circ} : a \in C\}; \\ \Lambda_C &= \{a^{\circ}a : a \in C\}. \end{aligned}$$

For $aa^{\circ}$, $bb^{\circ} \in I$, denote $aa^{\circ}$ and $bb^{\circ}$ by $x$ and $y$ respectively. Then $x^{\circ} = a^{\circ\circ}a^{\circ}$, $y^{\circ} = b^{\circ\circ}b^{\circ}$ and $aa^{\circ} = xx^{\circ}$, $yy^{\circ} = bb^{\circ}$. It follows from (2) of Theorem 4 that $xy \in I_C$ and so $aa^{\circ}bb^{\circ} \in I$. This shows that $I$ is a subsemigroup. So is $\Lambda$. Again by (1) of Theorem 4, we obtain that $R_{C^{\circ}} \cap L_{C^{\circ}}$ is an inverse transversal of $S$. ■

A regular semigroup $S$ is said to be *E-solid* ( see [3], [14]) if the subsemigroup $< E(S) >$ generated by the idempotents of $S$ is completely regular (i.e. is the union of its maximal subgroups). For our purpose now we require the following characterisation of such semigroups.

**Lemma 5.** (see [3]) *A regular semigroup $S$ is E-solid if and only if*

$$\mathcal{L}\mid_{E(S)} \circ \mathcal{R}\mid_{E(S)} = \mathcal{R}\mid_{E(S)} \circ \mathcal{L}\mid_{E(S)}. \qquad ■$$

In what follows we shall require the following simple observation: if $S$ is regular and $e$, $f$ are idempotents of $S$ such that $e\mathcal{L}f\mathcal{R}g$ and $eg = ge$ then $e = f = g$. In fact, we have

$$f = fegf = fgef = ge$$

and so

$$e = ef = ege = ge$$

and

$$g = fg = geg = ge,$$

whence $e = f = g$.

**Theorem 6.** *Let $S$ be E-solid and let $T$ be an inverse subsemigroup of $S$. If $V(T) = S$ then $T$ is an inverse transversal of $S$.*

**Proof.** As in Theorem 3, Let

$$\begin{aligned} P &= e \in S : \exists e' \in V(e) \cap T, \text{ such that } ee' = e, \\ Q &= e \in S : \exists e' \in V(e) \cap T, \text{ such that } e'e = e. \end{aligned}$$

Let $e \in P$ and let $e'$, $e" \in V(e) \cap T$. Then we may assume that $ee' = e$. Now since $e"e\mathcal{L}e\mathcal{R}ee"$ there exists $f \in E(S)$ such that $e"e\mathcal{R}f\mathcal{L}ee"$ . But $e"e\mathcal{R}e"\mathcal{L}ee"$ and so

$e"\mathcal{H}f$. Now let $e''' \in V(e") \cap T$. Then $e'''e"\mathcal{L}e"\mathcal{R}e"e'''$ and therefore $e'''e"\mathcal{L}f\mathcal{R}e"e'''$ since $e"\mathcal{H}f$. Since $e'''e"$, $e"e''' \in E(T)$ we deduce from the above observation that $f = e'''e" = e"e''' \in E(T)$. Now since $e'\mathcal{L}e\mathcal{L}e"e\mathcal{R}f$ with $e'$, $f \in E(T)$ we deduce similarly that $e' = e"e = f$, so that $e\mathcal{L}e' = f\mathcal{R}e"$. It follows that $e" = e"e = e'$ and therefore $| V(e) \cap T | = 1$ for every $e \in P$. Similarly, $| V(e) \cap T | = 1$ for every $e \in Q$, and it follows by Theorem 3 that $T$ is an inverse transversal of $S$. ∎

A regular semigroup $S$ is said to be *locally inverse* if for every $e \in E(S)$ the subsemigroup $eSe$ is inverse. It is well-known that $S$ is locally inverse if and only if $| S(e,f) | = 1$ for all $e, f \in E(S)$; equivalently, if and only if $\omega^l(e) = \{f \in E(S) : fe = f\}$ is a left normal band and $\omega^r(e) = \{f \in E(S) : ef = f\}$ is a right normal band for every $e \in E(S)$ [9].

**Theorem 7.** *Let $S$ be a locally inverse semigroup and let $e \in E(S)$. Then $V(eSe)$ is a regular subsemigroup with $eSe$ as an inverse transversal.*

**Proof.** If $a, b \in V(eSe)$ then there exist $a'$, $b' \in eSe$ with $a' \in V(a)$ and $b' \in V(b)$. If $g \in S(a'a, bb')$ then $b'ga' = eb'ga'e \in V(ab) \cap eSe$. Thus $V(eSe)$ is a subsemigroup of $S$; clearly it is regular. Now let

$$\begin{aligned} P &= \{f \in V(eSe) : \exists f' \in V(f) \cap eSe, \text{such that } ff' = f\}; \\ Q &= \{f \in V(eSe) : \exists f' \in V(f) \cap eSe, \text{such that } f'f = f\}. \end{aligned}$$

If $f \in P$ then $ff' = f$ with $f' \in V(f) \cap eSe$, which gives $efef' = efe$ and $f'efe = f'$. Consequently, $efe \in E(eSe)$ and therefore $f' = efe$. Hence $f = ff' = fefe$ and therefore $fe = f$ whence $f \in \omega^l(e)$. Conversely, if $f \in \omega^l(e)$ then $fe = f$ and consequently $efe \in E(eSe)$. It is easy to check that $efe \in V(f) \cap eSe$. Since $fefe = fe = f$ it follows that $f \in P$. Hence $P = \omega^l(e)$ and so that $P$ is a left normal band. Dually, $Q$ is a right normal band. The result now follows from Theorem 3. ∎

If $S$ is a regular semigroup then a subset $I$ of $S$ is said to be a *quasi-ideal* of $S$ if $ISI \subseteq I$. It follows by [4, Proposition 1.3] that if $S$ has an inverse transversal that is a quasi-ideal then $S$ is locally inverse. If $S$ is a locally inverse semigroup with an inverse transversal $S^\circ$ then $S^\circ$ is a quasi-ideal of $S$.We shall now determine necessary and sufficient conditions under which an inverse subsemigroup that is a quasi-ideal is an inverse transversal. That this is not so in general is illustrated as follows.

**Example.** et $S = \mu^\circ(G; I, \Lambda; P)$ be a completely 0-simple semigroup in which $G = \{e\}$, $I = \{1, 2, 3\}$, $\Lambda = \{1, 2\}$ and $P$ is the sandwich matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Then $T = \{(2, e, 1), (2, e, 2), (3, e, 1), (3, e, 2), 0\}$ is an inverse subsemigroup of $S$ and $V(T) = S$. However, $T$ is not an inverse transversal of $S$ since

$$(2, e, 1), (2, e, 2) \in V((1, e, 1)) \cap T.$$

For a subset $K$ of a regular semigroup $S$ let

$$RegK = \{x \in K : V(x) \cap K\}.$$

Then for a locally inverse semigroup we have the following

**Theorem 8.** *Let $S$ be a locally inverse semigroup and let $T$ be an inverse subsemigroup of $S$ that is also a quasi-ideal. Then $T$ is an inverse transversal of $V(T)$ if and only if $E(ST)$ and $E(TS)$ are subsemigroups. In this case, $V(T) = RegST \cdot RegTS$.*

**Proof.** Suppose that $T$ is an inverse transversal of $V(T)$. Let

$$\begin{aligned} I_T &= \{aa^\circ \mid a \in V(T), a^\circ \in V(a) \cap T\}, \\ \Lambda_T &= \{a^\circ a \mid a \in V(T), a^\circ \in V(a) \cap T\}. \end{aligned}$$

It follows from Theorem 2 that $I_T$ and $\Lambda_T$ are subsemigroups of $V(T)$. Clearly, $I_T \subseteq E(ST)$ and $\Lambda_T \subseteq E(TS)$. Conversely, if $ax \in E(ST)$ for $a \in S$ and $x \in T$ then $x^\circ xax \in V(ax) \cap T$ and $ax = axx^\circ xax$, where $x^\circ$ denotes the inverse of $x$ in $T$. It follows that $ax \in I_T$ and so $E(ST) \subseteq I_T$. Therefore $I_T = E(ST)$. Similarly, $\Lambda_T = E(TS)$. As required.

Suppose now that $E(ST)$ and $E(TS)$ are subsemigroups. Clearly, $V(T)$ is a regular subsemigroup. Let

$$\begin{aligned} P &= \{e \in V(T) : \exists e' \in V(E) \cap T \textit{ such that } ee' = e\}; \\ Q &= \{e \in V(T) : \exists e' \in V(E) \cap T \textit{ such that } e'e = e\}. \end{aligned}$$

Then $P = E(ST)$ and $Q = E(TS)$. It follows from Theorem 3 that $T$ is an inverse transversal of $V(T)$.

Now let $a \in V(T)$. Then $a = aa^\circ a^{\circ\circ} a^\circ a$ for $a^\circ \in V(a) \cap T$. It follows from $aa^\circ a^{\circ\circ} \in RegST$ and $a^\circ a \in RegTS$ that $a \in RegST \cdot RegTS$. Conversely, let $ab \in RegST \cdot RegTS$ with $a \in RegST$ and $b \in RegTS$. Then there exist $a' \in V(a) \cap ST$ and $b' \in V(b) \cap TS$. Let $g \in S(a,b)$. Then $b'ga' \in V(ab) \cap T$ and so $ab \in V(T)$. Consequently, $V(T) = RegST \cdot RegTS$. ■

The following result relates to [4, Proposition 1.4].

**Theorem 9.** *Let $S$ be a rectangular band of inverse semigroups $S_{i,\lambda} (i \in I, \lambda \in \Lambda)$ and suppose that $(\alpha, \beta) \in I \times \Lambda$. Then $S_{\alpha,\beta}$ is an inverse transversal of $SS_{\alpha,\beta}S$.*

**Proof.** If $a \in V(S_{\alpha,\beta})$ then there exists $a' \in V(a) \cap S_{\alpha,\beta}$. Thus $a = aa'a \in SS_{\alpha,\beta}S$ and so $V(S) \subseteq SS_{\alpha,\beta}S$. Conversely, if $a \in SS_{\alpha,\beta}S$ then $a = bxyc$ for $b, c \in S$ and $x, y \in S_{\alpha,\beta}$. Suppose that $b \in S_{i,\lambda}$ and $c \in S_{j,\mu}$. Then we have $bx \in S_{i,\beta}$ and $yc \in S_{\alpha,\mu}$. Let $(bx)' \in V(bx) \cap S_{i,\beta}$ and $(yc)' \in V(yc) \cap S_{\alpha,\mu}$, and $g \in S(bx, yc)$. Then $(yc)'g(bx)' \in V(a) \cap S_{\alpha,\beta}$ so $SS_{\alpha,\beta}S \subseteq V(S_{\alpha,\beta})$ and therefore $SS_{\alpha,\beta}S = V(S_{\alpha,\beta})$.

With the symbols of $P$ and $Q$ in Theorem 3 it is easy to see that $P = E(SS_{\alpha,\beta})$ and $Q = E(S_{\alpha,\beta}S)$. Suppose now that $e \in E(SS_{\alpha,\beta})$ and let $e \in E(S_{i,\beta})$ and $x \in V(e) \cap S_{\alpha,\beta}$. Then $ex \in E(S_{i,\beta})$. By the hypothesis, $e = ex$ and so $x = xe \in E(S_{\alpha,\beta})$, which means that $V(e) \cap S_{\alpha,\beta} \subseteq E(S_{\alpha,\beta})$ for every $e \in P$. Similarly, $V(e) \cap S_{\alpha,\beta} \subseteq E(S_{\alpha,\beta})$ for every $e \in Q$. It follows from (3) of Theorem 3 that $S_{\alpha,\beta}$ is an inverse transversal of $SS_{\alpha,\beta}S$. ■

**Corollary.** *If $S$ is a rectangular band of inverse semigroups $S_{i,\lambda}$ then $S$ has inverse transversals if and only if there exists $S_{\alpha,\beta}$ such that $S = SS_{\alpha,\beta}S$.* ■

If $S$ is a locally inverse semigroup and if $E(S) = (L_\lambda, R_i; M_{i,\lambda}; \phi_{i,\lambda}, \psi_{i,\lambda}; I, \Lambda)$ then, as established in [9], $S$ is a rectangular band of semigroups $S_{i,\lambda}$, and $1_{i,\lambda}S1_{i,\lambda}$ is a maximal inverse subsemigroup of $S_{i,\lambda}$.

In this situation, we have the following

**Theorem 10.** *If $S$ is a locally inverse semigroup whose form as given above then $S$ has inverse transversals if and only if there exists $(\alpha, \beta) \in I \times \Lambda$ such that the following conditions are satisfied:*

(i) $S = S1_{\alpha,\beta}S$;

(ii) *both $S1_{\alpha,\beta}$ and $1_{\alpha,\beta}S$ are regular.*

**Proof.** Suppose that $S$ has inverse transversals. Let $S^\circ$ be an inverse transversal of $S$. Then there exists $(\alpha, \beta) \in I \times \Lambda$ such that $S^\circ \subseteq S_{\alpha,\beta}$. It follows that $S_{\alpha,\beta}$ is regular and so that $S_{\alpha,\beta} = 1_{\alpha,\beta}S1_{\alpha,\beta} = S^\circ$. Since $a = aa^\circ a$ for every $a \in S$, we obtain that

$$S = SS^\circ S = S1_{\alpha,\beta}S1_{\alpha,\beta}S \subseteq S1_{\alpha,\beta}S.$$

Thus $S = S1_{\alpha,\beta}S$. On account of $1_{\alpha,\beta}S1_{\alpha,\beta} \subseteq 1_{\alpha,\beta}S$, $S1_{\alpha,\beta}$, we can see that $1_{\alpha,\beta}S$ and $S1_{\alpha,\beta}$ are regular.

Conversely, suppose that now $S$ satisfies conditions (i) and (ii). Then we have $x = a1_{\alpha,\beta}b$ for every $x \in S$. Let $u \in V(a1_{\alpha,\beta}) \cap S1_{\alpha,\beta}$ and $v \in V(1_{\alpha,\beta}b) \cap 1_{\alpha,\beta}S$. It is easy to see that $1_{\alpha,\beta}u1_{\alpha,\beta} \in V(a1_{\alpha,\beta})$ and $1_{\alpha,\beta}v1_{\alpha,\beta} \in V(1_{\alpha,\beta}b)$. Let $g \in S(a1_{\alpha,\beta}, 1_{\alpha,\beta}b)$. Then

$$1_{\alpha,\beta}u1_{\alpha,\beta}g1_{\alpha,\beta}v1_{\alpha,\beta} \in V(a1_{\alpha,\beta}b) = V(x).$$

Thus $x \in V(1_{\alpha,\beta}S1_{\alpha,\beta})$ and so $S = V(1_{\alpha,\beta}S1_{\alpha,\beta})$.Let

$$P = e \in S : ee' = e \ for \ some \ e' \in V(e) \cap (1_{\alpha,\beta}S1_{\alpha,\beta});$$

$$Q = e \in S : e'e = e \ for \ some \ e' \in V(e) \cap (1_{\alpha,\beta}S1_{\alpha,\beta}).$$

It is easy to see that $P \subseteq E(S1_{\alpha,\beta})$ and $Q \subseteq E(1_{\alpha,\beta}S)$. If $e \in P$ then $e = e1_{\alpha,\beta}$ and $e' = 1_{\alpha,\beta}e1_{\alpha,\beta}$. Let $e,\ f \in P$. On account of the fact that $1_{\alpha,\beta}S1_{\alpha,\beta}$ is a inverse semigroup, we obtain that

$$ef1_{\alpha,\beta}e1_{\alpha,\beta}f1_{\alpha,\beta} \in V(ef) \cap 1_{\alpha,\beta}S1_{\alpha,\beta}$$

and

$$V(1_{\alpha,\beta}e1_{\alpha,\beta}f1_{\alpha,\beta}) \in V(ef) \cap 1_{\alpha,\beta}S1_{\alpha,\beta}.$$

Thus $P$ is a subsemigroup. Similarly, so is $Q$. It follows from Theorem 3 that $1_{\alpha,\beta}S1_{\alpha,\beta}$ is an inverse transversal of $S$. ■

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## References


[1] T. S. Blyth and R. McFadden, *Regular semigroups with a multiplicative inverse transversal,* Proc. Roy. Soc. Edinburgh **92A** (1982), 253–270.

[2] K. Byleen, *Regular 4-spiral semigroups, idempotent-generated semigroups and Rees construction,* Semigroup Forum **22** (1981), 97–100.

[3] T. E. Hall, *On regular semigroups,* J. Algebra **2** (1973), 1–24.

[4] D. B. McAlister and R. McFadden, *Regular semigroups with inverse transversals,* Quart. J. Math. Oxford **34** (1983), 459–474.

[5] D. B. McAlister and R. McFadden, *Maximum idempotents in naturally ordered regular semigroups,* Proc. Edinburgh Math. Soc. **26** (1983) 455–474.

[6] D. B. McAlister and T. S. Blyth, *Split orthodox semigroups,* J. Algebra **51** (1978) 451–525.

[7] R. McFadden and T. S. Blyth, *Naturally ordered regular semigroups with a greatest idempotent,* Proc. Roy. Soc. Edinburgh **91A** (1981) 107–122.

[8] K. S. S, Nambooripad, *Structure of regular semigroups I,* Mem. Amer. Math. Soc. **22** (1979), 224 pages.

[9] F. Pastijn, *The structure of pseudo-inverse semigroups,* Trans. Amer. Math. Soc. **273** (1982), 631–655.

[10] F. Pastijn, *Rectangular bands of inverse semigroups,* Simon Stevin **56** (1982), 3–95.

[11] T. Saito, *Stucture of regular semigroups with a quasi-ideal inverse transversal,* Semigroup Forum **31** (1985), 291–296.

[12] T. Saito, *A note on regular semigroups with inverse transversals,* Semigroup Forum **33** (1986), 149–152.

[13] T. Saito, *Construction of regular semigroups with inverse transversals,* Proc. Edinburgh Math. Soc. **32** (1989), 41–51.

[14] M. Yamada, *Structure of quasi-orthodox semigroups,* Mem. Fac. Sc. Shimane Univ. **14** (1980), 1–18.



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