manuscripta math. 102, 85 – 99 (2000)

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Nonexistence results of solutions of semilinear differential inequalities on the Heisenberg group

Received: 17 June 1999

Abstract. Denoting $\Delta_{\mathbf{H}}$ the Laplacian operator on the $(2N+1)$-dimensional Heisenberg group $\mathbf{H}^N$, we prove some nonexistence results for solutions of inequalities of the three types

$$\Delta_{\mathbf{H}}(au) + |u|^p \leq 0, \tag{EI}$$
$$\partial_t u - \Delta_{\mathbf{H}}(au) - |u|^p \leq 0, \tag{PI}$$
$$\partial_{tt} u - \Delta_{\mathbf{H}}(au) - |u|^p \leq 0, \tag{HI}$$

in $\mathbf{H}^N$ and $\mathbf{H}^N \times \mathbf{R}_+$, with $a \in L^\infty$, when $1 < p \leq p_0$, where p_0 depends on N and the type of equation.

1. Introduction

The $(2N+1)$-dimensional Heisenberg group $\mathbf{H}^N$ is the space $\mathbf{R}^{2N+1} = \left\{(x, y, \tau) \in \mathbf{R}^N \times \mathbf{R}^N \times \mathbf{R}\right\}$ equipped with the group operation

$$\eta \circ \tilde{\eta} = \left(x + \tilde{x}, y + \tilde{y}, \tau + \tilde{\tau} + 2\sum_{i=1}^{N}(x_i\tilde{y}_i - y_i\tilde{x}_i)\right) \tag{1.1}$$

where

$$\eta = (x, y, \tau) = (x_1, \dots, x_N, y_1, \dots, y_N, \tau),$$
$$\tilde{\eta} = (\tilde{x}, \tilde{y}, \tilde{\tau}) = (\tilde{x}_1, \dots, \tilde{x}_N, \tilde{y}_1, \dots, \tilde{y}_N, \tilde{\tau}).$$

This group multiplication endows $\mathbf{H}^N$ with a structure of Lie group. The Laplacian $\Delta_{\mathbf{H}}$ over $\mathbf{H}^N$ is obtained from the vectors fields $X_i = \partial_{x_i} + 2y_i\partial_t$ and $Y_i = \partial_{y_i} - 2x_i\partial_t$ by the following

$$\Delta_{\mathbf{H}} = \sum_{i=1}^{N}(X_i \circ X_i + Y_i \circ Y_i)\,. \tag{1.2}$$

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Mathematics Subject Classification (2000): 35J60, 35K60, 35L60

Explicit computation gives the explicit expression

$$\Delta_{\mathbf{H}}u = \sum_{i=1}^{N} \left(\partial^2_{x_i x_i} u + \partial^2_{y_i y_i} u + 4y_i \partial^2_{x_i \tau} u - 4x_i \partial^2_{y_i \tau} u + 4(x_i^2 + y_i^2)\partial^2_{\tau\tau} u\right). \tag{1.3}$$

The operator $\Delta_{\mathbf{H}}$ is a degenerate elliptic operator satisfying the Hormander condition of order one. It is invariant with respect to the left multiplication in the group since

$$\Delta_{\mathbf{H}} \left(u(\eta_0 \circ \eta)\right) = \left(\Delta_{\mathbf{H}} u\right) (\eta_0 \circ \eta) \tag{1.4}$$

$\left(\forall (\eta_0, \eta) \in \mathbf{H}^N \times \mathbf{H}^N\right)$. On $\mathbf{H}^N$ it is natural to define a distance from η to the origin by

$$|\eta|_{\mathbf{H}} = \left(\tau^2 + \sum_{i=1}^{N} (x_i^2 + y_i^2)^2\right)^{1/4}. \tag{1.5}$$

This distance is the analogous of the parabolic distance for the heat operator in $\mathbf{R}_+ \times \mathbf{R}^N$. Its role is pointed out by the fact that if $u(\eta) = u(|\eta|_{\mathbf{H}})$ then

$$\Delta_{\mathbf{H}} u(\rho) = a(\eta) \left(\frac{d^2 u}{d\rho^2} + \frac{Q-1}{\rho}\frac{du}{d\rho}\right), \tag{1.6}$$

with $\rho = |\eta|_{\mathbf{H}}$, $a(\eta) = \rho^{-2} \sum_{i=1}^{N} (x_i^2 + y_i^2)$ and $Q = 2N + 2$. This last number Q is called the homogeneous dimension of $\mathbf{H}^N$.

Non existence results for positive solutions of nonlinear elliptic inequalities of the type

$$\Delta_{\mathbf{H}} u + |\eta|_{\mathbf{H}}^{\gamma} u^p \leq 0 \tag{1.7}$$

were studied by Garofalo and Lanconelli [5] under restrictive assumptions on u, and later on by Birindelli, Capuzzo Dolcetta and Cutri [2] under a much less restrictive assumption, namely only the positivity of u. In this last paper the authors proved that if $\gamma > -2$ and $1 < p \leq (Q+\gamma)/(Q-2)$ there exists no positive solution to (1.7) defined in whole $\mathbf{H}^N$. The techniques upon which this result is based on the use of specific test functions. In this paper we modify their approach and first prove the same type of result without any sign assumption on u. More precisely, let a be a bounded and measurable function defined in $\mathbf{H}^N$, then we prove that there exists no locally integrable function u defined in whole $\mathbf{H}^N$, such that $u \in L^p_{\mathrm{loc}}(\mathbf{H}^N, |\eta|_{\mathbf{H}}^{\gamma}\, d\eta)$, satisfying

$$\Delta_{\mathbf{H}}(au) + |\eta|_{\mathbf{H}}^{\gamma} |u|^p \leq 0, \tag{1.8}$$

whenever $\gamma > -2$ and $1 < p \leq (Q+\gamma)/(Q-2)$. As in [11] our method is essentially a dimensional analysis were the type of the operator is not fundamental but only its scaling invariance. We extend this result to elliptic Hamiltonian systems, parabolic and hyperbolic inequalities such as

$$\begin{cases} \Delta_{\mathbf{H}}(a_1 u) + |\eta|_{\mathbf{H}}^{\gamma_1} |v|^{p_1} \leq 0 \\ \Delta_{\mathbf{H}}(a_2 v) + |\eta|_{\mathbf{H}}^{\gamma_2} |u|^{p_2} \leq 0, \end{cases} \tag{1.9}$$

$$\partial_t u \geq \Delta_{\mathbf{H}}(au) + |\eta|_{\mathbf{H}}^{\gamma} |u|^p , \tag{1.10}$$

$$\partial_{tt} u \geq \Delta_{\mathbf{H}}(au) + |\eta|_{\mathbf{H}}^{\gamma} |u|^p . \tag{1.11}$$

For example, in the case of inequality (1.10) we prove that no weak solution u exists provided

$$\int_{\mathbf{R}^{2N+1}} u(\eta, 0) d\eta \geq 0 , \quad \gamma > -2 \quad \text{and} \quad 1 < p \leq (Q + \gamma + 2)/Q.$$

While in the case of inequality (1.11) the non-existence holds if

$$\int_{\mathbf{R}^{2N+1}} \partial_t u(\eta, 0) d\eta \geq 0 , \quad \gamma > -2 \quad \text{and} \quad 1 < p \leq (Q + \gamma + 1)/(Q - 1).$$

These results have to be compared with similar results valid for the semilinear heat equation and wave equation ([4], [9], [7]).

2. Elliptic inequalities

Let a be a bounded and measurable function defined, in $\mathbf{R}^{2N+1}$ γ and $p > 1$ real numbers. We identify points in $\mathbf{H}^N$ with points in $\mathbf{R}^{2N+1}$. We also recall that the natural Haar measure in $\mathbf{H}^N$ is identical to the Lebesgue measure $d\eta = dxdyd\tau$ in $\mathbf{R}^{2N+1} = \mathbf{R}^N \times \mathbf{R}^N \times \mathbf{R}$.

Definition 1. *A weak solution u of the differential inequality*

$$\Delta_{\mathbf{H}}(au) + |\eta|_{\mathbf{H}}^{\gamma} |u|^p \leq 0 \tag{2.1}$$

in $\mathbf{R}^{2N+1}$ is a locally integrable function such that $u \in L^p_{\text{loc}}(\mathbf{R}^{2N+1}, |\eta|_{\mathbf{H}}^{\gamma} d\eta)$ which satisfies

$$\int_{\mathbf{R}^{2N+1}} \left(au \Delta_{\mathbf{H}} \zeta + |\eta|_{\mathbf{H}}^{\gamma} |u|^p \zeta \right) d\eta \leq 0, \tag{2.2}$$

for any $\zeta \in C_c^2 \left(\mathbf{R}^{2N+1}\right)$, $\zeta \geq 0$.

Theorem 2.1. *Assume $\gamma > -2$ and $1 < p \leq (Q + \gamma)/(Q - 2)$, then there exists no solution u of inequality (2.1) defined in $\mathbf{R}^{2N+1}$.*

Proof. Let u be such a weak solution and ζ be a smooth nonnegative test function. From (2.2)

$$\int_{\mathbf{R}^{2N+1}} |\eta|_{\mathbf{H}}^{\gamma} |u|^p \zeta d\eta \leq - \int_{\mathbf{R}^{2N+1}} au\Delta_{\mathbf{H}}\zeta d\eta. \tag{2.3}$$

We choose ζ such that

$$\int_{\mathbf{R}^{2N+1}} |\Delta_{\mathbf{H}}\zeta|^{p'} \zeta^{1-p'} |\eta|_{\mathbf{H}}^{\gamma(1-p')} d\eta < \infty, \tag{2.4}$$

where $p' = p/(p-1)$ and such a choice is possible by taking ζ constant near 0. Then

$$\begin{aligned} &\left| \int_{\mathbf{R}^{2N+1}} au\Delta_{\mathbf{H}}\zeta d\eta \right| \\ &\leq \|a\|_{L^\infty} \left(\int_{\mathbf{R}^{2N+1}} |\Delta_{\mathbf{H}}\zeta|^{p'} \zeta^{1-p'} |\eta|_{\mathbf{H}}^{\gamma(1-p')} d\eta \right)^{1/p'} \left(\int_{\mathbf{R}^{2N+1}} |\eta|_{\mathbf{H}}^{\gamma} |u|^p \zeta d\eta \right)^{1/p} \\ &\leq \frac{1}{2} \int_{\mathbf{R}^{2N+1}} |\eta|_{\mathbf{H}}^{\gamma} |u|^p \zeta d\eta + C \int_{\mathbf{R}^{2N+1}} |\Delta_{\mathbf{H}}\zeta|^{p'} \zeta^{1-p'} |\eta|_{\mathbf{H}}^{\gamma(1-p')} d\eta . \end{aligned} \tag{2.5}$$

In the sequel C denotes a constant which may vary from line to line but is independent of the terms which will take part in any limit processing. Therefore the inequality

$$\int_{\mathbf{R}^{2N+1}} |\eta|_{\mathbf{H}}^{\gamma} |u|^p \zeta d\eta \leq C \int_{\mathbf{R}^{2N+1}} |\Delta_{\mathbf{H}}\zeta|^{p'} \zeta^{1-p'} |\eta|_{\mathbf{H}}^{\gamma(1-p')} d\eta \tag{2.6}$$

follows from (2.2), (2.5). Now we take

$$\zeta(\eta) = \zeta(x, y, \tau) = \varphi\left(\frac{\tau^\kappa + |x|^\mu + |y|^\mu}{R^2} \right)$$

where $\varphi \in C_c^\infty(\mathbf{R}_+)$ satisfies $0 \leq \varphi \leq 1$ and

$$\varphi(r) = \begin{cases} 0 & \text{if } r \geq 2, \\ 1 & \text{if } 0 \leq r \leq 1, \end{cases} \tag{2.7}$$

and R is a positive parameter, $\kappa > 0$ and $\mu > 0$ will be determined later on. Then

$$\begin{aligned} \Delta_{\mathbf{H}}\zeta &= \sum_{i=1}^N \left(\partial^2_{x_i x_i}\zeta + \partial^2_{y_i y_i}\zeta + 4y_i \partial^2_{x_i \tau}\zeta - 4x_i \partial^2_{y_i \tau}\zeta + 4(x_i^2 + y_i^2)\partial^2_{\tau\tau}\zeta \right) \\ &= \left(\mu\left(|x|^{2\mu-2} + |y|^{2\mu-2} \right) + 4\kappa^2 \left(|x|^2 + |y|^2 \right) \tau^{2\kappa-2} \right) \left(\frac{d^2\varphi}{dR^2} \circ \rho \right) R^{-4} \\ &\quad + \left(\mu(\mu + N - 2) \left(|x|^{\mu-2} + |y|^{\mu-2} \right) + 4\kappa(\kappa - 1) \left(|x|^2 + |y|^2 \right) \tau^{\kappa-2} \right) \\ &\quad \times \left(\frac{d\varphi}{dR} \circ \rho \right) R^{-2}, \end{aligned} \tag{2.8}$$

where $\rho = R^{-2}\left(\tau^\kappa + |x|^\mu + |y|^\mu\right)$. In order to estimate the right-hand side of (2.8) we perform the change of variables $(x, y, \tau) = \eta \longmapsto \tilde\eta = (\tilde x, \tilde y, \tilde\tau)$

$$\begin{cases} R^{-2}\tau^\kappa = \tilde\tau^\kappa \\ R^{-2}|x|^\mu = |\tilde x|^\mu \\ R^{-2}|y|^\mu = |\tilde y|^\mu \end{cases} \Leftrightarrow \begin{cases} \tau = R^{2/\kappa}\tilde\tau \\ x = R^{2/\mu}\tilde x \\ y = R^{2/\mu}\tilde y \end{cases} \tag{2.9}$$

Let $\tilde\rho(\tilde x, \tilde y, \tilde\tau) = \tilde\tau^\kappa + |\tilde x|^\mu + |\tilde y|^\mu$ and

$$\Omega = \left\{(\tilde x, \tilde y, \tilde\tau) \in \mathbf{R}^N \times \mathbf{R}^N \times \mathbf{R}_+ : \tilde\tau^\kappa + |\tilde x|^\mu + |\tilde y|^\mu \le 2\right\}.$$

Then

$$\begin{aligned} \Delta_{\mathbf{H}}\zeta = {} & R^{-4/\mu}\mu\left(|\tilde x|^{2\mu-2} + |\tilde y|^{2\mu-2}\right)\left(\frac{d^2\varphi}{dR^2}\circ\tilde\rho\right) \\ & + R^{-4/\mu}\mu(\mu+N-2)\left(|\tilde x|^{\mu-2} + |\tilde y|^{\mu-2}\right)\left(\frac{d\varphi}{dR}\circ\rho\right) \\ & + R^{4/\mu-4/\kappa}4\kappa^2\left(|\tilde x|^2 + |\tilde y|^2\right)\tau^{2\kappa-2}\left(\frac{d^2\varphi}{dR^2}\circ\tilde\rho\right) \\ & + R^{4/\mu-4/\kappa}4\kappa(\kappa-1)\left(|\tilde x|^2 + |\tilde y|^2\right)\tau^{\kappa-2}\left(\frac{d\varphi}{dR}\circ\rho\right). \end{aligned} \tag{2.10}$$

Since $d\eta = R^{4N/\mu+2/\tau}d\tilde\eta$, we derive from (2.10) that

$$\begin{aligned} & \int_{\mathbf{R}^{2N+1}} |\Delta_{\mathbf{H}}\zeta|^{p'}\zeta^{1-p'}|\eta|_{\mathbf{H}}^{\gamma(1-p')}\,d\eta \\ & \le C\left(R^{-4p'/\mu+4N/\mu+2/\kappa} + R^{4p'/\mu-4p'/\kappa+4N/\mu+2/\kappa}\right) \\ & \quad \times\left(R^{2\gamma(1-p')/\mu} + R^{\gamma(1-p')/\kappa}\right) \\ & \quad \times\int_\Omega\left(\left|\frac{d^2\varphi}{dR^2}\circ\tilde\rho\right|^{p'} + \left|\frac{d^2\varphi}{dR^2}\circ\tilde\rho\right|^{p'}\right)(\varphi\circ\tilde\rho)^{1-p'}d\tilde\eta, \end{aligned} \tag{2.11}$$

where C depends on μ and κ. Finally (2.6) reads as

$$\begin{aligned} \int_{\mathbf{R}^{2N+1}} |\eta|_{\mathbf{H}}^\gamma |u|^p\,\zeta\,d\eta \le C & \left(R^{-4p'/\mu+4N/\mu+2/\kappa} + R^{4p'/\mu-4p'/\kappa+4N/\mu+2/\kappa}\right) \\ & \times\left(R^{2\gamma(1-p')/\mu} + R^{\gamma(1-p')/\kappa}\right). \end{aligned} \tag{2.12}$$

Introducing the different exponents

$$\begin{cases} \alpha_1 = -4p'/\mu + 4N/\mu + 2/\kappa + 2\gamma(1-p')/\mu \\ \alpha_2 = -4p'/\mu + 4N/\mu + 2/\kappa + \gamma(1-p')/\kappa \\ \alpha_3 = 4p'/\mu - 4p'/\kappa + 4N/\mu + 2/\kappa + 2\gamma(1-p')/\mu \\ \alpha_4 = 4p'/\mu - 4p'/\kappa + 4N/\mu + 2/\kappa + \gamma(1-p')/\kappa \end{cases} \tag{2.13}$$

the problem is reduced to optimise the parameters μ and κ in order to have $\alpha_i \leq 0$ for $i = 1, \ldots, 4$. Setting $\theta = \mu/\kappa > 0$ we are left to finding the conditions under which there holds

$$\begin{cases} 4N \leq 4p' - 2\theta + 2\gamma(p'-1) \\ 4N \leq 4p' - 2\theta + \gamma(p'-1)\theta \\ 4N \leq -4p' + 4p'\theta - 2\theta + 2\gamma(p'-1) \\ 4N \leq -4p' + 4p'\theta - 2\theta + \gamma(p'-1)\theta. \end{cases} \tag{2.14}$$

If we call $\theta \longmapsto A_i(\theta)$ ($i = 1, \ldots, 4$) the linear functions appearing in the right-hand side inequalities (2.14), we see that those linear functions take the same value $A^* = 2(2+\gamma)(p'-1)$ for the value $\theta = 2$ (which is the optimal choice). Therefore (2.14) will be satisfied if and only if

$$2N \leq 2(2+\gamma)(p'-1) \Leftrightarrow p \leq (2N+2+\gamma)/(2N) = (Q+\gamma)/(Q-2). \tag{2.15}$$

In the case where $1 < p < (Q+\gamma)/(Q-2)$, we take $\theta > 0$ such as all the inequalities (2.14) are strict. Therefore all the exponents α_i are negative for $i = 1, \ldots, 4$. Since $\lim_{R\to\infty} \zeta(\eta) = \lim_{R\to\infty} \zeta_R(\eta) = 1$, we let R go to infinity in (2.12) and deduce that

$$\int_{\mathbf{R}^{2N+1}} |\eta|_{\mathbf{H}}^{\gamma} |u|^p \, d\eta = 0. \tag{2.16}$$

In the case where $p = (Q+\gamma)/(Q-2)$, we take $\mu = 2$, $\kappa = 1$ and first deduce from (2.12) that

$$\int_{\mathbf{R}^{2N+1}} |\eta|_{\mathbf{H}}^{\gamma} |u|^p \, d\eta < \infty. \tag{2.17}$$

We set $\Omega_R^* = \{(\tilde{x}, \tilde{y}, \tilde{\tau}) \in \mathbf{R}^N \times \mathbf{R}^N \times \mathbf{R}_+ : \; R^2 \leq \tau + |x|^2 + |y|^2 \leq 2R^2\}$. Since $\varphi(r)$ is constant for $r \in [0, \infty) \setminus (1, 2)$, we have

$$\begin{aligned} \left| \int_{\mathbf{R}^{2N+1}} au \Delta_{\mathbf{H}} \zeta \, d\eta \right| &= \left| \int_{\Omega_R^*} au \Delta_{\mathbf{H}} \zeta \, d\eta \right| \\ &\leq \|a\|_{L^\infty} \left(\int_{\Omega_R^*} |\Delta_{\mathbf{H}} \zeta|^{p'} \zeta^{1-p'} |\eta|_{\mathbf{H}}^{\gamma(1-p')} \, d\eta \right)^{1/p'} \\ &\quad \times \left(\int_{\Omega_R^*} |\eta|_{\mathbf{H}}^{\gamma} |u|^p \zeta \, d\eta \right)^{1/p}, \end{aligned} \tag{2.18}$$

and we derive from (2.3), (2.18) that

$$\int_{\mathbf{R}^{2N+1}} |\eta|_{\mathbf{H}}^{\gamma} |u|^p \zeta d\eta \leq \|a\|_{L^\infty} \left(\int_{\mathbf{R}^{2N+1}} |\Delta_{\mathbf{H}} \zeta|^{p'} \zeta^{1-p'} |\eta|_{\mathbf{H}}^{\gamma(1-p')} d\eta \right)^{1/p'}$$
$$\times \left(\int_{\Omega_R^*} |\eta|_{\mathbf{H}}^{\gamma} |u|^p \zeta d\eta \right)^{1/p}$$
$$\leq C \left(\int_{\Omega_R^*} |\eta|_{\mathbf{H}}^{\gamma} |u|^p d\eta \right)^{1/p}. \tag{2.19}$$

(we use here (2.12) with $\alpha_i = 0$ for $i = 1, \ldots, 4$). It follows from the integrability of $|\eta|_{\mathbf{H}}^{\gamma} |u|^p$ in $\mathbf{R}^{2N+1}$ that

$$\lim_{R \to \infty} \int_{\Omega_R^*} |\eta|_{\mathbf{H}}^{\gamma} |u|^p d\eta = 0. \tag{2.20}$$

Therefore the right-hand side of (2.18) goes to 0 when $R \to \infty$ and again (2.2-2.16) follows. □

Remark 2.2. It is known [2] that whenever $p > (Q + \gamma)/(Q - 2)$, there exist positive solutions of (2.1) in $\mathbf{R}^{2N+1}$.

Definition 2. *Let a_1 and a_2 be two bounded measurable functions in $\mathbf{R}^{2N+1}$. A weak solution (u, v) of the system of differential inequalities*

$$\begin{cases} \Delta_{\mathbf{H}}(a_1 u) + |\eta|_{\mathbf{H}}^{\gamma_1} |v|^{p_1} \leq 0 \\ \Delta_{\mathbf{H}}(a_2 v) + |\eta|_{\mathbf{H}}^{\gamma_2} |u|^{p_2} \leq 0 \end{cases} \tag{2.21}$$

in $\mathbf{R}^{2N+1}$ is a couple of locally integrable functions (u, v) such that

$$v \in L^{p_1}_{\text{loc}}(\mathbf{R}^{2N+1}, |\eta|_{\mathbf{H}}^{\gamma_1} d\eta) \text{ and}$$
$$u \in L^{p_2}_{\text{loc}}(\mathbf{R}^{2N+1}, |\eta|_{\mathbf{H}}^{\gamma_2} d\eta),$$

which satisfy

$$\int_{\mathbf{R}^{2N+1}} \left(a_1 u \Delta_{\mathbf{H}} \zeta + |\eta|_{\mathbf{H}}^{\gamma_1} |v|^{p_1} \zeta \right) d\eta \leq 0, \tag{2.22}$$

$$\int_{\mathbf{R}^{2N+1}} \left(a_2 v \Delta_{\mathbf{H}} \zeta + |\eta|_{\mathbf{H}}^{\gamma_2} |u|^{p_2} \zeta \right) d\eta \leq 0, \tag{2.23}$$

for any $\zeta \in C_c^2\left(\mathbf{R}^{2N+1}\right)$, $\zeta \geq 0$.

Theorem 2.3. *Assume*

$$\gamma_j > -2 \quad \text{and} \quad 1 < p_j \leq (Q + \gamma_j)/(Q - 2) \quad \text{for } j = 1, 2,$$

then there exists no solution (u, v) of inequality (2.21) defined in $\mathbf{R}^{2N+1}$.

Proof. Following the scheme of proof of Theorem 2.1, we take a nonnegative test function ζ satisfying

$$\begin{aligned}&\int_{\mathbf{R}^{2N+1}} |\Delta_{\mathbf{H}}\zeta|^{p_1'}\,\zeta^{1-p_1'}\,|\eta|_{\mathbf{H}}^{\gamma_1(1-p_1')}\,d\eta\\ &+\int_{\mathbf{R}^{2N+1}} |\Delta_{\mathbf{H}}\zeta|^{p_2'}\,\zeta^{1-p_2'}\,|\eta|_{\mathbf{H}}^{\gamma_2(1-p_2')}\,d\eta<\infty,\end{aligned}\tag{2.24}$$

and get

$$\begin{aligned}\int_{\mathbf{R}^{2N+1}} |\eta|_{\mathbf{H}}^{\gamma_1}\,|v|^{p_1}\,\zeta\,d\eta &\le \left|\int_{\mathbf{R}^{2N+1}} a_1 u\Delta_{\mathbf{H}}\zeta\,d\eta\right|,\\ &\le \|a_1\|_{L^\infty}\left(\int_{\mathbf{R}^{2N+1}} |\Delta_{\mathbf{H}}\zeta|^{p_2'}\,\zeta^{1-p_2'}\,|\eta|_{\mathbf{H}}^{\gamma_2(1-p_2')}\,d\eta\right)^{1/p_2'}\\ &\quad\times\left(\int_{\mathbf{R}^{2N+1}} |\eta|_{\mathbf{H}}^{\gamma_2}\,|u|^{p_2}\,\zeta\,d\eta\right)^{1/p_2}\\ &\le \frac{1}{2}\int_{\mathbf{R}^{2N+1}} |\eta|_{\mathbf{H}}^{\gamma_2}\,|u|^{p_2}\,\zeta\,d\eta\\ &\quad+C\int_{\mathbf{R}^{2N+1}} |\Delta_{\mathbf{H}}\zeta|^{p_2'}\,\zeta^{1-p_2'}\,|\eta|_{\mathbf{H}}^{\gamma_2(1-p_2')}\,d\eta\,,\end{aligned}\tag{2.25}$$

and

$$\begin{aligned}\int_{\mathbf{R}^{2N+1}} |\eta|_{\mathbf{H}}^{\gamma_2}\,|u|^{p_2}\,\zeta\,d\eta &\le \left|\int_{\mathbf{R}^{2N+1}} a_2 v\Delta_{\mathbf{H}}\zeta\,d\eta\right|\\ &\le \|a_2\|_{L^\infty}\left(\int_{R^{2N+1}} |\Delta_{\mathbf{H}}\zeta|^{p_1'}\,\zeta^{1-p_1'}\,|\eta|_{\mathbf{H}}^{\gamma_1(1-p_1')}\,d\eta\right)^{1/p_1'}\\ &\quad\times\left(\int_{\mathbf{R}^{2N+1}} |\eta|_{\mathbf{H}}^{\gamma_1}\,|u|^{p_1}\,\zeta\,d\eta\right)^{1/p_1}\\ &\le \frac{1}{2}\int_{\mathbf{R}^{2N+1}} |\eta|_{\mathbf{H}}^{\gamma_1}\,|u|^{p_1}\,\zeta\,d\eta\\ &\quad+C\int_{\mathbf{R}^{2N+1}} |\Delta_{\mathbf{H}}\zeta|^{p_1'}\,\zeta^{1-p_1'}\,|\eta|_{\mathbf{H}}^{\gamma_1(1-p_1')}\,d\eta,\end{aligned}\tag{2.26}$$

Therefore

$$\begin{aligned}&\int_{\mathbf{R}^{2N+1}} |\eta|_{\mathbf{H}}^{\gamma_1}\,|v|^{p_1}\,\zeta\,d\eta+\int_{\mathbf{R}^{2N+1}} |\eta|_{\mathbf{H}}^{\gamma_2}\,|u|^{p_2}\,\zeta\,d\eta\\ &\le 2C\int_{\mathbf{R}^{2N+1}} |\Delta_{\mathbf{H}}\zeta|^{p_1'}\,\zeta^{1-p_1'}\,|\eta|_{\mathbf{H}}^{\gamma_1(1-p_1')}\,d\eta\\ &\quad+2C\int_{\mathbf{R}^{2N+1}} |\Delta_{\mathbf{H}}\zeta|^{p_2'}\,\zeta^{1-p_2'}\,|\eta|_{\mathbf{H}}^{\gamma_2(1-p_2')}\,d\eta,\end{aligned}\tag{2.27}$$

We take now

$$\zeta(\eta)=\zeta(x,y,\tau)=\varphi\left(\frac{\tau^\kappa+|x|^\mu+|y|^\mu}{R^2}\right),$$

then perform the change of variables (2.9) and finally derive

$$\begin{aligned}
&\int_{\mathbf{R}^{2N+1}} |\eta|_{\mathbf{H}}^{\gamma_1} |v|^{p_1} \zeta\, d\eta + \int_{\mathbf{R}^{2N+1}} |\eta|_{\mathbf{H}}^{\gamma_2} |u|^{p_2} \zeta\, d\eta \\
\le{}& C \left(R^{-4p_1'/\mu + 4N/\mu + 2/\kappa} + R^{4p_1'/\mu - 4p_1'/\kappa + 4N/\mu + 2/\kappa} \right) \\
&\cdot \left(R^{2\gamma_1(1-p_1')/\mu} + R^{\gamma_1(1-p_1')/\kappa} \right) \\
&+ C \left(R^{-4p_2'/\mu + 4N/\mu + 2/\kappa} + R^{4p_2'/\mu - 4p_2'/\kappa + 4N/\mu + 2/\kappa} \right) \\
&\cdot \left(R^{2\gamma_2(1-p_2')/\mu} + R^{\gamma_2(1-p_2')/\kappa} \right) \\
\le{}& C \left(\sum_1^4 (R^{\alpha_i^1} + R^{\alpha_i^2}) \right),
\end{aligned} \tag{2.28}$$

where

$$\begin{cases}
\alpha_1^j = -4p_j'/\mu + 4N/\mu + 2/\kappa + 2\gamma_j(1-p_j')/\mu \\
\alpha_2^j = -4p_j'/\mu + 4N/\mu + 2/\kappa + \gamma_j(1-p_j')/\kappa \\
\alpha_3^j = 4p_j'/\mu - 4p_j'/\kappa + 4N/\mu + 2/\kappa + 2\gamma_j(1-p_j')/\mu \\
\alpha_4^j = 4p_j'/\mu - 4p_j'/\kappa + 4N/\mu + 2/\kappa + \gamma_j(1-p_j')/\kappa
\end{cases} \tag{2.29}$$

for $j = 1, 2$. Optimizing the α_k^j in order to have them nonpositive and setting $\theta = \mu/\kappa$ transforms this series of inequalities into

$$\begin{cases}
4N \le 4p_j' - 2\theta + 2\gamma_j(p_j' - 1) \\
4N \le 4p_j' - 2\theta + \gamma_j(p_j' - 1)\theta \\
4N \le -4p_j' + 4p_j'\theta - 2\theta + 2\gamma_j(p_j' - 1) \\
4N \le -4p_j' + 4p_j'\theta - 2\theta + \gamma_j(p_j' - 1)\theta
\end{cases} \tag{2.30}$$

For $\theta = 2$ those inequalities reduce to

$$2N \le 2(2 + \gamma_j)(p_j' - 1) \quad (j = 1, 2), \tag{2.31}$$

which gives $p_j \le (Q + \gamma_j)/(Q - 2)$. The end of the proof is as in Theorem 2.1. □

3. Parabolic and hyperbolic inequalities

We first consider parabolic type inequalities of the following type

$$\partial_t u \ge \Delta_{\mathbf{H}}(au) + |\eta|_{\mathbf{H}}^{\gamma} |u|^p \tag{3.1}$$

in $\mathbf{R}^{2N+1} \times \mathbf{R}_+ = \mathbf{R}_+^{2N+1,1}$ where $a = a(\eta, t)$ is a bounded and measurable function.

Definition 3. A weak solution u of the differential inequality (3.1) in $\mathbf{R}_+^{2N+1,1}$ with initial data $u_0 \in L^1_{\mathrm{loc}}(\mathbf{R}^{2N+1})$ is a locally integrable function such that $u \in L^p_{\mathrm{loc}}(\mathbf{R}_+^{2N+1,1}, |\eta|^\gamma_{\mathbf{H}}\, d\eta dt)$ which satisfies

$$\int_0^\infty \int_{\mathbf{R}^{2N+1}} \left(u\partial_t\zeta + au\Delta_{\mathbf{H}}\zeta + |\eta|^\gamma_{\mathbf{H}}\, |u|^p\, \zeta\right) d\eta dt + \int_{\mathbf{R}^{2N+1}} u_0(\eta)\zeta(\eta,0)d\eta \leq 0, \tag{3.2}$$

for any $\zeta \in C^2_c\left(\mathbf{R}_+^{2N+1,1}\right)$, $\zeta \geq 0$.

Theorem 3.1. *Assume*

$$\gamma > -2\,, \quad 1 < p \leq (Q+2+\gamma)/Q \quad \textit{and} \quad \int_{\mathbf{R}^{2N+1}} u_0(\eta)d\eta \geq 0.$$

Then there exists no solution u of inequality (3.1) defined in $\mathbf{R}_+^{2N+1,1}$.

Proof. If u be such a such solution and ζ is a smooth nonnegative test function, we get

$$\int_{\mathbf{R}^{2N+1}} |\eta|^\gamma_{\mathbf{H}}\, |u|^p\, \zeta\, d\eta \leq -\int_{\mathbf{R}^{2N+1}} au\Delta_{\mathbf{H}}\zeta\, d\eta. \tag{3.3}$$

We choose ζ such that

$$\int_0^\infty \int_{\mathbf{R}^{2N+1}} \left(|\partial_t\,\zeta|^{p'} + |\Delta_{\mathbf{H}}\zeta|^{p'}\right) \zeta^{1-p'}\, |\eta|^{\gamma(1-p')}_{\mathbf{H}}\, d\eta dt < \infty. \tag{3.4}$$

Then it follows from (3.2) that

$$\begin{aligned}
&\int_0^\infty \int_{\mathbf{R}^{2N+1}} |\eta|^\gamma_{\mathbf{H}}\, |u|^p\, \zeta\, d\eta dt + \int_{\mathbf{R}^{2N+1}} u_0(\eta)\zeta(\eta,0)d\eta \\
&\leq -\int_0^\infty \int_{\mathbf{R}^{2N+1}} (u\partial_t\zeta + au\Delta_{\mathbf{H}}\zeta)\, d\eta dt \\
&\leq \left(\int_0^\infty \int_{\mathbf{R}^{2N+1}} \left(|\partial_t\,\zeta|^{p'} + \|a\|_{L^\infty}\, |\Delta_{\mathbf{H}}\zeta|^{p'}\right) \zeta^{1-p'}\, |\eta|^{\gamma(1-p')}_{\mathbf{H}}\, d\eta dt\right)^{1/p} \\
&\quad \times \left(\int_0^\infty \int_{\mathbf{R}^{2N+1}} |\eta|^\gamma_{\mathbf{H}}\, |u|^p\, \zeta\, d\eta dt\right)^{1/p} \\
&\leq \frac{1}{2}\int_0^\infty \int_{\mathbf{R}^{2N+1}} |\eta|^\gamma_{\mathbf{H}}\, |u|^p\, \zeta\, d\eta dt \\
&\quad + C\int_0^\infty \int_{\mathbf{R}^{2N+1}} \left(|\partial_t\,\zeta|^{p'} + \|a\|_{L^\infty}\, |\Delta_{\mathbf{H}}\zeta|^{p'}\right) \zeta^{1-p'}\, |\eta|^{\gamma(1-p')}_{\mathbf{H}}\, d\eta dt,
\end{aligned} \tag{3.5}$$

Therefore

$$\begin{aligned}
&\int_0^\infty \int_{\mathbf{R}^{2N+1}} |\eta|^\gamma_{\mathbf{H}}\, |u|^p\, \zeta\, d\eta dt + 2\int_{\mathbf{R}^{2N+1}} u_0(\eta)\zeta(\eta,0)d\eta \\
&\leq 2C\int_0^\infty \int_{\mathbf{R}^{2N+1}} \left(|\partial_t\,\zeta|^{p'} + \|a\|_{L^\infty}\, |\Delta_{\mathbf{H}}\zeta|^{p'}\right) \zeta^{1-p'}\, |\eta|^{\gamma(1-p')}_{\mathbf{H}}\, d\eta dt.
\end{aligned} \tag{3.6}$$

Taking

$$\zeta(\eta, t) = \zeta(x, y, \tau, t) = \varphi\left(\frac{\tau^\kappa + |x|^\mu + |y|^\mu + t^\sigma}{R^2}\right)$$

where $\kappa > 0$, $\mu > 0$ and $\sigma > 0$ have to be optimised, and using the fact that

$$\begin{aligned}&\int_0^\infty \int_{\mathbf{R}^{2N+1}} |\partial_t \zeta|^{p'} \zeta^{1-p'} |\eta|_{\mathbf{H}}^{\gamma(1-p')} d\eta dt \\ &\leq C R^{-2p'/\sigma+2/\kappa+2/\sigma+4N/\mu} \left(R^{2\gamma(1-p')/\mu} + R^{\gamma(1-p')/\kappa}\right)\end{aligned} \tag{3.7}$$

we get (with the same notations as in Section 2)

$$\begin{aligned}&\int_0^\infty \int_{\mathbf{R}^{2N+1}} |\eta|_{\mathbf{H}}^{\gamma} |u|^p \zeta d\eta dt + 2\int_{\mathbf{R}^{2N+1}} u_0(\eta)\zeta(\eta, 0) d\eta \\ &\leq C R^{4N/\mu+2/\kappa+2/\sigma} \\ &\quad\times \left(R^{-4p'/\mu} + R^{4p'/\mu-4p'/\kappa} + R^{-2p'/\sigma}\right)\left(R^{2\gamma(1-p')/\mu} + R^{\gamma(1-p')/\kappa}\right) \\ &\quad\times \int_\Omega \left(\left|\frac{d\varphi}{dR}\circ\tilde{\rho}\right|^{p'} + \left|\frac{d^2\varphi}{dR^2}\circ\tilde{\rho}\right|^{p'}\right)(\varphi\circ\tilde{\rho})^{1-p'} d\tilde{\eta}.\end{aligned} \tag{3.8}$$

Setting $\theta = \mu/\kappa > 0$ and $\omega = \mu/\sigma > 0$, we are left to finding the conditions under which there holds

$$\begin{cases} 4N \leq 4p' - 2\theta - 2\omega + 2\gamma(p'-1) \\ 4N \leq 4p' - 2\theta - 2\omega + \gamma(p'-1)\theta \\ 4N \leq -4p' + 4p'\theta - 2\theta - 2\omega + 2\gamma(p'-1) \\ 4N \leq -4p' + 4p'\theta - 2\theta - 2\omega + \gamma(p'-1)\theta \\ 4N \leq 2p'\omega - 2\theta - 2\omega + 2\gamma(p'-1) \\ 4N \leq 2p'\omega - 2\theta - 2\omega + \gamma(p'-1)\theta \end{cases} \tag{3.9}$$

For $\theta = 2 = \omega$ those inequalities reduce to

$$4N \leq 4p' - 8 + 2\gamma(p'-1) \Leftrightarrow p \leq (Q+\gamma+2)/Q. \tag{3.10}$$

Since the functions in the right-hand side of (3.9) are linear functions, this is the optimal choice for θ and ω in order to have (3.9). When $1 < p < (Q+\gamma+2)/Q$, all the exponents of R in (3.8) are negative. Letting R go to infinity and using Fatou lemma yields

$$\begin{aligned}&\int_0^\infty \int_{\mathbf{R}^{2N+1}} |\eta|_{\mathbf{H}}^{\gamma} |u|^p d\eta dt + 2\int_{\mathbf{R}^{2N+1}} u_0(\eta) d\eta \\ &\leq \liminf_{R\to\infty}\left(\int_0^\infty \int_{\mathbf{R}^{2N+1}} |\eta|_{\mathbf{H}}^{\gamma} |u|^p \zeta d\eta dt + 2\int_{\mathbf{R}^{2N+1}} u_0(\eta)\zeta(\eta, 0) d\eta\right)\end{aligned} \tag{3.11}$$

which is zero from (3.8) and a nonzero u cannot exists since

$$\int_{\mathbf{R}^{2N+1}} u_0(\eta) d\eta \geq 0.$$

When $p = (Q + \gamma + 2)/Q$, we deduce from (3.8) that

$$\int_0^\infty \int_{\mathbf{R}^{2N+1}} |\eta|_{\mathbf{H}}^\gamma |u|^p \, d\eta dt < \infty.$$

Using the multiplicative form of (3.5), and the fact that $\varphi \equiv 1$ for $0 \le R \le 1$ implies the result as in the proof of Theorem 2.1. □

Remark 3.2. The integrability condition on u_0 can be weakened and replaced by

$$\limsup_{R\to\infty} \int_{\mathbf{R}^{2N+1}} u_0(\eta)\varphi\left((\tau + |x|^2 + |y|^2)/R^2\right) dxdyd\tau \ge 0.$$

We now consider hyperbolic type inequalities of the following type

$$\partial_{tt} u \ge \Delta_{\mathbf{H}}(au) + |\eta|_{\mathbf{H}}^\gamma |u|^p \tag{3.12}$$

in $\mathbf{R}^{2N+1} \times \mathbf{R}_+ = \mathbf{R}_+^{2N+1,1}$ where $a = a(\eta, t)$ is a bounded and measurable function.

Definition 4. A weak solution u of the differential inequality (3.1) in $\mathbf{R}_+^{2N+1,1}$ with initial data $u_0, u_1 \in L^1_{\text{loc}}(\mathbf{R}^{2N+1})$ and is a locally integrable function such that $u \in L^p_{\text{loc}}(\mathbf{R}_+^{2N+1,1}, |\eta|_{\mathbf{H}}^\gamma \, d\eta dt)$ which satisfies

$$\begin{aligned}
&\int_0^\infty \int_{\mathbf{R}^{2N+1}} \left(-u\partial_{tt}\zeta + au\Delta_{\mathbf{H}}\zeta + |\eta|_{\mathbf{H}}^\gamma |u|^p \, \zeta\right) d\eta dt \\
&- \int_{\mathbf{R}^{2N+1}} u_0(\eta)\partial_t\zeta(\eta, 0)d\eta + \int_{\mathbf{R}^{2N+1}} u_1(\eta)\zeta(\eta, 0)d\eta \le 0,
\end{aligned} \tag{3.13}$$

for any $\zeta \in C_c^2\left(\mathbf{R}_+^{2N+1,1}\right)$, $\zeta \ge 0$.

Theorem 3.3. *Assume*

$$\gamma > -2\,, \quad 1 < p \le (Q + 1 + \gamma)/(Q - 1) \quad \textit{and} \quad \int_{\mathbf{R}^{2N+1}} u_1(\eta)d\eta \ge 0,$$

then there exists no solution u of inequality (3.12) defined in $\mathbf{R}_+^{2N+1,1}$.

Proof. The proof follows the scheme of the one of Theorem 3.1, therefore we shall only indicate the modifications. If u is such a solution and ζ a smooth nonnegative test function such that

$$\int_0^\infty \int_{\mathbf{R}^{2N+1}} \left(|\partial_{tt}\,\zeta|^{p'} + |\Delta_{\mathbf{H}}\zeta|^{p'}\right) \zeta^{1-p'} \, |\eta|_{\mathbf{H}}^{\gamma(1-p')} \, d\eta dt < \infty, \tag{3.14}$$

we obtain

$$\begin{aligned}
&\int_0^\infty \int_{\mathbf{R}^{2N+1}} |\eta|_{\mathbf{H}}^\gamma |u|^p \zeta \, d\eta dt + \int_{\mathbf{R}^{2N+1}} u_1(\eta)\zeta(\eta,0) d\eta \\
&- \int_{\mathbf{R}^{2N+1}} u_0(\eta)\partial_t \zeta(\eta,0) d\eta \\
\le& \int_0^\infty \int_{\mathbf{R}^{2N+1}} (u\partial_{tt}\zeta - au\Delta_{\mathbf{H}}\zeta)\, d\eta dt \\
\le& \left(\int_0^\infty \int_{\mathbf{R}^{2N+1}} \left(|\partial_{tt}\zeta|^{p'} + \|a\|_{L^\infty} |\Delta_{\mathbf{H}}\zeta|^{p'}\right) \zeta^{1-p'} |\eta|_{\mathbf{H}}^{\gamma(1-p')} d\eta dt\right)^{1/p} \\
&\times \left(\int_0^\infty \int_{\mathbf{R}^{2N+1}} |\eta|_{\mathbf{H}}^\gamma |u|^p \zeta \, d\eta dt\right)^{1/p} \\
\le& \frac{1}{2}\int_0^\infty \int_{\mathbf{R}^{2N+1}} |\eta|_{\mathbf{H}}^\gamma |u|^p \zeta \, d\eta dt \\
+& C\int_0^\infty \int_{\mathbf{R}^{2N+1}} \left(|\partial_{tt}\zeta|^{p'} + \|a\|_{L^\infty} |\Delta_{\mathbf{H}}\zeta|^{p'}\right) \zeta^{1-p'} |\eta|_{\mathbf{H}}^{\gamma(1-p')} d\eta dt. \quad (3.15)
\end{aligned}$$

If we take the test function ζ such that

$$\int_{\mathbf{R}^{2N+1}} u_0(\eta)\partial_t(\eta,0) d\eta \le 0, \tag{3.16}$$

we derive from (3.15)–(3.16)

$$\begin{aligned}
&\int_0^\infty \int_{\mathbf{R}^{2N+1}} |\eta|_{\mathbf{H}}^\gamma |u|^p \zeta \, d\eta dt + 2\int_{\mathbf{R}^{2N+1}} u_1(\eta)\zeta(\eta,0) d\eta \le \\
&2C\int_0^\infty \int_{\mathbf{R}^{2N+1}} \left(|\partial_{tt}\zeta|^{p'} + \|a\|_{L^\infty} |\Delta_{\mathbf{H}}\zeta|^{p'}\right) \zeta^{1-p'} |\eta|_{\mathbf{H}}^{\gamma(1-p')} d\eta dt. \quad (3.17)
\end{aligned}$$

We take

$$\zeta(\eta,t) = \zeta(x,y,\tau,t) = \varphi\left(\frac{\tau^\kappa + |x|^\mu + |y|^\mu + t^\sigma}{R^2}\right)$$

where $\kappa > 0$, $\mu > 0$ and $\sigma > 1$. Notice that the choice $\sigma > 1$ insures (3.16). Moreover

$$\begin{aligned}
&\int_0^\infty \int_{\mathbf{R}^{2N+1}} |\partial_{tt}\zeta|^{p'} \zeta^{1-p'} |\eta|_{\mathbf{H}}^{\gamma(1-p')} d\eta dt \\
&\le CR^{-4p'/\sigma + 2/\kappa + 2/\sigma + 4N/\mu}\left(R^{2\gamma(1-p')/\mu} + R^{\gamma(1-p')/\kappa}\right). \quad (3.18)
\end{aligned}$$

Then (3.8) is replaced by

$$\begin{aligned}
&\int_0^\infty \int_{\mathbf{R}^{2N+1}} |\eta|_{\mathbf{H}}^\gamma |u|^p \zeta \, d\eta dt + 2\int_{\mathbf{R}^{2N+1}} u_1(\eta)\zeta(\eta,0) d\eta \\
\le& CR^{4N/\mu + 2/\kappa + 2/\sigma} \\
&\times \left(R^{-4p'/\mu} + R^{4p'/\mu - 4p'/\kappa} + R^{-4p'/\sigma}\right)\left(R^{2\gamma(1-p')/\mu} + R^{\gamma(1-p')/\kappa}\right) \\
&\times \int_\Omega \left(\left(\frac{d^2\varphi}{dR^2}\circ\rho\right)^{p'}\right)(\varphi\circ\rho)^{1-p'} d\tilde{\eta}, \quad (3.19)
\end{aligned}$$

and (3.9) by

$$\begin{cases} 4N \leq 4p' - 2\theta - 2\omega + 2\gamma(p'-1) \\ 4N \leq 4p' - 2\theta - 2\omega + \gamma(p'-1)\theta \\ 4N \leq -4p' + 4p'\theta - 2\theta - 2\omega + 2\gamma(p'-1) \\ 4N \leq -4p' + 4p'\theta - 2\theta - 2\omega + \gamma(p'-1)\theta \\ 4N \leq 4p'\omega - 2\theta - 2\omega + 2\gamma(p'-1) \\ 4N \leq 4p'\omega - 2\theta - 2\omega + \gamma(p'-1)\theta \end{cases} \tag{3.20}$$

with $\theta = \mu/\kappa > 0$ and $\omega = \mu/\sigma > 0$. If we choose $\theta = 2$ and $\omega = 1$, all the 4-linear functions on the right-hand side of (3.20) take the same value $4p' + 2\gamma(p'-1) - 6$, and the condition of optimality is therefore reduced to

$$4N \leq 4p' + 2\gamma(p'-1) - 6 \Leftrightarrow p \leq (Q+1+\gamma)/(Q-1) \tag{3.21}$$

which is the condition of the theorem. As for the condition $\sigma > 1$, it is assumed since μ/σ is involved for the value 1. Taking for example $\mu = \sigma = 2$ and $\kappa = 1$ yields $\theta = 2$ and $\omega = 1$. It follows from this choice that

$$\int_0^\infty \int_{\mathbf{R}^{2N+1}} |\eta|_{\mathbf{H}}^\gamma |u|^p \, d\eta dt + 2\int_{\mathbf{R}^{2N+1}} u_1(\eta) d\eta \leq$$
$$\liminf_{R\to\infty} \left(\int_0^\infty \int_{\mathbf{R}^{2N+1}} |\eta|_{\mathbf{H}}^\gamma |u|^p \, \zeta d\eta dt + 2\int_{\mathbf{R}^{2N+1}} u_1(\eta)\zeta(\eta,0) d\eta \right), \tag{3.22}$$

which is zero from (3.19) if $1 < p < (Q+1+\gamma)/(Q-1)$, or

$$\int_0^\infty \int_{\mathbf{R}^{2N+1}} |\eta|_{\mathbf{H}}^\gamma |u|^p \, d\eta dt + 2\int_{\mathbf{R}^{2N+1}} u_1(\eta) d\eta < \infty, \tag{3.23}$$

when $p = (Q+1+\gamma)/(Q-1)$, and therefore

$$\int_0^\infty \int_{\mathbf{R}^{2N+1}} |\eta|_{\mathbf{H}}^\gamma |u|^p \, d\eta dt + 2\int_{\mathbf{R}^{2N+1}} u_1(\eta) d\eta = 0,$$

as above. □

Remark 3.4. As in remark 3.2 we can replace the condition on u_1 by the weaker one

$$\limsup_{R\to\infty} \int_{R^{2N+1}} u_1(\eta)\varphi\left((\tau + |x|^2 + |y|^2)/R^2\right) dx dy d\tau \geq 0.$$

Remark 3.5. The analysis of the ODE's (3.24)

$$\frac{du}{dt} = u^p \quad \text{and} \quad \frac{d^2u}{dt^2} = u^p \quad \text{on } \mathbf{R}_+, \tag{3.24}$$

shows that the condition on the sign of the initial data is natural.

Acknowledgements. This work was prepared while the first author was visiting the Université François Rabelais in Tours on a position of invited professor. The authors are grateful to the Université François Rabelais and the Laboratoire de Mathématiques et Physique Théorique for warm hospitality and for giving the opportunity to prepare this article in good conditions.

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