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Existence result for minimal hypersurfaces with a prescribed finite number of planar ends

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Abstract. Paralleling what has been done for minimal surfaces in $\mathbb{R}^3$, we develop a gluing procedure to produce, for any $k \geq 2$ and any $n \geq 3$ complete immersed minimal hypersurfaces of $\mathbb{R}^{n+1}$ which have k planar ends. These surfaces are of the topological type of a sphere with k punctures and they all have finite total curvature

1. Introduction

Among the different tools designed to produce minimal surfaces in $\mathbb{R}^3$, the Weierstrass representation Theorem, which is probably the most popular, has been extensively used [9,10]. The main advantage of this method is that we have at hand an explicit local parameterization of the surface we are interested in. The main drawback is that the global geometric properties of the surfaces (such as embededness) are extremely hard to derive.

In a completely opposite direction, tools coming from nonlinear analysis have been useful either to produce new minimal surfaces [3,14], or to study the properties of the moduli space of such surfaces [11,12]. For example, the existence results which are based on perturbation arguments, have lead to examples [3,14] which would have been hard to find with the former technic. The main advantage of this type of constructions is that the geometry is usually well controlled.

Paralleling what is done for minimal surfaces, a gluing procedure has been developed to produce both compact and non compact complete constant mean curvature surfaces in $\mathbb{R}^3$. This was first achieved by N. Kapouleas [2] and was also considered by R. Mazzeo and the second author in [5] and even more recently in [6]. In this last paper became apparent that, in most of these constructions, the use of appropriately designed weighted Hölder spaces could simplify a lot the technicalities of the proofs. Moreover, it showed how Green's function played a central rôle in the construction, in particular stressing the fact that the local geometry of the surfaces at the point where the gluing is done is not relevant and that only global properties of the surfaces are of interest.

In higher dimension, the Weierstrass representation Theorem is not available anymore to produce minimal hypersurfaces and thus it is tempting to use the per-

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turbation arguments to produce some nontrivial examples of complete minimal hypersurfaces. In this perspective we prove the

Theorem 1.1. *Let $n \geq 3$. For all $k \geq 2$, there exists a smooth $k\,(n+1)$ dimensional manifold of complete immersed minimal hypersurfaces of $\mathbb{R}^{n+1}$ which have k planar ends. These surfaces are of the topological type of a sphere with k punctures and they all have finite total curvature.*

The structure of the proof of the result follows closely the proof of [6]. Thus, this paper is more intended to show first that the strategy developed in [6] for compact constant mean curvature surfaces with boundary can be easily adapted to our situation and also to derive all the relevant estimates and technical results which are needed for the machinery to work.

Let us briefly describe the general strategy of the proof : The proof of Theorem 1.1 is by induction. For $k = 2$, we have at our disposal the n-catenoid C_1 (which generalizes in higher dimension the well known catenoid), notice that in dimension $n + 1 \geq 4$ the n-catenoids have planar ends. Now, suppose M is a k ended nondegenerate minimal hypersurface, we choose any point $p \in M$ and remove from M a small disk centered at p. Then, we “glue” on another half n-catenoid which has been rescaled by a factor ε. The resulting hypersurface is then perturbed and, as a result, we obtain a one parameter family of minimal hypersurfaces with $k + 1$ planar ends.

Next, we prove that for ε small enough these surfaces are nondegenerate. In particular, this shows that the hypersurfaces we have produced actually belong to a smooth $(k + 1)\,(n + 1)$- dimensional family of such hypersurfaces.

Organization of the paper : Part 1 includes Sects. 3, 4 and 5, while Part 2 includes Sects. 6, 7 and 8. These two parts are completely independent and results of both parts are summarized in Sect. 5.2 and Sect. 8.2 respectively. Next, the results of the two parts are used in Part 3 which includes Sects. 9 and 10.

2. Notation

In this brief section we record some notation which will frequently used throughout the rest of the paper. First, $\lambda : \mathbb{R} \longrightarrow [0, 1]$ will denote a smooth cutoff function satisfying

$$\lambda \equiv 1 \quad \text{if} \quad t > 1 \quad \text{and} \quad \lambda \equiv 0 \quad \text{if} \quad t < 0.$$

Let us denote by $e_j(\theta)$, $j \in \mathbb{N}$ the eigenfunctions of the Laplacian on S^{n-1} with corresponding eigenvalue λ_j, that is $\Delta_{S^{n-1}} e_j = -\lambda_j\, e_j$, with $\lambda_j \leq \lambda_{j+1}$, which are normalized by

$$\int_{S^{n-1}} e_j^2\, d\theta = 1.$$

Furthermore, we will always assume that these are counted with multiplicity, namely

$$\lambda_0 = 0, \quad \lambda_1 = \ldots = \lambda_n = n - 1, \ldots .$$

We define some orthogonal projections π_{I} and π_{II} on $L^2(S^{n-1})$ as follows: if

$$\phi = \sum_{j\in\mathbb{N}} a_j\, e_j \in L^2(S^{n-1}),$$

we set

$$\pi_{\mathrm{I}}(\phi) \equiv \sum_{j\leq n} a_j\, e_j \quad \text{and} \quad \pi_{\mathrm{II}}(\phi) \equiv \sum_{j\geq n+1} a_j\, e_j \in L^2(S^{n-1}).$$

Finally, we define the continuous linear operator

$$D_\theta : \sum_{j\in\mathbb{N}} a_j\, e_j \in H^1(S^{n-1}) \longrightarrow \sum_{j\in\mathbb{N}} \gamma_j\, a_j\, e_j \in L^2(S^{n-1}),$$

where by definition

$$\gamma_j \equiv \sqrt{\left(\frac{n-2}{2}\right)^2 + \lambda_j}. \tag{2.1}$$

In other words, D_θ corresponds to the operator $\left(\left(\frac{n-2}{2}\right)^2 - \Delta_{S^{n-1}}\right)^{1/2}$. It could be useful to give another interpretation of D_θ. To this aim, we define for all $\phi \in H^1(S^{n-1})$, the function u as the unique solution of

$$\begin{cases} \Delta u = 0 & \text{in} \quad B_1 \\ u = \phi & \text{on} \quad \partial B_1. \end{cases}$$

If we set $v \equiv r^{\frac{2-n}{2}}\, u$, we obtain $D_\theta \phi = \partial_r v_{|\partial B_1}$. Hence, $D_\theta\phi$ is related to the normal derivative on the boundary of the harmonic extension of ϕ in the unit ball.

To keep the notations short, we set

$$\Delta_0 \equiv \partial_{ss} + \Delta_{S^{n-1}} - \left(\frac{n-2}{2}\right)^2$$

which acts on functions defined on $\mathbb{R} \times S^{n-1}$.

3. Minimal hypersurfaces which are graphs over a cylinder

We define the unit n-catenoid C_1 which is a minimal hypersurface of revolution and give some isothermal type parameterization of C_1. We also derive an almost explicit formula for the mean curvature of any hypersurface close to C_1.

3.1. Minimal hypersurfaces of revolution

To begin with, let us concentrate on hypersurfaces of revolution (say around the x_{n+1} axis). Such an hypersurface can be parameterized by

$$(t_1, t_2) \times S^{n-1} \ni (t, \theta) \longrightarrow (\rho(t)\,\theta, t) \in \mathbb{R}^{n+1},$$

where the function ρ is assumed to be defined and positive in some interval (t_1, t_2). In which case, the first fundamental form is given by

$$\mathbb{I}_\rho \equiv (1 + \dot\rho^2)\, dt^2 + \rho^2\, d\theta^2,$$

where $\cdot$ denotes differentiation with respect to t and where $d\theta^2$ is the first fundamental form of S^{n-1}. Up to some multiplicative constant, the volume functional is then given by

$$E_\rho \equiv \int_{t_1}^{t_2} \sqrt{1 + \dot\rho^2}\, \rho^{n-1}\, dt.$$

The associated Euler–Lagrange equation reads

$$\ddot\rho\, \rho - (n-1)\,(1 + \dot\rho^2) = 0, \tag{3.1}$$

and, whenever ρ is a positive solution of this equation, the corresponding hypersurface generated by ρ is minimal. All solutions of (3.1) are given by $\rho(t) \equiv \alpha\, \rho_0((t - t_0)/\alpha)$ where $t_0 \in \mathbb{R}$ and where $\alpha > 0$ is some constant and where ρ_0 is the solution of (3.1) with initial data $\rho_0(0) = 1$ and $\dot\rho_0(0) = 0$. In particular ρ_0 satisfies

$$1 + \dot\rho_0^2 = \rho_0^{2n-2}. \tag{3.2}$$

Definition 3.1. *The* unit n-catenoid *is defined to be the hypersurface of revolution which is generated by ρ_0 and is denoted by C_1.*

The properties of ρ_0 are summarized in the

Lemma 3.1. *The function ρ_0 is even, strictly increasing for $t > 0$ and defined over some maximal interval $(-T^*, T^*)$, where*

$$T^* \equiv \int_1^\infty \frac{dx}{(x^{2n-2} - 1)^{1/2}} > 0.$$

Furthermore

$$\lim_{t \uparrow T^*} (T^* - t)\, \rho_0^{n-2} = \frac{1}{n-2},$$

which implies in particular that ρ_0 tends to ∞ when t tends to T^* or to $-T^*$.

The fact that ρ_0 tends to $+\infty$ as the parameter t tends to $\pm T^*$ reflects the fact that the n-catenoid has two planar ends.

It will be very convenient to use a conformal parameterization of the unit n-catenoid C_1. To this aim, we define the functions $s \to \phi(s)$ and $s \to \psi(s)$ by the formulæ

$$\phi = \rho_0 \circ \psi$$

and

$$\psi' = \phi^{2-n} \quad \text{with} \quad \psi(0) = 0,$$

where this time $'$ denotes differentiation with respect to s. Using (3.2), one sees that ϕ is the unique non-constant $\mathcal{C}^2$ solution of

$$\phi'^2 + \phi^{4-2n} = \phi^2 \quad \text{with} \quad \phi(0) = 1. \tag{3.3}$$

It is not hard to check the

Lemma 3.2. *The function ψ is odd and is a diffeomorphism from $\mathbb{R}$ into $(-T^*, T^*)$. The function ϕ is even and defined on all $\mathbb{R}$. In addition, there exists $a > 0$ such that*

$$\begin{aligned} e^{-s}\,\phi(s) &= a\,(1 + \mathcal{O}(e^{(2-2n)s})) \quad \text{as} \quad s \to +\infty \\ e^{s}\,\phi(s) &= a\,(1 + \mathcal{O}(e^{(2n-2)s})) \quad \text{as} \quad s \to -\infty \end{aligned} \tag{3.4}$$

From now on, we will always assume that the unit n-catenoid C_1 is parameterized by

$$X_0 : (s, \theta) \in \mathbb{R} \times S^{n-1} \longrightarrow (\phi(s)\,\theta, \psi(s)) \in \mathbb{R}^{n+1}. \tag{3.5}$$

The rationale for this change of parameterization, is that, in these coordinates, the mean curvature of any surface close to C_1 can be computed almost explicitely, or at least takes a simple form, as we will see in the next paragraph.

Notice that the lower part of the n-catenoid, which is the image of $(-\infty, 0) \times S^{n-1}$ by X_0, can also be parameterized as a graph over the $x_{n+1} = 0$ hyperplane by

$$\mathbb{R}^n \setminus \overline{B_1} \ni x \longrightarrow (x, u_0(x)) \in \mathbb{R}^{n+1},$$

where u_0 is the unique (negative, decreasing) solution of

$$r^{n-1}\,\partial_r u_0 + \left(1 + (\partial_r u_0)^2\right)^{1/2} = 0 \quad \text{with} \quad \lim_{r\to\infty} u_0 = -T^*.$$

It is an easy exercise to see that the function u_0 has the following expansion as $r \equiv |x|$ tends to ∞

$$u_0(x) = -\,T^* + \frac{r^{2-n}}{n-2} + \mathcal{O}(r^{4-3n}).$$

3.2. The mean curvature operator for hypersurfaces close to C_1

Let us assume that the orientation of C_1 is chosen so that the unit normal vector field is given by

$$N_0(s,\theta) = \frac{1}{\phi(s)} \left(\psi'(s)\,\theta, -\phi'(s)\right). \tag{3.6}$$

All surfaces close enough to C_1 can be parameterized (at least locally) as normal graphs over C_1, namely

$$X = X_0 + w\, N_0,$$

for some small function w. We have the

Proposition 3.1. *The hypersurface parameterized by X is minimal if and only if the function w is a solution of the following nonlinear elliptic partial differential equation*

$$\begin{aligned} \mathcal{L}_0\, w = \; & Q_2\left(s, \frac{w}{\phi}, \nabla\left(\frac{w}{\phi}\right), \nabla^2\left(\frac{w}{\phi}\right)\right) \\ & + \phi^{n-1}\, Q_3\left(s, \frac{w}{\phi}, \nabla\left(\frac{w}{\phi}\right), \nabla^2\left(\frac{w}{\phi}\right)\right), \end{aligned} \tag{3.7}$$

where

$$\mathcal{L}_0 = \partial_s\left(\phi^{n-2}\,\partial_s\right) + \phi^{n-2}\,\Delta_{S^{n-1}} + n(n-1)\,\phi^{-n},$$

is the linearized mean curvature operator about C_1, where

$$(q_1, q_2, q_3) \longrightarrow Q_2(s, q_1, q_2, q_3),$$

is homogeneous of degree 2 and where

$$(q_1, q_2, q_3) \longrightarrow Q_3(s, q_1, q_2, q_3),$$

collects all the higher order nonlinear terms, that is

$$Q_3(s,0,0,0) = 0, \quad \nabla_{q_i} Q_3(s,0,0,0) = 0 \quad \textit{and} \quad \nabla^2_{q_i\,q_j} Q_3(s,0,0,0) = 0.$$

Furthermore, the coefficients Q_2 on the one hand, and the partial derivatives at any order of Q_3, with respect to the q_i's, computed at any point of some neighborhood $\mathcal{V}$ of $(0,0,0)$ on the other hand, are bounded functions of s and so are the derivatives of any order of these functions, uniformly in $\mathcal{V}$.

We will write for short

$$Q_i\left(\frac{w}{\phi}\right) \equiv Q_i\left(s, \frac{w}{\phi}, \nabla\left(\frac{w}{\phi}\right), \nabla^2\left(\frac{w}{\phi}\right)\right) \quad \text{for} \quad i = 2, 3.$$

Though this is not apparent in the notation, $Q_i(\cdot)$ depends on s.

Proof. For simplicity in the notations, we set

$$\tilde{N}_0 \equiv \phi\, N_0 \quad \text{and} \quad \tilde{w} \equiv \frac{w}{\phi},$$

so that we now have the parameterization $X = X_0 + \tilde{w}\, \tilde{N}_0$. Granted these definitions, the first fundamental form of the hypersurface parameterized by X reads

$$\begin{aligned}\mathbb{I} = {} & \phi^2\,(ds^2 + d\theta^2) + 2\,\phi^{3-n}\,\tilde{w}\,((1-n)\,ds^2 + d\theta^2) \\ & + 2\phi\,\phi'\,\tilde{w}\,(\tilde{w}_s\,ds^2 + \sum_i \tilde{w}_i\,ds\,d\theta_i) + \phi^{4-2n}\,\tilde{w}^2\,(n(n-2)\,ds^2 + d\theta^2) \\ & + \phi^2\,\tilde{w}^2\,ds^2 + \phi^2\,(\tilde{w}_s^2\,ds^2 + 2\,\tilde{w}_s\,\sum_i \tilde{w}_i\,ds\,d\theta_i + \sum_{i,j} \tilde{w}_i\,\tilde{w}_j\,d\theta_i\,d\theta_j),\end{aligned}$$

where we have set $\tilde{w}_s \equiv \partial_s \tilde{w}$ and $\tilde{w}_i \equiv \partial_{\theta_i} \tilde{w}$ for all $i = 1, \dots, n$. Using the well known formula

$$\det\,(I + A) = 1 + \mathrm{Tr}(A) + \frac{1}{2}\left(\mathrm{Tr}(A)^2 - \mathrm{Tr}(A^2)\right) + \mathcal{O}(|A|^3),$$

we establish

$$\begin{aligned}\det \mathbb{I} = {} & \phi^{2n}\left(1 + 2\phi'\phi^{-1}\tilde{w}\tilde{w}_s + \tilde{w}_s^2 + |\nabla_\theta \tilde{w}|^2 + \tilde{w}^2\big(1 - (n^2 - n + 1)\phi^{2-2n}\big)\right) \\ & + \phi^{1+n}\,\tilde{Q}_3(\tilde{w}) + \phi^{2n}\,\tilde{Q}_4(\tilde{w}),\end{aligned}$$

where $\tilde{Q}_3$ is homogeneous of degree 3 and where $\tilde{Q}_4$ collects all the higher order terms. Observe that the Taylor's coefficients of $\tilde{Q}_i$ are constant coefficients polynomials in $1/\phi$ and ϕ'/ϕ, hence these coefficients are bounded functions of s and so are the derivatives of any order of these functions by virtue of (3.3).

Changing back $\tilde{w}$ into w/ϕ, we obtain the volume functional

$$\begin{aligned}\mathcal{E}(w) = {} & \int \phi^n \left(1 + \phi^{-2}\,|\nabla w|^2 - n(n-1)\,\phi^{-2n}\,w^2\right. \\ & \left. \qquad + \phi^{1-n}\,\tilde{Q}_3\left(\frac{w}{\phi}\right) + \tilde{Q}_4\left(\frac{w}{\phi}\right)\right)^{1/2}.\end{aligned}$$

The critical points of which satisfy the nonlinear elliptic equation

$$\partial_s(\phi^{n-2}\partial_s w) + \phi^{n-2}\Delta_{S^{n-1}} w + n(n-1)\phi^{-n} w = Q_2\left(\frac{w}{\phi}\right) + \phi^{n-1}\,Q_3\left(\frac{w}{\phi}\right),$$

where Q_2 is homogeneous of degree 2 and where Q_3 collects all the higher order terms. Hence, the hypersurface parameterized by X is minimal if and only if w is a solution of (3.7). The properties of Q_2 and Q_3 follow at once from the analyticity of $x \to (1+x)^{1/2}$ and the properties of $\tilde{Q}_3$ and $\tilde{Q}_4$. This ends the proof of the Proposition. □

We close this section by noticing that, the conjugate operator

$$\mathcal{L} = \phi^{\frac{2-n}{2}} \, \mathcal{L}_0 \, \phi^{\frac{2-n}{2}},$$

takes the simple form

$$\mathcal{L} = \partial_{ss} + \Delta_{S^{n-1}} - \left(\frac{n-2}{2}\right)^2 + \frac{n(3n-2)}{4} \, \phi^{2-2n}. \tag{3.8}$$

And, using this notation together with (3.7), we see that the hypersurface parameterized by

$$X_w = X_0 + w \, \phi^{\frac{2-n}{2}} \, N_0,$$

is minimal if and only if w is a solution of

$$\mathcal{L} w = \phi^{\frac{2-n}{2}} \, Q_2 \left(\phi^{-\frac{n}{2}} \, w\right) + \phi^{\frac{n}{2}} \, Q_3 \left(\phi^{-\frac{n}{2}} \, w\right). \tag{3.9}$$

4. Mapping properties of the linearized mean curvature operator

We define the indicial roots of $\mathcal{L}$, the linearized mean curvature operator about the n-catenoid, and give the expression of all the $2\,(n+1)$ Jacobi fields which arise from geometric transformation of C_1. Next, we prove that, when restricted to any eigenspace e_j, the operator $\mathcal{L}$ satisfies the maximum principle provided $j \geq n+1$. Finally, some right inverse for $\mathcal{L}$ is constructed on any half n-catenoid. Similar results are also proved for the operator Δ_0.

4.1. Indicial roots and Jacobi fields

We start with the study of Δ_0, since this is the easiest. If we project the operator Δ_0 over the eigenspaces spanned by e_j, we obtain the sequence of operators

$$\partial_{ss} - \lambda_j - \left(\frac{n-2}{2}\right)^2, \quad j \in \mathbb{N}.$$

The indicial roots of Δ_0 at both $+\infty$ or $-\infty$ are given by $\pm\gamma_j$, where γ_j has been defined in (2.1). It is easy to see that these indicial roots all appear as the asymptotic behavior at $\pm\infty$ of the solutions of the homogeneous problem $\Delta_0 w = 0$, since $e^{\pm\gamma_j s} \, e_j$ solves $\Delta_0(e^{\pm\gamma_j s} \, e_j) = 0$.

Paralleling what we have done for Δ_0, we may now project the operator $\mathcal{L}$ over the eigenspaces spanned by e_j. This time, we obtain the sequence of operators

$$L_j = \partial_{ss} - \lambda_j - \left(\frac{n-2}{2}\right)^2 + \frac{n(3n-2)}{4} \, \phi^{2-2n}, \quad j \in \mathbb{N}.$$

The indicial roots of $\mathcal{L}$ at both $+\infty$ or $-\infty$ are again given by $\pm\gamma_j$. All these indicial roots also appear as the asymptotic behavior at $\pm\infty$ of the solutions of the homogeneous problem $\mathcal{L} w = 0$.

It is possible to determine explicitely some Jacobi fields, i.e. solutions of the homogeneous problem $\mathcal{L}w = 0$, in terms of the functions ϕ and ψ. These Jacobi fields correspond to explicit one-parameter geometric transformation of C_1, say $\xi \to C(\xi)$ with $C(0) = C_1$. For all ξ small enough, $C(\xi)$ can be written (at least locally) as a normal graph over C_1 and differentiation with respect to ξ gives rise to one Jacobi field.

Using the above procedure, if one considers, as a one parameter family of transformation of C_1, the translation along the x_{n+1} axis one finds the Jacobi field

$$\Psi^{0,+} \equiv \phi^{\frac{n-4}{2}} \phi', \tag{4.1}$$

which corresponds to the indicial root $\gamma_0 = \frac{n-2}{2}$ at $+\infty$ and $-\gamma_0 = \frac{2-n}{2}$ at $-\infty$. While dilation of C_1 gives the Jacobi field

$$\Psi^{0,-} \equiv \phi^{\frac{n-4}{2}} \left(\phi\, \psi' - \psi\, \phi'\right), \tag{4.2}$$

which also corresponds to the indicial root $\gamma_0 = \frac{n-2}{2}$ at $+\infty$ and $-\gamma_0 = \frac{2-n}{2}$ at $-\infty$. Notice that, in order to obtain the Jacobi fields corresponding to $-\gamma_0$ at $+\infty$ (or to γ_0 at $-\infty$), it is enough to take a linear combination of $\Psi^{0,+}$ and $\Psi^{0,-}$. For example, $T^* \, \Psi^{0,+} + \Psi^{0,-}$ corresponds to the indicial root $-\gamma_0$ at both $\pm\infty$ and $T^* \, \Psi^{0,+} - \Psi^{0,-}$ corresponds to the indicial root γ_0 at both $\pm\infty$.

Next, translating C_1 in a direction orthogonal to the axis yields the linearly independent Jacobi fields

$$\Psi^{j,-} \equiv \phi^{\frac{n-4}{2}} \psi' e_j, \quad \text{for} \quad j = 1, \dots, n, \tag{4.3}$$

which correspond to the indicial root $-\gamma_j = -\frac{n}{2}$ at $+\infty$ and $\gamma_j = \frac{n}{2}$ at $-\infty$. Finally rotating C_1 in a direction orthogonal to the axis leads to the linearly independent Jacobi fields

$$\Psi^{j,+} \equiv \phi^{\frac{n-4}{2}} \left(\phi\, \phi' + \psi\, \psi'\right) e_j, \quad \text{for} \quad j = 1, \dots, n, \tag{4.4}$$

which correspond to the indicial root $\gamma_j = \frac{n}{2}$ at $+\infty$ and $-\gamma_j = -\frac{n}{2}$ at $-\infty$. The derivation of these formulæ is quite standard and left to the reader. Details of the derivation are given for example in [5] in the framework of Delaunay surfaces, see also [6].

Notice that the indicial roots of $\mathcal{L}_0$ are given by $\frac{2-n}{2} \pm \gamma_j$ at $+\infty$, while they are given by $\frac{n-2}{2} \pm \gamma_j$ at $-\infty$.

4.2. Bounded solutions of $\mathcal{L}w = 0$ and $\Delta_0 w = 0$

Our first result is simply the

Proposition 4.1. *Assume that w is a bounded solution of $\Delta_0 w = 0$ in $(s_1, s_2) \times S^{n-1}$ (with boundary data $w = 0$ on $\{s_i\} \times S^{n-1}$ if any of the s_i is finite). Then $w \equiv 0$.*

Proof. The potential in Δ_0 being negative, the result is straightforward when both s_i are finite. In the general case, we decompose $w = \sum_{j\in\mathbb{N}} w_j\, e_j$, we see that $v \equiv w_j\, e_j$ is a linear combination of $e^{\pm\gamma_j s}\, e_j$ and therefore cannot be bounded unless $w \equiv 0$. □

Now we want to prove the following simple looking result

Proposition 4.2. *Assume that w is a bounded solution of $\mathcal{L}w = 0$ in $(s_1, s_2) \times S^{n-1}$ (with boundary data $w = 0$ on $\{s_i\} \times S^{n-1}$ if any of the s_i is finite). Further assume that, for each fixed $s \in (s_1, s_2)$, $w(s, \cdot)$ is orthogonal to $e_1, \dots, e_n$ in the L^2-sense on S^{n-1}. Then $w \equiv 0$.*

Before we proceed with the proof of this result let us notice, even though the result looks as simple as the previous one, this time it is *a priori* not obvious at all to conclude that

$$L_j = \partial_{ss} - \lambda_j - \left(\frac{n-2}{2}\right)^2 + \frac{n(3n-2)}{4}\, \phi^{2-2n},$$

satisfies the maximum principle for all $j \geq n+1$. For $n = 3$, or for $n \geq 4$ and j large enough, the potential in L_j is negative and the result is straightforward. Unfortunately, for $n \geq 4$ and j small, the potential in L_j is positive for s close to 0, thus nothing can be concluded using a direct argument.

Notice that the assumption that, for each fixed $s \in (s_1, s_2)$, $w(s, \cdot)$ is orthogonal to $e_1, \dots, e_n$ in the L^2-sense on S^{n-1} cannot be weakened. For example, for $j = 1, \dots, n$, the function $\Psi^{j,-}\, e_j$ is a Jacobi field which is bounded on all $\mathbb{R} \times S^{n-1}$. Furthermore, it is easy to see that there exists $s_0 \in \mathbb{R}$ such that $T^*\, \Psi^{0,+}(s_0) + \Psi^{0,-}(s_0) = 0$. Therefore $T^*\, \Psi^{0,+} + \Psi^{0,-}$ is a Jacobi field which is bounded (and has 0 boundary data) in $[s_0, +\infty) \times S^{n-1}$.

Proof. Even though we are interested in the operator $\mathcal{L}$, the proof is easier when using the operator $\mathcal{L}_0$. There is no loss of generality in doing so since, whenever w is a solution of $\mathcal{L}w = 0$, then $\tilde{w} \equiv \phi^{\frac{2-n}{2}}\, w$ solves $\mathcal{L}_0 \tilde{w} = 0$.

To begin with, we assume that both s_i are finite. Considering the eigenfunction decomposition of $\tilde{w} = \sum_{j\geq n+1} \tilde{w}_j\, e_j$, we see that $v \equiv \tilde{w}_j\, e_j$ is a solution of $\mathcal{L}_0 v = 0$ in $(s_1, s_2) \times S^{n-1}$, with $v = 0$ on $\{s_1, s_2\} \times S^{n-1}$. We multiply $\mathcal{L}_0 v = 0$ by v and integrate by parts the result over $(s_1, s_2) \times S^{n-1}$ to obtain

$$\int v'^2\, \phi^{n-2} + \lambda_j \int v^2\, \phi^{n-2} = n(n-1) \int v^2\, \phi^{-n}, \tag{4.5}$$

where all integrals are understood over $(s_1, s_2) \times S^{n-1}$ and where, as usual, $'$ denotes differentiation with respect to the variable s.

We proceed with some auxiliary computation. First, using (3.1) and (3.2), we obtain

$$\frac{d}{dt}\left(\frac{\dot{\rho}_0}{\rho_0}\right) = (n-2)\, \rho_0^{2n-4} + \rho_0^{-2},$$

and, since $ds = \rho_0^{n-2}\, dt$, this becomes

$$\frac{d}{ds}\left(\frac{\dot\rho_0}{\rho_0}\circ\psi\right) = (n-2)\,\phi^{n-2} + \phi^{-n}.$$

We now multiply this equality by v^2 and integrate the result over $(s_1, s_2)\times S^{n-1}$ to obtain

$$(n-2)\int v^2\,\phi^{n-2} + \int v^2\,\phi^{-n} = \int v^2\,\frac{d}{ds}\left(\frac{\dot\rho_0}{\rho_0}\circ\psi\right).$$

Next, we integrate the right-hand side by parts and apply Cauchy–Schwarz inequality. This yields

$$\begin{aligned}(n-2)&\int v^2\,\phi^{n-2} + \int v^2\,\phi^{-n}\\ &\le 2\left(\int v'^2\,\phi^{n-2}\right)^{1/2}\left(\int v^2\left(\frac{\dot\rho_0}{\rho_0}\circ si\right)^2\phi^{2-n}\right)^{1/2}.\end{aligned}$$

Finally, we use (3.3) to conclude that

$$\begin{aligned}(n-2)&\int v^2\,\phi^{n-2} + \int v^2\,\phi^{-n}\\ &\le 2\left(\int v'^2\,\phi^{n-2}\right)^{1/2}\left(\int v^2\,\phi^{n-2} - \int v^2\,\phi^{-n}\right)^{1/2}.\end{aligned}$$

In order to simplify the exposition, we define

$$A = \int v^2\phi^{n-2},\quad B = \int v^2\,\phi^{-n}\quad\text{and}\quad C = \int v'^2\,\phi^{n-2}.$$

The previous inequality, together with (4.5), can be translated into

$$C + \lambda_j\,A = n(n-1)\,B\quad\text{and}\quad (n-2)\,A + B \le 2\,C^{1/2}\,(A-B)^{1/2}.$$

In addition, since $\phi > 1$ for all $s \neq 0$, we see that $A \ge B$. If v is not identically 0, then $D \equiv A/B \ge 1$ has to satisfy

$$((n-2)D+1)^2 \le 4\,\big(n(n-1) - \lambda_j\,D\big)\,(D-1)\,.$$

Since $\lambda_j \ge 2n$ for all $j \ge n+1$, this would also imply that

$$((n-2)D+1)^2 \le 4\,(n(n-1) - 2n\,D)\,(D-1)\,.$$

However, it is an easy exercise to see that this inequality never holds. Since we have reached a contradiction, this proves that $v \equiv 0$ and the result is therefore complete in the case where both s_i are finite.

In the case where s_1 or s_2 is not finite, the proof is identical to what we have done, though we now have to justify all the integrations. But, the inspection of the indicial roots of both $\mathcal{L}_0$ and $\mathcal{L}$ for $j \ge n+1$ allows to conclude that, if $w = \sum_{j\ge n+1} w_j\,e_j$ is a bounded solution of $\mathcal{L}w = 0$, then $v \equiv \phi^{\frac{2-n}{2}}\,w_j\,e_j$ is also bounded and decays sufficiently fast at $\pm\infty$ in order to justify all the previous integrations. □

This result can also be understood as

Corollary 4.1. *The n-catenoid C_1 is non degenerate.*

The precise definition of nondegeneracy will be given in Sect. 9.1.

Proof. Assume that $\delta < -\frac{n}{2}$ and that w is a solution of $\mathcal{L}w = 0$ which is bounded by ϕ^δ. We decompose w into $w = \sum_{j\in\mathbb{N}} w_j\, e_j$. The fact that $w_j \equiv 0$ for all $j \geq n+1$ follows directly from Proposition 4.2. Thus, $\sum_{j\leq n} w_j\, e_j$ has to be a linear combination of all the Jacobi fields given in (4.1), ..., (4.4) and it is easy to see that these can't be bounded by ϕ^δ unless $w \equiv 0$. □

4.3. The linearized mean curvature operator on a half n-catenoid

As in [5], the analysis of the mapping properties of $\mathcal{L}$ or Δ_0 is easy to do in some weighted Hölder spaces we are now going to define.

Definition 4.1. *For all $\delta \in \mathbb{R}$ and for all $S \in \mathbb{R}$, the space $\mathcal{C}^{k,\alpha}_\delta([S, +\infty) \times S^{n-1})$ is defined to be the space of functions $w \in \mathcal{C}^{k,\alpha}([S, +\infty) \times S^{n-1})$ for which the following norm is finite*

$$\|w\|_{k,\alpha,\delta} \equiv \sup_{s\geq S} |e^{-\delta s}\, w|_{k,\alpha\,([s,s+1]\times S^{n-1})}.$$

Here $|\ |_{k,\alpha\,([s,s+1]\times S^{n-1})}$ denotes the usual Hölder norm in $[s, s+1] \times S^{n-1}$.

To begin with, we investigate the mapping properties of $\mathcal{L}$ when defined between the above weighted spaces since this is the hardest case. These mapping properties crucially depend on the choice of δ. We prove the

Proposition 4.3. *Assume that $\delta \in \left(-\frac{n+2}{2}, -\frac{n}{2}\right)$ and $\alpha \in (0,1)$ are fixed. There exists some constant $c > 0$ and, for all $S \in \mathbb{R}$, there exists an operator*

$$\mathcal{G}_S : \mathcal{C}^{0,\alpha}_\delta([S, +\infty) \times S^{n-1}) \longrightarrow \mathcal{C}^{2,\alpha}_\delta([S, +\infty) \times S^{n-1}),$$

such that, for all $f \in \mathcal{C}^{0,\alpha}_\delta([S+\infty) \times S^{n-1})$, the function $w = \mathcal{G}_S(f)$ is the unique solution of

$$\begin{cases} \mathcal{L}w = f & \text{in} \quad [S, +\infty) \times S^{n-1} \\ w \in \operatorname{Span}\{e_0, \dots, e_n\} & \text{on} \quad \{S\} \times S^{n-1}, \end{cases}$$

which belongs to the space $\mathcal{C}^{2,\alpha}_\delta([S, +\infty) \times S^{n-1})$. Furthermore, $\|w\|_{2,\alpha,\delta} \leq c\,\|f\|_{0,\alpha,\delta}$. Finally, if, for each fixed $s \in [S, +\infty)$, the function $f(s,\cdot)$ is orthogonal to $e_0, \dots, e_n$ in the L^2-sense on S^{n-1}, then so is $w = \mathcal{G}_S(f)$.

Before, we proceed with the proof of this Proposition, let us emphasize that, in the last estimate, the constant c is independent of S. This is one of the reasons which forces the choice of the parameter δ in the interval $(-\frac{n+2}{2}, -\frac{n}{2})$. Another reason is that we want to use this result to perturb any n- catenoid, since we want this perturbation to be at least bounded, this implies that we need to take $\delta \leq \frac{n-2}{2}$.

Notice that, if $\delta = \pm\gamma_j$, it is easy to see that $\mathcal{L}$, defined between the above spaces, does not even have closed range and the result is certainly not true. Now, if we assume that $\delta \in (-\frac{n}{2}, \frac{2-n}{2}) \cup (\frac{2-n}{2}, \frac{n-2}{2})$, the existence of $\mathcal{G}_S$ remains true for all but a finite number of S. However, in this later case, we do not obtain a uniform bound for the norm of $\mathcal{G}_S$. Finally, if we take $-\gamma_{j+1} < \delta < -\gamma_j$, then the result still holds but more freedom is needed on the boundary data, namely $w \in \mathrm{Span}\{e_0, \dots, e_j\}$. Therefore, the interval $(-\frac{n+2}{2}, -\frac{n}{2})$ can be understood as the first interval for which a uniform bound on the norm of the inverse is available.

Proof. Uniqueness of $\mathcal{G}_S$ follows from a simple modification of the proof of Corollary 4.1. We therefore concentrate our attention on the existence of $\mathcal{G}_S$. We consider the eigenfunction decomposition of f

$$f = \sum_{j\in\mathbb{N}} f_j\, e_j,$$

and adopt the notation $f = f_{\mathrm{I}} + f_{\mathrm{II}}$, where $f_{\mathrm{I}} = \pi_{\mathrm{I}}(f)$ and $f_{\mathrm{II}} = \pi_{\mathrm{II}}(f)$ correspond to the decomposition of f into the projection onto the first $n+1$ eigenmodes and the higher order eigenmodes. We look for a solution w which will also be decomposed as

$$w = \sum_{j\in\mathbb{N}} w_j\, e_j,$$

and again we set $w = w_{\mathrm{I}} + w_{\mathrm{II}}$ where $w_{\mathrm{I}} = \pi_{\mathrm{I}}(w)$ and $w_{\mathrm{II}} = \pi_{\mathrm{II}}(w)$.

Step 1. To begin with we are going to prove that, given f_{II} there exists w_{II} solution of $\mathcal{L} w_{\mathrm{II}} = f_{\mathrm{II}}$ in $(S, +\infty) \times S^{n-1}$ with $w_{\mathrm{II}} = 0$ on $\{S\} \times S^{n-1}$ and

$$\sup_{[S,+\infty)\times S^{n-1}} |e^{-\delta s}\, w_{\mathrm{II}}| \leq c \sup_{[S,+\infty)\times S^{n-1}} |e^{-\delta s}\, f_{\mathrm{II}}|,$$

for some constant which does not depend on f_{II}, nor on S. Our problem being linear, we may always assume that

$$\sup_{[S,+\infty)\times S^{n-1}} |e^{-\delta s}\, f_{\mathrm{II}}| = 1.$$

For all $j \geq n+1$, it follows from Proposition 4.2 that, when restricted to the space of functions w such that $w(s,\cdot)$ is orthogonal to $e_0, \dots, e_n$ in the L^2-sense on S^{n-1}, the operator $\mathcal{L}$ is injective over $(S, S') \times S^{n-1}$. As a consequence, for all $S' > S+1$ we are able to solve $\mathcal{L} v_{\mathrm{II}} = f_{\mathrm{II}}$, in $(S', S) \times S^{n-1}$, with $v_{\mathrm{II}} = 0$ on $\{S, S'\} \times S^{n-1}$.

We claim that, there exists some constant $c > 0$ independent of $S' > S+1$ and of f_{II} such that

$$\sup_{(S',S)\times S^{n-1}} |e^{-\delta s}\, v_{\mathrm{II}}| \leq c.$$

We argue by contradiction and assume that the result is not true. In this case, there would exist sequences $S'_i > S_i + 1$, a sequence of functions $f_{\mathrm{II},i}$ satisfying

$$\sup_{(S_i, S'_i)\times S^{n-1}} |e^{-\delta s} f_{\mathrm{II},i}| = 1,$$

and a sequence $v_{\mathrm{II},i}$ of solutions of $\mathcal{L} v_{\mathrm{II},i} = f_{\mathrm{II},i}$, in $(S_i, S'_i) \times S^{n-1}$, with $v_{\mathrm{II},i} = 0$ on $\{S_i, S'_i\} \times S^{n-1}$ such that

$$A_i \equiv \sup_{(S_i, S'_i)\times S^{n-1}} |e^{-\delta s}\, v_{\mathrm{II},i}| \longrightarrow +\infty.$$

Let us denote by $(s_i, \theta_i) \in (S_i, S'_i) \times S^{n-1}$, a point where the above supremum is achieved. We now distinguish a few cases according to the behavior of the sequence s_i (which, up to a subsequence can always be assumed to converge in $[-\infty, +\infty]$). Up to some subsequence, we may also assume that the sequences $S'_i - s_i$ (resp. $s_i - S_i$) converges to $S^* \in (0, +\infty]$ (resp. to $S_* \in [-\infty, 0)$).

Notice that the sequence $s_i - S_i$ remains bounded away from 0. Indeed, since $v_{\mathrm{II},i}$ and $(\partial_{ss} + \Delta_{S^{n-1}})\, v_{\mathrm{II},i}$ are both bounded by a constant (independent of i) times $e^{\delta S_i} A_i$ in $[S_i, S_i + 1] \times S^{n-1}$ and since $v_{\mathrm{II},i} = 0$ on $\{S_i\} \times S^{n-1}$, we can conclude that the gradient of $v_{\mathrm{II},i}$ is also uniformly bounded by a constant times $e^{\delta S_i} A_i$ in $[S_i, S_i + 1/2] \times S^{n-1}$. As a consequence the above supremum cannot be achieved at a point which is too close to S_i. Similarly the sequence $S'_i - s_i$ also remains bounded away from 0.

We define the sequence of rescaled functions

$$\tilde{v}_{\mathrm{II},i}(s, \theta) \equiv \frac{e^{-\delta s_i}}{A_i}\, v_{\mathrm{II},i}(s + s_i, \theta).$$

Case 1. Assume that the sequence s_i converges to $s_* \in \mathbb{R}$. After the extraction of some subsequence, if this is necessary, we may assume that the sequence $\tilde{v}_{\mathrm{II},i}$ converges to some nontrivial solution of

$$\mathcal{L} v_{\mathrm{II}} = 0,$$

in $(S_*, S^*) \times S^{n-1}$, with boundary condition $v_{\mathrm{II}} = 0$, if either S_* or S^* is finite. Furthermore

$$\sup_{(S_*, S^*)\times S^{n-1}} |e^{-\delta s}\, v_{\mathrm{II}}| = 1. \tag{4.6}$$

We now decompose v_{II} into

$$v_{\mathrm{II}} = \sum_{j \geq n+1} v_j\, e_j.$$

If S_* is not finite, the inspection of the indicial roots shows that, necessarily, v_j is bounded in (S_*, S^*) together with the fact that $-\frac{n+2}{2} < \delta$. But, applying Proposition 4.2, this implies that $v_j = 0$ for all $j \geq n+1$, contradicting (4.6).

Case 2. Assume that the sequence s_i converges to $-\infty$ and thus $S^* = +\infty$. After the extraction of some subsequence, if this is necessary, we may assume that the sequence $\tilde{v}_{\mathrm{II},i}$ converges to some nontrivial solution of

$$\Delta_0 \, v_{\mathrm{II}} = 0,$$

in $(S_*, +\infty) \times S^{n-1}$, with boundary condition $v_{\mathrm{II}} = 0$, if S_* is finite. Furthermore

$$\sup_{(S_*,\infty)\times S^{n-1}} |e^{-\delta s} \, v_{\mathrm{II}}| = 1, \tag{4.7}$$

but both cases, S_* finite or not, are easy to rule out using the eigenfunction decomposition of v_{II}

$$v_{\mathrm{II}} = \sum_{j \geq n+1} v_j \, e_j .$$

Indeed, v_j has to be a linear combination of $e^{\pm\gamma_j s}$ and, since we have assumed that $\delta \in (-\frac{n+2}{2}, -\frac{n}{2})$ it is easy to see that $v_j \equiv 0$, contradicting (4.7).

Case 3. Assume that the sequence s_i converges to $+\infty$ and thus $S_* = -\infty$. This case being similar to Case 2, we shall omit it.

Now that the proof of the claim is finished, we may pass to the limit $S' \to +\infty$ and obtain a solution of $\mathcal{L} w_{\mathrm{II}} = f_{\mathrm{II}}$, in $(S, +\infty) \times S^{n-1}$, with $w_{\mathrm{II}} = 0$ on $\{S\} \times S^{n-1}$, which satisfies

$$\sup_{(S,+\infty)\times S^{n-1}} |e^{-\delta s} \, w_{\mathrm{II}}| \leq c,$$

for some constant $c > 0$ independent of S.

Step 2. We now turn our attention to the case $j = 0, \ldots, n$. This time, just by solving the associated ordinary differential equations, we are able to find for all $S' > S + 1$ a function v_j defined in $(-\infty, S']$ which is a solution of

$$\begin{cases} L_j v_j = f_j & \text{in} \quad (S, S') \\ L_j v_j = 0 & \text{in} \quad (-\infty, S), \end{cases}$$

with $v_j = \partial_s v_j = 0$ at S'.

The problem being linear, we may assume that

$$\sup_{[S,+\infty)\times S^{n-1}} |e^{-\delta s} \, f_{\mathrm{I}}| = 1.$$

We claim that there exists some constant $c > 0$, independent of S and S', such that

$$\forall j = 0, \ldots, n, \qquad \sup_{(-\infty,S')} |e^{-\delta s} \, v_j| \leq c.$$

We argue by contradiction and assume that the result is not true. There would exist sequences $S_i' > S_i + 1$, a sequence of functions $f_{\mathrm{I},i}$ satisfying

$$\sup_{(S_i,S_i')\times S^{n-1}} |e^{-\delta s} \, f_{\mathrm{I},i}| = 1,$$

and, finally, a sequence of solutions $v_{\mathrm{I},i}$ of

$$\begin{cases} \mathcal{L} v_{\mathrm{I},i} = f_{\mathrm{I},i} & \text{in} \quad (S_i, S_i') \times S^{n-1} \\ \mathcal{L} v_{\mathrm{I},i} = 0 & \text{in} \quad (-\infty, S_i) \times S^{n-1}, \end{cases}$$

with $v_{\mathrm{I},i} = \partial_s v_{\mathrm{I},i} = 0$ on $\{S_i'\} \times S^{n-1}$ such that

$$A_i \equiv \sup_{(-\infty, S_i') \times S^{n-1}} |e^{-\delta s}\, v_{\mathrm{I},i}| \longrightarrow +\infty.$$

Notice that $\mathcal{L} v_{\mathrm{I},i} = 0$ in $(-\infty, S_i) \times S^{n-1}$. Thus, in this range, the function $v_{\mathrm{I},i}$ is a linear combination of the functions $\Psi^{j,\pm}$, for $j = 0, \dots, n$. Since we have chosen $\delta < -\frac{n}{2}$, the above supremum is finite and achieved.

Let us denote by $(s_i, \theta_i) \in (-\infty, S_i') \times S^{n-1}$, a point where the above supremum is achieved and distinguish a few case according to the behavior of the sequence s_i (which, up to a subsequence can always be assumed to converge). Up to some subsequence, we may also assume that the sequence $S_i' - s_i$ converges to $S^* \in (0, +\infty)$. (Again, notice that $S_i' - s_i$ stays bounded away from 0).

We define the sequence of rescaled functions

$$\tilde{v}_{\mathrm{I},i}(s, \theta) \equiv \frac{e^{-\delta s_i}}{A_i}\, v_{\mathrm{I},i}(s + s_i, \theta).$$

Case 1. Assume that the sequence s_i converges to $s_* \in \mathbb{R}$. After the extraction of some subsequence, if this is necessary, we may assume that the sequence $\tilde{v}_{\mathrm{I},i}$ converges to some nontrivial solution of

$$\mathcal{L} v_{\mathrm{I}} = 0, \tag{4.8}$$

in $(-\infty, S^*) \times S^{n-1}$, with boundary condition $v_{\mathrm{I}} = \partial_s v_{\mathrm{I}} = 0$ at S^* if S^* is finite. Furthermore

$$\sup_{(-\infty, S^*) \times S^{n-1}} |e^{-\delta s}\, v_{\mathrm{I}}| = 1.$$

Necessarily, $S^* = +\infty$. Otherwise, we readily obtain $v_{\mathrm{I}} = 0$, which contradicts the previous equality. Now, the function v_{I} is a linear combination of the functions $\Psi^{j,\pm}$, for $j = 0, \dots, n$, and since we have chosen $\delta \in (-\frac{n+2}{2}, -\frac{n}{2})$, it is easy to see that none of the linear combinations of the $\Psi^{j,\pm}$ decays fast enough, unless it is identically zero. This is clearly in contradiction with the equality following (4.8).

Case 2. Assume that the sequence s_i converges to $-\infty$. After the extraction of some subsequence, if this is necessary, we may assume that the sequence $\tilde{v}_{\mathrm{I},i}$ converges to some nontrivial solution of

$$\Delta_0\, v_{\mathrm{I}} = 0,$$

in $\mathbb{R}$. Furthermore

$$\sup_{\mathbb{R} \times S^{n-1}} |e^{-\delta s}\, v_{\mathrm{I}}| = 1. \tag{4.9}$$

But $v_{\rm I}$ is a linear combination of $e^{\pm\gamma_j s}\, e_j$, for $j = 0, \dots, n$, and, since $\delta \in (-\frac{n+2}{2}, -\frac{n}{2})$, we see that $v_{\rm I}$ has to be identically zero, which contradicts (4.9).

Case 3. Assume that the sequence s_i converges to $+\infty$. This case being similar to Case 2, we shall omit it.

Since we have ruled out every possible situation which would contradict our claim, the proof of the claim is complete. We may pass to the limit $S' \to +\infty$ and obtain a solution of

$$\mathcal{L} w_{\rm I} = f_{\rm I},$$

in $(S, +\infty) \times S^{n-1}$ such that

$$\sup_{[S,+\infty)\times S^{n-1}} |e^{-\delta s}\, w_{\rm I}| \leq c,$$

for some constant $c > 0$ independent of S.

To complete the proof of the Proposition, it suffices to apply Schauder's estimates in order to get the relevant estimates for all the derivatives. □

A similar result holds for Δ_0. The proof being a straightforward modification of the previous proof, we omit it.

Proposition 4.4. *Assume that* $\delta \in (-\frac{n+2}{2}, -\frac{n}{2})$ *and* $\alpha \in (0,1)$ *are fixed. Then, there exists some constant* $c > 0$ *and, for all* $S \in \mathbb{R}$, *there exists an operator*

$$G_S : \mathcal{C}^{0,\alpha}_\delta\big([S, +\infty) \times S^{n-1}\big) \longrightarrow \mathcal{C}^{2,\alpha}_\delta\big([S, +\infty) \times S^{n-1}\big),$$

such that for all $f \in \mathcal{C}^{0,\alpha}_\delta\big([S, +\infty) \times S^{n-1}\big)$, *the function* $w = G_S(f)$ *is the unique solution of*

$$\begin{cases} \Delta_0 w = f & \text{in} \quad [S, +\infty) \times S^{n-1} \\ w \in \mathrm{Span}\{e_0, \dots, e_n\} & \text{on} \quad \{S\} \times S^{n-1}, \end{cases}$$

which belongs to $\mathcal{C}^{2,\alpha}_\delta\big([S, +\infty) \times S^{n-1}\big)$. *Furthermore,* $\|w\|_{2,\alpha,\delta} \leq c\, \|f\|_{0,\alpha,\delta}$.

We will need a supplement to the previous Proposition. Indeed the next result is not Corollary of the previous Proposition since, this time, the weight parameter does not belong to $(-\frac{n+2}{2}, -\frac{n}{2})$.

Proposition 4.5. *There exists* $c > 0$ *such that, for all* $S \in \mathbb{R}$ *and all* $g_{\rm II} \in \pi_{\rm II}\big(\mathcal{C}^{2,\alpha}(S^{n-1})\big)$, *there exists a unique* $w_0 \in \mathcal{C}^{2,\alpha}_{-\frac{n+2}{2}}\big([S, +\infty) \times S^{n-1}\big)$ *solution of*

$$\begin{cases} \Delta_0 w_0 = 0 & \text{in} \quad (S, +\infty) \times S^{n-1} \\ w_0 = g_{\rm II} & \text{on} \quad \{S\} \times S^{n-1}. \end{cases}$$

Furthermore, we have

$$||w_0||_{2,\alpha,-\frac{n+2}{2}} \leq c\, e^{\frac{n+2}{2}S}\, ||g_{\rm II}||_{2,\alpha},$$

Proof. Again we decompose $g_{\mathrm{II}} = \sum_{j \geq n+1} g_j \, e_j$ and define for all $J \geq n+1$ the function $w_0^J \equiv \sum_{j=n+1}^{J} g_j \, e^{\gamma_j (S-s)} \, e_j$, which clearly solves $\Delta_0 w_0 = 0$, in $[S, +\infty) \times S^{n-1}$.

First of all, let us consider all the eigenfrequencies for which $\gamma_j = \frac{n+2}{2}$. They correspond to the indices $j = n+1, \dots, \frac{n(n+1)}{2}$. We set, $J_0 \equiv \frac{n(n+1)}{2}$. Obviously, we have

$$|w_0^{J_0}(s, \theta)| \leq c \, e^{\frac{n+2}{2}(S-s)} \, ||g_{\mathrm{II}}||_{2,\alpha},$$

for some constant independent of S.

Next, it is easy to see that, for all $J \geq J_0$, we have

$$|(w_0^J - w_0^{J_0})(s, \theta)| \leq c_J \, e^{\frac{n+2}{2}(S-s)} \, ||g_{\mathrm{II}}||_{2,\alpha},$$

for some constant c_J which is independent of S (but may depend on J). It remains to prove that c_J does not depend on J. The proof of this fact is again by contradiction and is very close to the proof of Proposition 4.3, so we omit it.

Once this estimate is proved the estimates for the derivatives follow from Schauder's estimates as usual. □

Thanks to Proposition 4.3 and Proposition 4.2, if $\delta \in \left(-\frac{n+2}{2}, -\frac{n}{2}\right)$, then for all $g_{\mathrm{II}} \in \pi_{\mathrm{II}} \left(\mathcal{C}^{2,\alpha}(S^{n-1})\right)$, we may define $w \equiv \mathcal{P}_S(g_{\mathrm{II}}) \in \mathcal{C}^{2,\alpha}_{\delta}\left([S, +\infty) \times S^{n-1}\right)$ be the unique solution of

$$\begin{cases} \mathcal{L} w = 0 & \text{in} \quad (S, +\infty) \times S^{n-1} \\ w = g_{\mathrm{II}} & \text{on} \quad \{S\} \times S^{n-1}. \end{cases}$$

Furthermore, we have

$$||\mathcal{P}_S(g_{\mathrm{II}})||_{2,\alpha,\delta} \leq c \, e^{-\delta S} \, ||g_{\mathrm{II}}||_{2,\alpha}, \tag{4.10}$$

for some constant $c > 0$ which is independent of S. Further information concerning the operator $\mathcal{P}_S$ is provided by the following Proposition in which we compare the Neumann data of $\mathcal{P}_S(g_{\mathrm{II}})$ with $D_\theta \, g_{\mathrm{II}}$.

Proposition 4.6. *Assume that $\delta \in \left(-\frac{n+2}{2}, -\frac{n}{2}\right)$ and $\alpha \in (0,1)$ are fixed. There exists $c > 0$ such that, for all $S \in \mathbb{R}$ and for all $g_{\mathrm{II}} \in \pi_{\mathrm{II}} \left(\mathcal{C}^{2,\alpha}(S^{n-1})\right)$*

$$||\partial_s \, \mathcal{P}_S(g_{\mathrm{II}})(S, \cdot) + D_\theta g_{\mathrm{II}}||_{1,\alpha} \leq c \, e^{\left(\frac{n+2}{2} + \delta\right) S} \, ||g_{\mathrm{II}}||_{2,\alpha}.$$

Proof. To begin with, we define, thanks to Proposition 4.5, the function w_0 which is the unique solution of $\Delta_0 w_0 = 0$ in $(S, +\infty) \times S^{n-1}$ with $w_0 = g_{\mathrm{II}}$ on $\{S\} \times S^{n-1}$, which belongs to $\mathcal{C}^{2,\alpha}_{-\frac{n+2}{2}}\left([S, +\infty) \times S^{n-1}\right)$. In addition, we know that

$$\|w_0\|_{2,\alpha,-\frac{n+2}{2}} \leq c \, e^{\frac{2+n}{2} S} \, ||g_{\mathrm{II}}||_{2,\alpha},$$

for some $c > 0$ independent of S. Notice that

$$D_\theta g_{\mathrm{II}} = -\partial_s w_0(S, \cdot).$$

We now set

$$f \equiv \mathcal{L} w_0 = \frac{n(3n-2)}{4} \phi^{2-2n} w_0.$$

The previous estimate yields

$$\|f\|_{0,\alpha,\delta} \leq c\, e^{\frac{n+2}{2} S} \, ||g_{\mathrm{II}}||_{2,\alpha}.$$

To conclude, it suffices to take $w = w_0 - \mathcal{G}_S(f)$ and apply Proposition 4.3. □

5. Minimal hypersurfaces which are close to a half *n*-catenoid

In this section we prove the existence of an infinite dimensional family of minimal hypersurfaces, which are normal graphs over a truncated n-catenoid. This infinite dimensional family is parameterized by the boundary data. Furthermore, we define and investigate the properties of the Cauchy data mapping associated to this family of minimal hypersurfaces.

5.1. Minimal hypersurfaces close to a half n-catenoid

Now and hereafter, we set for all $\varepsilon \in (0, 1]$

$$s_\varepsilon \equiv \frac{1}{(n-1)(3n-2)} \log \varepsilon < 0 \quad \text{and} \quad r_\varepsilon \equiv \varepsilon^{\frac{1}{n-1}} \phi(s_\varepsilon). \tag{5.1}$$

At this point, these choices may seem quite arbitrary but they will be commented and justified in Sect. 9.3. For the time being, let us notice that, as ε tends to 0

$$r_\varepsilon \sim \varepsilon^{\frac{3}{3n-2}} \quad \text{or also} \quad \varepsilon \sim r_\varepsilon^{n-\frac{2}{3}}.$$

We use the parameterization (3.5) for the unit n-catenoid. Its outer unit normal N_0 is then given by (3.6). Let us define a smooth, nonincreasing function $\xi_\varepsilon : \mathbb{R} \longrightarrow [-1, 1]$ by

$$\xi_\varepsilon(s) = \left(1 - \lambda^2(s - s_\varepsilon - 1)\right) - \lambda^2(s - s_\varepsilon - 1)\, \frac{\phi'(s)}{\phi(s)}.$$

Thus, $\xi_\varepsilon = 1$ for $s \leq s_\varepsilon + 1$ and $\xi_\varepsilon = -\dfrac{\phi'}{\phi}$ for $s \geq s_\varepsilon + 2$. Now, consider the vector field

$$N_\varepsilon(s, \theta) = \left(\sqrt{1 - \xi_\varepsilon^2(s)}\, \theta, \xi_\varepsilon(s)\right),$$

this is a perturbation of the unit normal N_0, and in fact, using the estimate of Lemma 3.2, we have for all $k \geq 0$

$$\left| \nabla^k \left(N_\varepsilon \cdot N_0 - 1 \right) \right| \leq c_k \, e^{(2n-2)s_\varepsilon},$$

in $[s_\varepsilon, s_\varepsilon + 2] \times S^{n-1}$, as ε tends to 0.

We now look for all minimal hypersurfaces close to the unit n- catenoid, rescaled by a factor $\varepsilon^{\frac{1}{n-1}}$, which admit the parameterization

$$X_w \equiv \varepsilon^{\frac{1}{n-1}} \, X_0 + w \, \phi^{\frac{2-n}{2}} \, N_\varepsilon,$$

for $(s, \theta) \in [s_\varepsilon, +\infty) \times S^{n-1}$ and for some small function w. The reason why we have scaled the unit n- catenoid by $\varepsilon^{\frac{1}{n-1}}$ will also be explained in Sect. 9.3.

It follows from (3.9) that such an hypersurface is minimal if and only if w satisfies a nonlinear equation of the form

$$\mathcal{L} w = \bar{Q}_\varepsilon(w),$$

where

$$\bar{Q}_\varepsilon(w) = L_\varepsilon w + \varepsilon^{\frac{1}{n-1}} \, \phi^{\frac{2-n}{2}} \, \bar{Q}_{2,\varepsilon} \left(\phi^{-\frac{n}{2}} \, \varepsilon^{-\frac{1}{n-1}} \, w \right) + \varepsilon^{\frac{1}{n-1}} \, \phi^{\frac{n}{2}} \, \bar{Q}_{3,\varepsilon} \left(\phi^{-\frac{n}{2}} \, \varepsilon^{-\frac{1}{n-1}} \, w \right).$$

Here $\bar{Q}_{2,\varepsilon}$ and $\bar{Q}_{3,\varepsilon}$ enjoy properties which are similar to those enjoyed by Q_2 and Q_3 in Proposition 3.1. Observe in addition that the bounds on the coefficients of $\bar{Q}_{2,\varepsilon}$ or on the partial derivatives of $\bar{Q}_{3,\varepsilon}$ are independent of ε. We even have $Q_2 = \bar{Q}_{2,\varepsilon}$ and $Q_3 = \bar{Q}_{3,\varepsilon}$ in $[s_\varepsilon + 2, +\infty) \times S^{n-1}$. In particular, there exists $c > 0$ such that, for all $\varepsilon \in (0, 1)$ we have

$$\left| \bar{Q}_{2,\varepsilon}(w) \right|_{0,\alpha([s,s+1]\times S^{n-1})} \leq c \, |w|^2_{2,\alpha([s,s+1]\times S^{n-1})}. \tag{5.2}$$

for all function $w \in \mathcal{C}^{2,\alpha}([s, s+1] \times S^{n-1})$. Similarly, there exist $c_0 > 0$ and $c > 0$ such that

$$\left| \bar{Q}_{3,\varepsilon}(w) \right|_{0,\alpha([s,s+1]\times S^{n-1})} \leq c \, |w|^3_{2,\alpha([s,s+1]\times S^{n-1})}. \tag{5.3}$$

provided $|w|_{2,\alpha\,([s,s+1]\times S^{n-1})} \leq c_0$.

The linear operator L_ε represents the difference between the linearized mean curvature operator for hypersurfaces parameterized using the vector field N_0 and those parameterized using the vector field N_ε. This operator L_ε has coefficients which are supported in $[s_\varepsilon, s_\varepsilon + 2] \times S^{n-1}$ and which are bounded by a constant times $e^{(2n-2)s_\varepsilon}$ in $\mathcal{C}^{0,\alpha}([s_\varepsilon, s_\varepsilon + 2] \times S^{n-1})$. The details of the derivation of this formula can be found, for example, in [6].

Now, given $h_{\mathrm{II}} \in \pi_{\mathrm{II}} \left(\mathcal{C}^{2,\alpha}(S^{n-1}) \right)$, we want to solve the boundary value problem

$$\begin{cases} \mathcal{L} w = \bar{Q}_\varepsilon(w) & \text{in} \quad (s_\varepsilon, +\infty) \times S^{n-1} \\ \pi_{\mathrm{II}} w = g_{\mathrm{II}} & \text{on} \quad \{s_\varepsilon\} \times S^{n-1}, \end{cases} \tag{5.4}$$

where we have set

$$g_{\mathrm{II}} \equiv \phi^{\frac{n-2}{2}}(s_\varepsilon)\, h_{\mathrm{II}}.$$

A solution will produce a minimal hypersurface whose boundary is parameterized by

$$\theta \in S^{n-1} \longrightarrow \left(\varepsilon^{\frac{1}{n-1}}\,\phi(s_\varepsilon)\,\theta,\, \varepsilon^{\frac{1}{n-1}}\,\psi(s_\varepsilon) + w(s_\varepsilon, \theta)\,\phi^{\frac{2-n}{2}}(s_\varepsilon)\right), \tag{5.5}$$

and whose end is asymptotic to a n-catenoid. Notice that the boundary of this hypersurface is a graph over a sphere of radius r_ε in the $x_{n+1} = 0$ hyperplane. This is the reason why we have modified N_0 into N_ε.

Naturally, for small g_{II} the existence of a solution of (5.5) follows at once from the inverse function theorem, using Proposition 4.3. However, since we want to have more information about the range of validity of the inverse function theorem, we prefer to use a standard fixed point argument to establish the existence of w. First, we fix $\delta \in (-\frac{2+n}{2}, -\frac{n}{2})$, $\alpha \in (0, 1)$ and we define

$$\tilde{w} \equiv \mathcal{P}_{s_\varepsilon}(g_{\mathrm{II}}). \tag{5.6}$$

We know from (4.10) that

$$\|\tilde{w}\|_{2,\alpha,\delta} \leq c\, e^{-\delta s_\varepsilon}\, \|g_{\mathrm{II}}\|_{2,\alpha}.$$

Then, if we write $w = \tilde{w} + v$, we must find a function $v \in \mathcal{C}^{2,\alpha}_\delta([s_\varepsilon, +\infty) \times S^{n-1}$ such that

$$\begin{cases} \mathcal{L}v = \bar{Q}_\varepsilon(\tilde{w} + v) & \text{in} \quad (s_\varepsilon, +\infty) \times S^{n-1} \\ \pi_{\mathrm{II}} v = 0 & \text{on} \quad \{s_\varepsilon\} \times S^{n-1}. \end{cases}$$

To obtain a solution of this equation, it is enough to find a fixed point of the mapping

$$\mathcal{N}_\varepsilon(v) \equiv \mathcal{G}_{s_\varepsilon}\left(\bar{Q}_\varepsilon(\tilde{w} + v)\right).$$

Notice that, although this is not explicit in the notation, this operator depends on h_{II}.

Proposition 5.1. *Fix $\delta \in \left(-\frac{n+2}{2}, -\frac{n}{2}\right)$ and $\alpha \in (0,1)$. For all $\kappa > 0$ there exist constants $c_\kappa > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$ and for all $h_{\mathrm{II}} \in \pi_{\mathrm{II}}\left(\mathcal{C}^{2,\alpha}(S^{n-1})\right)$ satisfying*

$$||h_{\mathrm{II}}||_{2,\alpha} \leq \kappa\, r_\varepsilon^2, \tag{5.7}$$

the mapping $\mathcal{N}_\varepsilon$ is a contraction mapping in the ball

$$B \equiv \left\{v : ||v||_{2,\alpha,\delta} \leq c_\kappa\, e^{(\frac{3n-2}{2} - \delta)s_\varepsilon}\, r_\varepsilon^2\right\},$$

and hence has a unique fixed point in this ball.

Again the choice $||h_{\mathrm{II}}||_{2,\alpha} \leq \kappa\, r_\varepsilon^2$ will be commented and justified in Sect. 9.3.

Proof. In order to prove the result, we have to show that there exists a constant c_κ such that

$$||\mathcal{N}_\varepsilon(0)||_{2,\alpha,\delta} \leq \frac{c_\kappa}{2}\, e^{\left(\frac{3n-2}{2}-\delta\right)s_\varepsilon}\, r_\varepsilon^2$$

and also that

$$||\mathcal{N}_\varepsilon(v_2) - \mathcal{N}_\varepsilon(v_1)||_{2,\alpha,\delta} \leq \frac{1}{2}||v_2 - v_1||_{2,\alpha,\delta},$$

for all $v_1\, v_2 \in B$.

In order to derive the first estimate we first obtain, from the properties of L_ε and $\bar{Q}_{2,\varepsilon}$ that

$$\begin{aligned}\|L_\varepsilon \tilde{w}\|_{0,\alpha,\delta} &\leq c\, e^{\left(\frac{3n-2}{2}-\delta\right)s_\varepsilon}\, \|h_{\mathrm{II}}\|_{2,\alpha}\\ &\leq c\,\kappa\, e^{\left(\frac{3n-2}{2}-\delta\right)s_\varepsilon}\, r_\varepsilon^2,\end{aligned}$$

$$\begin{aligned}\|\varepsilon^{\frac{1}{n-1}}\, \phi^{\frac{2-n}{2}}\, \bar{Q}_{2,\varepsilon}(\phi^{-\frac{n}{2}}\, \varepsilon^{-\frac{1}{n-1}}\, \tilde{w})\|_{0,\alpha,\delta} &\leq c\, e^{(4-4n-2\delta)s_\varepsilon}\, \|h_{\mathrm{II}}\|^2_{2,\alpha}\\ &\leq c\,\kappa^2\, e^{(2n-2-2\delta)s_\varepsilon}\, r_\varepsilon^2,\end{aligned}$$

where all constants do not depend on κ nor on ε. Finally, it follows from (5.3) that

$$\begin{aligned}\|\varepsilon^{\frac{1}{n-1}}\, \phi^{\frac{n}{2}}\, \bar{Q}_{3,\varepsilon}(\phi^{-\frac{n}{2}}\, \varepsilon^{-\frac{1}{n-1}}\, \tilde{w})\|_{0,\alpha,\delta} &\leq c\, e^{\left(\frac{14-13n}{2}-\delta\right)s_\varepsilon}\, \|h_{\mathrm{II}}\|^3_{2,\alpha}\\ &\leq c\,\kappa^3\, e^{\left(\frac{11n-10}{2}-\delta\right)s_\varepsilon}\, r_\varepsilon^2,\end{aligned}$$

for some constant which does not depend on κ nor on ε provided

$$\|\varepsilon^{-\frac{1}{n-1}}\, \phi^{-\frac{n}{2}}\, \tilde{w}\|_{2,\alpha,0} \leq c_0.$$

Observe that this condition is fulfilled provided ε is chosen small enough.

Taking advantage from the fact that the norm of $\mathcal{G}_{S_\varepsilon}$ is bounded independently of ε we conclude that

$$\begin{aligned}||\mathcal{N}_\varepsilon(0)||_{2,\alpha,\delta} &\leq c\, e^{\left(\frac{3n-2}{2}-\delta\right)s_\varepsilon}\, r_\varepsilon^2\left(\kappa + \kappa^2\, e^{\left(\frac{n}{2}-1-\delta\right)s_\varepsilon} + \kappa^3\, e^{(4n-4)s_\varepsilon}\right)\\ &\leq \tilde{c}\,\kappa\, e^{\left(\frac{3n-2}{2}-\delta\right)s_\varepsilon}\, r_\varepsilon^2\end{aligned}$$

for all ε small enough, say $\varepsilon \in (0, \varepsilon_0]$. It remains to define $c_\kappa = 2\,\tilde{c}\,\kappa$ in order for the stated estimate for $\mathcal{N}_\varepsilon(0)$ to hold.

The second estimate is obtained by reducing ε_0 if necessary and is left to the reader. □

Remark 5.1. Observe that, reducing ε_0 if this is necessary, we can assume that the mapping $h_{\mathrm{II}} \to v$, where v is the fixed poi of $\mathcal{N}_\varepsilon$ is continuous.

5.2. The first Cauchy data mapping

We summarized what we have obtained so far. Let us fix $\delta \in (-\frac{n+2}{2}, -\frac{n}{2})$ and $\alpha \in (0,1)$. Then, for all ε sufficiently small and for all $h_{\mathrm{II}} \in \pi_{\mathrm{II}}(\mathcal{C}^{2,\alpha}(S^{n-1}))$ satisfying (5.7), we have been able to find a minimal hypersurface parameterized by

$$X_w \equiv \varepsilon^{\frac{1}{n-1}} X_0 + w\, \phi^{\frac{2-n}{2}} N_\varepsilon \quad \text{in} \quad [s_\varepsilon, +\infty) \times S^{n-1},$$

with $\phi^{\frac{2-n}{2}} \pi_{\mathrm{II}} w = h_{\mathrm{II}}$ on $\{s_\varepsilon\} \times S^{n-1}$ and with $w \in \mathcal{C}^{2,\alpha}_\delta([s_\varepsilon, +\infty) \times S^{n-1})$. In particular, the end of this hypersurface is asymptotic to a rescaled n-catenoid. Furthermore, by definition of N_ε, we know that, for all $(s,\theta) \in [s_\varepsilon, s_\varepsilon + 1) \times S^{n-1}$,

$$X_w \equiv \left(\varepsilon^{\frac{1}{n-1}} \phi\, \theta, \varepsilon^{\frac{1}{n-1}} \psi + w\, \phi^{\frac{2-n}{2}} \right).$$

Now, we can translate this hypersurface along the x_{n+1} axis by the amount $-\varepsilon^{\frac{1}{n-1}} \psi(s_\varepsilon)$. The resulting hypersurface will be denoted $C_\varepsilon(h_{\mathrm{II}})$. If we perform the change of variable

$$r = \varepsilon^{\frac{1}{n-1}} \phi(s),$$

we see that near its boundary, this hypersurface is a graph over the $x_{n+1} = 0$ hyperplane

$$x \in B_{r_\varepsilon} \setminus B_{r_\varepsilon/2} \longrightarrow (x, U_{\varepsilon, h_{\mathrm{II}}}(x)) \in C_\varepsilon(h_{\mathrm{II}}).$$

Definition 5.1. *The first Cauchy data mapping $\mathcal{S}_\varepsilon(h_{\mathrm{II}}) \in \mathcal{C}^{2,\alpha}(S^{n-1}) \times \mathcal{C}^{1,\alpha}(S^{n-1})$ is defined by*

$$\mathcal{S}_\varepsilon(h_{\mathrm{II}})(\theta) \equiv (U_{\varepsilon,h_{\mathrm{II}}}(r_\varepsilon\, \theta), r_\varepsilon\, \partial_r U_{\varepsilon,h_{\mathrm{II}}}(r_\varepsilon\, \theta)),$$

where we recall that, by definition, $r_\varepsilon \equiv \varepsilon^{\frac{1}{n-1}} \phi(s_\varepsilon)$.

Since

$$\frac{dr}{r} = \frac{\phi'(s)}{\phi(s)}\, ds,$$

we obtain an expression of $S_\varepsilon(h_{\mathrm{II}})$ in terms of ϕ, ψ and w the solution of (5.4)

$$\mathcal{S}_\varepsilon(h_{\mathrm{II}}) = \left(\phi^{\frac{2-n}{2}}(s_\varepsilon)\, w(s_\varepsilon, \cdot), \frac{\phi(s_\varepsilon)}{\phi'(s_\varepsilon)} \left(\varepsilon^{\frac{1}{n-1}} \psi'(s_\varepsilon) + \partial_s\big(\phi^{\frac{2-n}{2}}\, w\big)(s_\varepsilon, \cdot)\right)\right).$$

We also define

$$\mathcal{S}_0\,(h_{\mathrm{II}}) \equiv \left(h_{\mathrm{II}}, -\varepsilon\, r_\varepsilon^{2-n} - \frac{n-2}{2}\, h_{\mathrm{II}} + D_\theta h_{\mathrm{II}} \right).$$

The comparison between these two mappings plays a key rôle in our construction.

Proposition 5.2. *The mappings* $\mathcal{S}_\varepsilon$ *and* $\mathcal{S}_0$ *are continuous. Furthermore, there exists a constant* $c > 0$ *and for all* $\kappa > 0$, *there exists* $\varepsilon_0 > 0$ *such that, for all* $\varepsilon \in (0, \varepsilon_0]$ *and for all* $\|h_{\mathrm{II}}\|_{2,\alpha} \leq \kappa\, r_\varepsilon^2$, *we have*

$$||(\mathcal{S}_\varepsilon - \mathcal{S}_0)(h_{\mathrm{II}})||_{\mathcal{C}^{2,\alpha}\times\mathcal{C}^{1,\alpha}} \leq c\, r_\varepsilon^2.$$

It is very important that, in this Proposition, the constant c does not depend on $\varepsilon \in (0, \varepsilon_0]$ and also does not depend on the constant κ which, later on, will be chosen large.

Proof. The statement about continuity is straightforward and is left to the reader. The estimate follows from the fact that $w = \tilde{w} + v$, where $\tilde{w}$ has been defined in (5.6) and where v is given by Proposition 5.1.

First, notice that

$$\varepsilon^{\frac{1}{n-1}}\, \psi'(s_\varepsilon) = \varepsilon^{\frac{1}{n-1}}\, \phi^{2-n}(s_\varepsilon) = \varepsilon\, r_\varepsilon^{2-n},$$

and, from Lemma 3.2, we know that

$$\left| \frac{\phi(s_\varepsilon)}{\phi'(s_\varepsilon)} + 1 \right| \leq c\, e^{(2n-2)s_\varepsilon}.$$

Thus

$$\left| \frac{\phi(s_\varepsilon)}{\phi'(s_\varepsilon)}\, \varepsilon^{\frac{1}{n-1}}\, \psi'(s_\varepsilon) + \varepsilon\, r_\varepsilon^{2-n} \right| \leq c\, e^{6(n-1)s_\varepsilon} \leq c\, r_\varepsilon^2,$$

for some constant c which obviously does not depend on κ! Now

$$\partial_s(\phi^{\frac{2-n}{2}}\, \tilde{w})(s_\varepsilon, \cdot) = \frac{2-n}{2}\, \frac{\phi'(s_\varepsilon)}{\phi(s_\varepsilon)}\, h_{\mathrm{II}} + \phi^{\frac{2-n}{2}}(s_\varepsilon)\, \delta_s \tilde{w}(s_\varepsilon, \cdot),$$

and using Proposition 4.6, we obtain

$$\left\| \frac{\phi(s_\varepsilon)}{\phi'(s_\varepsilon)}\, \partial_s(\phi^{\frac{2-n}{2}}(s_\varepsilon)\, \tilde{w}(s_\varepsilon, \cdot)) + \frac{n-2}{2}\, h_{\mathrm{II}} - D_\theta h_{\mathrm{II}} \right\|_{1,\alpha} \leq c_\kappa\, e^{(\frac{n+2}{2}+\delta)s_\varepsilon}\, r_\varepsilon^2.$$

Finally, using Proposition 5.1, we also have

$$\|\phi^{\frac{2-n}{2}}\, v\|_{2,\alpha} + \|\partial_s(\phi^{\frac{2-n}{2}}\, v)\|_{1,\alpha} \leq c_\kappa\, e^{(2n-2)s_\varepsilon}\, r_\varepsilon^2.$$

The result follows at once from these estimates choosing ε_0 sufficiently small depending on κ. □

6. Minimal hypersurfaces which are graphs over an hyperplane

We are now concerned with both the mean curvature and the linearized mean curvature operator for hypersurfaces which are graphs over the $x_{n+1} = 0$ hyperplane. We also give a list of assumptions which will be needed to ensure that all the results in Sect. 6–8 do hold uniformly.

We will proceed by what we call the "analytic" and "geometric" modifications of an hypersurface. The first of these modifications is intended to transform a regular hypersurface into a singular hypersurface, which looks like the graph of Green's function near its pole, and thus will be close to the lower end of an n-catenoid. The second modification is intended to remedy to the fact that, when we have solved (5.4) and when we will solve (8.1), we do not prescribe the $n+1$ first eigenmodes of the eigenfunction decomposition of the boundary data and, in doing so, we have "lost" some degrees of freedom.

6.1. The mean curvature operator for graphs

Assume we are given some function u, defined in some open regular domain Ω of $\mathbb{R}^n$, which is at least of class $\mathcal{C}^2$. We may then define an hypersurface Σ_0 as the graph of u

$$\overline{\Omega} \ni x \longrightarrow (x, u(x)) \in \mathbb{R}^{n+1}.$$

With respect to this parameterization, the first fundamental form of this hypersurface is given by

$$\mathbb{I}_u = \sum_{i,j} (\delta_{ij} + \partial_{x_i} u \, \partial_{x_j} u) \, dx_i \, dx_j .$$

Since we have

$$\det \mathbb{I}_u = 1 + |\nabla u|^2,$$

the volume functional can be defined by

$$E_u \equiv \int (1 + |\nabla u|^2)^{1/2} \, dx,$$

and the associated Euler–Lagrange equation is then given by

$$H_u \equiv \operatorname{div} \left(\frac{\nabla u}{(1 + |\nabla u|^2)^{1/2}} \right) = 0. \tag{6.1}$$

6.2. The linearized mean curvature operator

All hypersurfaces sufficiently $\mathcal{C}^1$-close to Σ_0 can also be parameterized as vertical graphs over the hyperplane $x_{n+1} = 0$. Namely

$$x \longrightarrow (x, u(x) + w(x)) \in \mathbb{R}^{n+1}, \tag{6.2}$$

for some (sufficiently regular) function w. It follows from (6.1) that the linearized mean curvature operator about Σ_0 is given explicitly by

$$\Lambda_u w \equiv \operatorname{div}\left(\frac{\nabla w}{(1+|\nabla u|^2)^{1/2}} - \frac{\nabla u \cdot \nabla w}{(1+|\nabla u|^2)^{3/2}}\nabla u\right). \tag{6.3}$$

In order to state properly the next properties of Λ_u, we need to introduce the following weighted spaces

Definition 6.1. *For all regular open subsets $\Omega \subset \mathbb{R}^n$ with $0 \in \Omega$, for all $k \in \mathbb{N}$, $\alpha \in (0,1)$ and $\nu \in \mathbb{R}$, the space $\mathcal{C}^{k,\alpha}_\nu(\overline{\Omega} \setminus \{0\})$ is defined to be the space of functions $w \in \mathcal{C}^{k,\alpha}_{loc}(\overline{\Omega} \setminus \{0\})$ for which the following norm is finite*

$$\|w\|_{k,\alpha,\nu} \equiv |w|_{k,\alpha,\overline{\Omega}\setminus B_{r_0}} + \sup_{0<2r\leq r_0} r^{-\nu}\,[w]_{k,\alpha,[2r,r]},$$

where, by definition

$$[w]_{k,\alpha,[2r,r]} \equiv \sum_{j=0}^{k} r^j \sup_{r\leq|x|\leq 2r} |\nabla^j w| + r^{k+\alpha} \sup_{r\leq|x_i|\leq 2r, x_i\neq x_j} \frac{|\nabla^k w(x_1) - \nabla^k w(x_2)|}{|x_1 - x_2|^\alpha}$$

and where $r_0 > 0$ is fixed in such a way that $B_{r_0} \subset \overline{\Omega}$.

We will also need the

Definition 6.2. *For all $\bar{r} < r_0$, the space $\mathcal{C}^{k,\alpha}_\nu(\overline{\Omega} \setminus B_{\bar{r}})$ is defined to be the space of restrictions to $\overline{\Omega} \setminus B_{\bar{r}}$ of functions $w \in \mathcal{C}^{k,\alpha}_\nu(\overline{\Omega} \setminus \{0\})$, endowed with the induced norm.*

Let Σ_0 be an hypersurface given as a graph

$$\overline{\Omega} \ni x \longrightarrow (x, u(x)) \in \Sigma_0 \subset \mathbb{R}^{n+1}.$$

In the subsequent sections, we will need some technical assumptions, which will ensure that all the results will hold uniformly in α, u and Ω and will only depend on the constants r_0, η_0 and η_ν which are defined below. The importance of these assumptions will become clear within the subsequent sections.

(H.1) $B_{r_0/2} \subset \Omega \subset B_{2r_0}$.

(H.2) $u(0) = 0$ and $\nabla u(0) = 0$. Stated differently, 0 belongs to Σ_0 and the tangent space at 0 is always the hyperplane $x_{n+1} = 0$. Notice that there is no loss of generality in assuming so since these assumptions can always be fulfilled modulo some suitable rigid motion.

(H.3) $\|u\|_{\mathcal{C}^{2,\alpha}(\overline{B_{2r_0}})} \leq \eta_0$ and $\|u\|_{\mathcal{C}^{3,\alpha}(\overline{B_{r_0}})} \leq \eta_0$.

(H.4) The operator Λ_u defined from $[\mathcal{C}^{2,\alpha}(\overline{\Omega})]_{\mathcal{D}}$ into $\mathcal{C}^{0,\alpha}(\overline{\Omega})$ is an isomorphism where by definition

$$[\mathcal{C}^{2,\alpha}(\overline{\Omega})]_{\mathcal{D}} \equiv \left\{ w \in \mathcal{C}^{2,\alpha}(\overline{\Omega}) \ : \ w = 0 \quad \text{on} \quad \partial\Omega \right\}.$$

Moreover $\|\Lambda_u^{-1}\|_{(\mathcal{C}^{0,\alpha},\mathcal{C}^{2,\alpha})} \leq \eta_0$ where η_0.

(H.5) Assume that $\nu \in (-n, 1-n)$ is fixed. For all $r < r_0$, there exists an operator $\Gamma_{u,r}$ defined from $\mathcal{C}^{0,\alpha}_{\nu-2}(\overline{\Omega} \setminus B_r)$ into $[\mathcal{C}^{2,\alpha}_{\nu}(\overline{\Omega} \setminus B_r)]_{\mathcal{D},n}$, such that $\Lambda_u \circ \Gamma_{u,r} = Id$. Here by definition

$$\begin{aligned}[\mathcal{C}^{2,\alpha}_{\nu}(\overline{\Omega} \setminus B_r)]_{\mathcal{D},n} \equiv \ & \left\{ w \in \mathcal{C}^{2,\alpha}_{\nu}(\overline{\Omega} \setminus B_r) : \quad w = 0 \quad \text{on} \quad \partial\Omega, \right. \\ & \left. \text{and} \quad \pi_{\mathrm{II}}(w) = 0 \quad \text{on} \quad \partial B_r \right\}.\end{aligned}$$

Moreover $\|\Gamma_{u,r}\|_{\left(\mathcal{C}^{0,\alpha}_{\nu-2},\mathcal{C}^{2,\alpha}_{\nu}\right)} \leq \eta_\nu$, where η_ν does not depend on $r < r_0$.

Remark 6.1. Though this will never be explicit in the statements of the results, all the bounds we will obtain in Sect. 6–8 will not depend on u or Ω satisfying the assumptions above but will only depend on r_0, η_0 and η_ν.

We can now state the

Lemma 6.1. *Assume (H.1), (H.2) and (H.3) hold. The linearized mean curvature operator Λ_u can be expanded as*

$$\Lambda_u = \mathrm{div}\,(\nabla + \Lambda'_u), \tag{6.4}$$

and where Λ'_u is a first order partial differential operator without any zero order terms and all of whose coefficients are bounded functions in $\mathcal{C}^{1,\alpha}_2(\overline{\Omega} \setminus \{0\}) \cap \mathcal{C}^{2,\alpha}_2(\overline{B_{r_0/2}} \setminus \{0\})$.

Proof. This follows directly from (6.3). □

It will also be convenient to notice that

Lemma 6.2. *Assume (H.1), (H.2) and (H.3) hold. Then, the expression of the mean curvature H_{u+w} of the hypersurface parameterized by (6.2) is given by*

$$H_{u+w} = H_u + \Lambda_u w - \mathrm{div}\left(r\, Q'_u(\nabla w) + Q''_u(\nabla w)\right), \tag{6.5}$$

where $q \to Q'_u(q)$ is homogeneous of degree 2 and $q \to Q''_u(q)$ collects all the higher order nonlinear terms. That is

$$Q''_u(0) = 0 \quad \nabla_q Q''_u(0) = 0 \quad \text{and} \quad \nabla^2_{q\,q} Q''_u(0) = 0.$$

Moreover, the coefficients of Q'_u on the one hand and all partial derivatives at any order of Q''_u, with respect to q computed at any point of some neighborhood $\mathcal{V}$ of 0 on the other hand, are functions which are bounded in $\mathcal{C}^{1,\alpha}_0(\overline{\Omega} \setminus \{0\}) \cap \mathcal{C}^{2,\alpha}_0(\overline{B_{r_0/2}} \setminus \{0\})$, uniformly in $\mathcal{V}$.

Proof. This follows directly from (6.1). □

Note that, here and elsewhere, r stands for $r(x) \equiv |x|$.

6.3. Analytic modification of a hypersurface using Green's function

The fact that (H.4) is fulfilled implies that we are able to solve

$$\Lambda_u \gamma_0 = -(n-2)\,|S^{n-1}|\,\delta_0, \quad \text{in} \quad \Omega, \tag{6.6}$$

with $\gamma_0 = 0$ on $\partial\Omega$, where $|S^{n-1}|$ is the volume of the unit sphere.

Using (6.4), the following Lemma is a simple exercise, which is left to the reader.

Lemma 6.3. *Assume that (H.1)–(H.4) hold and that γ_0 is the solution of (6.6). Then, there exists $c > 0$ such that, for all $k \leq 3$,*

$$\begin{aligned} \left|\nabla^k \left(\gamma_0 - r^{2-n}\right)\right| &\leq c\, r^{4-n-k}, && \textit{if} \quad n \geq 5, \\ \left|\nabla^k \left(\gamma_0 - r^{-2}\right)\right| &\leq c\, r^{-k}\, \log 1/r, && \textit{if} \quad n = 4, \\ \left|\nabla^k \left(\gamma_0 - r^{-1} - a_0\right)\right| &\leq c\, r^{1-k}\, \log 1/r, && \textit{if} \quad n = 3, \end{aligned}$$

in $B_{r_0} \setminus \{0\}$, for some constant $a_0 \in \mathbb{R}$.

For all $\varepsilon > 0$, we can define Σ_ε to be the hypersurface parameterized by

$$\Omega \setminus \{0\} \ni x \longrightarrow \left(x, u(x) + \frac{\varepsilon}{n-2}\, \gamma_0(x)\right) \in \mathbb{R}^{n+1} \quad \text{if} \quad n \geq 4$$

and by

$$\Omega \setminus \{0\} \ni x \longrightarrow \left(x, u(x) + \frac{\varepsilon}{n-2}\, (\gamma_0(x) - a_0)\right) \in \mathbb{R}^{4} \quad \text{if} \quad n = 3.$$

We finally compare the mean curvature of the hypersurface Σ_ε with the mean curvature of the initial hypersurface Σ_0.

Proposition 6.1. *Assume that (H.1)–(H.4) hold. The derivatives of H_ε, the mean curvature of Σ_ε, can be estimated by*

$$|\nabla^k (H_\varepsilon - H_0)| \leq c\, \left(r^{-k} \left(\varepsilon^2\, r^{2-2n} + \varepsilon^3\, r^{2-3n}\right)\right), \quad \textit{for all} \quad \varepsilon^{\frac{3}{3n-2}} \leq r,$$

where H_0 is the mean curvature of Σ_0 and where $k = 0, 1$ and where $c > 0$ does not depend on $\varepsilon \in (0, \varepsilon_0]$.

Proof. The result follows at once from (6.5), with $w = \varepsilon\, \gamma_0$. □

6.4. *Geometric modifications of the hypersurface* Σ_ε

We recall that we have defined

$$r_\varepsilon \equiv \varepsilon^{\frac{1}{n-1}} \, \phi(s_\varepsilon) \sim \varepsilon^{\frac{3}{3n-2}}.$$

We now perform some geometric transformations of the surface Σ_ε by applying some rigid motion and also by modifying the parameter ε.

First, in the definition of Σ_ε, we change the scaling parameter ε into $\varepsilon + e$, for some parameter $e \in (-\varepsilon, \varepsilon)$. Then, for all $R \in \mathbb{R}^n$, $R \neq 0$, we apply the rigid motion corresponding to a rotation of angle $|R|$ in the plane spanned by the vectors $(0, 1)$ and $(R/|R|, 0)$. For $R \neq 0$, this transformation can be described analytically by

$$\begin{aligned} &\mathbb{R}^{n+1} \ni (x, x_{n+1}) \\ &\quad \to (x^\perp, 0) + \cos|R| \, (x^\parallel, x_{n+1}) - \frac{\sin|R|}{|R|} \, (x_{n+1} \, R, -R \cdot x^\parallel) \in \mathbb{R}^{n+1}, \end{aligned}$$

where by definition $x^\parallel \equiv \dfrac{x \cdot R}{|R|^2} \, R$ and $x^\perp \equiv x - x^\parallel$. Finally, we perform a translation of vector $(T, d) \in \mathbb{R}^n \times \mathbb{R}$.

We denote by $\mathcal{A} = (T, R, d, e) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$ the set of parameters and by $\Sigma_{\varepsilon,\mathcal{A}}$ the resulting hypersurface. By definition, the norm of $\mathcal{A}$ is given by

$$\|\mathcal{A}\| \equiv \varepsilon \, r_\varepsilon^{1-n} \, |T|_{\mathbb{R}^n} + r_\varepsilon \, |R|_{\mathbb{R}^n} + |d| + r_\varepsilon^{2-n} \, |e|.$$

We now compare the geometrically and analytically "modified" hypersurface $\Sigma_{\varepsilon,\mathcal{A}}$ with the initial hypersurface Σ_0.

Proposition 6.2. *Assume that (H.1)–(H.4) hold. Let* $\kappa > 0$ *be given. There exists* $c_\kappa > 0$ *and* $\varepsilon_0 > 0$ *such that, for all* $\varepsilon \in (0, \varepsilon_0]$, *if*

$$\|\mathcal{A}\| \leq \kappa \, r_\varepsilon^2,$$

then, the hypersurface $\Sigma_{\varepsilon,\mathcal{A}}$ *can be locally parameterized as a vertical graph over the initial hypersurface* Σ_0

$$\overline{B_{r_0/2}} \setminus B_{r_\varepsilon/2} \ni x \longrightarrow \big(x, u(x) + w_{\varepsilon,\mathcal{A}}(x)\big) \in \Sigma_{\varepsilon,\mathcal{A}}, \tag{6.7}$$

where the function $w_{\varepsilon,\mathcal{A}}$ *satisfies* $|\nabla^k w_{\varepsilon,\mathcal{A}}(x)| \leq c_\kappa \, \left(r^{-k} \left(r_\varepsilon \, r + \varepsilon \, r^{2-n}\right)\right)$, *for all* $k \leq 3$.

Again, the restriction $\|\mathcal{A}\| \leq \kappa \, r_\varepsilon^2$ will be commented in Sect. 9.3.

Proof. We restrict our attention to the proof of the estimates for $w_{\varepsilon,\mathcal{A}}$, leaving the estimates of the derivatives of these functions to the reader. In the proof c_κ will denote some constant which depends on κ but which does not depend on ε provided ε is chosen small enough.

We define new coordinates $\tilde{x}$ which are the orthogonal projection of a point of the modified hypersurface over the $x_{n+1} = 0$ hyperplane. Namely

$$\left| \tilde{x} - T - x^{\perp} - \cos|R|\, x^{\parallel} + \sin|R| \, \frac{R}{|R|} \, \left(u(x) + (\varepsilon + e)\, \frac{r^{2-n}}{n-2} \right) \right| \leq c_{\kappa}\, |\sin R|\, \varepsilon\, r^{4-n}. \quad (6.8)$$

(In dimension $n = 3, 4$, the term $\varepsilon\, r^{4-n}$ on the right hand side has to be replaced by $\varepsilon\, r^{4-n} \log 1/r$, but this is irrelevant for the subsequent computations). It follows easily from our choices that there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0]$, if $r_\varepsilon/4 \leq r \leq r_0$, then

$$r/2 < \tilde{r} < 2\, r, \quad (6.9)$$

where we have set $\tilde{r} = |\tilde{x}|$. In particular this yields

$$|\tilde{x} - x| \leq c_\kappa \, \left(r_\varepsilon^{5/3} + r_\varepsilon\, \tilde{r}^2 \right),$$

provided $r_\varepsilon/2 \leq \tilde{r} \leq r_0/2$. These expansions, together with (6.9), imply (with little work) that

$$\begin{aligned} \left| \frac{\sin|R|}{|R|}\, R \cdot x \right| &\leq c_\kappa\, r_\varepsilon\, \tilde{r} \\ |\cos|R|\, u(x) - u(\tilde{x})| &\leq c_\kappa \left(r_\varepsilon^{5/3}\, \tilde{r} + r_\varepsilon\, \tilde{r}^3 \right) \\ \left| \cos|R|\, (\varepsilon + e)\, \frac{r^{2-n}}{n-2} \right| &\leq c\, \varepsilon\, r^{4-n} + c_\kappa\, \varepsilon\, \tilde{r}^{2-n} \\ |d| &\leq c_\kappa r_\varepsilon^2. \end{aligned}$$

Therefore, the modified surface can be parameterized by (6.7), with $w_{\varepsilon,\mathcal{A}}$ satisfying the desired estimates. □

We will also need the

Proposition 6.3. *Assume that (H.1)–(H.4) hold. There exists* $c > 0$ *and, for all* $\kappa > 0$ *be given, there exists* $\varepsilon_0 > 0$ *(depending on* κ*) such that, for all* $\varepsilon \in (0, \varepsilon_0]$ *and for all* $r \in [r_\varepsilon/2, 2\, r_\varepsilon]$*, the parameterization of* $\Sigma_{\varepsilon,\mathcal{A}}$ *has the following expansion*

$$x \longrightarrow \left(x, e\, \frac{r^{2-n}}{n-2} + \left(e\, \frac{r^{2-n}}{n-2} + d + R \cdot x + \varepsilon\, r^{-n}\, T \cdot x \right) + \bar{w}_{\varepsilon,\mathcal{A}}(x) \right), \quad (6.10)$$

where, for all $k \leq 3$*, the function* $\bar{w}_{\varepsilon,\mathcal{A}}$ *satisfies* $|\nabla^k \bar{w}_{\varepsilon,\mathcal{A}}(x)| \leq c\, r_\varepsilon^{2-k}$.

Note that, and this will be very important, in the estimate for $\bar{w}_{\varepsilon,\mathcal{A}}$ the constant $c > 0$ does not depend on κ provided ε is chosen small enough.

Proof. Keeping the notations of the previous proof, for all $\tilde r \in [r_\varepsilon/4, 4\, r_\varepsilon]$, we can write

$$|x - \tilde x + T| \leq c_\kappa\, r_\varepsilon^{7/3}.$$

In particular, we obtain

$$\left| r^{2-n} - \tilde r^{2-n} - (n-2)\, \tilde r^{-n}\, T \cdot \tilde x \right| \leq c_\kappa\, r_\varepsilon^{(10-3n)/3}.$$

Now, we use these expansions to get

$$\begin{aligned} \left| \sin |R|\, \frac{R}{|R|} \cdot x - R \cdot \tilde x \right| &\leq c_\kappa\, r_\varepsilon^{8/3}, \\ |\cos |R|\, u(x)| &\leq c\, r_\varepsilon^2 + c_\kappa\, r_\varepsilon^{8/3}, \\ \left| \cos |R|\, (\varepsilon + e)\, r^{2-n} - (\varepsilon + e)\, \tilde r^{2-n} - (n-2)\, \varepsilon\, \frac{T \cdot \tilde x}{\tilde r^n} \right| &\leq c_\kappa\, r_\varepsilon^{8/3}, \\ \varepsilon\, r^{4-n} &\leq c_\kappa r_\varepsilon^{10/3}. \end{aligned}$$

Notice that, and this is important, in the second estimate the first constant c does not depend on κ. The relevant estimates for $\bar w_{\varepsilon,\mathcal{A}}$ then follow at once choosing ε_0 sufficiently small depending on κ. □

7. Mapping properties of the linearized mean curvature operator about $\Sigma_{\varepsilon,\mathcal{A}}$

In this section, we derive for $\Sigma_{\varepsilon,\mathcal{A}}$ the counterpart of Proposition 4.3. To begin with, we define $\Omega_{\mathcal{A}}$ to be the projection onto the hyperplane $x_{n+1} = 0$ of the image of Σ_0 by the geometric transformations described in the previous section. More precisely, this set does not depend on ε, e nor on d and is just the projection of the image of $\Sigma_0 = u(\Omega)$ by the affine mapping

$$\mathbb{R}^n \ni x \longrightarrow x^\perp + \cos |R|\, x^\parallel + T \in \mathbb{R}^n.$$

With a slight abuse of notation, we still denote by $\Sigma_{\varepsilon,\mathcal{A}}$ the hypersurface parameterized by

$$\overline{\Omega}_{\mathcal{A}} \setminus B_{r_\varepsilon} \ni x \longrightarrow (x, u(x) + w_{\varepsilon,\mathcal{A}}(x)),$$

where the function $w_{\varepsilon,\mathcal{A}}$ is the one defined in Proposition 6.2. Thus, $\Sigma_{\varepsilon,\mathcal{A}}$ is the singular surface constructed in the previous section, which has been truncated.

The linearized mean curvature operator about $\Sigma_{\varepsilon,\mathcal{A}}$ now reads

$$\Lambda_{\varepsilon,\mathcal{A}} = \Lambda_u + \operatorname{div} \Lambda'_{\varepsilon,\mathcal{A}}, \tag{7.1}$$

in $\Omega_{\mathcal{A}} \setminus B_{r_\varepsilon}$, where $\Lambda'_{\varepsilon,\mathcal{A}}$ is a first order partial differential operator, all of whose coefficients have, for $k = 0, 1, 2$, their k-th partial derivatives bounded by a constant (which depends on κ) times $r^{-k}\, (\varepsilon\, r^{2-n} + r_\varepsilon\, r + \varepsilon^2\, r^{2-2n})$, provided $\|\mathcal{A}\| \leq \kappa\, r_\varepsilon^2$. This last statement just follows from differentiating (6.5) with respect to w at $w_{\varepsilon,\mathcal{A}}$, together with the expansion for $w_{\varepsilon,\mathcal{A}}$ which is given in Proposition 6.2.

Our main result, in this section, is the

Proposition 7.1. *Assume that (H.1)–(H.5) hold. Fix* $\nu \in (-n, 1-n)$, $\alpha \in (0,1)$. *Then, for all* $\kappa > 0$, *there exists* $\varepsilon_0 > 0$ *and all* $\varepsilon \in (0, \varepsilon_0]$, *there exists an operator*

$$\Gamma_{\varepsilon,\mathcal{A}} : \mathcal{C}^{0,\alpha}_{\nu-2}(\overline{\Omega_{\mathcal{A}}} \setminus B_{r_\varepsilon}) \longrightarrow \mathcal{C}^{2,\alpha}_{\nu}(\overline{\Omega_{\mathcal{A}}} \setminus B_{r_\varepsilon}),$$

such that, for all $f \in \mathcal{C}^{0,\alpha}_{\nu-2}(\overline{\Omega_{\mathcal{A}}} \setminus B_{r_\varepsilon})$, *the function* $w = \Gamma_{\varepsilon,\mathcal{A}}(f)$ *is a solution of the problem*

$$\begin{cases} \Lambda_{\varepsilon,\mathcal{A}} w = f & in \quad \Omega_{\mathcal{A}} \setminus B_{r_\varepsilon} \\ \pi_{\mathrm{II}}(w) = 0 & on \quad \partial B_{r_\varepsilon} \\ w = 0 & on \quad \partial \Omega_{\mathcal{A}}. \end{cases}$$

In addition $||\Gamma_{\varepsilon,\mathcal{A}}(f)||_{2,\alpha,\nu} \leq c\, ||f||_{0,\alpha,\nu-2}$, *for some constant* $c > 0$ *independent of* κ, *of* α, *of* ε *and independent of* $\mathcal{A}$ *such that* $||\mathcal{A}|| \leq \kappa\, r_\varepsilon^2$.

Notice that, in contrast with Proposition 4.3, we do not have uniqueness of $\Gamma_{\varepsilon,\mathcal{A}}$ but we can choose this operator in such a way that its norm stays bounded independently of ε.

Proof. Given the construction of $\Omega_{\mathcal{A}}$ one can build $\Theta_{\mathcal{A}} : \overline{\Omega} \to \overline{\Omega_{\mathcal{A}}}$ a $\mathcal{C}^{2,\alpha}$ diffeomorphism such that $\Theta_{\mathcal{A}}(x) = x$ in $B_{r_0/4}$. Moreover, $\|\Theta_{\mathcal{A}} - \mathrm{I}\|_{\mathcal{C}^{2,\alpha}} \leq c\, r_\varepsilon$ for some constant $c > 0$ depending on κ but independent of ε.

Using this diffeomorphism, we define $\tilde{\Lambda}_{\varepsilon,\mathcal{A}}$ by the formula

$$\tilde{\Lambda}_{\varepsilon,\mathcal{A}}\,(w \circ \Theta_{\mathcal{A}}) \equiv (\Lambda_{\varepsilon,\mathcal{A}} w) \circ \Theta_{\mathcal{A}},$$

which is a well defined operator from $\mathcal{C}^{2,\alpha}_{\nu}(\overline{\Omega} \setminus B_{r_\varepsilon})$ into $\mathcal{C}^{0,\alpha}_{\nu-2}(\overline{\Omega} \setminus B_{r_\varepsilon})$. Moreover, using (7.1) as well as the properties of $\Theta_{\mathcal{A}}$, we have

$$||(\Lambda_u - \tilde{\Lambda}_{\varepsilon,\mathcal{A}})w||_{0,\alpha,\nu-2} \leq c_\kappa\, r_\varepsilon^{2/3}\, ||w||_{2,\alpha,\nu}.$$

It is now easy to see that, provided ε is chosen small enough, and granted (H.5), the result follows from a simple perturbation argument. □

Fix $\kappa > 0$ and assume that $\|\mathcal{A}\| \leq \kappa\, r_\varepsilon^2$. For all $h_{\mathrm{II}} = \sum_{j \geq n+1} h_j\, e_j \in \pi_{\mathrm{II}}\big(\mathcal{C}^{2,\alpha}(S^{n-1})\big)$, we define in $\Omega_{\mathcal{A}} \setminus B_{r_\varepsilon}$ the function

$$w_0 \equiv \lambda \left(\frac{2r_0 - 8r}{r_0} \right) \sum_{j \geq n+1} \left(\frac{r}{r_\varepsilon} \right)^{\frac{2-n}{2} - \gamma_j} h_j\, e_j,$$

which satisfies $\Delta w_0 = 0$ in $B_{r_0/2} \setminus B_{r_\varepsilon}$. Arguing as in the proof of Proposition 4.5, it is easy to see that

$$\|w_0\|_{2,\alpha,-n} \leq c\, r_\varepsilon^n\, \|h\|_{2,\alpha},$$

for some constant $c > 0$ which does not depend on ε. Furthermore, thanks to the previous result, we see that the function w defined by $w \equiv -\Gamma_{\varepsilon,\mathcal{A}}(\Lambda_{\varepsilon,\mathcal{A}} w_0) + w_0$ solves

$$\begin{cases} \Lambda_{\varepsilon,\mathcal{A}} w = 0 & \text{in} \quad \Omega_{\mathcal{A}} \setminus B_{r_\varepsilon} \\ \pi_{\mathrm{II}}(w) = h_{\mathrm{II}}(\cdot/r_\varepsilon) & \text{on} \quad \partial B_{r_\varepsilon} \\ w = 0 & \text{on} \quad \partial\Omega_{\mathcal{A}}. \end{cases}$$

This allows to define an operator

$$\Pi_{\varepsilon,\mathcal{A}} : h_{\mathrm{II}} \in \pi_{\mathrm{II}} \left(\mathcal{C}^{2,\alpha}(S^{n-1}) \right) \longrightarrow w \in \mathcal{C}^{2,\alpha}_{\nu}(\overline{\Omega_{\mathcal{A}}} \setminus B_{r_\varepsilon}),$$

and, for all $\nu \in (-n, 1-n)$, we have

$$||(\Pi_{\varepsilon,\mathcal{A}}(h_{\mathrm{II}})||_{2,\alpha,\nu} \leq c_\kappa \, r_\varepsilon^{-\nu} \, ||h_{\mathrm{II}}||_{2,\alpha}. \tag{7.2}$$

We can now state the counterpart of Proposition 4.6.

Proposition 7.2. *Assume that (H.1)-(H.5) hold. Fix $\nu \in (-n, 1-n)$ and $\alpha \in (0, 1)$. Then, for all $\kappa > 0$ there exist $c_\kappa > 0$ and $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0]$, we have*

$$\left\| r_\varepsilon \, \partial_r \Pi_{\varepsilon,\mathcal{A}}(h_{\mathrm{II}})(r_\varepsilon\theta) + \frac{n-2}{2} h_{\mathrm{II}} + D_\theta h_{\mathrm{II}} \right\|_{1,\alpha} \leq c_\kappa \, (r_\varepsilon^{n+\nu} + r_\varepsilon^{2/3}) \, ||h_{\mathrm{II}}||_{2,\alpha}.$$

Proof. The proof is identical to the proof of Proposition 4.6, therefore we omit it. □

Notice that, for the time being, everything we have done holds for *any* hypersurface satisfying (H.1)–(H.5), whether this hypersurface is minimal or not. This shows that the local geometry of the hypersurface is completely hidden by the analytic modification we have done.

8. Minimal hypersurfaces close to $\Sigma_{\varepsilon,\mathcal{A}}$

In this section, we proceed exactly as in Sect. 5 and prove the existence of an infinite dimensional family of minimal hypersurfaces which are graphs over the modified hypersurfaces $\Sigma_{\varepsilon,\mathcal{A}}$, provided the initial hypersurface Σ_0 is assumed to be minimal. Again, this family will be parameterized by its boundary data. We will defined and study the Cauchy data mapping associated to these hypersurfaces.

8.1. Minimal hypersurfaces close to $\Sigma_{\varepsilon,\mathcal{A}}$

We keep the notations of the last section and assume from now on that (H.1)–(H.5) hold. We will also assume that Σ_0 is a minimal hypersurface. The mean curvature of a surface which is parameterized by

$$\overline{\Omega_{\mathcal{A}}} \setminus B_{r_\varepsilon} \ni x \longrightarrow (x, u(x) + w_{\varepsilon,\mathcal{A}}(x) + w(x)),$$

for some real valued function w, is given by

$$H = H_{\varepsilon,\mathcal{A}} + \Lambda_{\varepsilon,\mathcal{A}} w - \operatorname{div} \mathcal{Q}_{\varepsilon,\mathcal{A}}(w),$$

where $H_{\varepsilon,\mathcal{A}}$ is the mean curvature of the hypersurface $\Sigma_{\varepsilon,\mathcal{A}}$ and where $\mathcal{Q}_{\varepsilon,\mathcal{A}}(w)$ collects all the nonlinear terms. Near the origin, it follows from (6.5) with w replaced by $w_{\varepsilon,\mathcal{A}} + w$ that we have

$$\mathcal{Q}_{\varepsilon,\mathcal{A}}(w) \equiv (r + \varepsilon\, r^{1-n})\, Q'_{\varepsilon,\mathcal{A}}(\nabla w) + Q''_{\varepsilon,\mathcal{A}}(\nabla w),$$

where $q \to Q'_{\varepsilon,\mathcal{A}}(q)$ is homogeneous of degree 2 and $q \to Q''_{\varepsilon,\mathcal{A}}(q)$ satisfies

$$Q''_{\varepsilon,\mathcal{A}}(0) = 0 \quad \nabla_q Q''_{\varepsilon,\mathcal{A}}(0) = 0 \quad \text{and} \quad \nabla^2_{qq} Q''_{\varepsilon,\mathcal{A}}(0) = 0.$$

Moreover, for all κ, there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0]$, the coefficients of $Q'_{\varepsilon,\mathcal{A}}$ on the one hand and all partial derivatives of $Q''_{\varepsilon,\mathcal{A}}$, with respect to q, computed at any point of some small fixed neighborhood $\mathcal{V}$ of 0, on the other hand are functions whose norm in $\mathcal{C}^{1,\alpha}_0(\overline{\Omega_{\mathcal{A}}} \setminus B_{r_\varepsilon})$ are bounded, uniformly in $\mathcal{V}$, by some constant independent κ, of $\mathcal{A}$ and of $\varepsilon \in (0, \varepsilon_0]$.

Given $h_{\mathrm{II}} \in \pi_{\mathrm{II}}\left(\mathcal{C}^{2,\alpha}(S^{n-1})\right)$, we want to solve the boundary value problem

$$\begin{cases} \Lambda_{\varepsilon,\mathcal{A}} w = -H_{\varepsilon,\mathcal{A}} + \operatorname{div} \mathcal{Q}_{\varepsilon,\mathcal{A}}(w) & \text{in} \quad \Omega_{\mathcal{A}} \setminus B_{r_\varepsilon} \\ \pi_{\mathrm{II}}(u + w_m + w) = h_{\mathrm{II}}(\cdot/r_\varepsilon) & \text{on} \quad \partial B_{r_\varepsilon} \\ w = 0 & \text{on} \quad \partial\Omega_{\mathcal{A}}. \end{cases} \tag{8.1}$$

This will produce a minimal hypersurface which is a graph over $\Omega_{\mathcal{A}} \setminus B_{r_\varepsilon}$ and which has boundary values on $\partial B_{r_\varepsilon}$ given by

$$\theta \in S^{n-1} \longrightarrow (r_\varepsilon\theta, u(r_\varepsilon\theta) + w_{\varepsilon,\mathcal{A}}(r_\varepsilon\theta) + w(r_\varepsilon\theta)) \in \mathbb{R}^{n+1}.$$

In particular, this "inner" boundary is a graph over the sphere of radius r_ε in the $x_{n+1} = 0$ hyperplane.

Let us fix $\nu \in (-n, 1-n)$. For all $h_{\mathrm{II}} \in \pi_{\mathrm{II}}(\mathcal{C}^{2,\alpha}(S^{n-1}))$ with $\|h_{\mathrm{II}}\|_{2,\alpha} \leq \kappa\, r_\varepsilon^2$, we define

$$\tilde{w} = \Pi_{\varepsilon,\mathcal{A}}(h_{\mathrm{II}} - \pi_{\mathrm{II}} \bar{w}_{\varepsilon,\mathcal{A}}(r_\varepsilon \cdot)) - \Gamma_{\varepsilon,\mathcal{A}}(H_{\varepsilon,\mathcal{A}}).$$

Recall from (6.7) and (6.10), that $\pi_{\mathrm{II}}(u + w_{\varepsilon,\mathcal{A}}) = \pi_{\mathrm{II}} \bar{w}_{\varepsilon,\mathcal{A}}$ on $\partial B_{r_\varepsilon}$. We know from (7.2), that

$$||\Pi_{\varepsilon,\mathcal{A}}(h_{\mathrm{II}} - \pi_{\mathrm{II}}(\bar{w}_{\varepsilon,\mathcal{A}}))||_{2,\alpha,\nu} \leq c\, r_\varepsilon^{-\nu}\, ||h_{\mathrm{II}} - \pi_{\mathrm{II}}(\bar{w}_{\varepsilon,\mathcal{A}})||_{2,\alpha}, \tag{8.2}$$

and also from Proposition 6.3 that

$$||\pi_{\mathrm{II}}(\bar{w}_{\varepsilon,\mathcal{A}})||_{2,\alpha} \leq c\, r_\varepsilon^2, \tag{8.3}$$

for some constant c which is independent of κ, provided ε is chosen small enough.

Moreover, the estimate of $H_{\varepsilon,\mathcal{A}}$ the mean curvature of the modified surface has been obtained in Proposition 6.1 and this, together with Proposition 7.1, yields

$$\|\Gamma_{\varepsilon,\mathcal{A}}(H_{\varepsilon,\mathcal{A}})\|_{2,\alpha,\nu} \leq c\, r_\varepsilon^{2-\nu},$$

for some constant $c > 0$ which does not depend on κ, nor on $\mathcal{A}$, provided ε is taken small enough.

Remark 8.1. In dimension $n = 3$, one needs to impose $\nu \in (-8/3, -2)$ for the last estimate to hold.

Setting $w = \tilde{w} + v$, it remains to find $v \in \mathcal{C}^{2,\alpha}_\nu(\Sigma_{\varepsilon,\mathcal{A}})$ such that

$$\begin{cases} \Lambda_{\varepsilon,\mathcal{A}} v = \operatorname{div} \mathcal{Q}_{\varepsilon,\mathcal{A}}(\tilde{w} + v) & \text{in} \quad \Omega_{\mathcal{A}} \setminus B_{r_\varepsilon} \\ \pi_{\mathrm{II}}(v) = 0 & \text{on} \quad \partial B_{r_\varepsilon} \\ v = 0 & \text{on} \quad \partial \Omega_{\mathcal{A}}. \end{cases}$$

As before, it is enough to find a fixed point of the mapping

$$\mathcal{M}_{\varepsilon,\mathcal{A}}(v) = \Gamma_{\varepsilon,\mathcal{A}}(\mathcal{Q}_{\varepsilon,\mathcal{A}}(\tilde{w} + v)).$$

Though this is not explicit in the notation, this operator depends on h_{II}.

Proposition 8.1. *Assume that* $\nu \in (-n, 1-n)$ *(or* $\nu \in -8/3, -2)$ *when* $n = 3$*) and that* $\alpha \in (0,1)$ *are fixed. For all* $\kappa > 0$, *there exist* $c_\kappa > 0$ *and* $\varepsilon_0 > 0$ *such that, for all* $\varepsilon \in (0, \varepsilon_0]$, *if* $h_{\mathrm{II}} \in \pi_{\mathrm{II}}(\mathcal{C}^{2,\alpha}(S^{n-1}))$ *is fixed with*

$$||h_{\mathrm{II}}||_{2,\alpha} \leq \kappa\, r_\varepsilon^2, \tag{8.4}$$

then $\mathcal{M}_{\varepsilon,\mathcal{A}}$ *is a contraction mapping on the ball*

$$B \equiv \left\{ v \,:\, ||v||_{2,\alpha,\nu} \leq c_\kappa\, r_\varepsilon^{10/3-\nu} \right\},$$

and thus has a unique fixed point in this ball.

The restriction $||h_{\mathrm{II}}||_{2,\alpha} \leq \kappa\, r_\varepsilon^2$ is the one which we have already encountered in Proposition 5.1 and will be commented and justified in Sect. 9.3.

Proof. We have to prove that

$$||\mathcal{M}_{\varepsilon,\mathcal{A}}(0)||_{2,\alpha,\nu} \leq \frac{c_\kappa}{2}\, r_\varepsilon^{10/3-\nu},$$

for some constant $c_\kappa > 0$ and

$$||\mathcal{M}_{\varepsilon,\mathcal{A}}(v_2) - \mathcal{M}_{\varepsilon,\mathcal{A}}(v_1)||_{2,\alpha,\nu} \leq \frac{1}{2}||v_2 - v_1||_{2,\alpha,\nu},$$

provided v_1 and v_2 belong to B.

The first inequality follows from (8.2) and (8.3) together with the properties of $Q'_{\varepsilon,\mathcal{A}}$ and $Q''_{\varepsilon,\mathcal{A}}$. We get

$$\|\mathrm{div}((r + \varepsilon\, r^{1-n})\, Q'_{\varepsilon,\mathcal{A}}(\tilde{w}))\|_{0,\alpha,\nu-2} \le \tilde{c}_\kappa\, r_\varepsilon^{10/3-\nu}$$

and also that

$$\|\mathrm{div}(Q''_{\varepsilon,\mathcal{A}}(\tilde{w}))\|_{0,\alpha,\nu-2} \le \tilde{c}_\kappa\, r_\varepsilon^{4-\nu},$$

provided ε is chosen small enough, say $\varepsilon \in (0, \varepsilon_0]$, where the constant $\tilde{c}_\kappa > 0$ depends on κ. The existence of c_κ follows from Proposition 7.1.

The second inequality is obtained by reducing ε_0 if necessary and is left to the reader. □

8.2. Second Cauchy data mapping

Let us summarize what we have proved in the last sections. We fix $\nu \in (-n, 1-n)$ (or $\nu \in (-8/3, -2)$ if $n = 3$) and $\alpha \in (0, 1)$. For all ε small enough, for all $\mathcal{A}$ satisfying $\|\mathcal{A}\| \le \kappa\, r_\varepsilon^2$ and for all $h_{\mathrm{II}} \in \pi_{\mathrm{II}}(\mathcal{C}^{2,\alpha}(S^{n-1}))$ satisfying (8.4), we have been able to find a minimal hypersurface close to $\Sigma_{\varepsilon,\mathcal{A}}$, which can be parameterized by

$$\Omega_{\mathcal{A}} \setminus B_{r_\varepsilon} \ni x \longrightarrow (x, u(x) + w_{\varepsilon,\mathcal{A}}(x) + w(x)),$$

where w_m is the function defined in Proposition 6.2 and where w is the solution of (8.1). This hypersurface has two boundaries one of which is (up to a rigid motion) the boundary of Σ_0 and will be called the "outer" boundary. The other boundary is a graph over $\partial B_{r_\varepsilon}$ and will be referred to as the "inner" boundary.

This hypersurface is now translated along the x_{n+1} axis by the amount $-\,\varepsilon\, r_\varepsilon^{2-n}/(n-2)$. The resulting surface is denoted $\Sigma_\varepsilon(\mathcal{A}, h_{\mathrm{II}})$ and is parameterized by

$$\overline{\Omega_{\mathcal{A}}} \setminus B_{r_\varepsilon} \ni x \longrightarrow (x, V_{\varepsilon,\mathcal{A},h_{\mathrm{II}}}(x)) \in \Sigma_\varepsilon(\mathcal{A}, h_{\mathrm{II}}).$$

Definition 8.1. *The second Cauchy data mapping is defined by*

$$\mathcal{T}_\varepsilon(\mathcal{A}, h_{\mathrm{II}})(\theta) \equiv (V_{\varepsilon,\mathcal{A},h_{\mathrm{II}}}(r_\varepsilon\theta), r_\varepsilon\, \partial_r V_{\varepsilon,\mathcal{A},h_{\mathrm{II}}}(r_\varepsilon\theta)).$$

For notational convenience we set

$$\mathcal{F} \equiv \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \pi_{\mathrm{II}}\left(\mathcal{C}^{2,\alpha}(S^{n-1})\right),$$

which is endowed with the norm

$$\|(\mathcal{A}, w)\|_{\mathcal{F}} \equiv ||\mathcal{A}|| + ||w||_{2,\alpha}.$$

The domain of $\mathcal{T}_\varepsilon$ is just a subset of $\mathcal{F}$.

Thanks to the result of Proposition 6.3, we have the expression of $\mathcal{T}_\varepsilon(\mathcal{A}, h_{\rm II})$ in terms of $w_{\varepsilon,\mathcal{A}}$ and the solution w of (8.1)

$$\mathcal{T}_\varepsilon(\mathcal{A}, h_{\rm II}) = \Big((w^0_{\varepsilon,\mathcal{A}} + \bar{w}_{\varepsilon,\mathcal{A}} + w)(r_\varepsilon \cdot), -\varepsilon\, r_\varepsilon^{2-n} + r_\varepsilon\, \partial_r (w^0_{\varepsilon,\mathcal{A}} + \bar{w}_{\varepsilon,\mathcal{A}} + w)(r_\varepsilon \cdot)\Big),$$

where we have set

$$w^0_{\varepsilon,\mathcal{A}}(x) \equiv \frac{e}{n-2}\, r^{2-n} + d + R \cdot x + \varepsilon\, r^{-n}\, T \cdot x.$$

We also define

$$\mathcal{T}_0(\mathcal{A}, h_{\rm II}) \equiv \left(w^0_{\varepsilon,\mathcal{A}}(r_\varepsilon \cdot) + h_{\rm II}, -\varepsilon\, r_\varepsilon^{2-n} + r_\varepsilon\, \partial_r w^0_{\varepsilon,\mathcal{A}}(r_\varepsilon \cdot) - \frac{n-2}{2} h_{\rm II} - D_\theta h_{\rm II}\right).$$

The counterpart of Proposition 5.2 is given by the

Proposition 8.2. *The mappings $\mathcal{T}_\varepsilon$ and $\mathcal{T}_0$ are continuous. Furthermore, there exists $c > 0$ and, for all $\kappa > 0$, there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0]$, we have the estimate*

$$||(\mathcal{T}_\varepsilon - \mathcal{T}_0)(\mathcal{A}, h_{\rm II})||_{\mathcal{C}^{2,\alpha} \times \mathcal{C}^{1,\alpha}} \leq c\, r_\varepsilon^2.$$

Again, it is important in the last Proposition that the constant $c > 0$ does not depend on κ.

Proof. The proof is identical to the one of Proposition 5.2 and is therefore omitted. □

9. The gluing procedure

Starting from an orientable minimal hypersurface M with k ends, we build a minimal hypersurface with $k+1$ ends. We start by removing from M a small disk and thus obtain a non compact piece M_{r_0} and a compact piece N_{r_0} which are both minimal hypersurfaces.

Then, we define a family of minimal hypersurfaces $M_{r_0}(h)$ which are close to M_{r_0} and which are parameterized by their boundary value h. We also define a family of minimal hypersurfaces $N_{r_0}(h)$ which are close to N_{r_0} and which are also paraterized by their boundary value h.

Next, we apply the program of Sects. 6, 7 and 8 to $N_{r_0}(h)$. This produces a family of minimal hypersurfaces $N_{r_0,\varepsilon}(h, \mathcal{A}, h_{\rm II})$.

Collecting the families of minimal hypersurfaces $M_{r_0}(h)$, $N_{r_0,\varepsilon}(h, \mathcal{A}, h_{\rm II})$ together with $C_\varepsilon(h_{\rm II})$, the family of minimal hypersurfaces defined in Sect. 5, we now look for h, $h_{\rm II}$ and $\mathcal{A}$ such that the Cauchy data at the boundaries of $M_{r_0}(h)$, of $N_{r_0,\varepsilon}(h, \mathcal{A}, h_{\rm II})$ and of $C_\varepsilon(h_{\rm II})$ match.

This will end the construction of a minimal hypersurface with $k+1$ ends.

9.1. Preliminaries

Assume that M is an orientable minimal hypersurface with k planar ends $E_1, \dots, E_k$. Up to some rigid motion each end E_i can be parameterized as a normal graph over a properly rescaled half n- catenoid

$$[S_i, +\infty) \times S^{n-1} \ni (s, \theta) \longrightarrow a_i \, X_0(s, \theta) + w_i(s, \theta) \, \phi^{\frac{2-n}{2}}(s) \, N_0(s, \theta) \in E_i,$$

where $a_i \in (0, +\infty)$ and where $w_i \in \mathcal{C}^{2,\alpha}_\delta([S_j, +\infty) \times S^{n-1})$ for any $\delta \in (-\frac{2+n}{2}, -\frac{n}{2})$. As we have shown in section Sect. 4, for each end, there are $2\,(n+1)$ linearly independent Jacobi fields which correspond to the $2\,(n+1)$ different geometric transformations and which are associated with the indicial roots $\pm\gamma_0, \dots, \pm\gamma_n$. We shall denote them by

$$\Psi_i^{j,\pm} \quad \text{for} \quad j = 0, \dots, n \quad \text{and} \quad i = 1, \dots, k.$$

Notice that, for each end, we do not consider the linearized mean curvature $\mathcal{L}_{M,0}$ but rather its conjugate, as defined in (3.8). This conjugation can easily be made globally since the functions ϕ which are defined on the ends E_i can be extended to a global smooth function $\phi > 0$ on M and then $\mathcal{L}_M \equiv \phi^{\frac{2-n}{2}} \, \mathcal{L}_{M,0} \, \phi^{\frac{2-n}{2}}$.

We decompose M into slightly overlapping pieces which are a compact piece M^c and the ends E_i. Furthermore, we ask that, for each $i = 1, \dots, k$, the set $M^c \cap E_i$ is diffeomorphic to $[0, 1] \times S^{n-1}$. With this decomposition, we give the

Definition 9.1. *The function space $\mathcal{E}^{k,\alpha}_\mu(M)$ is defined to be the space of all functions $w \in \mathcal{C}^{k,\alpha}(M)$ for which the following norm is finite*

$$|w|_{k,\alpha,\delta} \equiv \sum_{i=1}^{k} \|w_{|E_i}\|_{k,\alpha,\delta} + \|w_{|M^c}\|_{k,\alpha,M^c}.$$

where $\| \; \|_{k,\alpha,\delta}$ is the norm defined in Definition 4.1. Notice that we have identified w on E_i with a function on $[S_i, +\infty) \times S^{n-1}$ via the graph representation of E_i.

Definition 9.2. *The deficiency space is defined by*

$$\mathcal{K} \equiv \oplus_{i=1,\dots,k} \mathrm{Span}\{\, \lambda(\cdot - S_i) \, \Psi_i^{j,\pm} \; : \; j = 0, \dots, n\}$$

Following (almost word for word) the proof of the “Linear Decomposition Lemma” in [4] or in [8], we can prove the

Theorem 9.1. *Fix $\delta \in (-\frac{2+n}{2}, -\frac{n}{2})$. Assume that the operator $\mathcal{L}_M$ from $\mathcal{E}^{2,\alpha}_\delta(M)$ into $\mathcal{E}^{0,\alpha}_\delta(M)$ is injective. Then $\mathcal{L}_M$ from $\mathcal{E}^{2,\alpha}_{-\delta}(M)$ into $\mathcal{E}^{0,\alpha}_{-\delta}(M)$ is surjective with kernel dimension $k\,(n+1)$.*

Furthermore, if $\mathcal{K}_0$ denotes the trace of the kernel over $\mathcal{K}$, that is $\mathcal{K}_0$ is a $k\,(n+1)$ dimensional subspace of $\mathcal{K}$ such that

$$Ker\, \mathcal{L}_M \subset \mathcal{K}_0 \oplus \mathcal{E}^{2,\alpha}_\delta(M),$$

and if $\mathcal{K}_1$ *is a* $k\,(n+1)$ *dimensional subspace of* $\mathcal{K}$ *such that*

$$\mathcal{K} = \mathcal{K}_0 \oplus \mathcal{K}_1,$$

then

$$\mathcal{L}_M : \mathcal{E}^{2,\alpha}_\delta M) \oplus \mathcal{K}_1 \longrightarrow \mathcal{E}^{0,\alpha}_\delta(M),$$

is an isomorphism.

We can now give the precise definition of nondegeneracy.

Definition 9.3. *We will say that a minimal hypersurface* M *is nondegenerate, if the linearized mean curvature operator*

$$\mathcal{L}_M : \mathcal{E}^{2,\alpha}_\delta(M) \longrightarrow \mathcal{E}^{0,\alpha}_\delta(M),$$

is injective for all $\delta \in (-\infty, -\frac{n}{2})$.

For example, we have seen in Corollary 4.1 that C_1, the unit n-catenoid, is nondegenerate.

From now on, we will assume that M is nondegenerate. We choose any point $p \in M$. Without loss of generality, we can assume that $p = 0$ and that the tangent space of M at 0 is the hyperplane $x_{n+1} = 0$, since this can always be achieved by a suitable rigid motion. Provided $r_0 > 0$ is chosen small enough, M can be parameterized near 0 as a graph

$$B_{4r_0} \ni x \longrightarrow (x, u_0(x)).$$

For all $r \le 4r_0$, we will denote by N_r the graph of u_0 over B_r and we define $M_r \equiv M \setminus N_r$.

Definition 9.4. *For all* $\delta \in \mathbb{R}$, *all* $k \in \mathbb{N}$ *and all* $r < r_0$, *the space* $\mathcal{E}^{k,\alpha}_\delta(\overline{M_r})$ *is defined to be the space of restrictions to* $\overline{M_r}$ *of functions* $w \in \mathcal{E}^{k,\alpha}_\delta(M)$, *endowed with the induced norm.*

We modify the normal vector field on M so that it becomes equal to $(0, 1)$ in N_{2r_0} and equal to the normal vector field in M_{4r_0}. Of course r_0 is assumed to be chosen small enough so that the modified vector field is transversal along M. The linearized mean curvature operator with respect to this vector field will be denoted by $\mathcal{L}^*_M$. Notice that, $\mathcal{L}^*_M = \mathcal{L}_M$ in M_{4r_0} and also that $\mathcal{L}^* = \Lambda_{u_0}$ in N_{2r_0}.

Reducing r_0 if this is necessary, we can assume that :

(P.1) For any fixed $\delta \in (-\frac{2+n}{2}, -\frac{n}{2})$, the operator $\mathcal{L}^*_M$ defined from $\mathcal{E}^{2,\alpha}_\delta(M) \oplus \mathcal{K}_1$ into $\mathcal{E}^{0,\alpha}_\delta(M)$ is an isomorphism.

(P.2) For any fixed $\delta \in (-\frac{2+n}{2}, -\frac{n}{2})$, the operator $\mathcal{L}^*_M$ defined from $[\mathcal{E}^{2,\alpha}_\delta(\overline{M_{r_0}}) \oplus \mathcal{K}_1]_\mathcal{D}$ into $\mathcal{E}^{0,\alpha}_\delta$ is an isomorphism. Here we have set

$$[\mathcal{E}^{2,\alpha}_\delta(\overline{M_{r_0}}) \oplus \mathcal{K}_1]_\mathcal{D} \equiv \{w \in \mathcal{E}^{2,\alpha}_\delta(\overline{M_{r_0}}) \oplus \mathcal{K}_1 \ : \ w = 0 \quad \text{on} \quad \partial M_{r_0}\}.$$

(P.3) The operator $\mathcal{L}^*_M = \Lambda_{u_0}$ from $[\mathcal{C}^{2,\alpha}(\overline{B_{r_0}})]_{\mathcal{D}}$ into $\mathcal{C}^{0,\alpha}(\overline{B_{r_0}})$ is an isomorphism.

(P.4) For some fixed $\nu \in (-n, 1-n)$ and for all $r < r_0/2$, there exists an operator $\Gamma_{u_0,r}$ defined from $\mathcal{C}^{0,\alpha}_{\nu-2}(\overline{B_{r_0}} \setminus B_r)$ into $[\mathcal{C}^{2,\alpha}_{\nu}(\overline{B_{r_0}} \setminus B_r)]_{\mathcal{D},n}$ such that $\Lambda_{u_0} \circ \Gamma_{u_0,r} = Id$. Furthermore, the norm of $\Gamma_{u_0,r}$ is bounded independently of $r < r_0/2$.

The fact that Properties (P.1)–(P.3) do hold provided r_0 is chosen small enough is standard and follows from simple perturbation arguments. The last property also follows from a perturbation argument using the fact that

$$\|(\Lambda_u - \Delta) w\|_{0,\alpha,\nu-2} \leq c\, r_0^2\, \|w\|_{2,\alpha,\nu},$$

as can easily seen using Lemma 6.1, together with the Lemma

Lemma 9.1. *Assume that $\nu \in (-n, 1-n)$ is fixed and that $0 < r < r_0/2$. Then, there exists some operator*

$$G_{r_0,r} : \mathcal{C}^{0,\alpha}_{\nu-2}((\overline{B_{r_0}} \setminus B_r)) \longrightarrow \mathcal{C}^{2,\alpha}_{\nu}((\overline{B_{r_0}} \setminus B_r))$$

such that, for all $f \in \mathcal{C}^{0,\alpha}_{\nu-2}((\overline{B_{r_0}} \setminus B_r))$, the function $w = G_{r_0,r}(f)$ is a solution of the problem

$$\begin{cases} \Delta w = f & \textit{in} \quad B_{r_0} \setminus B_r \\ \pi_{\mathrm{II}} w = 0 & \textit{on} \quad \partial B_r \\ w = 0 & \textit{on} \quad \partial B_{r_0}. \end{cases}$$

In addition, we have $||G_{r_0,r}(f)||_{2,\alpha,\nu} \leq c\, ||f||_{0,\alpha,\nu-2}$, for some constant $c > 0$ independent of r.

Proof. We set $t = \log r$ so that Δ now reads

$$\Delta = e^{-2t}\, (\partial_{tt} + (n-2)\, \partial_t + \Delta_{S^{n-1}}).$$

Next, conjugate this operator by

$$e^{\frac{n+2}{2} t}\, \Delta\, e^{\frac{2-n}{2} t} = \Delta_0,$$

which is now defined from $\mathcal{C}^{2,\alpha}_{\delta}([S, \tilde{S}] \times S^{n-1})$ into $\mathcal{C}^{0,\alpha}_{\delta}([S, \tilde{S}] \times S^{n-1})$, with $\delta = \nu + \frac{n-2}{2}$. The proof of the result is now similar to the proof of Proposition 4.4 or Proposition 4.3, the fact that we prescribe 0 boundary data at $s = \tilde{S}$ does not introduce any new difficulty. □

Property (P.3) being fulfilled, we may apply the inverse function theorem to produce a minimal hypersurface $N^h_{r_0}$ which is close to N_{r_0} and whose boundary data are given by $u_0 + h$ on ∂B_{r_0}, for any sufficiently small function $h \in \mathcal{C}^{2,\alpha}(\partial B_{r_0})$, say $\|h\|_{\mathcal{C}^{2,\alpha}} \leq \eta_1$.

Similarly, property (P.2) being fulfilled, we may apply the inverse function theorem to produce a minimal hypersurface $M_{r_0}(h)$ which is close to M_{r_0} and whose boundary data are given by $u_0 + h$ on ∂B_{r_0}, for any sufficiently small function $h \in \mathcal{C}^{2,\alpha}(\partial B_{r_0})$, say $\|h\|_{\mathcal{C}^{2,\alpha}} \leq \eta_1$.

While, the application of the inverse function theorem to produce $N_{r_0}(h)$ is standard, the application of the inverse function Theorem to produce $M_{r_0}(h)$ is technically more involved due to the presence of the deficiency subspace, we refer to [4,8] and also [11] for the details.

Now we translate along the x_{n+1} axis and rotate the hypersurface $N_{r_0}(h)$ so that the resulting hypersurface $\tilde{N}_{r_0}(h)$ contains 0, and that the tangent space of $\tilde{N}_{r_0}(h)$ at 0 is the $x_{n+1} = 0$ hyperplane.

We set $\Sigma_0 \equiv \tilde{N}_{r_0}(h)$ and the projection of Σ_0 over the $x_{n+1} = 0$ hyperplane is denoted Ω, so that Σ_0 is a graph over Ω corresponding to some function u. Reducing r_0 and η_1, if this is necessary, we see, thanks to (P.1)-(P.4), that (H.1)-(H.5) can be fulfilled independently of h such tha t $\|h\|_{\mathcal{C}^{2,\alpha}} \leq \eta_1$. Now, for fixed $\kappa > 0$ and all $\varepsilon > 0$ small enough, we can apply the program of Sect. 6, 7 and 8 to Σ_0, Ω and u. This produces a sequence of family of hypersurfaces

$$\Sigma_0 \longrightarrow \Sigma_\varepsilon \longrightarrow \Sigma_{\varepsilon,\mathcal{A}} \longrightarrow \Sigma_{\varepsilon,\mathcal{A},h_{\mathrm{II}}},$$

indexed by $\|\mathcal{A}\| \leq \kappa\, r_\varepsilon^2$ and $\|h_{\mathrm{II}}\|_{2,\alpha} \leq \kappa\, r_\varepsilon^2$. Notice that, though it is not explicit in the notation, all these hypersurfaces depend on h through u and Ω.

9.2. Matching the Cauchy data

We now have at our disposal $M_{r_0}(h)$ and the hypersurface $\Sigma_{\varepsilon,\mathcal{A}}(h_{\mathrm{II}})$ which has two boundaries, one of which is equal (up to a rigid motion $\mathcal{R}$) to the boundary of $N_{r_0}(h)$. We perform this rigid motion $\mathcal{R}$ on $\Sigma_{\varepsilon,\mathcal{A}}(h_{\mathrm{II}})$ and denote by $N_{r_0,\varepsilon}(h, \mathcal{A}, h_{\mathrm{II}})$ the corresponding hypersurface. Thus we now have $\partial M_{r_0}(h)$ which is equal to the outer boundary of $N_{r_0,\varepsilon}(h, \mathcal{A}, h_{\mathrm{II}})$.

We can also perform the same rigid motion $\mathcal{R}$ on $C_\varepsilon(h_{\mathrm{II}})$. The resulting hypersurface will still be denoted by $C_\varepsilon(h_{\mathrm{II}})$. Notice that even though we have performed this rigid motion, the boundary of $C_\varepsilon(h_{\mathrm{II}})$ and the inner boundary of $N_{r_0,\varepsilon}(h, \mathcal{A}, h_{\mathrm{II}})$ are both graphs over the image of $r_\varepsilon\, S^{n-1} \times \{0\}$ by the rigid motion $\mathcal{R}$, whose eigenfunction decomposition match except for the coefficients corresponding to the first $n+1$ eigenfunctions e_j.

Our aim will be now to find h, h_{II} and $\mathcal{A}$ in such a way that

$$M_\varepsilon \equiv M_{r_0}(h) \cup N_{r_0,\varepsilon}(h, \mathcal{A}, h_{\mathrm{II}}) \cup C_\varepsilon(h_{\mathrm{II}}),$$

is a $\mathcal{C}^{1,\alpha}$ hypersurface.

By construction, both $M_{r_0}(h)$ and $N_{r_0,\varepsilon}(h, \mathcal{A}, h_{\mathrm{II}})$ are graphs over the $x_{n+1} = 0$ hyperplane near their common boundary, say

$$x \in B_{2r_0} \setminus B_{r_0} \longrightarrow (x, u_0(x) + w_h(x))$$

for $M_{r_0}(h)$ and

$$x \in B_{r_0} \setminus B_{r_0/2} \longrightarrow (x, u_0(x) + \tilde{w}_{h,\mathcal{A},h_{\mathrm{II}},\varepsilon}(x))$$

for $N_{3r_0,\varepsilon}(h, \mathcal{A}, h_{\mathrm{II}})$. We may define the mapping

$$\mathcal{U}_\varepsilon(h, \mathcal{A}, h_{\mathrm{II}}) \equiv r_0\, \partial_r \left(w_h(r_0\theta) - \tilde{w}_{h,\mathcal{A},h_{\mathrm{II}},\varepsilon}(r_0\theta)\right) \in \mathcal{C}^{1,\alpha}(\partial B_{r_0}),$$

for all $(h, \mathcal{A}, h_{\mathrm{II}}) \in \mathcal{C}^{2,\alpha}(\partial B_{r_0}) \times \mathcal{F}$ such that $\|h\|_{2,\alpha} \leq \eta_1$, $\|\mathcal{A}\| \leq \kappa\, r_\varepsilon^2$ and $\|h_{\mathrm{II}}\|_{2,\alpha} \leq \kappa\, r_\varepsilon^2$.

We also define for all $h \in \mathcal{C}^{2,\alpha}(\partial B_{r_0})$

$$\mathcal{U}_0(h) \equiv r_0\, \partial_r \left(w_h^0(r_0\theta) - \tilde{w}_h^0(r_0\theta) \right) \in \mathcal{C}^{1,\alpha}(\partial B_{r_0}),$$

where w_h^0 is the (unique) solution of $\mathcal{L}_M^* w_h^0 = 0$ in M_{r_0} such that $w_h^0 = h$ on ∂B_{r_0} which belongs to $\mathcal{E}_\delta^{2,\alpha}(\overline{M_{r_0}}) \oplus \mathcal{K}_1$ and where $\tilde{w}_h^0$ is the (unique) solution of $\Lambda_{u_0} \tilde{w}_h^0 = 0$ in B_{r_0} such that $\tilde{w}_h^0 = h$ on ∂B_{r_0} which belongs to $\mathcal{C}^{2,\alpha}(\overline{B_{r_0}})$. In other words, $\mathcal{U}_0$ is the difference of the two Dirichlet to Neumann mappings corresponding to the operator $\mathcal{L}_M^*$ defined in M_{r_0} and N_{r_0}. It is well known that these later are linear first order elliptic differential operator with principal symbol $a(x, \xi) = -|\xi| + \mathcal{O}(r_0)$ and $b(x, \xi) = |\xi| + \mathcal{O}(r_0)$ respectively.

Notice that, since we have assumed (P.1) to hold, we know that $\mathcal{L}_M^*$ is an isomorphism from $\mathcal{E}_\delta^{2,\alpha}(M) \oplus \mathcal{K}_1$ into $\mathcal{E}_\delta^{0,\alpha}(M)$. In particular, this implies that $\mathcal{U}_0$, defined from $\mathcal{C}^{2,\alpha}(\partial B_{r_0})$ into $\mathcal{C}^{1,\alpha}(\partial B_{r_0})$, is also an isomorphism. Indeed, $\mathcal{U}_0$ is a linear first order elliptic pseudo-differential operator with principal symbol $c(x, \xi) = -2\,|\xi| + \mathcal{O}(r_0)$. Therefore, in order to check that $\mathcal{U}_0$ is an isomorphism, it is enough to prove that it is injective. Now if we assume that $\mathcal{U}_0(h) = 0$ then the function w defined by $w \equiv w_h^0$ in M_{r_0} and $w \equiv \tilde{w}_h^0$ in N_{r_0} is a global solution of $\mathcal{L}_M^* w = 0$ in M, and furthermore, w belongs to $\mathcal{E}_\delta^{2,\alpha}(M) \oplus \mathcal{K}_1$. Thus $w \equiv 0$ (thanks to (P.1)) and, as a consequence, $h \equiv 0$.

Finally, following the construction in Sects. 6, 7 and 8, we find that

$$\|\mathcal{U}_\varepsilon(h, \mathcal{A}, h_{\mathrm{II}}) - \mathcal{U}_0(h)\|_{\mathcal{C}^{1,\alpha}} \leq c\,(\|h\|^2_{\mathcal{C}^{2,\alpha}} + r_\varepsilon^{n-2/3}), \tag{9.1}$$

for some constant $c > 0$ which does not depend on κ, provided ε is chosen small enough. The estimate on the right hand side can be justified easily if one follows carefully the different steps in the construction of $\mathcal{U}_\varepsilon(h, \mathcal{A}, h_{\mathrm{II}})$. Indeed, in the first step of the construction, we solve some nonlinear elliptic equation in M_{r_0} and in N_{r_0} (with boundary data h). This produced some Cauchy data which can be expanded as the sum of a linear term, which corresponds to $\mathcal{U}_0(h)$ plus higher order terms, which can be bounded by a constant times $\|h\|^2_{\mathcal{C}^{2,\alpha}}$. In the second step, we perform the analytic modification of $N_{r_0}(h)$ by introducing the parameter ε, this produces a change in the Cauchy data which can be bounded by a constant times $\varepsilon \sim r_\varepsilon^{n-2/3}$. Finally, we perform the geometric transformation (which does not change the Cauchy data) and then we change the Dirichlet data on the inner boundary but this only changes the ou ter Cauchy data map by a function bounded by a constant times $\kappa\, r_\varepsilon^{2-\nu}$, for ε small enough. Recall that $\nu \in (-n, 1-n)$ is fixed, hence this last term can be bounded by a constant times $r_\varepsilon^{n-2/3}$ provided ε is small enough.

We define the operator

$$\begin{aligned} \mathbf{C}_\varepsilon : \mathcal{C}^{2,\alpha}(\partial B_{r_0}) \times \mathbb{R}^{2n+2} \times \pi_{\mathrm{II}}(\mathcal{C}^{2,\alpha}(S^{n-1})) \\ \longrightarrow \mathcal{C}^{1,\alpha}(\partial B_{r_0}) \times \mathcal{C}^{2,\alpha}(S^{n-1}) \times \mathcal{C}^{1,\alpha}(S^{n-1}), \end{aligned}$$

by

$$\mathbf{C}_\varepsilon(h, \mathcal{A}, h_{\mathrm{II}}) \equiv (\mathcal{U}_\varepsilon(h, \mathcal{A}, h_{\mathrm{II}}), \mathcal{T}_\varepsilon(h, \mathcal{A}, h_{\mathrm{II}}) - \mathcal{S}_\varepsilon(h_{\mathrm{II}})).$$

Notice that, in the notation, we have made explicit the dependence of both $\mathcal{T}_\varepsilon$ and $\mathcal{S}_\varepsilon$ on h. Also observe that the norm of $(h, \mathcal{A}, h_{\mathrm{II}})$ in $\mathcal{C}^{2,\alpha}(\partial B_{r_0}) \times \mathbb{R}^{2n+2} \times \pi_{\mathrm{II}}(\mathcal{C}^{2,\alpha}(S^{n-1}))$ is given by

$$\|h\|_{\mathcal{C}^{2,\alpha}} + \|\mathcal{A}\| + \|h_{\mathrm{II}}\|_{\mathcal{C}^{2,\alpha}}.$$

We also define

$$\mathbf{C}_0(h, \mathcal{A}, h_{\mathrm{II}}) \equiv (\mathcal{U}_0(h), \mathcal{T}_0(\mathcal{A}, h_{\mathrm{II}}) - \mathcal{S}_0(h_{\mathrm{II}})).$$

This last linear operator can also be written as

$$\mathbf{C}_0(h, \mathcal{A}, h_{\mathrm{II}}) = (\mathcal{U}_0(h), w^0_{\mathcal{A}}(r_\varepsilon \cdot), r_\varepsilon\, \partial_r w^0_{\mathcal{A}}(r_\varepsilon \cdot) - 2D_\theta h_{\mathrm{II}}),$$

where we recall that, for $x = r\,\theta$

$$w^0_{\mathcal{A}}(x) = \frac{e}{n-2}\, r^{2-n} + d + r\, R \cdot \theta + \varepsilon\, r^{1-n}\, T \cdot \theta.$$

Let us observe that both $\mathbf{C}_\varepsilon$ and $\mathbf{C}_0$ have range in a proper subspace of $\mathcal{C}^{1,\alpha}(\partial B_{r_0}) \times \mathcal{C}^{2,\alpha}(S^{n-1}) \times \mathcal{C}^{1,\alpha}(S^{n-1})$, namely in

$$\mathcal{C}^{1,\alpha}(\partial B_{r_0}) \times \big(\mathrm{Span}\{e_j \,:\, j = 0, \dots, n\}\big) \times \mathcal{C}^{1,\alpha}(S^{n-1}).$$

Also observe that $\mathbf{C}_0$ is an isomorphism from $\mathcal{C}^{2,\alpha}(\partial B_{r_0}) \times \mathbb{R}^{2n+2} \times \pi_{\mathrm{II}}(\mathcal{C}^{2,\alpha}(S^{n-1}))$ into its range $\mathcal{C}^{1,\alpha}(\partial B_{r_0}) \times \big(\mathrm{Span}\{e_j \,:\, j = 0, \dots, n\}\big) \times \mathcal{C}^{1,\alpha}(S^{n-1})$. Moreover the norm of its inverse is bounded independently of $\varepsilon \in (0, 1)$.

We denote by $\mathcal{B}^\alpha_\kappa$ the ball of radius $\kappa\, r_\varepsilon^2$ in $\mathcal{C}^{2,\alpha}(\partial B_{r_0}) \times \mathbb{R}^{2n+2} \times \pi_{\mathrm{II}}(\mathcal{C}^{2,\alpha}(S^{n-1}))$. It follows from our previous analysis that, for fixed $\kappa > 0$, the mapping $\mathbf{C}_\varepsilon$ is well defined in $\mathcal{B}^\alpha_\kappa$ provided the parameter ε is small enough.

Now, we prove the

Proposition 9.1. *There exist $\kappa > 0$ and $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0]$, the mapping $\mathbf{C}_\varepsilon$ has a zero in $\mathcal{B}^\alpha_\kappa$.*

This zero of $\mathbf{C}_\varepsilon$ produces a $\mathcal{C}^{1,\alpha}$ hypersurface M_ε which is the union of minimal hypersurfaces and which has $k+1$ ends of catenoidal type. It is then a simple exercise to see, thanks to regularity theory, that M_ε is in fact a $\mathcal{C}^\infty$ minimal hypersurface with $k + 1$ ends of planar type.

Proof. Collecting the results of Proposition 5.2 and Proposition 8.2 together with (9.1), we see that there exists $\kappa_0 > 0$ such that, for all $\kappa > 0$, there exists ε_0 (depending on κ) such that the image of $\mathcal{B}^\alpha_\kappa$ by $\mathbf{C}_0^{-1}\, (\mathbf{C}_\varepsilon - \mathbf{C}_0)$, is included in $\mathcal{B}^\alpha_{\kappa_0}$. Here we have also used the fact that $\mathbf{C}_0$ is an isomorphism whose inverse is bounded independently of ε.

To conclude, we want to use Schauder's fixed point Theorem which will ensure the existence of at least one fixed point of $\mathbf{C}_0^{-1}\, (\mathbf{C}_\varepsilon - \mathbf{C}_0)$ and hence, at least one zero of $\mathbf{C}_\varepsilon$ in $\mathcal{B}_\kappa^\alpha$, provided $\kappa > \kappa_0$. However, since $\mathbf{C}_0^{-1}(\mathbf{C}_\varepsilon - \mathbf{C}_0)$ is not compact it is not possible to apply directly Schauder's Theorem. This is the reason why we introduce a family of smoothing operators $\mathbf{D}^q$, for all $q > 1$, which satisfy for fixed $0 < \alpha' < \alpha < 1$

$$\|\mathbf{D}^q f\|_{\mathcal{C}^{2,\alpha}(S^{n-1})} \leq c_0 \, \|f\|_{\mathcal{C}^{2,\alpha}(S^{n-1})}. \tag{9.2}$$

and

$$\|f - \mathbf{D}^q f\|_{\mathcal{C}^{2,\alpha'}(S^{n-1})} \leq c_0 \, q^{\alpha'-\alpha} \, \|f\|_{\mathcal{C}^{2,\alpha}(S^{n-1})}. \tag{9.3}$$

for some constant $c_0 > 0$ which does not depend on $q > 1$. The existence of such smoothing operators is available in [1]. To keep the notation short, we use the same notation for the smoothing operator defined on $\mathcal{C}^{2,\alpha}(\partial B_{r_0}) \times \mathbb{R}^{2n+2} \times \pi_{\mathrm{II}}(\mathcal{C}^{2,\alpha}(S^{n-1}))$ and acting on both function spaces.

Now we fix $\kappa > c_0 \, \kappa_0$. For all $q > 1$, we may apply Schauder's fixed point Theorem to $\mathbf{D}^q \, \mathbf{C}_0^{-1}(\mathbf{C}_\varepsilon - \mathbf{C}_0)$ to obtain the existence of $(h_q, \mathcal{A}_q, h_{\mathrm{II},q})$ fixed point of $\mathbf{D}^q \, \mathbf{C}_0^{-1}(\mathbf{C}_\varepsilon - \mathbf{C}_0)$ in $\mathcal{B}_\kappa^\alpha$, provided ε is chosen small enough, say $\varepsilon \in (0, \varepsilon_0]$.

Since $(h_q, \mathcal{A}_q, h_{\mathrm{II},q})$ has norm bounded uniformly in q, we may extract a sequence $q_j \to +\infty$ such that $(h_{q_j}, \mathcal{A}_{q_j}, h_{\mathrm{II},q_j})$ converges in $\mathcal{C}^{2,\alpha'}(\partial B_{r_0}) \times \mathbb{R}^{2n+2} \times \pi_{\mathrm{II}}(\mathcal{C}^{2,\alpha'}(S^{n-1}))$ for some fixed $\alpha' < \alpha$. Thanks to the continuity of $\mathbf{C}_\varepsilon$, $\mathbf{C}_0$ and C_0^{-1} (with respect to the $\mathcal{C}^{2,\alpha'}$ and $\mathcal{C}^{1,\alpha'}$ topology) and also to (9.3), the limit of this sequence is a fixed point of the mapping $\mathbf{C}_0^{-1}(\mathbf{C}_\varepsilon - \mathbf{C}_0)$ and hence, produces a zero of $\mathbf{C}_\varepsilon$, for all $\varepsilon \in (0, \varepsilon_0]$. This completes our proof. □

The induction process will then be complete once we will have proven that M_ε is nondegenerate for all ε small enough.

9.3. *Determination of the gluing region*

As promised, we now comment the different choices of s_ε, r_ε. From what we have seen in Lemma 6.3, the modified hypersurface Σ_ε is parameterized, near 0, by

$$x \longrightarrow \Big(x, u(x) + \frac{\varepsilon}{n-2} \, r^{2-n} + \mathcal{O}\big(\varepsilon \, r^{4-n}\big)\Big). \tag{9.4}$$

(In dimension $n = 3, 4$, an extra $\log r$ is needed in the last term of this expression but this is irrelevant for the determination of the gluing region).

Now consider C_ε the image of the n-catenoid C_1 by the homothety of magnitude $\varepsilon^{\frac{1}{n-1}}$. After a suitable translation, the lower end of this catenoid can be parameterized by

$$x \longrightarrow \Big(x, \frac{\varepsilon}{n-2} \, r^{2-n} + \mathcal{O}\big(\varepsilon^3 \, r^{4-3n}\big)\Big). \tag{9.5}$$

Comparing (9.4) with (9.5) and using the fact that $u(x) = \mathcal{O}(r^2)$, we see that the distance between the two hypersurfaces (measured along the x_{n+1} axis) can be estimated by

$$\mathcal{O}\left(r^2 + \varepsilon \, r^{4-n} + \varepsilon^3 \, r^{4-3n}\right).$$

(Again, $\log r$ term are needed for the second terms when $n = 3, 4$). It is easy to see that this quantity is minimal for $r \sim \varepsilon^{3/(3n-2)}$.

We are now in a position to justify the definitions (5.1). Indeed the parameters s_ε and r_ε are chosen so that

$$r_\varepsilon = \varepsilon^{\frac{1}{n-1}}\, \phi(s_\varepsilon) \sim \varepsilon^{\frac{3}{3n-2}}.$$

Finally, we see that the distance between the two hypersurfaces measured along the x_{n+1} axis is bounded by a constant times r_ε^2. This is the reason why in Sect. 5.1 and in Sect. 8.1 we were just interested only in perturbing the boundary data by some function h_{II} whose norm is bounded by a constant κ times r_ε^2, see (8.4). Looking closely at (6.10), we see that, provided $\|\mathcal{A}\|$ is bounded by a constant κ times r_ε^2, the geometric transformations associated to $\mathcal{A}$ only induce a perturbation on the boundary of $\Sigma_{\varepsilon,\mathcal{A}}$ which, up to some error which is bounded by a fixed constant times r_ε^2, is linear in $\mathcal{A}$. This shows that it should be enough to choose κ large enough to obtain some solution to our problem.

10. Nondegeneracy of the solutions constructed

In this last section, we prove that the hypersurfaces we have obtained in Sect. 9 are nondegenerate, for all ε small enough. In particular, this will imply that the hypersurface M_ε belongs to a smooth $(k+1)$ $(n+1)$ dimensional family of minimal hypersurfaces with $k+1$ ends.

Starting with a hypersurface M which is nondegenerate, we wish to show that the family of hypersurfaces M_ε constructed in the previous section are also nondegenerate, provided ε is small enough. The proof is by contradiction.

Assume that, for a sequence ε_i tending to 0, the operator $\mathcal{L}_{M_{\varepsilon_i}}$ is not injective on $\mathcal{E}^{2,\alpha}_\delta(M_{\varepsilon_i})$, for some $\delta \in (-\frac{n+2}{2}, -\frac{n}{2})$. If this is so, there exists, for each i, some nontrivial function $w_i \in \mathcal{E}^{2,\alpha}_\delta(M_{\varepsilon_i})$ such that $\mathcal{L}_{M_{\varepsilon_i}} w_i = 0$.

By construction, we may decompose

$$M_{\varepsilon_i} \equiv M_{r_0}(h_i) \cup N_{r_0,\varepsilon_i}(h_i, \mathcal{A}_i, h_{\mathrm{II},i}) \cup C_{\varepsilon_i}(h_{\mathrm{II},i}),$$

Moreover, we may decompose $M_{r_0}(h_i)$ into the union of a compact piece M_i^c and k planar ends $E_{i,1}, \dots, E_{i,k}$. We may also ask that, as i tends to $+\infty$ the different pieces of the decomposition of $M_{r_0}(h_i)$ converge to the corresponding decomposition for M_{r_0} (the convergence being understood on compact regions of M_{r_0}) . We define on each M_{ε_i} some weight function $q_i > 0$, as follows:

- $q_i \sim 1$ on M_i^c,
- $q_i \sim e^{\delta s}$ on each end $E_{i,1}, \dots, E_{i,k}$,
- $q_i \sim r^{-\delta}$ in $N_{r_0,\varepsilon_i}(h_{\varepsilon_i}, \mathcal{A}_i, h_{\mathrm{II},i})$,
- $q_i \sim r_{\varepsilon_i}^{-\delta}\, e^{\delta(s - s_{\varepsilon_i})}$ in $C_{\varepsilon_i}(h_{\mathrm{II},i})$,

where $f \sim g$ means that $1/2 \le f/g \le 2$.

Finally, we define in M a weight function $q_\infty > 0$ such that $q_\infty = e^{\delta s}$ on each end $E_{i,1}, \ldots, E_{i,k}$. Moreover, we can ask that, as i tends to $+\infty$, the sequence of functions q_i converges to q_∞ uniformly on compact subsets of $M \setminus \{0\}$.

Next, we normalize the sequence w_i so that

$$\sup_{M_{\varepsilon_i}} q_i^{-1} w_i = 1.$$

The indicial roots of $\mathcal{L}_{M_{\varepsilon_i}}$ at each end are given by $\pm\gamma_j$. Hence any bounded solution of $\mathcal{L}_{M_{\varepsilon_i}} w = 0$ which belongs to the space $\mathcal{E}^{2,\alpha}_\delta(M_{\varepsilon_i})$ decays like$e^{-\frac{n+2}{2}s}$ at each end. This implies that the above supremum is achieved (say at some point $p_i \in M_{\varepsilon_i}$). We now distinguish a few cases according to the behavior of the sequence p_i.

Case 1. Assume that, up to a subsequence, the sequence p_i converges to some point $p_\infty \in M \setminus \{0\}$. Extracting some subsequences, if this is necessary, we find that the sequence w_i converges uniformly on any compact of $M \setminus \{0\}$ to a nontrivial solution of

$$\mathcal{L}_M w_\infty = 0.$$

Moreover, w_∞ is bounded by a constant times q_∞. In particular, this implies that the singularity at 0 is removable. But, this is impossible thanks to the fact that we have assumed M nondegenerate.

Case 2. Assume that, up to a subsequence, the sequence $|p_i|$ tends to $+\infty$ and, for example, that $p_i \in E_j$ for some fixed $j = 1, \ldots, k$. Then, p_i corresponds to some parameters (s_i, θ_i), with $s_i \to +\infty$. We consider the sequence of rescaled functions

$$\tilde{w}_i(s, \theta) \equiv e^{-\delta s_i} w_i(s + s_i, \theta),$$

and up to a subsequence we may assume that this new sequence converges, uniformly on compact, to a nontrivial solution of

$$\Delta_0 w_\infty = 0,$$

in all $\mathbb{R} \times S^{n-1}$. Moreover, w_∞ is bounded by $e^{\delta s}$. But this case is easy to rule out since $\delta \in (-\frac{n+2}{2}, -\frac{n}{2})$.

Case 3. Assume that, up to a subsequence, the sequence p_i tends to 0 or belongs to $C_{\varepsilon_i}(h_{\mathrm{II},i})$. In this last case it seems that we would have to distinguish two subcases according to whether p_i remains in the annular region $N_{r_0,\varepsilon_i}(h_i, \mathcal{A}_i, h_{\mathrm{II},i})$ or belongs to the truncated n-catenoid $C_{\varepsilon_i}(h_{\mathrm{II},i})$. However, it is easy to see that $N_{r_0,\varepsilon_i}(h_i, \mathcal{A}_i, h_{\mathrm{II},i})$ is a normal graph over the end of the n- catenoid C_ε which has been truncated. Hence, in either subcases, the point p_i corresponds to some parameters (s_i, θ_i) for the n-catenoid C_{ε_i}. In addition, s_i is less than s_{ε_i} if p_i belongs to the annular region $N_{r_0,\varepsilon_i}(h_i, \mathcal{A}_i, h_{\mathrm{II},i})$ and s_i is greater than s_{ε_i} if p_i belongs to the truncated n-catenoid $C_{\varepsilon_i}(h_{\mathrm{II},i})$.

Now, the main observation is that, the weight function q_i is designed in such a way that, if $C_{\varepsilon_i}(h_{\mathrm{II},i})$ is considered to be a normal graph over the truncated end of the n-catenoid C_ε, then $q_i \sim r_{\varepsilon_i}^\delta \, e^{\delta(s-s_{\varepsilon_i})}$ here. Hence the two subcases can be treated as one and we can consider the sequence of rescaled functions

$$\tilde{w}_i(s,\theta) \equiv r_{\varepsilon_i}^\delta \, e^{\delta(s_{\varepsilon_i}-s_i)} \, w_i(s+s_i,\theta).$$

Up to a subsequence, we may assume that this new se quence converges either to a nontrivial solution of

$$\Delta_0 \, w_\infty = 0 \quad \text{on} \quad \mathbb{R} \times S^{n-1},$$

if $|s_i|$ tends to ∞, or to a nontrivial solution of

$$\mathcal{L} \, w_\infty = 0 \quad \text{on} \quad \mathbb{R} \times S^{n-1},$$

if s_i converges to $s_* \in \mathbb{R}$. Moreover, w_∞ is bounded by a constant times $e^{\delta s}$. Again, this is not possible thanks to the choice of δ.

We have ruled out every possible case, which is the desired contradiction. The proof of the nondegeneracy is therefore complete.

Since we know that M_ε are nondegenerate for all ε small enough, we can apply the inverse mapping theorem like in [8,4] or [11], to prove that M_ε belongs to a $(k+1)\,(n+1)$-dimensional manifold of hypersurfaces with $k+1$ planar ends.

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