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On manifolds whose tangent bundle contains an ample subbundle

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Introduction

Let X be a complex projective manifold of dimension n and let E be a vector bundle of rank r, or equivalently a locally free $\mathcal{O}_X$-sheaf of rank r. The bundle E is called ample if the relative hyperplane line bundle $O_{\mathbf{P}(E)}(1)$ over its projectivisation $\mathbf{P}(E) = Proj_X(Sym(E))$ is ample. We will assume that E is a subsheaf of the tangent sheaf TX, that is there exists an injective morphism $E \hookrightarrow TX$. In this paper we will prove the following:

Theorem. *If E is an ample locally free subsheaf of TX then $X \cong \mathbf{P}^n$ and $E \cong \mathcal{O}(1)^{\oplus r}$ or $E \cong T\mathbf{P}^n$.*

The characterization of $\mathbf{P}^n$ as the only manifold whose tangent bundle is ample was conjectured by R. Hartshorne. The Hartshorne conjecture was proved by S. Mori in a celebrated paper [Mo], which contained an amazing proof of the existence of rational curves on Fano manifolds. Building up on Mori's work a version of the present theorem was successively proved for $r = 1$ and $r = n, \quad n-1, \quad n-2$ by J. Wahl [Wa] and, respectively, by F. Campana and T. Peternell [C-P]. Moreover Campana and Peternell posed a question with the above characterization of $\mathbf{P}^n$ which generalizes previous results.

The proof of the main theorem will apply rational curves on X. Our notation is consistent with the book of J. Kollár ([Ko]). In particular $\mathrm{Hom}(\mathbf{P}^1, X)$ denotes the scheme parameterizing morphisms from $\mathbf{P}^1$ to X and $\mathrm{Hom}(\mathbf{P}^1, X; 0 \to x)$ the scheme parameterizing morphisms sending $0 \in \mathbf{P}^1$ to $x \in X$. By $F : \mathrm{Hom}(\mathbf{P}^1, X) \times \mathbf{P}^1 \to X$ we denote the evaluation morphism $F(f, p) = f(p)$. For any family of morphisms $V \subset \mathrm{Hom}(\mathbf{P}^1, X)$

by F_V we denote the restriction of F to V and by $\mathrm{Locus}(V)$ the closure of the image of F_V. We say that V is unsplit if the image of V in $\mathrm{Chow}(X)$, via the natural morphism $[f] \mapsto [f(\mathbf{P}^1)]$, is proper.

If G is a vector bundle over $\mathbf{P}^1$ with G^+ we denote its positive part, that is the sub-bundle $[\mathrm{im}(H^0(G(-1)) \otimes \mathcal{O} \to G(-1)] \otimes \mathcal{O}(1)$.

Prologue

Let X and E be as in the introduction: we therefore assume that E is ample and moreover of rank $r > 1$, as the case $r = 1$ is set by Wahl's result.

By a theorem of Y. Miyaoka (see [Mi] or [K-al], (9.0.2)) X is uniruled. So we can choose a closed irreducible component $V \subset \mathrm{Hom}(\mathbf{P}^1, X)$ which covers X (that is $\mathrm{Locus}(V) = X$) and which is a generically unsplit family (see [Ko], IV. 2.4). Moreover for a general $f \in V$ we have $f^*TX = \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus d} \oplus \mathcal{O}^{\oplus(n-d-1)}$ where $d = \deg(f^*(-K_X)) - 2$, see ([Ko], IV 2.8, 2.9 and 2.10).

Lemma (0.1). *For any $f \in V$ the pull-back f^*E is isomorphic either to $\mathcal{O}(1)^{\oplus r}$ or to $\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus(r-1)}$. In particular the family of curves parametrized by V is unsplit.*

Proof. For a general $f \in V$ the pull-back f^*E is an ample subbundle of $f^*TX = \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus(d)} \oplus \mathcal{O}^{\oplus(n-d-1)}$ and thus it is as in the lemma. Since E is ample this is true also for all $f \in V$. Since $\deg(f^*E) = r$ or $\deg(f^*E) = r + 1$ and $r > 1$, and for any ample bundle $\mathcal{E}$ over a rational curve we have $\deg(\mathcal{E}) \geq \mathrm{rank}(\mathcal{E})$, it follows that no curve from V can be split into a sum of two or more rational curves, hence V is unsplit.

The family V defines a relation of *rational connectedness with respect to* V, which we shall call rc$V$ relation for short, in the following way: $x_1,\ x_2 \in X$ are in the rc$V$ relation if there exists a chain of rational curves parametrized by morphisms from V which joins x_1 and x_2. The rc$V$ relation is an equivalence relation and its equivalence classes can be parametrized generically by an algebraic set. More precisely, we have the following result due to Campana [Ca] and, independently, to Kollár-Miyaoka-Mori [KMM2].

Theorem (0.2). *(see [Ko], IV.4.16). There exist an open subset $X^0 \subset X$ and a proper surjective morphism with connected fibers $\varphi^0 : X^0 \to Z^0$ onto a normal variety, such that the fibers of φ^0 are equivalence classes of the rcV relation.*

We shall call the morphism φ^0 an rc$V$ fibration. If Z_0 is just a point then we will call X a rationally connected manifold with the respect to the family V, in short an rc$V$ manifold. In what follows we shall analyze X using the notions of rc$V$ relation and rc$V$ fibration. The key is the following observation.

Lemma (0.3). *Let X, E and V be as above and moreover assume that $\varphi^0 : X^0 \to Z^0$ is an rcV fibration. Then E is tangent to a general fiber of φ^0. That is, if X_g is a general fiber of φ^0, then the injection $E_{|X_g} \to TX_{|X_g}$ factors through TX_g, that is we have injection $E_{|X_g} \hookrightarrow TX_g$.*

Proof. Choose a general X_g (in particular smooth) and let moreover $x \in X_g$ and $f \in V_x = V \cap \mathrm{Hom}(\mathbf{P}^1, X; 0 \to x)$ be general as well. Then $\mathrm{Locus}(V_x) \subset X_g$ and thus, by [Ko] II.3.4, $(f^*TX)_p^+ \subset (f^*TX_g)_p$ for every $p \in \mathbf{P}^1 \setminus \{0\}$. This implies that $E_{|X_g} \to TX_{|X_g}$ factors to $E_{|X_g} \to TX_g$ generically and since the map $TX_g \to TX_{|X_g}$ has cokernel which is torsion free (it is the normal sheaf which is locally free) this yields $E_{|X_g} \hookrightarrow TX_g$, a sheaf injection.

The argument for the proof of the main theorem will go as follows. First we assume that X is a rc$V$ manifold and for $f \in V$ the pullback f^*E is either $\mathcal{O}(1)^{\oplus r}$ or $\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus(r-1)}$ and we prove that $X \cong \mathbf{P}^n$ and $E \cong \mathcal{O}(1)^{\oplus r}$ or $E \cong T\mathbf{P}^n$, respectively. This is taken care of in the two subsequent sections: these two cases are reduced to Wahl's and Mori's theorems, respectively. In the last section we assume that the rc$V$ fibration is non-trivial: then we prove that it can be extended in codimension 1 so that we can produce a projective bundle $X_B \to B$ over a smooth curve B with an ample vector bundle $E_B \hookrightarrow T_{X_B/B}$. This is impossible, as observed by Campana and Peternell.

Trivial projective bundles over rc$V$ manifolds

In this section we assume that X is a smooth projective manifold which is rationally connected with respect to an unsplit family V of rational curves (rc$V$ manifold). We let E be a rank r vector bundle on X. We begin by observing some general facts.

Proposition (1.1). *Let X be a projective manifold and $V \subset Hom(\mathbf{P}^1, X)$ an unsplit family of rational curves whose locus is the whole X. Then X is a rcV manifold if and only if the Picard number $\rho(X)$ is 1.*

Proof. The "only if" part follows, for instance, from IV. 3.13.3 in [Ko] and the assumption on the unsplitteness of the family V. Note that this part is true even if X is not smooth. The "if" part (which will not be needed in the paper) is proved in [KMM1]. (For other similar results see also [K-S]).

Let us note that rationally connectedness and $\rho(X) = 1$ imply that X is Fano, so in fact $Pic X = \mathbf{Z}$. The following result concerns vector bundles of higher rank.

Proposition (1.2). *In the above notation suppose moreover that there exists an integer a such that for any $f \in V$ we have $f^*E = \mathcal{O}(a)^{\oplus r}$. Then there exists a (uniquely defined) line bundle L over X such that $\deg f^*L = a$ and $E \cong L^{\oplus r}$.*

Proof. The argument applies induction with respect to r. Let us consider the projectivisation $p : \mathbf{P}(E) \to X$ with the relative $\mathcal{O}(1)$ bundle which we will denote by $\mathcal{L}$. We note that for any $f \in V$ and $y \in p^{-1}(f(0))$ we have a unique lift-up $\hat{f} : \mathbf{P}^1 \to \mathbf{P}(E)$ such that $p \circ \hat{f} = f$ and $\deg(\hat{f}^*(\mathcal{L})) = a$, and $\hat{f}(0) = y$. That is, since $\mathbf{P}(f^*E) = \mathbf{P}^1 \times \mathbf{P}^{r-1}$, the morphism $\hat{f}$ is obtained by composing $\mathbf{P}(f^*E) \to \mathbf{P}(E)$ with the morphism $\mathbf{P}^1 \to \mathbf{P}^1 \times \{y\} \subset \mathbf{P}^1 \times \mathbf{P}^{r-1}$. Thus, for a generic f we have $\hat{f}^*T\mathbf{P}(E) = f^*TX \oplus \mathcal{O}^{\oplus(r-1)}$. We can choose an irreducible $\hat{V} \subset \mathrm{Hom}(\mathbf{P}^1, \mathbf{P}(E))$ which parameterizes these lift-ups, that is, via the natural morphism $p_* : \mathrm{Hom}(\mathbf{P}^1, \mathbf{P}(E)) \to \mathrm{Hom}(\mathbf{P}^1, X)$, defined by $p_*(\hat{f}) = p \circ \hat{f}$, the component $\hat{V}$ dominates V.

Claim (1.2.1). *The morphism $p_* : \hat{V} \to V$ is proper and thus surjective, moreover $\hat{V}$ is an unsplit family.*

Proof. We use valuative criterion of properness [Hartshorne, II.4.7]. Let Δ be a spectrum of a discrete valuation ring (or a germ of a smooth curve in the analytic context) with a closed point δ and the general point Δ^0. Then for any family of morphims $F_\Delta : \Delta \times \mathbf{P}^1 \to X$ coming from $\Delta \to V$ we have $\mathbf{P}(F_\Delta^*(E)) = \Delta \times \mathbf{P}^1 \times \mathbf{P}^{r-1}$. Take now $\hat{F}_{\Delta_0} : \Delta^0 \times \mathbf{P}^1 \to \mathbf{P}(E)$, coming from a lift-up $\Delta^0 \to \hat{V}$ of $\Delta \to V$. By the construction $\hat{F}_{\Delta_0}$ is the composition of $\mathbf{P}(F_\Delta^*(E)) \to \mathbf{P}(E)$ with the product $id \times \psi_0 : \Delta_0 \times \mathbf{P}^1 \to (\Delta_0 \times \mathbf{P}^1) \times \mathbf{P}^{r-1}$, for some $\psi_0 : \Delta_0 \to \mathbf{P}^{r-1}$. The morphism ψ_0 extends to $\psi : \Delta \to \mathbf{P}^{r-1}$, thus $\hat{F}_{\Delta_0}$ extends to $\hat{F}_\Delta$ which is the composition of $\mathbf{P}(F_\Delta^*(E)) \to \mathbf{P}(E)$ with the product $id \times \psi$, hence p_* is proper.

The proof of unsplitting of $\hat{V}$ is similar. Namely let W and $\hat{W}$ denote the image of V and $\hat{V}$ in $\mathrm{Chow}(X)$ and $\mathrm{Chow}(\mathbf{P}(E))$, respectively. Let $p_* : \hat{W} \to W$ denote, by abuse of notation, the push-forward map. Similarly as above we prove that p_* is proper. Therefore $\hat{W}$ is proper and $\hat{V}$ is unsplit.

To proceed with the proof of the proposition let us consider the rc$\hat{V}$ fibration of $\mathbf{P}(E)$. Let $Y \subset \mathbf{P}(E)$ be a general fiber of this fibration. It is projective and smooth, and by the Proposition (1.1) $\rho(Y) = 1$. By the surjectivity of $p_* : \hat{V} \to V$ and rational connectedness of X, the restriction map $p_Y : Y \to X$ is surjective and, since $\rho(Y) = 1$, it has no positive dimensional fiber, so it is a finite morphism. Moreover, the restriction of $\mathcal{L}$ to Y, call it $\mathcal{L}_Y$, has intersection equal a with any curve from $\tilde{V} = \hat{V} \cap \mathrm{Hom}(\mathbf{P}^1, Y)$.

Let us consider the pull-back $\tilde{p} : \mathbf{P}(p_Y^*E) \to Y$ with the induced morphism $\tilde{p}_Y : \mathbf{P}(p_Y^*E) \to \mathbf{P}(E)$ such that $p \circ \tilde{p}_Y = p_Y \circ \tilde{p}$. By the universal property of the fiber product the projective bundle $\tilde{p}$ admits a section $s : Y \to \mathbf{P}(p_Y^*E)$ such that $\tilde{p}_Y \circ s$ is the embedding of Y into $\mathbf{P}(E)$. This gives us a sequence of bundles over Y:

$$0 \longrightarrow E' \longrightarrow p_Y^*(E) \longrightarrow \mathcal{L}_Y \longrightarrow 0$$

where E' is a bundle of rank $r-1$ and over Y it satisfies the required assumptions with respect to the family $\tilde{V}$. Thus, by the inductive assumption,

$E' \cong \mathcal{L}_Y^{\oplus(r-1)}$. But because Y is Fano, $H^1(Y, \mathcal{O}_Y) = 0$ and thus the above sequence of vector bundles splits; therefore $p_Y^*(E) \cong \mathcal{L}_Y^{\oplus r}$.

Now we shall be done by the following.

Lemma (1.2.2). *Let X be a Fano manifold with $p : \mathbf{P}(E) \to X$ a projectivisation of a rank r bundle. Suppose that $\Psi : Y \to X$ is a finite morphism. If $\mathbf{P}(\Psi^*(E)) \cong Y \times \mathbf{P}^{r-1}$ then $\mathbf{P}(E) \cong X \times \mathbf{P}^{r-1}$.*

Proof. By $\mathcal{L}$ let us denote the relative $\mathcal{O}(1)$ over $\mathbf{P}(E)$. We claim that $r\mathcal{L} - p^*\det E$ is nef and $(r\mathcal{L} - p^*\det E)^r = 0$ over $\mathbf{P}(E)$. This follows because the pull-back of $r\mathcal{L} - p^*\det E$ to $\mathbf{P}(\Psi^*(E))$ has these features. By the same reason $r\mathcal{L} - p^*(\det E + K_X) = -K_{\mathbf{P}(E)}$ is ample and therefore $\mathbf{P}(E)$ is a Fano manifold and by Kawamata-Shokurov base-point-freeness $r\mathcal{L} - p^*\det E$ defines a contraction, $\varphi : \mathbf{P}(E) \to Z$, onto a normal projective variety of dimension $r - 1$. Any fiber of φ is mapped, via p, surjectively onto X, with no positive dimensional fiber. Let T be a general fiber of φ. Then, T is smooth and by adjunction we find out that

$$K_T = (K_{\mathbf{P}(E)})_{|T} = (p^*K_X + (p^*(\det E) - r\mathcal{L}))_{|T} = (p^*K_X)_{|T}$$

and therefore the restriction $p_{|T} : T \to X$ is unramified. Since X, being Fano, is simply-connected, it follows that T is a section of p. Thus we conclude that $Z \cong \mathbf{P}^{r-1}$ and $\mathbf{P}(E) \cong X \times \mathbf{P}^{r-1}$.

As an immediate consequence we get the following

Proposition (1.3). *Let X be a rationally connected manifold with respect to an unsplit family V and let E be an ample vector bundle on X which is a subsheaf of TX. If deg$(f^*E) =$ rank(E) for some (hence for any) $f \in V$, then $X \cong \mathbf{P}^n$ and $E \cong \mathcal{O}(1)^{\oplus r}$.*

Proof. By the above splitting result (Proposition (1.2)) we reduce the situation to the case $r = 1$, that is the Wahl's theorem (in the special case of $\rho(X) = 1$).

Tangent cone to a family of curves

Let X be a smooth projective variety and $V \subset \mathrm{Hom}(\mathbf{P}^1, X)$, a closed irreducible component as before; assume that V is generically unsplit so that for a general $[f] \in V$ we have $f^*TX = \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus(d)} \oplus \mathcal{O}^{\oplus(n-d-1)}$; in particular for a general $[f]$ the map f is an immersion.

Fix a general $x \in X$ and consider $V_x := V \cap \mathrm{Hom}(\mathbf{P}^1, X; 0 \mapsto x)$. Let t be a local coordinate around $0 \in \mathbf{P}^1$. Consider a derivative map $\Phi_x : V_x \to P(T_xX) = P((f^*TX)_0)$ which is defined at $[f] \in V_x$, if f is an immersion at 0, by $\Phi_x([f]) = [(Tf)_0(\partial/\partial t)]$, c.f. [Mori79, pp. 602–603]. In the formula $Tf : T\mathbf{P}^1 \to f^*TX$ is the tangent map and T_xX is identified naturally, via f^*, with $(f^*TX)_0$.

By P we denote the "natural projectivisation" (that is vector spaces modulo homotheties) in opposition to "Grothendieck projectivisation" (that is projective spectrum of the symmetric algebra of a vector space) which we denote by $\mathbf{P}$. Using the latter formalism the map Φ_x is the value over 0 of the natural section of $\mathbf{P}(f^*\Omega_X)$ obtained by the surjective (at 0) morphism of derivatives: $Df : f^*\Omega_X \longrightarrow \Omega_{\mathbf{P}^1} \cong \mathcal{O}(-2)$.

We define $S_x \subset P(T_xX)$ as the closure of the image of the map Φ_x and we call it tangent cone of curves from V at the point x. J.-M. Hwang and N. Mok call this variety of minimal rational tangents [H-M]. The name of tangent cone follows from the fact that S_x is (at least around $[f]$) the tangent cone to $\mathrm{Locus}(V_x)$. Indeed, let $\pi : \widehat{X_x} \to X$ be the blow-up of X at x with the exceptional divisor $E_x = P(T_xX)$. Consider $\hat{f} : \mathbf{P}^1 \to \widehat{C} \subset \widehat{X_x}$, the lift-up of f, then $\hat{f}^*(T\widehat{X_x}) = \mathcal{O}(2) \oplus \mathcal{O}^{\oplus(d)} \oplus \mathcal{O}(-1)^{\oplus(n-d-1)}$. Thus $\mathrm{Hom}(\mathbf{P}^1, \widehat{X_x})$ is smooth at $[\hat{f}]$ and of dimension $d+3$. Moreover, by [Ko], II.3.4, the evaluation morphism $\hat{F} : \mathrm{Hom}(\mathbf{P}^1, \widehat{X_x}) \times \mathbf{P}^1 \to \widehat{X_x}$ is an immersion along $[\hat{f}] \times \mathbf{P}^1$ and moreover, by definition, $\hat{F}([\hat{f}], 0) = \Phi_x([f])$. On the other hand if we take an irreducible component $\hat{V}$ of $\mathrm{Hom}(\mathbf{P}^1, \widehat{X_x})$ which contains $[\hat{f}]$ then $\mathrm{Locus}(\hat{V})$ outside of E_x coincides with a component of $\mathrm{Locus}(V_x)$. Thus around $\Phi_x([f])$ we get $S_x = E_x \cap \widetilde{\mathrm{Locus}(V_x)}$, with $\widetilde{\mathrm{Locus}(V_x)}$ denoting the strict transform of $\mathrm{Locus}(V_x)$, so S_x is the tangent cone to $\mathrm{Locus}(V_x)$.

For our purposes we need the following observation which follows from the above discussion (see also [Hw], Proposition (2.3)).

Lemma (2.1). *The projectivised tangent space of the tangent cone S_x at $\Phi_x([f])$ is equal to $P((f^*TX)_0^+) \subset P((f^*TX)_0) = P(T_xX)$.*

Proof. By [Ko] II.3.4 the tangent space to $\mathrm{Locus}(V_x)$ at $f(p)$ for $p \neq 0$ is the image of the evaluation of sections of the twisted pull-back of TX which is $\mathrm{Im}(T\hat{F})_p = (f^*TX)_p^+ \subset (f^*TX)_p = T_{f(p)}X$. Thus passing with p to 0 we get the result.

Lemma (2.2). *Let $V \subset Hom(\mathbf{P}^1, X)$ be as above and moreover suppose that $\mathcal{E} \hookrightarrow TX$ is a reflexive subsheaf with a torsionfree cokernel. If for a general $[f] \in V$ the tangent map $Tf : T\mathbf{P}^1 \to f^*TX$ factors to an injection $T\mathbf{P}^1 \hookrightarrow f^*\mathcal{E}$, then $(f^*TX)^+ \hookrightarrow f^*\mathcal{E}$.*

Proof. We choose a general f which is an immersion at $0 \to x$. Then $\Phi_x([f]) \in P(\mathcal{E}_x) = P((f^*\mathcal{E})_0) \subset P(T_xX) = P((f^*TX)_0)$ and the same holds for morphisms in a neighborhood of $[f]$ in V_x. Thus around $\Phi_x([f])$ the tangent cone S_x is contained in $P(\mathcal{E}_x) = P((f^*\mathcal{E})_0)$, so is its tangent space $P((f^*TX)_0^+)$.

Now we use the Lemma (2.2) to conclude the case $f^*E \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^{r-1}$ which we have singled out in our preliminary discussion. In view of Mori result we will be done by the following

Proposition (2.3). *Let X be a manifold which is rationally connected with respect to some unsplit family $V \subset Hom(\mathbf{P}^1, X)$. Assume that E is an ample vector bundle admitting a sheaf injection $E \to TX$ and for a general $[f] \in V$ we have $f^*E \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^{r-1}$. Then $E \cong TX$.*

Proof. Comparing the splitting type of f^*E and f^*TX we see that the tangent map $Tf : T\mathbf{P}^1 \to f^*TX$ factors to a vector bundle (nowhere degenerate) injection $T\mathbf{P}^1 \to f^*E$. In other words, we have surjective morphism $(f^*E)^* \to \Omega_{\mathbf{P}^1} \cong \mathcal{O}(-2)$. Thus the values of $\Phi_{f(p)}$ at any point $p \in \mathbf{P}^1$ are in $P(E)$. This holds also for small deformations of the morphism f at any point $p \in \mathbf{P}^1$ and therefore (the component of) tangent cone $S_{f(p)}$ is contained in $P(E_{f(p)})$. Thus, in view of the previous lemma, we have the inclusion $(f^*TX)^+ \subset f^*E$ and hence, by the splitting type of f^*TX, we conclude that $f^*E = (f^*TX)^+$ and therefore $\deg(f^*E) = \deg(f^*(-K_X))$. Since $\rho(X) = 1$ it follows that $\det(E) = -K_X$.

The embedding $E \hookrightarrow TX$ gives rise to a non-trivial morphism $\det(E) \to \Lambda^r TX$ and thus to a non-zero section of $\Lambda^r TX \otimes K_X$. We use dualities to have the equalities:

$$h^0(X, \Lambda^r TX \otimes K_X) = h^n(X, \Omega^r_X) = h^r(X, \Omega^n_X)$$
$$= h^r(X, K_X) = h^{n-r}(X, \mathcal{O}_X)$$

and, since X is Fano, the latter number is non-zero only if $r = n$. Thus $\Lambda^r TX \otimes (\det E)^{-1} \cong \mathcal{O}_X$ so $E \hookrightarrow TX$ is nowhere degenerate, hence an isomorphism.

Extending rc$V$ fibrations in codimension 1

Let X, E be as in the main theorem; let V be the unsplit family and $\varphi^0 : X^0 \to Z^0$ be the rc$V$ fibration defined in the prologue. In this section we assume that $dim Z^0 \geq 1$ and we see that this will lead to a contradiction.

A general fiber of φ^0, call it X_g, is rationally connected and, as we have proved in Lemma (0.3), E is tangent to X_g, that is the injection $E_X \to TX_{X_g}$ factors to $E_{X_g} \to TX_g$.

By the result of the previous sections X_g is isomorphic to $\mathbf{P}^k$ and E_{X_g} is either $TX_g = T\mathbf{P}^k$ or $\mathcal{O}(1)^{\oplus r}$. We can shrink Z^0 and X^0 so that $\varphi^0 : X^0 \to Z^0$ is a projective space bundle in étale or analytic topology.

Take now $\tilde{Z}$ an irreducible component of Hilb(X) which contains the point corresponding to a general fiber X_g. Over $\tilde{Z}$ there exists a universal flat family $\tilde{X} \subset \tilde{Z} \times X$ with projections $\tilde{\varphi} : \tilde{X} \to \tilde{Z}$ and $\tilde{\beta} : \tilde{X} \to X$. The family $\tilde{X} \to \tilde{Z}$ extends $X^0 \to Z^0$, that is we have an inclusion $X^0 \hookrightarrow \tilde{X}$ and $Z^0 \hookrightarrow \tilde{Z}$ such that $\tilde{\varphi}$ extends φ^0.

Lemma (3.1). *For any $z \in \tilde{Z}$ the fiber $\tilde{X}_z := \tilde{\varphi}^{-1}(z)$ is irreducible.*

Proof. This follows from the fact that any two points in $\tilde{X}_z$ can be joint by an irreducible curve parametrized by a morphism from V. Indeed, take $x_1, x_2 \in \tilde{X}_z$, then there exists a 1-parameter family of $X_t \cong \mathbf{P}^k$ whose limit is $\tilde{X}_z$ with points $x_1^t, x_2^t \in X_t$ whose limits are x_1 and x_2, respectively. Now x_1^t can be joint with x_2^t by a line C_t parametrized by a $[f_t] \in V$. Since V is unsplit the limit curve is irreducible and the claim follows.

The morphism $\tilde{\beta}$ is birational with exceptional Locus $E(\tilde{\beta})$ which is of codimension 1, by Zariski main theorem. If we set $X^* = \tilde{X} \setminus E(\tilde{\beta})$ then, via $\tilde{\beta}$, we have inclusion $X^* \subset X$ and $X \setminus X^*$ is of codimension ≥ 2. Let φ^* be the restriction of $\tilde{\varphi}$ to X^* and let $Z^* = \varphi^*(X^*)$.

Lemma (3.2). *The morphism $\varphi^* : X^* \to Z^*$ is proper.*

Proof. Since $\tilde{\varphi}$ is proper, it is enough to show that if $E(\beta)$ meets a fiber of $\tilde{\varphi}$ then it contains all such a fiber. Let $E(\beta) = \bigcup E_i$ be a decomposition into irreducible components. Since $\tilde{\varphi}$ extends φ^0 defined on a subset of X it follows that none of E_i meets a generic fiber of $\tilde{\varphi}$ so $dim(\tilde{\varphi}(E_i)) \leq dim Z - 1 = n - k - 1$. As the dimension of any fiber of $\tilde{\varphi}_{|E_i}$ is $\leq k$ it follows that, actually, all fibers of $\tilde{\varphi}_{|E_i}$ are of dimension k. But all fibers of $\tilde{\varphi}$ are irreducible and of dimension k, hence our claim.

Let $\hat{\varphi} : \hat{X} \to \hat{Z}$ be the normalization of $\tilde{\varphi} : \tilde{X} \to \tilde{Z}$, with the induced morphism $\hat{\beta} : \hat{X} \to X$. Since $X^* \subset \tilde{X}$ is smooth it lifts up to $X^* \subset \hat{X}$. The restriction $\hat{\varphi}$ to X^*, call it $\hat{\varphi}^*$, is proper and call its image $\hat{Z}^*$.

Lemma (3.4). *Outside a subset of codimension ≥ 2 the morphism $\hat{\varphi}^*$ is a $\mathbf{P}^k$-bundle (in the analytic topology).*

Proof. This is a result of Fujita. Let B be a curve obtained by intersection of $dim Z - 1$ general very ample divisor on $\hat{Z}$ and let $B^* = B \cap \hat{Z}^*$. By Bertini B^* is smooth and $X_B^* = (\hat{\varphi}^*)^{-1}(B) \cap X^*$ is smooth as well. Moreover, the induced morphism $\hat{\varphi}_B : X_B^* \to B^*$ is generically projective bundle, so it is a projective bundle by [Fu], Lemma (2.12). Thus, by Bertini, outside of codimension ≥ 2 the morphism $\hat{\varphi}^*$ is smooth and with fibers equal to $\mathbf{P}^k$.

Let $\varphi' : X' \to Z'$ be the restriction of $\hat{\varphi}$ which is a $\mathbf{P}^k$-bundle. By the above lemmas the codimension of $X \setminus X'$ is ≥ 2 and therefore we can take a general smooth projective curve B' in X which is contained in X' and is not contained in fibers of φ'. Then $B = \varphi'(B')$ is projective and contained in Z', and since our choice was general it is moreover smooth. Let $X_B = (\varphi')^{-1}(B)$. Again, by the generality of the choice we can assume that $E_{|X_B} \to TX_{|X_B}$ is a subsheaf inclusion which is generically of maximal rank.

Lemma (3.5). *In the above notation $E_{|X_B}$ is a locally free subsheaf of $T_{X_B/B}$.*

Proof. The vector bundle E is tangent to a general fiber of φ^0, as proved in the Lemma (0.3). Therefore the sheaf inclusion $E_{|X_B} \hookrightarrow TX_{|X_B}$ generically factors via $E_{|X_B} \to T_{X_B/B}$. Since the cokernel of the composition $T_{X_B/B} \to TX_B \to TX_{|X_B}$ is torsion free (it is actually locally free) the inclusion $E_{|X_B} \hookrightarrow T_{X_B/B}$ over X_B follows.

Conclusion. We arrive to a contradiction by applying the following result, which is due to Campana and Peternell, to the $\mathbf{P}^k$ bundle $f^* : X_B \to B$ and to the ample vector bundle $E_{|X_B}$.

Lemma. *([C-P], Lemma (1.2)) Let X be a n-dimensional projective manifold, $\varphi : X \to Y$ a $\mathbf{P}^k$ bundle $(k < n)$ of the form $X = \mathbf{P}(V)$ with a vector bundle V on Y. Then the relative tangent sheaf $T_{X/Y}$ does not contain an ample locally free subsheaf.*

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