Invent. math. 143, 449–497 (2001)
Digital Object Identifier (DOI) 10.1007/s002220000112

Inventiones mathematicae

The infinitesimal 16th Hilbert problem in the quadratic case

Lubomir Gavrilov

Laboratoire Emile Picard, UMR 5580, Université Paul Sabatier, 118, route de Narbonne, 31062 Toulouse Cedex, France (e-mail: gavrilov@picard.ups-tlse.fr)

Oblatum 23-III-2000 & 19-VI-2000
Published online: 11 October 2000 –

Abstract. Let $H(x, y)$ be a real cubic polynomial with four *distinct* critical values (in a complex domain) and let $X_H = H_y \frac{\partial}{\partial x} - H_x \frac{\partial}{\partial y}$ be the corresponding Hamiltonian vector field. We show that there is a neighborhood $\mathcal{U}$ of X_H in the space of all quadratic plane vector fields, such that any $X \in \mathcal{U}$ has at most two limit cycles.

Contents

1. Introduction

Let $H(x, y)$ be a real cubic polynomial with four *distinct* (real or complex) critical values and suppose that the quadratic Hamiltonian vector field

$$X_H = H_y \frac{\partial}{\partial x} - H_x \frac{\partial}{\partial y} \qquad (1)$$

has a center. We prove the following

Theorem 1. *There is a neighborhood $\mathcal{U}$ of X_H in the space of all quadratic vector fields, such that any $X \in \mathcal{U}$ has at most two limit cycles.*

Recall that the second part of the 16th Hilbert's problem [18] asks about "*the maximum number and the position of Poincaré's boundary cycles (*cycles limites*)*" for plane polynomial vector fields of degree n:

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y) \ . \tag{2}$$

Solving this problem, even in the case $n = 2$, at the present state of knowledge seems to be hopeless. The result of the paper may be considered as its infinitesimal version. To the end of this Introduction we shall explain in a non-formal way our strategy of proving Theorem 1. It is based mainly on the study of the zeros of suitable Abelian integrals.

Consider a small polynomial deformation

$$X_\varepsilon = X_H + \varepsilon Y_\varepsilon$$

of the polynomial Hamiltonian vector field X_H, where

$$Y_\varepsilon(x, y) = Y_1(x, y, \varepsilon)\frac{\partial}{\partial x} + Y_2(x, y, \varepsilon)\frac{\partial}{\partial y}, \quad X_H = H_y\frac{\partial}{\partial x} - H_x\frac{\partial}{\partial y}$$

are quadratic vector fields, X_H has a non-degenerate singular point which is a center, and moreover Y_ε depends analytically on ε.

Without loss of generality we assume that the center is located at the origin, $Y_\varepsilon(0, 0) \equiv (0, 0)$, and

$$H(x, y) = (x^2 + y^2)/2 + \text{ "higher order terms".} \tag{3}$$

Consider the continuous family of ovals

$$\gamma(h) \subset \{(x, y) \in \mathbb{R}^2 : H(x, y) = h\}$$

which tend to the origin in $\mathbb{R}^2$ as $h \to 0$, and are defined on a maximal open interval $(0, \tilde{h})$. Let l be a closed arc, contained in the open period annulus

$$\bigcup_{h \in (0, \tilde{h})} \gamma(h) \tag{4}$$

and transversal to the family of ovals $\gamma(h)$. For sufficiently small $|\varepsilon|$ the arc l is still transversal to the vector field X_ε, and can be parameterized by $h = H(x, y)|_l$. Therefore we can define, on a suitable open subset of l, the first return map $h \to P_\varepsilon(h)$ associated to the vector field X_ε and the arc l, as it is shown on Fig. 1. The limit cycles of the perturbed vector field X_ε correspond to the fixed points of the analytic map P_ε. It is well known [33, Pontryagin] that

$$P_\varepsilon(h) - h = -\varepsilon I_{Y_0}(h) + o(\varepsilon) \ , \tag{5}$$

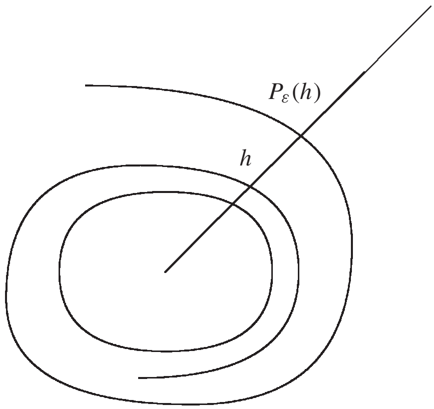


Fig. 1 The first return map $P_\varepsilon(h)$ associated to the vector field X_ε and the arc l

where the Pontryagin function $I_{Y_0}(h)$ is given by

$$I_{Y_0}(h) = \int\int_{\{H\leq h\}} div(Y_0)\, dx \wedge dy,$$
$$div(Y_0) = (Y_1(x, y, 0))_x + (Y_2(x, y, 0))_y \tag{6}$$

and

$$\lim_{\varepsilon\to 0} o(\varepsilon)/\varepsilon = 0$$

uniformly in h on any compact subset of $[0, \tilde{h})$.

In contrast to the first return map $P_\varepsilon(h)$, the Pontryagin function $I_{Y_0}(h)$ does not depend on the choice of the arc l. If in addition $H(x, y)$ has distinct critical values and $P_\varepsilon(h) \not\equiv h$, then $I_{Y_0}(h) \not\equiv 0$ [25, Il'yashenko]. It is easy to see in this case that the number of zeros of $I_{Y_0}(h)$ on the interval $[0, \tilde{h})$ provides an upper bound for the number of the limit cycles of X_ε which bifurcate from the open period annulus (4). The same holds true for the *closed* period annulus, provided that it is bounded by a homoclinic loop (as in the present paper) as proved by Roussarie [34]. As $Y_0(x, y)$ is a quadratic vector field, then the function $I_{Y_0}(h)$ can be written in the form

$$I_{Y_0}(h) = I_{\alpha\beta\gamma}(h) = \int\int_{\{H\leq h\}} (\alpha\, x + \beta\, y + \gamma)\, dx \wedge dy\ .$$

It is a complete elliptic integral and its qualitative behavior, as a function of the *complex variable* h is the main subject of the present paper.

Denote by $\{H \leq h\}$ the interior of the oval $\gamma(h)$, where $h \in [0, \tilde{h}]$. For $h = 0$ it reduces to a point and for $h = \tilde{h}$ it is the interior of a homoclinic loop containing the saddle point $(1, 0)$ of the vector field X_H. The *centroid* *point* $(\xi(h), \eta(h))$ of $\{H \leq h\}$ has coordinates

$$(\xi(h), \eta(h)) = \left(\frac{\iint_{\{H\leq h\}} x\, dx \wedge dy}{\iint_{\{H\leq h\}} dx \wedge dy}, \frac{\iint_{\{H\leq h\}} y\, dx \wedge dy}{\iint_{\{H\leq h\}} dx \wedge dy} \right), \ h \in [0, \tilde{h}] \,.$$

Consider the *centroid curve* (introduced first by Horozov and Iliev [19])

$$L = \{(\xi(h), \eta(h)) : h \in [0, \tilde{h}]\}$$

associated to the polynomial (3). A basic fact about L is that it is smooth, and this holds true even without any restriction on the cubic polynomial $H(x, y)$. Assuming this, we note that $I_{\alpha\beta\gamma}(h)$ has a zero of multiplicity k at $h \in [0, \tilde{h}]$ if and only if the intersection number of the affine line

$$\{(x, y) \in \mathbb{R}^2 : \alpha x + \beta y + \gamma = 0\}$$

with the centroid curve L at the point $(\xi(h), \eta(h))$ equals to k. We shall prove that the centroid curve L is convex, and when running it, the tangential vector rotates within an angle less than π. Clearly this implies that $I_{\alpha\beta\gamma}(h)$ has at most two zeros which on its turn proves Theorem 1 (after a series of well known steps).

The quadratic Hamiltonian vector field X_H can have either one center and one saddle, or two centers and two saddles, or one center and three saddles (recall that Hamiltonian function (3) has four distinct critical values). If X_H has one center and three saddles the convexity of the centroid curve was proved in [19] in the following way

Step 1. The second derivative of the Pontryagin function $I_{\alpha\beta\gamma}(h)$ can have at most two zeros on $h \in [0, \tilde{h}]$ and hence $I_{\alpha\beta\gamma}(h)$ has at most four zeros [12]. As this function always vanishes at the origin then this means that every line $l = \{(x, y) \in \mathbb{R}^2 : \alpha x + \beta y + \gamma = 0\}$ intersects at most three times the centroid curve L.

Step 2. A local analysis shows that the curvature of L in a neighborhood of its ends has the same sign and can not vanish. From this one deduces that if L is not convex, there exists an affine line l which intersects L at least four times. This contradicts Step 1.

If X_H has one center and one saddle point, the analogue of the principal result of [12], used in Step 1 above is proved in Theorem 4.1 below. It is based on the following observations

- The Abelian integral $\frac{d^2}{dh^2} I_{\alpha\beta\gamma}(h)$ can be continued analytically in the complex domain $\mathcal{D} = \mathbb{C} \backslash [\tilde{h}, \infty)$. This is not an obvious fact, because $H(x, y)$ has two complex conjugate critical values, but follows from its Dynkin diagram (Fig. 3).
- The zeros of $\frac{d^2}{dh^2} I_{\alpha\beta\gamma}(h)$ in the complex domain $\mathcal{D}$ are counted by making use of topological arguments (Picard-Lefschetz formula and the argument principle).

The Proof of Theorem 1 is completed in this case by reasoning as in Step 2 above.

All this does not work when the quadratic Hamiltonian vector field X_H has two centers and two saddles. It can be shown that in this case the exact upper bound for the number of the zeros of the second derivative of $I_{\alpha\beta\gamma}(h)$ is *four*. Instead of $L \subset \mathbb{R}^2$ we consider the dual centroid curve $L^* \subset \mathbb{RP}^2$, which is the set of lines $l = \{(x, y) \in \mathbb{R}^2 : \alpha x + \beta y + \gamma = 0\}$ tangent to L, in the projective space $\mathbb{RP}^2$ with homogeneous coordinates $[\alpha : \beta : \gamma]$. Consider on other hand the bifurcation set $\mathbf{B}$ of the zeros of $\frac{d^2}{dh^2} I_{\alpha\beta\gamma}(h)$ in the complex domain $\mathcal{D} = \mathbb{C}\backslash[0, \infty)$. A point $[\alpha : \beta : \gamma]$ belongs to $\mathbf{B}$ if and only if a zero of $\frac{d^2}{dh^2} I_{\alpha\beta\gamma}(h)$ bifurcates from the border of $\mathcal{D} = \mathbb{CP}\backslash[0, \infty] \subset \mathbb{CP}$. The dual projective curve L^* and the bifurcation set $\mathbf{B}$ live therefore in the same projective space. The proof of the convexity of L is split in the following steps.

Step 1. We compute explicitly the bifurcation set $\mathbf{B}$. It turns out that it is an union of four distinct projective lines and one segment (piece of a projective line).

Step 2. We determine the mutual position of L^* and $\mathbf{B}$. It turns out that L^* can intersect only three connected components of the complement to $\mathbf{B}$ in $\mathbb{RP}^2$.

Step 3. We prove that in each of the three connected components defined in Step 2, the Abelian integral $\frac{d^2}{dh^2} I_{\alpha\beta\gamma}(h)$ can have at most *two* zeros. This implies that the order of tangency of a line $l \in L^*$ is at most three. Equivalently, this shows that if the curvature $\kappa(h)$ of L vanishes at the point $(\xi(h), \eta(h)) \in L$, then this zero is simple.

Step 4. In this final step we use a deformation argument (as in [22]) to show that the curvature of L can not vanish. Namely, suppose that for some cubic Hamiltonian H the centroid curve L is convex. Such are for example the centro-symmetrical Hamiltonians, as it has been proved in [22]. Consider a continuous deformation of the Hamiltonian $H(x, y)$, and the associated deformation of the centroid curve L. As the curvature of L can not vanish at its ends (a local result!) then if the curvature $\kappa(h)$ acquires a zero, this zero is at least double which contradicts to Step 3.

The convexity of the centroid curve together with results of Roussarie [35] show that the vector field X_ε can have at most two zeros. This on its turn implies Theorem 1.

The paper is organized in the following way. In Sect. 2 we introduce the main notations used in the sequel, and establish several general results concerning the global Milnor bundle of the cubic polynomial $H(x, y)$. These results might be of independent interest and are based on [12–15]. In Sect. 3 we study the bifurcation set **B** of the zeros of the Abelian integral $\frac{d^2}{dh^2} I_{\alpha\beta\gamma}(h)$ in the complex domain $\mathcal{D} = \mathbb{C}\backslash[\tilde{h}, \infty)$. From this it is possible to obtain the exact number of the zeros for each connected component of the complement to **B**. We shall compute these numbers for some connected components in Sect. 4 (the others will not be used in the proof). The geometry of the centroid curve is studied in Sect. 5 where we materialize the strategy outlined above and prove the central result of the paper (Theorem 5.3). Theorem 1 is deduced from this in Sect. 6. In the last section we trace the historical origins of the problem we solved. We give a precise formulation of the infinitesimal 16th Hilbert problem for any n and emphasize its algebro-geometric content.

Theorem 1 has been previously announced in [11].

Acknowledgements. I thank Emil Horozov who introduced me to the subject, as well for the final reading of the manuscript. I am grateful to Iliya Iliev for his critics and scepticism which helped me to finish the paper. Finally I thank the referee and H. Żołądek for their suggestions and corrections concerning Sects. 6 and 7.

2. Notations and preliminary results

Let us suppose that the quadratic Hamiltonian vector field X_H (1) has a continuous family of periodic solutions $\delta_1(h) \subset \{(x, y) \in \mathbb{R}^2 : H(x, y) = h\}$ defined on a maximal open interval (h_1, h_2). It follows that the family $\delta(h)$ contains a center of X_H and is bounded by a loop (containing eventually several singular points). If we place the center at the origin of the coordinate system and a second critical point in $(1, 0)$, then the Hamiltonian $H(x, y)$ takes the following normal form [19]

$$H(x, y) = H(x, y; a, b) = \frac{1}{2}(x^2 + y^2) - \frac{1}{3}x^3 + a\,x\,y^2 + \frac{1}{3}b\,y^3\,. \qquad (7)$$

We shall also suppose, without loss of generality, that the interval of definition of $\delta(h)$ is $(0, \frac{1}{6})$ (of course X_H can have a second center).

Convention. To the end of this paper, if it is not explicitly mentioned otherwise, the cubic Hamiltonian $H(x, y)$ is supposed to have four distinct (real or complex) critical values. More explicitly, if H is given by (7) then we shall always suppose that $(a, b) \in \Omega_1^0 \cup \Omega_2^0 \cup \Omega_3^0$, where the open connected domains $\Omega_i^0 \subset \mathbb{R}^2$ are defined below.

If $H(x, y) = H(x, y; a, b)$ has four distinct critical values, then the continuous family of periodic solutions $\delta_1(h)$, $h \in (0, \frac{1}{6})$ is bounded by a saddle connection (a homoclinic solution containing one saddle point with coordinates $(0, 1)$).

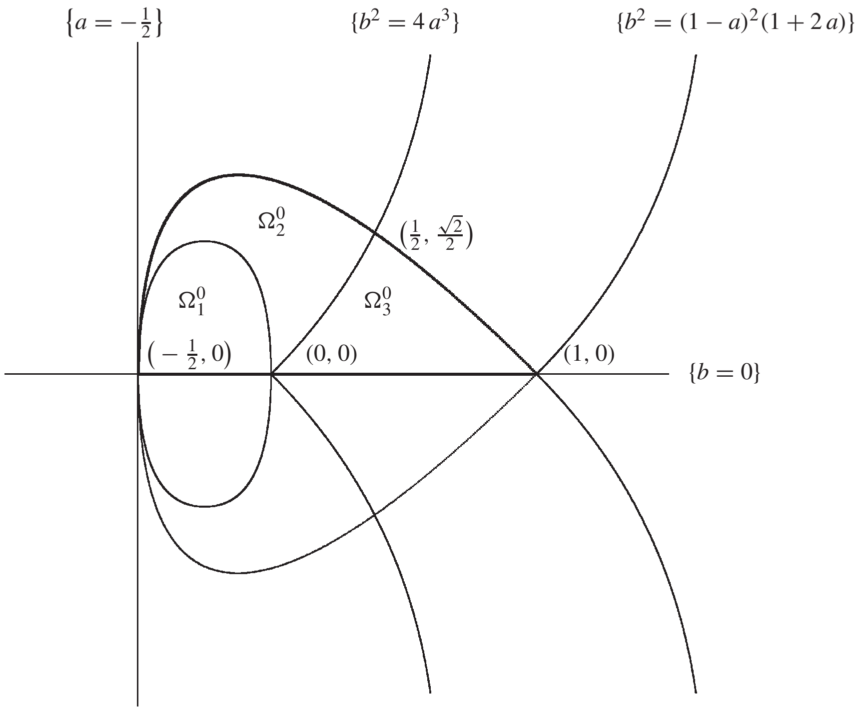


Fig. 2 The set D of real cubic polynomials $H(x,y;a,b)=\frac{1}{2}(x^2+y^2)-\frac{1}{3}x^3+axy^2+\frac{1}{3}by^3$ which do not have four distinct critical values

The set $D\subset\mathbb{R}^2\{a,b\}$ of real cubic Hamiltonians $H(x,y;a,b)$ which do not have four distinct critical values are easily computed

$$D=\{(a,b)\in\mathbb{R}^2:(2\,a+1)((1+2\,a)(1-a)^2-b^2)\times(4\,a(1+2\,a)+b^2)b(4\,a^3-b^2)=0\}\,.$$

$H(x,y;a,b)$ has less than three critical values if and only if $4\,a^3-b^2=0$ (one critical point has escaped to infinity). The set $\mathbb{R}^2\backslash D$ has 14 connected components (see Fig. 2). As $H(x,-y;a,-b)=H(x,y;a,b)$ then without loss of generality we shall suppose that $b\geq 0$. Let h_1,h_2,h_3,h_4 be the critical values of $H(x,y)$. We shall also suppose that $h_1=0$, $h_2=1/6$ and if the critical values h_3 and h_4 are real, then $h_3,h_4\geq 1/6$. The latter is equivalent (after an elementary computation) to the claim that if $(a,b)\notin D$ then (a,b) belongs to one of the three open connected domains Ω_3^0, Ω_1^0 and Ω_2^0 shown on Fig. 2. In each of them the vector field X_H has one center and three saddles, one center and one saddle, or two centers and two saddles respectively [19].

Through the paper we shall use the following standard notations introduced in [19]

$$X(h) = \iint_{H \le h} x dx \wedge dy, \quad Y(h) = \iint_{H \le h} y dx \wedge dy,$$

$$M(h) = \iint_{H \le h} dx \wedge dy, K(h) = \iint_{H \le h} x\, y\, dx \wedge dy$$

$$\omega_X = -xydx, \quad \omega_Y = -\frac{1}{2} y^2 dx, \quad \omega_M = -ydx, \quad \omega_K = -\frac{1}{2} x\, y^2 dx\ . \tag{8}$$

The Pontryagin function associated to an arbitrary quadratic perturbation of X_H ($\{H \le h\}$ denotes the interior of the oval $\delta(h)$) is

$$I_{\alpha\beta\gamma}(h) = \iint_{\{H \le h\}} (\alpha x + \beta y + \gamma)\, dx \wedge dy = \int_{\delta_1(h)} \alpha\, \omega_X + \beta\, \omega_Y + \gamma\, \omega_M\ . \tag{9}$$

Denote

$$\bar{\Gamma}_h = \left\{ [x, y, z] \in \mathbb{CP}^2 : H\left(\frac{x}{z}, \frac{y}{z}\right) = h \right\}$$

and

$$\Gamma_h = \{(x, y) \in \mathbb{C}^2 : H(x, y) = h\}\ .$$

If $h \neq h_i$, then both Γ_h and $\bar{\Gamma}_h$ are smooth, $\bar{\Gamma}_h$ is a compact Riemann surface of genus one which is also the compactification of the affine elliptic curve Γ_h. We have

$$\bar{\Gamma}_h = \Gamma_h \cup \infty_1 \cup \infty_2 \cup \infty_3$$

where ∞_i, $i = 1, 2, 3$, are three distinct points on $\bar{\Gamma}_h$, and therefore

$$rank H_1(\Gamma_h, \mathbb{Z}) = 4, \quad rank H_1(\bar{\Gamma}_h, \mathbb{Z}) = 2.$$

The (co)homology Milnor bundle associated to the global Milnor bundle of H has a canonical flat connexion, called the Gauss-Manin connection. For a section $\omega(h)$ of the cohomology bundle, the covariant derivative

$$\nabla : H^1_{DR}(\Gamma_h, \mathbb{C}) \to H^1_{DR}(\Gamma_h, \mathbb{C})\ .$$

defining the connexion satisfies the identity

$$\frac{d}{dh} \int_{\gamma(h)} \omega = \int_{\gamma(h)} \nabla \omega$$

where $\gamma(h)$ is any locally constant section of the homology bundle.

The restrictions of the polynomial one-forms (8) on Γ_h define meromorphic one-forms on $\bar{\Gamma}_h$, which are holomorphic on Γ_h. By abuse of notations we denote these one-forms by $\omega_M, \omega_X, \omega_Y, \omega_K$ too. They define global sections of the cohomology Milnor bundle and

$$\nabla\omega_M = -\frac{dx}{H_y}, \nabla\omega_X = -\frac{x\,dx}{H_y}, \nabla\omega_Y = -\frac{y\,dx}{H_y}, \nabla\omega_K = -\frac{x\,y\,dx}{H_y}\ .$$

Lemma 1 (Proposition 5, [15]). *For every fixed $h \neq h_i$ the differential one-forms $\nabla\omega_M$,$\nabla\omega_X$, $\nabla\omega_Y$ $\nabla\omega_K$ are holomorphic on the* affine *curve Γ_h and define a basis of $H^1_{DR}(\Gamma_h, \mathbb{C})$.*

Let h_i, $i = 1, ..., 4$, be the distinct critical values of H, where $h_1 = 0$, $h_2 = \frac{1}{6}$, and let (x_i, y_i) be the corresponding critical points. If $h_3, h_4 \in \mathbb{R}$ we shall always suppose that $\frac{1}{6} < h_3 < h_4$ and if $h_3 = \overline{h_4} \notin \mathbb{R}$ we shall suppose that $Im(h_3) > 0$. In the case when h_i is a critical value corresponding to a center (saddle point) of X_H we shall denote sometimes this value by h_i^c (respectively h_i^s). Similar convention will be used for the continuous families of vanishing cycles δ_i (see Fig. 3). The vector field X_H can have either

1. one center and one saddle ($(a, b) \in \Omega_1^0$), or
2. two centers and two saddles ($(a, b) \in \Omega_2^0$), or
3. one center and three saddles ($(a, b) \in \Omega_3^0$).

In each of these three cases H has its own Dynkin diagram. Namely, let $h = h_0$ be a fixed regular value with $Im(h_0) > 0$ and let l_1, l_2, l_3, l_4 be three mutually non-intersecting paths, connecting h_0 to h_i. In the cases 2, 3 the critical values are real and we shall suppose that l_i are contained in the upper half-plane $Im(h) > 0$ (except their ends which coincide with h_i). In the case 1 we shall suppose that the paths l_1, l_2, l_3 are contained in the upper half-plane $Im(h) > 0$ and for l_4 we shall suppose that it is contained in the domain $\mathcal{D} = \mathbb{C}\backslash[\frac{1}{6}, \infty)$. The paths l_i are shown on Fig. 3.

Denote by $\delta_i(h) \in H_1(\Gamma_h, \mathbb{Z})$ the continuous families of cycles which vanish at (x_i, y_i) as h tends to h_i along the path l_i. As $H(x, y)$ is a good polynomial then for all regular h the cycles $\delta_i(h)$ form a basis of the first integer homology group of the affine algebraic curve $\Gamma_h \subset \mathbb{C}^2$ [13, Theorem 2.4]. The families $\delta_i(h)$ define locally constant sections of the global homology Milnor bundle of $H(x, y)$ with base $\mathbb{C}\backslash\{h_1, h_2, h_3, h_4\}$ and fiber $H_1(\Gamma_h, \mathbb{Z})$. When the polynomial $H(x, y)$ has complex conjugate critical values ($(a, b) \in \Omega_1^0$) a different choice of paths leads, *a priori*, to different cycles $\delta_3(h_0)$ and $\delta_4(h_0)$. Here it is not so because the intersection indices (to be computed below) $< \delta_3, \delta_4 >, < \delta_3, \delta_1 >, < \delta_1, \delta_4 >$ are all equal to zero.

Definition 1. *The Dynkin diagram of $H(x, y)$ is the graph with vertices the cycles δ_i. Two cycles δ_i, δ_j, $i < j$, are connected by an edge (dotted edge) if the intersection index $< \delta_i, \delta_j >$ is equal to $+1$ (-1).*

The Dynkin diagram of H describes the intersection indices $< \delta_i, \delta_j >$, and hence the monodromy group of the global homology Milnor bundle. It depends, however, on the homotopy class of the non-intersecting paths l_i. The choice of paths on Fig. 3 is motivated by the proof of Theorem 4.1.

Lemma 2. *For a suitable orientation of the cycles δ_i the Dynkin diagram of the generic cubic polynomial $H(x, y)$ (7) is shown on Fig. 3.*

Corollary 1. *The function $I_{\alpha\beta\gamma}(h)$ (and hence $\frac{d^2}{dh^2} I_{\alpha\beta\gamma}(h)$) can be continued analytically in the complex domain*

$$\mathcal{D} = \mathbb{C} \setminus \left[\frac{1}{6}, \infty\right) = \mathbb{CP}^1 \setminus \left[\frac{1}{6}, \infty\right] .$$

Indeed, If H has four real critical values $h_1 = 0$, $h_2 = \frac{1}{6}$, h_3, h_4, then it is easy to check that h_3 and h_4 are contained in the interval $[\frac{1}{6}, \infty)$. If $h_3 = \overline{h_4} \notin \mathbb{R}$ then the cycle $\delta_1(h_0)$ has a zero intersection index with the cycles $\delta_3(h)$ and $\delta_4(h)$ (Fig. 3). Therefore the locally constant section $\delta_1(h)$ of the homology Milnor bundle is globally constant over $\mathcal{D} \setminus \{0, h_3, h_4\}$. $\square$

Proof of Lemma 2. For every generic cubic polynomial $H(x, y; a, b)$ we can find a continuous deformation $H(x, y; a(\lambda), b(\lambda))$, $\lambda \in [0, 1]$, in the space of all real cubic polynomials of the form (7) with the properties:

- $H(x, y; a, b) = H(x, y; a(0), b(0))$
- $H(x, y; a(\lambda), b(\lambda))$ is generic for $\lambda \in [0, 1)$
- $H(x, y; a(1), b(1))$ has four distinct critical points, and is one of the polynomials

$$H_1(x, y) = \frac{1}{2}(x^2 + y^2) - \frac{1}{3}x^3 - \frac{1}{3}x\, y^2$$

(one center and one saddle), or

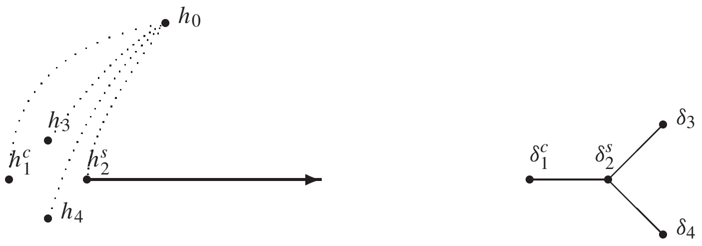
$$H_2(x, y) = \frac{1}{2}(x^2 + y^2) - \frac{1}{3}x^3 + \frac{1}{3}y^3$$

(two centers and two saddles), or

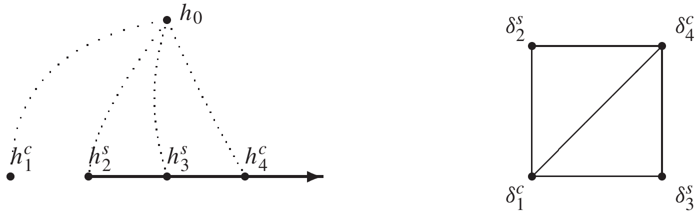
$$H_3(x, y) = \frac{1}{2}(x^2 + y^2) - \frac{1}{3}x^3 + x\, y^2$$

(one center and three saddles).

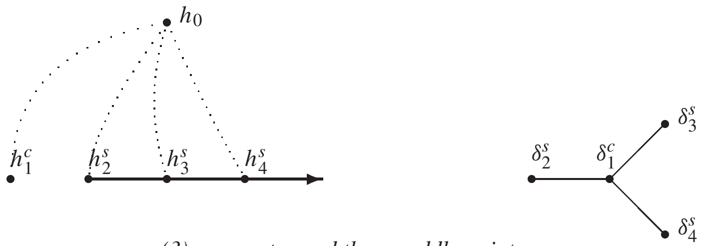
It follows that the Dynkin diagrams of $H(x, y; a(0), b(0))$ and $H(x, y; a(1), b(1))$ are the same. The critical points of the reversible polynomials $H_3(x, y)$ and $H_2(x, y)$ above are all real and in the both cases the saddle points are contained in the critical level set $\{(x, y) : H(x, y) = 1/6\}$. Therefore the result follows from A'Campo [1] or Husein-Zade [23,2].



(1) one center and one saddle point



(2) two centers and two saddle points



(3) one center and three saddle points

Fig. 3 Dynkin diagram of the generic real cubic polynomial $H(x, y; a, b) = \frac{1}{2}(x^2 + y^2) - \frac{1}{3}x^3 + axy^2 + \frac{1}{3}by^3$

Finally the polynomial $H_1(x, y)$ has two complex critical points, but we note that

$$H_1(1 - x, \sqrt{-3}\, y) = \frac{1}{6} - H_3(x, y)$$

and hence the Dynkin diagram in the case (1) is obtained from the Dynkin diagram in the case (3) (after exchanging δ_1^c and δ_2^s). $\square$

Let $H^0(\bar{\Gamma}_h, \Omega^1(\infty_1 + \infty_2 + \infty_3))$ be the vector space of meromorphic one forms on $\bar{\Gamma}_h$ having at most a simple pole at ∞_i, $i = 1, 2, 3$, and holomorphic on Γ_h.

Lemma 3. *The differential one-forms $\nabla\omega_M, \nabla\omega_X, \nabla\omega_Y$ form a basis of the vector space*

$$H^0(\bar{\Gamma}_h, \Omega^1(\infty_1 + \infty_2 + \infty_3))$$

for all $h \neq h_i$, $i = 1, 2, 3, 4$.

Indeed it is straightforward to check that $\nabla\omega_M, \nabla\omega_X, \nabla\omega_Y$ are holomorphic on Γ_h and have simple poles at ∞_i. It follows from Lemma 1 that these forms are also linearly independent, and from the Riemann-Roch formula that $dim H^0(\bar{\Gamma}_h, \Omega^1(\infty_1 + \infty_2 + \infty_3)) = 3$. □

An easy computation shows that the residues of $\nabla\omega_M, \nabla\omega_X, \nabla\omega_Y$ at ∞_i are constants (in h) and therefore the second derivatives $\nabla^2\omega_M, \nabla^2\omega_X, \nabla^2\omega_Y$ are differential forms with no residues. Such differential forms are called "of second kind" and they represent also cohomology classes in the first De Rham cohomology group $H^1_{DR}(\bar{\Gamma}_h, \mathbb{C})$ of $\bar{\Gamma}_h$.

Lemma 4. *The differential one-forms $\nabla^2\omega_M, \nabla^2\omega_X, \nabla^2\omega_Y$ generate the vector space $H^1_{DR}(\bar{\Gamma}_h, \mathbb{C})$ for all $h \neq h_i$, $i = 1, 2, 3, 4$.*

Note that $dim H^1_{DR}(\bar{\Gamma}_h, \mathbb{C}) = rank H_1(\bar{\Gamma}_h, \mathbb{Z}) = 2$. The proof of Lemma 4 follows essentially from [12, Lemma 5.2]. Namely, after a suitable rotation we put, as in [12] the cubic polynomial (7) in the following form

$$H(x, y) = \frac{x^2 + y^2}{2} + Ax^3 + Bx^2y + Cxy^2, \;\; C \neq 0, \;\; B^2 - 4AC \neq 0 \,. \tag{10}$$

The level set $\Gamma_h = \{H = h\}$ can be rewritten in the form

$$z^2 = (B^2 - 4AC)x^4 - 2(A + C)x^3 - x^2 + 4Chx + 2h \,. \tag{11}$$

where $z = H_y = (1 + 2Cx)y + Bx^2)$. In x, z coordinates we have

$$\omega_M = -\frac{z - Bx^2}{1 + 2Cx}dx, \;\; \omega_X = -\frac{x(z - Bx^2)}{1 + 2Cx}dx, \;\; \omega_Y = -\frac{1}{2}\left(\frac{z - Bx^2}{1 + 2Cx}\right)^2 dx$$

and hence

$$\nabla^2\omega_M = -\nabla^2\frac{z - Bx^2}{1 + 2Cx}dx = -\nabla^2\frac{z}{1 + 2Cx}dx = -\nabla\frac{dx}{z} = \frac{1 + 2Cx}{z^3}dx$$

$$\nabla^2\omega_X = -\nabla^2\frac{x(z - Bx^2)}{1 + 2Cx}dx = -\nabla^2\frac{xz}{1 + 2Cx}dx =$$
$$-\nabla\frac{x}{z}dx = \frac{x(1 + 2Cx)}{z^3}dx$$

$$\nabla^2 \omega_Y = -\frac{1}{2}\nabla^2 \left(\frac{z - Bx^2}{1+2Cx}\right)^2 dx = \nabla^2 \frac{Bx^2 z}{(1+2Cx)^2} dx$$
$$= \nabla \frac{Bx^2}{z(1+2Cx)} dx = -\frac{Bx^2}{z^3} dx \ .$$

As $H(x, y)$ has four distinct critical values, then $B \neq 0$, and hence the following identity holds in $H^1_{DR}(\bar{\Gamma}_h, \mathbb{C})$

$$Span\{\nabla^2\omega_M, \nabla^2\omega_X, \nabla^2\omega_Y\} = Span\left\{\frac{dx}{z^3}, \frac{x\,dx}{z^3}, \frac{x^2 dx}{z^3}\right\} . \tag{12}$$

Further linear rescaling of the variable z brings the curve (11) in the following normal form

$$\tilde{\Gamma}_h = \left\{(x, z) \in \mathbb{C}^2 : \frac{1}{2}z^2 = \frac{x^4}{4} - \frac{x^2}{2} + c_1 x + c_0\right\} , \tag{13}$$

where c_1 and c_0 are suitable real constants *depending linearly on h*. Note that the normalizations of the affine curves $\tilde{\Gamma}_h$ and Γ_h are isomorphic Riemann surfaces. We shall prove that the differentials

$$\frac{dx}{z^3}, \frac{x\,dx}{z^3}, \frac{x^2 dx}{z^3}$$

generate the first De Rham cohomology group of the normalized curve $\bar{\Gamma}_h$ with affine equation (13). Indeed, for every $\gamma \in H_1(\bar{\Gamma}_h, \mathbb{Z})$ we have the following system of Picard-Fuchs equations [12, Lemma 5.2]

$$\begin{aligned}
-\Delta(c_1, c_0) \int_\gamma \frac{dx}{z^3} &= A(c_1, c_0) \int_\gamma \frac{dx}{z} + B(c_1, c_0) \int_\gamma \frac{x^2 dx}{z} \\
-\Delta(c_1, c_0) \int_\gamma \frac{x^2 dx}{z^3} &= C(c_1, c_0) \int_\gamma \frac{dx}{z} - A(c_1, c_0) \int_\gamma \frac{x^2 dx}{z} \\
-\Delta(c_1, c_0) \int_\gamma \frac{x\,dx}{z^3} &= D(c_1, c_0) \int_\gamma \frac{dx}{z} + E(c_1, c_0) \int_\gamma \frac{x^2 dx}{z}
\end{aligned} \tag{14}$$

where $\Delta(c_1, c_0) = 4\,c_0\,(1-4\,c_0)^2 + 2\,c_1^2(1-36\,c_0) - 27\,c_1^4$ is the discriminant of the polynomial $\frac{x^4}{4} - \frac{x^2}{2} + c_1 x + c_0$ and

$$A(c_1, c_0) = 4\,c_0\,(1 - 4\,c_0) + 3\,c_1^2, \;\; B(c_1, c_0) = 4\,c_0 + 9\,c_1^2 - 1$$
$$C(c_1, c_0) = \left(3\,c_1^2 + 4\,c_0\right)(1 - 4\,c_0) - c_1^2, \;\; D(c_1, c_0) = c_1\left(16\,c_0 + 9\,c_1^2\right),$$
$$E(c_1, c_0) = c_1\,(12\,c_0 + 1) \ .$$

On the other hand it is well known that the differentials $\frac{dx}{z}$, $\frac{x\,dx}{z}$ and $\frac{x^2 dx}{z}$ form a basis of $H^1_{DR}(\Gamma_h, \mathbb{C})$ provided that $\Delta(c_1, c_0) \neq 0$ (the latter always holds true for generic polynomials H and regular values $h \neq h_i$). As $\frac{dx}{z}$ is of first kind, $\frac{x\,dx}{z}$ of third, and $\frac{x^2dx}{z}$ of second kind on $\bar{\Gamma}_h$, then $\frac{dx}{z}$ and $\frac{x^2dx}{z}$ form a basis of $H^1_{DR}(\bar{\Gamma}_h, \mathbb{C})$. It remains to check that the vectors (A, C, D) and $(B, -A, E)$ can not be co-linear. We have

$$det \begin{pmatrix} A(c_1, c_0) & B(c_1, c_0) \\ C(c_1, c_0) & -A(c_1, c_0) \end{pmatrix} = (1 - 4\,c_0)\,\Delta(c_1, c_0)$$

$$det \begin{pmatrix} A(c_1, c_0) & B(c_1, c_0) \\ D(c_1, c_0) & E(c_1, c_0) \end{pmatrix} = c_1\,\Delta(c_1, c_0)$$

and if $c_1 = 1 - 4\,c_0 = 0$ then $\Delta(c_1, c_0) = 0$. This completes the proof of Lemma 4. □

Let $\gamma_1(h)$, $\gamma_2(h)$ be independent locally constant *real* sections of the global homology Milnor bundle of the polynomial $H(x, y)$ and let

$$\omega = \alpha\,\nabla^2\omega_X + \beta\,\nabla^2\omega_Y + \gamma\,\nabla^2\omega_M$$

where α, β, γ, are fixed real constants

Lemma 5.

$$det \begin{pmatrix} \int_{\gamma_1(h)} \omega & \int_{\gamma_1(h)} \nabla^2\omega_M \\ \int_{\gamma_2(h)} \omega & \int_{\gamma_2(h)} \nabla^2\omega_M \end{pmatrix} = \frac{P_1(h)}{\prod_{i=1}^4 (h - h_i)}\,. \tag{15}$$

$$det \begin{pmatrix} \int_{\gamma_1(h)} \omega & \int_{\gamma_1(h)} \nabla\omega_M \\ \int_{\gamma_2(h)} \omega & \int_{\gamma_2(h)} \nabla\omega_M \end{pmatrix} = \frac{P_2(h)}{\prod_{i=1}^4 (h - h_i)}\,. \tag{16}$$

where $P_1(h)$ and $P_2(h)$ are suitable real polynomials of degree at most one and two respectively.

Proof. The Picard-Lefschetz formula implies that the Wronskians defined in (15), (16) are rational functions with simple poles at the critical values h_i, $i = 1...4$. The asymptotic analysis of the Abelian integrals shows that for $|h|$ sufficiently big $|\int_{\gamma_i(h)} \omega|$ and $|\int_{\gamma_i(h)} \nabla^2\omega_M|$ grow no faster than $|h|^{-4/3}$, and $|\int_{\gamma_i(h)} \nabla\omega_M|$ grows no faster than $|h|^{-1/3}$ (see the proof of Proposition 1). This shows that $\deg P_1 \leq |h|^{4/3}$, $\deg P_2 \leq |h|^{7/3}$. □

Corollary 2. *There exist real linear functions $\alpha(h)$, $\beta(h)$, $\gamma(h)$, which are unique up to multiplication by a non-zero constant, such that for every $h \neq h_i$ the differential one-form*

$$\alpha(h)\nabla^2\omega_X + \beta(h)\nabla^2\omega_Y + \gamma(h)\nabla^2\omega_M \tag{17}$$

represents the zero co-homology class in $H^1_{DR}(\bar{\Gamma}_h, \mathbb{C})$.

Indeed, the differential one-form

$$det \begin{pmatrix} \nabla^2\omega_X & \nabla^2\omega_Y & \nabla^2\omega_M \\ \int_{\gamma_1(h)} \nabla^2\omega_X & \int_{\gamma_1(h)} \nabla^2\omega_Y & \int_{\gamma_1(h)} \nabla^2\omega_M \\ \int_{\gamma_2(h)} \nabla^2\omega_X & \int_{\gamma_2(h)} \nabla^2\omega_Y & \int_{\gamma_2(h)} \nabla^2\omega_M \end{pmatrix}$$

is zero in $H^1_{DR}(\bar{\Gamma}_h, \mathbb{C})$ for every h. Multiplying it by $\prod_{i=1}^4 (h - h_i)$ and using the preceding Lemma we obtain the identity (17), where $\alpha(h)$, $\beta(h)$, $\gamma(h)$ are suitable real linear functions. The uniqueness of $\alpha(h)$, $\beta(h)$, $\gamma(h)$ follows from Lemma 4. $\square$

The next result shows that the curve

$$h \to [\alpha(h) : \beta(h) : \gamma(h)] : \mathbb{R} \to \mathbb{RP}^2 \tag{18}$$

is a projective line which is not reduced to a point.

Lemma 6. *Let $\delta(h)$ be a continuous family of vanishing cycles in the fibers of the Milnor fibration of $H(x, y)$. Then the Abelian integrals*

$$\int_{\delta(h)} \nabla^2\omega_X, \int_{\delta(h)} \nabla^2\omega_Y, \int_{\delta(h)} \nabla^2\omega_M$$

are functions in h which are linearly independent over $\mathbb{C}$.

Proof. If for some α, β, γ we have

$$\alpha \int_{\delta(h)} \nabla^2\omega_X + \beta \int_{\delta(h)} \nabla^2\omega_Y + \gamma \int_{\delta(h)} \nabla^2\omega_M \equiv 0$$

then

$$\alpha \int_{\delta(h)} \nabla\omega_X + \beta \int_{\delta(h)} \nabla\omega_Y + \gamma \int_{\delta(h)} \nabla\omega_M \equiv const.$$

As the Dynkin diagram of H is connected and the critical values of H are distinct, then as in the proof of [14, Prosition 3.2] we conclude that the above constant is equal to zero. Therefore

$$\int_{\delta(h)} \nabla\omega_X, \int_{\delta(h)} \nabla\omega_Y, \int_{\delta(h)} \nabla\omega_M$$

are also linearly dependent over $\mathbb{C}$ which contradicts to Lemma 1.

3. The bifurcation set of the zeros of the Abelian integral $\frac{d^2}{dh^2}I_{\alpha\beta\gamma}(h)$ in a complex domain

We already noted that for fixed $a, b, \alpha, \beta, \gamma$ the Abelian integral $\frac{d^2}{dh^2}I_{\alpha_i\beta_i\gamma_i}(h)$ is an analytic function in h in the complex domain $\mathcal{D} = \mathbb{CP}^1\backslash[\frac{1}{6}, \infty]$. The number of its zeros depends, however, on the parameters $a, b, \alpha, \beta, \gamma$.

Definition 2. *A point* $([\alpha_0 : \beta_0 : \gamma_0], a_0, b_0) \in \mathbb{RP}^2 \times \mathbb{R}^2$ *belongs to the bifurcation set* $\mathbf{B}$ *of the zeros of* $\frac{d^2}{dh^2}I_{\alpha\beta\gamma}(h)$ *in* $\mathcal{D}$ *if and only if there exists a sequence*

$$\alpha_i, \beta_i, \gamma_i, a_i, b_i, h_i \in \mathbb{R}^5 \times \mathcal{D}$$

such that

$$\lim_{i\to\infty}(\alpha_i, \beta_i, \gamma_i, a_i, b_i, h_i) = (\alpha_0, \beta_0, \gamma_0, a_0, b_0, h_0)$$

where $\frac{d^2}{dh^2}I_{\alpha_i\beta_i\gamma_i}(h_i) = 0$ *and* $h_0 \in [\frac{1}{6}, \infty]$. *In this case we shall say that a zero of* $\frac{d^2}{dh^2}I_{\alpha_0\beta_0\gamma_0}(h)$ *bifurcates from* $h = h_0$.

For every $(p, q) \in \mathbb{R}^2$ denote by $\mathbf{B}^{pq}$ the section

$$\mathbf{B}^{pq} = \mathbf{B} \cap \{a = p, b = q\} \subset \mathbb{RP}^2$$

and

$$\mathbf{B}_{reg} = \left\{([\alpha : \beta : \gamma], a, b) \in \mathbf{B} : (a, b) \in \cup_{i=1}^3 \Omega_i^0\right\}.$$

The purpose of this section is to describe the "regular part" $\mathbf{B}_{reg}$ of the bifurcation set $\mathbf{B}$.

Theorem 3.1. *The set* $\mathbf{B}_{reg}$ *is an union of five smooth analytic manifolds of co-dimension one denoted* l_2, l_3, l_4, l_∞ *and* Δ.

If the generic point (a, b) *is such that* X_H *has three saddles and one center, then the sections* $l_\infty^{ab} = B^{ab} \cap l_\infty$, $l_i^{ab} = B^{ab} \cap l_i$, $i = 2, 3, 4$ *are distinct projective lines,* $\Delta^{ab} = B^{ab} \cap \Delta$ *is a segment (a piece of a projective line) joining* l_∞^{ab} *and* l_2^{ab} *as on Fig. 4.*

If the generic point (a, b) *is such that* X_H *has one saddle and one center, then the sections* l_∞^{ab} *and* l_2^{ab} *are distinct projective lines,* $l_3^{ab} = \emptyset$, $l_4^{ab} = \emptyset$, *and* $\Delta^{ab} = B^{ab} \cap \Delta$ *is a segment (a piece of a projective line) joining* l_∞^{ab} *and* l_2^{ab}.

A point $([\alpha : \beta : \gamma], a, b) \in \mathbf{B}_{reg}$ *belongs to* l_2, l_3, l_4, l_∞ *or* Δ, *if and only if a zero of the corresponding Abelian integral* $\frac{d^2}{dh^2}I_{\alpha\beta\gamma}(h)$ *bifurcates from the point* h_0, *where* $h_0 = h_2$, *or* $h_0 = h_3 \in (1/6, \infty)$, *or* $h_0 = h_4 \in (1/6, \infty)$, *or* $h_0 = \infty$, *or* $h_0 \in (1/6, \infty)$ *respectively. The precise description of* l_i,l_∞, Δ *is given in Corollary 3,5,6.*

To prove the above Theorem we shall study each of the sets l_i^{ab}, l_∞^{ab}, Δ separately.

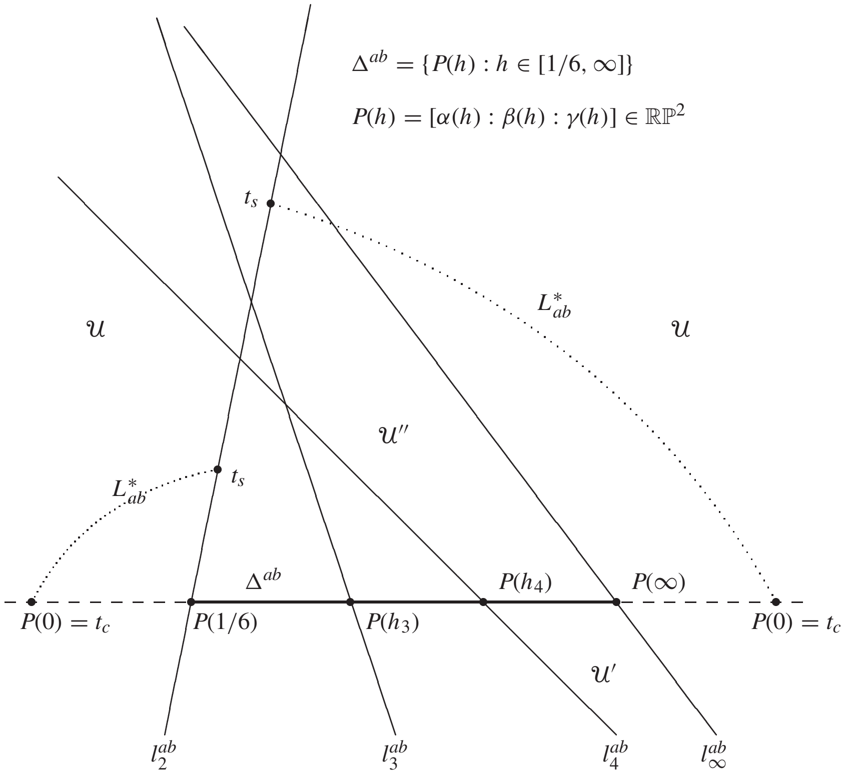


Fig. 4 The bifurcation set $\mathbf{B}^{ab} \subset \mathbb{RP}^2$ of the zeros of the Abelian integral $\frac{d^2}{dh^2} I_{\alpha\beta\gamma}(h)$

3.1. Bifurcations of zeros from $h_0 = \infty$

Proposition 1. *In a suitable neighborhood of $(\infty, a, b) \in \mathbb{CP}^1 \times \mathbb{C}^2$ the functions $h^{4/3}M''(h)$, $h^{4/3}X''(h)$, $h^{4/3}Y''(h)$ are holomorphic with respect to $a, b, h^{-1/3}$. If we denote*

$$\lim_{h\to\infty} h^{4/3}M''(h) = c_M^\infty, \quad \lim_{h\to\infty} h^{4/3}X''(h) = c_X^\infty, \quad \lim_{h\to\infty} h^{4/3}Y''(h) = c_Y^\infty$$

then c_X^∞, c_Y^∞, c_M^∞ are real constants, $c_M^\infty \neq 0$, and

$$c_X^\infty = \frac{b^2 - 2a^2(a-1)}{6(4a^3 - b^2)} c_M^\infty, \; c_Y^\infty = -\frac{b(a+1)}{6(4a^3 - b^2)} c_M^\infty \, .$$

Proof. To simplify the notations, let us fix first a, b. As the polynomial $H(x, y)$ has four critical points, then an elementary computation shows that its highest order homogeneous part

$$H^0 = -\frac{1}{3}x^3 + axy^2 + \frac{1}{3}by^3$$

is non-degenerate, in the sense that it has an isolated critical point at the origin (equivalently $b^2 - 4a^3 \neq 0$). Consider the deformation

$$H^h(x, y) = (H(x\,h^{1/3}, y\,h^{1/3}) - h\,h_0)/h = H^0(x, y) + \frac{x^2 + y^2}{2} h^{-1/3} - h_0$$

where $h \in [1, \infty]$. As the global Milnor number of $H^h(x, y)$ does not depend on h [13, Proposition 3.2], then [13, Theorem 2.5] implies that the smooth fibration

$$\begin{array}{lcc} [1, \infty] \times \cup_{h\in[1,\infty]}\{(x, y) \in \mathbb{C}^2 : H^h(x, y) = 0\} & \to & [1, \infty] \\ (h, \{(x, y) \in \mathbb{C}^2 : H^h(x, y) = 0\}) & \to & h \end{array} \tag{19}$$

is trivial, provided that 0 is a regular value of $H^h(x, y)$ for all h, such that $h \in [1, \infty]$. Clearly the last condition is satisfied if $|h_0|$ is sufficiently big. We conclude that the continuous family of cycles $\delta(h)$ is defined for all h such that $h/h_0 \in [1, \infty]$. In the same way we prove that, more generally, the fibration

$$\begin{array}{lcr} S^1 \times [1, \infty] \times \cup_{h\in[1,\infty],|s|=1}\{(x, y) \in \mathbb{C}^2 : H^{h\,s}(x, y) = 0\} & \to & S^1 \times [1, \infty] \\ : (s, h, \{(x, y) \in \mathbb{C}^2 : H^{h\,s}(x, y) = 0\}) & \to & (s, h) \end{array} \tag{20}$$

is *locally* trivial. Therefore we can define a continuous deformation of cycles along the circle

$$\{|h_0|\,s : s = e^{i\varphi}, 0 \leq \varphi \leq 2\pi\}$$

and hence an automorphism

$$l_* : H_1(\Gamma_h, \mathbb{Z}) \to H_1(\Gamma_h, \mathbb{Z})\ . \tag{21}$$

The local triviality of (20) shows that l_* is the operator of classical monodromy of the singularity $H^0 = -\frac{1}{3}x^3 + axy^2 + \frac{1}{3}by^3$ of type D_4. This implies that $(l_*)^3 = id$ and hence the Abelian integrals $M(h), X(h), Y(h)$ are single valued with respect to $h^{-1/3}$ in a punctured neighborhood of ∞. The triviality of (19) implies

$$\lim_{h\to\infty} h^{-2/3}M(h) = -\int_{\delta(\infty)} y\,dx,\ \lim_{h\to\infty} h^{-1}X(h) = -\int_{\delta(\infty)} xy\,dx,$$

$$\lim_{h\to\infty} h^{-1}Y(h) = -\frac{1}{2}\int_{\delta(\infty)} y^2\,dx,$$

where $\delta(\infty) \subset \{(x, y) \in \mathbb{C}^2 : H^0(x, y) = h_0\}$. In particular we get

$$c_M^\infty = \frac{2}{9}\int_{\delta(\infty)} y\,dx\ .$$

To prove that $c_M^\infty \neq 0$ we use that

$$\frac{2}{9}\int_{H^0=h} y\,dx = c_M^\infty \left(\frac{h}{h_0}\right)^{2/3}$$

and

$$\frac{2}{9}\int_{H^0=h} \frac{dx}{H_y^0} = c_M^\infty \frac{d}{dh}\left(\frac{h}{h_0}\right)^{2/3} .$$

The differential one-form dx/H_y^0 is holomorphic on the elliptic curve

$$\{(x, y) \in \mathbb{C}^2 : H^0(x, y) = h\},\ h \neq 0$$

and hence

$$\int_{H^0=h} \frac{dx}{H_y^0} \neq 0, \quad \text{if } h \neq 0 .$$

The above shows also that for fixed generic a, b the functions $h^{4/3}M''(h)$, $h^{4/3}X''(h)$, $h^{4/3}Y''(h)$ are holomorphic with respect to h in a neighborhood of $\infty \in \mathbb{CP}^1$. Finally we note that if a, b are not fixed, but belong to a sufficiently small neighborhood $U \subset \mathbb{C}^2$ of some generic point (a_0, b_0), then the trivial fibration (19) can be replaced by a similar trivial fibration with a base $[1, \infty] \times U$, and (20) by a locally trivial fibration with a base $S^1 \times [1, \infty] \times U$. The same arguments as above show that the functions $h^{4/3}M''(h)$, $h^{4/3}X''(h)$, $h^{4/3}Y''(h)$, are bounded and single valued in a neighborhood of $(\infty, a_0, b_0) \in \mathbb{CP}^1 \times \mathbb{C}^2$, and hence can be continued to holomorphic functions.

Finally to obtain explicit formulae for c_X, c_Y we use the Picard-Fuchs equation satisfied by $X(h), Y(h), M(h), K(h)$. Namely, from [19, Lemma 3.3] we have

$$\begin{aligned}&(6\,h\,(4\,a^3 - b^2) + a + 1)Y''(h) + (b^2 + 8\,a^2 + 4\,a)\,K''(h)\\ &+ (a + 1)\,b\,M'(h) = 0\end{aligned}$$

$$\begin{aligned}&(6\,h - 1)\,b\,(4\,a^3 - b^2)\,X''(h) + a\,(4\,a^3 - 2\,a^2 - 2\,a - b^2)\,Y''(h)\\ &+ [(4\,a^2 + 3\,a + 1)(4\,a^3 - b^2) - 8\,a^3(a + 1)^2]\,K''(h)\\ &+ b\,(2\,a^3 - 2\,a^2 - b^2)\,M' = 0 .\end{aligned}$$

As $K''(h)$ is meromorphic at ∞ with respect to $h^{-1/3}$ and $h^{2/3}K''(h) = O(1)$ then the result follows. □

Corollary 3. $l_\infty = \{([\alpha : \beta : \gamma], a, b) \in \mathbb{RP}^2 \times \mathbb{R}^2 : \alpha\, c_X^\infty + \beta\, c_Y^\infty + \gamma\, c_M^\infty = 0\}$

Corollary 4. *For a fixed generic a, b let*

$$\frac{d^2}{dh^2} I_{\alpha\beta\gamma}(h) = h^{-4/3}\left(c_1^\infty + c_2^\infty\, h^{-1/3} + c_3^\infty\, h^{-2/3} + c_4^\infty\, h^{-1} + \ldots\right) \quad (22)$$

and consider a parameterized curve $s \to ([\alpha(s), \beta(s), \gamma(s)], a(s), b(s))$*,* $s \in (-\varepsilon, \varepsilon)$ *which intersects* **B** *transversally at* $([\alpha(0), \beta(0), \gamma(0)], a(0), b(0)) \in l_\infty$*. If the coefficient* c_2 *does not vanish, then a* simple zero *of* $\frac{d^2}{dh^2} I_{\alpha\beta\gamma}(h)$ *bifurcates from* $h = \infty$ *as the parameter* s *crosses* $s = 0$*.*

It follows that a multiple zero of $\frac{d^2}{dh^2} I_{\alpha\beta\gamma}(h)$ bifurcates from ∞ only for those $\alpha, \beta, \gamma, a, b$, with $c_1^\infty = c_2^\infty = 0$.

Proposition 2. *For every fixed generic* a, b *there is an unique* $[\alpha : \beta : \gamma] \in \mathbb{RP}^2$ *such that* $c_1^\infty = c_2^\infty = 0$ *in (22).*

Proof. Consider the Abelian integrals

$$I_{\alpha\beta\gamma}(h) = \int_{\delta_1^c(h)} \alpha\, \omega_X + \beta\, \omega_Y + \gamma\, \omega_M \ , \ J_{\alpha\beta\gamma}(h) = \int_{l_* \delta_1^c(h)} \alpha\, \omega_X + \beta\, \omega_Y + \gamma\, \omega_M \ ,$$

where l_* is the operator of the classical monodromy (21). As $< \delta_1, l_* \delta_1 > \neq 0$, then $\delta_1(h)$ and $l_* \delta_1(h)$ form a basis of $H_1(\bar{\Gamma}_h, \mathbb{Z})$ for regular values of h. It follows from Lemma 5 that for any fixed $[\tilde{\alpha} : \tilde{\beta} : \tilde{\gamma}]$ holds

$$\tilde{W}(h) = det \begin{pmatrix} \frac{d^2}{dh^2} I_{\alpha\beta\gamma}(h) & \frac{d^2}{dh^2} I_{\tilde{\alpha},\tilde{\beta},\tilde{\gamma}}(h) \\ \frac{d^2}{dh^2} J_{\alpha\beta\gamma}(h) & \frac{d^2}{dh^2} J_{\tilde{\alpha},\tilde{\beta},\tilde{\gamma}}(h) \end{pmatrix} = \frac{P_1(h)}{\delta(h)} \tag{23}$$

where $P_1(h)$ is a polynomial in h of degree at most one, and $\delta(h)$ is the discriminant of $H(x, y)$ (a real polynomial in h of degree 4). If the claim of Proposition 2 does not hold true, then there exist $[\alpha : \beta : \gamma]$, $[\tilde{\alpha} : \tilde{\beta} : \tilde{\gamma}]$ and such that in a neighborhood of $h = \infty$ holds

$$\frac{d^2}{d\, h^2} I_{\alpha\beta\gamma}(h) = c_4^\infty h^{-7/3} + \dots, \ \frac{d^2}{d\, h^2} I_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}}(h) = \tilde{c}_5^\infty h^{-8/3} + \dots .$$

It follows that

$$\tilde{W}(h) = det \begin{pmatrix} c_4^\infty\, h^{-7/3} + \dots & \tilde{c}_5^\infty h^{-8/3} + \dots \\ c_4^\infty h^{-7/3} e^{-2\pi i/3} + \dots & \tilde{c}_5^\infty h^{-8/3} e^{2\pi i/3} + \dots \end{pmatrix} = O(h^{-15/3})$$

which is only possible if $P_1(h) \equiv 0$ in (23). As the integrals $I_{\alpha\beta\gamma}(h)$ and $I_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}}(h)$ may be supposed linearly independent (Lemma 6) then the De Rham theorem implies that the one-forms $\omega = \alpha\, \nabla^2 \omega_X + \beta\, \nabla^2 \omega_Y + \gamma\, \nabla^2 \omega_M$ and $\tilde{\omega} = \tilde{\alpha}\, \nabla^2 \omega_X + \tilde{\beta}\, \nabla^2 \omega_Y + \tilde{\gamma}\, \nabla^2 \omega_M$ restricted to Γ_h are cohomological for every regular value of h, and therefore there is a relation

$$a(h)\omega + b(h)\tilde{\omega} = 0 \in H^1_{DR}(\Gamma_h, \mathbb{Z}), \ a(h)/b(h) \not\equiv 0 \ .$$

According to Corollary 2 the functions $a(h)$ and $b(h)$ are linear in h. Replacing eventually ω and $\tilde{\omega}$ by their suitable linear combination we can always suppose that

$$h\, \omega + \tilde{\omega} = 0 \in H^1_{DR}(\Gamma_h, \mathbb{Z}). \tag{24}$$

where

$$\frac{d^2}{dh^2} I_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}}(h) = h^{-4/3}\big(\tilde{c}_4^{\infty}\, h^{-3/3} + ...\big),$$

and hence

$$\frac{d^2}{dh^2} I_{\alpha\beta\gamma}(h) = h^{-4/3}\big(c_7^{\infty}\, h^{-6/3} + ...\big). \tag{25}$$

By Lemma 4 one can always find $\bar{\alpha}, \bar{\beta}, \bar{\gamma} \in \mathbb{R}$ such that

$$\overline{W}(h) = det \begin{pmatrix} \frac{d^2}{dh^2} I_{\alpha\beta\gamma}(h) & \frac{d^2}{dh^2} I_{\bar{\alpha},\bar{\beta},\bar{\gamma}}(h) \\ \frac{d^2}{dh^2} J_{\alpha\beta\gamma}(h) & \frac{d^2}{dh^2} J_{\bar{\alpha},\bar{\beta},\bar{\gamma}}(h) \end{pmatrix} = \frac{P_1(h)}{\delta(h)} \not\equiv 0 \tag{26}$$

On the other hand (25) implies

$$\overline{W}(h) = det \begin{pmatrix} h^{-4/3}\big(c_7^{\infty}\, h^{-6/3} + ...\big) & \bar{c}_1 h^{-4/3} + ... \\ h^{-4/3}\big(c_7^{\infty}\, h^{-6/3} + ...\big) & \bar{c}_1 h^{-4/3} e^{-2\pi i/3} + ... \end{pmatrix} = O(h^{-14/3})$$

which contradicts to $P_1(h) \not\equiv 0$ in (26). □

3.2. Bifurcations of zeros from the critical values h_2, h_3, h_4

Proposition 3. *In a suitable neighborhood of $(h_i, a, b) \in \mathbb{C}^3$ holds*

$$I_{\alpha\beta\gamma}(h) = -\frac{\log(h - h_i)}{2\pi\sqrt{-1}}\, f(h, a, b, \alpha, \beta, \gamma) + g(h, a, b, \alpha, \beta, \gamma)$$

where f and g are linear in α, β, γ, holomorphic in h, a, b, and

$$f(h, a, b, \alpha, \beta, \gamma) = \int_{\delta_i(h)} \alpha\, \omega_X + \beta\, \omega_Y + \gamma\, \omega_M\, .$$

Proof. In a sufficiently small neighborhood of (h_i, a, b) the Abelian integral

$$f(h, a, b, \alpha, \beta, \gamma) = \int_{\delta_i(h)} \alpha\, \omega_X + \beta\, \omega_Y + \gamma\, \omega_M$$

is a bounded, single valued, and hence holomorphic function in h, a, b. We note also that $f(h_i, a, b, \alpha, \beta, \gamma) = 0$. Consider now the function

$$g(h, a, b, \alpha, \beta, \gamma) = I_{\alpha\beta\gamma}(h) + \frac{\log(h - h_i)}{2\pi\sqrt{-1}}\, f(h, a, b, \alpha, \beta, \gamma)\, .$$

In a small neighborhood of (h_i, a, b) the cycle $\delta_1(h)$ can be represented by a loop on the affine algebraic curve Γ_h which is bounded, and therefore $I_{\alpha\beta\gamma}(h)$ remains bounded too. The function

$$\frac{\log(h - h_i)}{2\pi\sqrt{-1}}\, f(h, a, b, \alpha, \beta, \gamma)$$

is also bounded, because $f(h_i, a, b, \alpha, \beta, \gamma) = 0$. Finally the Picard-Lefschetz formula shows that $g(h, a, b, \alpha, \beta, \gamma)$ is single valued and hence holomorphic. □

Corollary 5.

$$l_i = \left\{([\alpha : \beta : \gamma], a, b) \in \mathbb{RP}^2 \times \mathbb{R}^2 \ : \alpha\, x_i + \beta\, y_i + \gamma = 0\right\}$$

where (x_i, y_i) *is the critical point of* $H(x, y)$ *associated to the critical value* h_i, $i = 2, 3, 4$.

Proof. From Proposition 3 we have that in a neighborhood of $h = 1/6$ holds

$$\frac{d^2}{dh^2} I_{\alpha\beta\gamma}(h) = \pm \frac{1}{2\pi\sqrt{-1}\left(h - \frac{1}{6}\right)} \int_{\delta_2^s(h)} \alpha\nabla\omega_X + \beta\nabla\omega_Y + \gamma\nabla\omega_M \quad (27)$$
$$\pm \frac{\log\left(h - \frac{1}{6}\right)}{2\pi\sqrt{-1}} \int_{\delta_2^s(h)} \alpha\, \nabla^2\omega_X + \beta\, \nabla^2\omega_Y + \gamma\, \nabla^2\omega_M + \text{ h.f.}$$

where $\delta_2^s(h)$ is the continuous family of cycles vanishing at the saddle point corresponding to the critical value $h = 1/6$, and "h.f." means a suitable holomorphic function in a neighborhood of $h = 1/6$. Therefore $([\alpha : \beta : \gamma], a, b) \in l_i$ if and only if

$$\int_{\delta_i(h_i)} \alpha\nabla\omega_X + \beta\nabla\omega_Y + \gamma\nabla\omega_M = 0$$

which, on its hand is equivalent to

$$\alpha\, x_i + \beta\, y_i + \gamma = 0\ .$$

Indeed, it is easy to check that

$$\lim_{h\to h_i} \int_{\delta(h)} \frac{dx \wedge dy}{d\,H} \neq 0$$

where $\frac{dx\wedge dy}{d\,H}$ is the Gel'fand-Leray form of $dx \wedge dy$. As

$$\frac{\alpha\, d\,\omega_X + \beta\, d\,\omega_Y + \gamma\, d\,\omega_M}{d\,H} = (\alpha\, x_i + \beta\, y_i + \gamma)\frac{dx \wedge dy}{d\,H}$$

then the Corollary is proved.

3.3. Bifurcations of zeros from the regular points on the interval $(1/6, \infty)$

We say that a point on the the interval $(1/6, \infty)$ is regular, if $h \neq h_i$. In a neighborhood of such a point the Abelian integral $\frac{d^2}{dh^2} I_{\alpha\beta\gamma}(h)$ is a holomorphic function in $h, a, b, \alpha, \beta, \gamma$. Let

$$I^{\pm}_{\alpha\beta\gamma}(h) = \int_{\delta_1^{\pm}(h)} \alpha\omega_X + \beta\omega_Y + \gamma\omega_M,\ h \in \left(\frac{1}{6}, h_3\right) \cup (h_3, h_4) \cup (h_4, \infty)$$

be the analytic continuation of the Abelian integral $I_{\alpha\beta\gamma}(h)$, $h \in \mathcal{D} = \mathbb{CP}^1 \backslash [\frac{1}{6}, \infty]$ along an arc such that $Im(h) > 0$ ($Im(h) < 0$) respectively. For $h \in (1/6, \infty)$ we have $\delta_1^+(h) = \overline{\delta_1^-(h)}$. It follows that if a zero bifurcates from a regular value $h_0 \in (1/6, \infty)$, then in fact a pair of complex conjugate zeros bifurcate from h_0 and in particular

$$\int_{\delta_1^+(h_0)} \alpha\nabla^2\omega_X + \beta\nabla^2\omega_Y + \gamma\nabla^2\omega_M = \int_{\delta_1^-(h_0)} \alpha\nabla^2\omega_X + \beta\nabla^2\omega_Y + \gamma\nabla^2\omega_M = 0\ .$$

On the other hand it is easy to check, by making use of the Picard-Lefschetz formula, that in each particular case on Fig. 3 the cycles $\delta_1^+(h)$, $\delta_1^-(h)$ form a base of $H_1(\overline{\Gamma_h}, \mathbb{Z})$, and hence the one-form

$$\alpha\, \nabla^2\omega_X + \beta\, \nabla^2\omega_Y + \gamma\, \nabla^2\omega_M$$

restricted to the curve $\overline{\Gamma_h}$ is cohomological to zero. The coefficients α, β, γ of this one-form can be explicitly computed. Indeed, it follows from Corollary 2 that for every regular value h of $H(x, y)$ there exists an *unique point* $[\alpha(h) : \beta(h) : \gamma(h)] \in \mathbb{RP}^2$ such that the cohomology class of the differential form

$$\alpha(h)\, \nabla^2\omega_X + \beta(h)\, \nabla^2\omega_Y + \gamma(h)\, \nabla^2\omega_M$$

is equal to zero and hence

Corollary 6.

$$\begin{aligned} \Delta = \{([\alpha : \beta : \gamma], a, b) \in \mathbb{RP}^2 \times \left\{ \cup_{i=1}^3 \Omega_i^0 \right\} :\\ \alpha = \alpha(h), \beta = \beta(h), \gamma = \gamma(h), h \in [1/6, \infty]\} \end{aligned}$$

where $\alpha(h)$, $\beta(h)$, $\gamma(h)$ *are real linear functions in* h.

Finally we shall prove that for every fixed $(a, b) \in \cup_{i=1}^3 \Omega_i^0$ the straight lines l_i^{ab}, l_∞^{ab} are distinct and do not contain Δ^{ab}. First we shall check that Δ^{ab} is not contained in any of these lines. The coordinates of the point $[\alpha(0) : \beta(0) : \gamma(0)] \in \Delta^{ab}$ are $[b : 1-a : 0]$ (see (33) and $[b : 1-a : 0] \in l_i^{ab}$ is equivalent to

$$b\, x_i + (1 - a)\, y_i = 0\ . \tag{28}$$

One easily shows that

$$y\left(-b^2+1-3\,a^2+2\,a^3\right)$$

belongs to the ideal generated by H_x, H_y and $b\,x+(1-a)\,y$. Therefore (28) is only possible if $y=0$ or $-b^2+1-3\,a^2+2\,a^3=0$. The second identity is impossible because $(a,b)\in\cup_{i=1}^3\Omega_i^0$. The line $y=0$ contains the critical points $(0,0)$ and $(1,0)$ which combined with (28) gives $b=0$ in contradiction with $(a,b)\in\cup_{i=1}^3\Omega_i^0$.

4. Upper bounds for the number of the zeros of the Abelian integral $\frac{d^2}{dh^2}I_{\alpha\beta\gamma}(h)$ in a complex domain

Theorem 4.1. *The maximum number of the zeros of the Abelian integral $\frac{d^2}{dh^2}I_{\alpha\beta\gamma}(h)$ in the complex domain $\mathcal{D}=\mathbb{C}\backslash[1/6,\infty)$ is equal to two if X_H has one saddle and one center, and four otherwise.*

Proof. Suppose first that X_H has one saddle and one center. This means that $H(x,y)$ has two Morse real critical points at $(0,0)$ and $(1,0)$, as well two complex conjugate Morse critical points. As $\nabla\omega_M$ is a holomorphic one-form on the elliptic curve $\bar{\Gamma}_h$, then $\int_{\delta_1(h)}\nabla\omega_M$ does not vanish, provided that $h\neq h_i$, and $\int_{\delta_1(0)}\nabla\omega_M=2\pi\neq 0$. We shall count the number of the zeros of the analytic function

$$F(h)=\frac{\int_{\delta_1^c(h)}\alpha\nabla^2\omega_X+\beta\nabla^2\omega_Y+\gamma\nabla^2\omega_M}{\int_{\delta_1(h)}\nabla\omega_M}$$

in the complex domain $\mathcal{D}$, by making use of the argument principle. Let R be a big enough constant and r a small enough constant. Denote by $\tilde{\mathcal{D}}$ the set obtained by removing the small disc $\{|h-h_2^s|<r\}$ from $\mathcal{D}\cap\{|h|<R\}$. To estimate the number of the zeros of $F(h)$ in $\tilde{\mathcal{D}}$ (and hence of $\frac{d^2}{dh^2}I_{\alpha\beta\gamma}(h)$ in $\mathcal{D}$) we shall evaluate the increment $\Delta_{\partial\tilde{\mathcal{D}}}Arg(F(h))$ of the argument of the function $F(h)$ along the boundary of $\tilde{\mathcal{D}}$, traversed in a positive direction. Then, according to the argument principle, we have that the number of the zeros of $F(h)$ in $\tilde{\mathcal{D}}$ equals to

$$\frac{\Delta_{\partial\tilde{\mathcal{D}}}Arg(F(h))}{2\,\pi}.$$

Let

$$\varphi\to h(\varphi):S^1\to\partial\tilde{\mathcal{D}},\ \ S^1=\mathbb{R}/2\,\pi\,\mathbb{Z}$$

be a parameterization of the oriented boundary of $\tilde{\mathcal{D}}$, which is analytic, except at $h=r$ and $h=R$. If R is big enough and r is small enough, then

$F(h) \neq 0$ along $\partial\tilde{\mathcal{D}}$ and hence the increase of the argument $\Delta_{\partial\tilde{\mathcal{D}}} Arg(F(h))$ divided by 2π equals to the degree of the map of oriented circles

$$S^1 \to S^1 : \varphi \mapsto z(\varphi) = \frac{F(h(\varphi))}{|F(h(\varphi))|} ,$$

where the orientation of $\mathbb{R}/2\pi\mathbb{Z}$ is induced by the orientation of $\partial\tilde{\mathcal{D}}$, and the orientation of $\{z \in \mathbb{C} : |z| = 1\}$ is induced by the orientation of $\mathbb{C}$.

Recall that if z_0 is a regular value of this map, then

$$\deg(S^1 \to S^1) = \sum_{\varphi \in z^{-1}(z_0)} \deg_\varphi(S^1 \to S^1)$$

where $\deg_\varphi(S^1 \to S^1)$ equals to $+1$ if $z'(\varphi) \in T_{z(\varphi)} S^1$ is positively oriented, and -1 otherwise. In this case we shall also say that the immersed curve $F(\partial\tilde{\mathcal{D}})$ crosses the half-line $\{z : z = z_0\rho, \rho > 0\}$ at $F(h(\varphi))$ in a positive (respectively negative) direction. This leads to the following equivalent formulation of the argument principle.

The number of the zeros of $F(h)$ in $\tilde{\mathcal{D}}$ equals to the number of the intersections of the immersed curve $F(\partial\tilde{\mathcal{D}})$ with the half-line $\{Im(z) = 0, \ Re(z) > 0\}$ counted with their signs. The sign is positive, if $F(\partial\tilde{\mathcal{D}})$ crosses $\{Im(z) = 0, \ Re(z) > 0\}$ in a positive direction, and negative otherwise.

Consider the auxiliary function $\tilde{F}(h) = (h - h_2)\, F(h)$. We shall study first the intersections of $\tilde{F}(S_r(h_2))$, $S_r(h_2) = \{h : |h - h_2^s| = r, h \neq h_2 + r\}$ with $\{z : Im(z) = 0\}$. In a neighborhood of $h = h_2$ we use (27) to obtain

$$\tilde{F}(h) = \frac{1}{\log(h - h_2)} \frac{\int_{\delta_2(h_2)} \alpha\nabla\omega_X + \beta\nabla\omega_Y + \gamma\nabla\omega_M}{\int_{\delta_2(h_2)} \nabla\omega_M} + O\left(\frac{1}{(\log(h - h_2))^2}\right). \tag{29}$$

The increase of the argument of $\tilde{F}(h)$ along $S_r(h_2)$ is close to zero and hence $\tilde{F}(S_r(h_2))$ crosses $\{z : Im(z) = 0\}$ exactly once (at $h = h_2 - r$). Moreover $\tilde{F}(S_r(h_2))$ crosses $\{z : Im(z) = 0\}$ in the same direction as the curve $G(S_r(h_2))$, $G(h) = 1/\log(h_2 - h)$ does. Finally we note that when h traverses $S_r(h_2)$ in a negative direction (which is induced by the positive orientation of $\partial\tilde{\mathcal{D}}$), the curve $G(S_r(h_2))$ crosses $\{z : Im(z) = 0\}$ exactly once in a *negative* direction. Indeed, suffice it to apply the argument principle to the function $G(h) = 1/\log(h_2 - h)$ in the complex domain $\mathcal{D}$. As $G(h)$ has a simple pole at $h = h_2 - 1 \in \mathcal{D}$ and has no zeros, and $Im G(h) \neq 0$ for $h \in [r, R]$, then the result follows.

For $h \in [r, R]$ denote by $\tilde{F}^+(h)$ (respectively $\tilde{F}^-(h)$ the value of $\tilde{F}(h)$ obtained by analytic continuation along an arc contained in $\mathcal{D} \cap \{z : Im(z) > 0\}$

($\mathcal{D} \cap \{z : Im(z) < 0\}$ respectively). (The same notation will be used for the locally constant sections $\delta(h)$ of the homology bundle.) The number of intersections of $\tilde{F}^+([r, R])$ with $\{z : Im(z) = 0\}$ equals to the number of the zeros of the imaginary part of $\tilde{F}(h)$ on the interval $[r, R]$. Let us denote

$$W_{\gamma_1\gamma_2}(\omega_1, \omega_2) = det \begin{pmatrix} \int_{\gamma_1(h)} \omega_1 & \int_{\gamma_1(h)} \omega_2\omega_M \\ \int_{\gamma_2(h)} \omega_1 & \int_{\gamma_2(h)} \omega_2\omega_M \end{pmatrix} \equiv \frac{P_1(h)}{\prod_{i=1}^4 (h - h_i)} .$$

For $h \in [r, R]$ we have

$$\begin{aligned} Im\tilde{F}(h) &= \frac{1}{2}(\tilde{F}^+(h) - \tilde{F}^-(h)) \\ &= \frac{1}{2}\left(\frac{\int_{\delta_1^+(h)} \alpha\nabla^2\omega_X + \beta\nabla^2\omega_Y + \gamma\nabla^2\omega_M}{\int_{\delta_1^+(h)} \nabla\omega_M}\right. \\ &\qquad \left. - \frac{\int_{\delta_1^-(h)} \alpha\nabla^2\omega_X + \beta\nabla^2\omega_Y + \gamma\nabla^2\omega_M}{\int_{\delta_1^-(h)} \nabla\omega_M}\right) \\ &= \frac{W_{\delta_1^+\delta_1^-}(\alpha\nabla^2\omega_X + \beta\nabla^2\omega_Y + \gamma\nabla^2\omega_M, \nabla\omega_M)}{2\int_{\delta_1^+(h)} \nabla\omega_M \int_{\delta_1^-(h)} \nabla\omega_M} . \end{aligned}$$

As $W_{\delta_1^+\delta_1^-}(\alpha\nabla^2\omega_X + \beta\nabla^2\omega_Y + \gamma\nabla^2\omega_M, \nabla\omega_M)$ is a polynomial of degree at most two in h divided by the discriminant of $H(x, y; a, b) - h$ (Lemma 5), then the imaginary part of $\tilde{F}(h)$ vanishes at most twice along the interval $[r, R]$.

Consider finally $\tilde{F}(h)$ restricted to

$$S_R(\infty) = \{h \in \mathbb{C} : |h| = R, h \neq R\} .$$

We have

$$\tilde{F}(h) = -\frac{\alpha\, c_X + \beta\, c_Y + \gamma\, c_M}{3\, c_M} + O(h^{-\frac{1}{3}}) .$$

Therefore the increase of the argument of $\tilde{F}(h)$ along $S_R(\infty)$ is close to zero. It follows that the immersed curve $\tilde{F}(S_R(\infty))$ crosses $\{z \in \mathbb{C} : Im(z) = 0\}$ exactly once (at $h = -R$).

Summing up the above data we conclude that $\tilde{F}(\partial\tilde{\mathcal{D}})$ crosses the line $\{z \in \mathbb{C} : Im(z) = 0\}$ at most six times, and at least one of the crossings is in a negative direction (at $h = h_2 - r$). The argument principle implies that $\tilde{F}(h)$ has at most two zeros in $\tilde{\mathcal{D}}$, and hence $F(h)$ has at most two zeros in $\mathcal{D}$.

Suppose now that $H(x, y)$ has four real critical points. The proof is as before, except that the critical values h_3 and h_4 belong now to the interval $(1/6, \infty)$. In a neighborhood of h_i, $i = 3, 4$, we have a formula similar to (29) (with h_2, δ_2 replaced by h_i, δ_i) which shows that the increase of the argument of $\tilde{F}(h)$ along the semi-circles $\{h = r\, e^{i\varphi} : 0 \le \varphi \le \pi\}$, $\{h = r\, e^{i\varphi} : \pi \le \varphi \le 2\pi\}$ is close to zero. Therefore the immersed curve $\tilde{F}(\partial\tilde{\mathcal{D}})$ crosses the line $\{z \in \mathbb{C} : Im(z) = 0\}$ at most ten times, and at least one of the crossings is in a negative direction. The argument principle implies that $\tilde{F}(h)$ has at most four zeros in $\tilde{\mathcal{D}}$, and hence $F(h)$ has at most four zeros in $\mathcal{D}$. □

The bound for the number of the zeros of the Abelian integral $\frac{d^2}{dh^2} I_{\alpha\beta\gamma}(h)$, found in Theorem 4.1, will be enough for the proof of the main result of the present paper, except in the case when the vector field X_H has two centers and two saddles. Let us denote by $\mathcal{U}$ the open connected component of the set

$$\left\{\mathbb{RP}^2 \times \left\{\Omega_2^0\right\}\right\} \backslash \mathbf{B}$$

which contains the set

$$\left\{([\alpha(0) : \beta(0) : \gamma(0)], a, b) : (a, b) \in \Omega_2^0\right\} .$$

Theorem 4.2. *If* $([\alpha : \beta : \gamma], a, b) \in \mathcal{U}$ *then the associated Abelian integral* $\frac{d^2}{dh^2} I_{\alpha\beta\gamma}(h)$ *has exactly one zero in the complex domain* $\mathcal{D} = \mathbb{C}\backslash[1/6, \infty)$.

Proof. The sections $\mathcal{U}^{ab}$ of $\mathcal{U}$ are shown on Fig. 4. As $\mathcal{U}^{ab}$ is connected, then $\mathcal{U}$ is connected too and Theorem 3.1 shows that it is enough to find the number of the zeros of a fixed Abelian integral $\frac{d^2}{dh^2} I_{\alpha\beta\gamma}(h)$ for a fixed Hamiltonian $H(x, y; a, b)$, where $([\alpha : \beta : \gamma], a, b) \in \mathcal{U}$. Namely let us put $\alpha = \alpha(0)$, $\beta = \beta(0)$, $\gamma = \gamma(0)$, where the functions $\alpha(h)$, $\beta(h)$, $\gamma(h)$ were defined in Corollary 2 and Corollary 6. Recall that the differential one-form

$$\omega(h) = \alpha(h)\nabla^2\omega_X + \beta(h)\nabla^2\omega_Y + \gamma(h)\nabla^2\omega_M$$

is homologous to zero on the level set Γ_h, i.e. $\int_{\delta_i(h)} \omega(h) \equiv 0$, $i = 1, 2, 3, 4$ (Corollary 2). We have $\omega(h) = \omega(0) + h\,\omega(\infty)$, where

$$\omega(\infty) = \alpha'(h)\nabla^2\omega_X + \beta'(h)\nabla^2\omega_Y + \gamma'(h)\nabla^2\omega_M$$

(the functions $\alpha(h)$, $\beta(h)$, $\gamma(h)$ are linear in h) and hence $\omega(0)$ is homologous to $-h\,\omega(\infty)$. Therefore the leading term of $\int_{\delta_i(h)} \omega(0)$ in a neighborhood of $h = 0$ vanishes which gives

$$\int_{\delta_1(0)} \alpha(0)\nabla^2\omega_X + \beta(0)\nabla^2\omega_Y + \gamma(0)\nabla^2\omega_M = 0 \tag{30}$$

$$\int_{\delta_1(0)} \alpha(0)\nabla\,\omega_X + \beta(0)\nabla\,\omega_Y + \gamma(0)\nabla\,\omega_M = 0\,. \tag{31}$$

The geometric interpretation of the above identities is that the line

$$t_c = \{(x, y) \in \mathbb{R}^2 : \alpha(0)\,x + \beta(0)\,y + \gamma(0) = 0\} \tag{32}$$

is tangent to the centroid curve $L_{ab} \subset \mathbb{R}^2$ at the point $(\xi(0), \eta(0))$. The equation of this line is computed in [19, Corollary 3.4] and we have

$$[\alpha(0) : \beta(0) : \gamma(0)] = [b : 1 - a : 0] \ . \tag{33}$$

Suppose in addition that $a = 0$. The vector field X_H has in this case a central symmetry and the properties of the centroid curve L_{ab} were studied in details in [22].

The zeros of the Abelian integral $\frac{d^2}{dh^2} I_{b10}(h)$ will be computed in two steps. First we shall show that the imaginary part of the function

$$F(h) = \frac{\frac{d^2}{dh^2} I_{b10}(h)}{\frac{d}{dh} I_{001}(h)}, \ 0 < b < 1$$

does not vanish on $(1/6, h_3) \cup (h_3, h_4) \cup (h_4, \infty)$. This would imply, as in the proof of Theorem 4.1, that $\frac{d^2}{dh^2} I_{b10}(h)$ has at most two zeros in the complex domain $\mathcal{D}$. At the second step we shall check that the Abelian integral $\frac{d^2}{dh^2} I_{b10}(h)$ has an odd number of zeros on $(-\infty, 1/6)$ which combined to $\frac{d^2}{dh^2} I_{b10}(0) = 0$ implies Theorem 4.2.

The number of the zeros of the imaginary part of $F(h)$ on $(1/6, \infty)$ is the same as the number of the zeros of the Wronskian

$$W_{\delta_1 \delta_2}(\omega(0), \nabla \omega_M) \ .$$

We are going to compute explicitly the above function. From [22, Sect. 6] we get the following identities (replacing λ by b)

$$-b^3 (6h - 1) X'' + (1 - 6hb^2) Y'' + (b - b^3) M' = 0 \tag{34}$$

$$b X'' - Y'' - 6bh M'' - 2b M' = 0 \tag{35}$$

where the Abelian integrals $X(h)$, $Y(h)$, $M(h)$ were defined in Sect. 2. The identity (34) shows that the vector

$$(-b^3 (6h - 1), 1 - 6hb^2, b - b^3)$$

is co-linear to the vector

$$\left(W_{\delta_1 \delta_2}\left(\nabla^2 \omega_Y, \nabla \omega_M\right), -W_{\delta_1 \delta_2}\left(\nabla^2 \omega_X, \nabla \omega_M\right), W_{\delta_1 \delta_2}\left(\nabla^2 \omega_X, \nabla^2 \omega_Y\right)\right)$$

and hence

$$\begin{aligned} W_{\delta_1 \delta_2}\left(\nabla^2 \omega_Y, \nabla \omega_M\right) &= -b^3 (6h - 1) k_1 \\ -W_{\delta_1 \delta_2}\left(\nabla^2 \omega_X, \nabla \omega_M\right) &= (1 - 6hb^2) k_1 \\ W_{\delta_1 \delta_2}\left(\nabla^2 \omega_X, \nabla^2 \omega_Y\right) &= (b - b^3) k_1 \ . \end{aligned}$$

In the same way we use (35) to obtain

$$W_{\delta_1\delta_2}\big(b\,\nabla^2\omega_X - \nabla^2\omega_Y, \nabla\,\omega_M\big) = -6\,b\,h\,k_2$$

$$W_{\delta_1\delta_2}\big(b\,\nabla^2\omega_X - \nabla^2\omega_Y, \nabla^2\omega_M\big) = -2\,b\,k_2$$

$$W_{\delta_1\delta_2}\big(\nabla^2\omega_M, \nabla\,\omega_M\big) = -k_2\ .$$

From the above formulae we get two different expressions for $W_{\delta_1\,\delta_2}(b\nabla^2\omega_X - \nabla^2\omega_Y, \nabla\,\omega_M)$ involving k_1 and k_2 respectively which gives

$$-6\,b\,h\,k_2 = b\,(1 - 6\,h\,b^2)\,k_1 + b^3\,(6\,h - 1)\,k_1\ .$$

It is easy to see that k_1 and k_2 are linear functions in h, divided by the discriminant $\prod_{i=1}^4 (h - h_i)$, and hence, up to multiplication by a non-zero constant in h we have

$$k_1 = \frac{6\,h}{\prod_{i=1}^4 (h - h_i)},\ \ k_2 = -\frac{12\,b^2\,h + b^2 + 1}{\prod_{i=1}^4 (h - h_i)}\ .$$

We obtain finally

$$W_{\delta_1\delta_2}(\omega(0), \nabla\,\omega_M) = W_{\delta_1\delta_2}\big(b\nabla^2\omega_X + \nabla^2\omega_Y, \nabla\,\omega_M\big)$$

$$= (b - b^3)\frac{6\,h}{\prod_{i=1}^4 (h - h_i)}$$

which shows that the imaginary part of $F(h)$ does not vanish on

$$(1/6, h_3) \cup (h_3, h_4) \cup (h_4, \infty)\ .$$

Repeating the proof of Theorem 4.1 we get that the Abelian integral

$$\frac{d^2}{dh^2} I_{b10}(h) = \int\!\!\int_{\{H \le h\}} (b\,x + y)\,dx\,dy$$

has at most two zeros in the complex domain $\mathcal{D}$. As the real analytic function $\frac{d^2}{dh^2} I_{b10}(h)$ already vanishes at $h = 0$, then if it had a second zero in $\mathcal{D}$, this zero would belong to the interval $(-\infty, 1/6)$. Proposition 1 implies that in a neighborhood of $h = \infty$ holds

$$X''(h) = -\frac{1}{6}\,c_M\,h^{-4/3} + o(h^{-4/3})$$

$$Y''(h) = \frac{1}{6\,b}\,c_M\,h^{-4/3} + o(h^{-4/3})$$

$$M'(h) = -3\,c_M\,h^{-1/3} + o(h^{-1/3})$$

and hence

$$F(h) = \frac{\frac{d^2}{dh^2} I_{b10}(h)}{\frac{d}{dh} I_{001}(h)} = \frac{b\,X''(h) + Y''(h)}{M'(h)} = \frac{1-b^2}{6\,b\,h} + o\left(\frac{1}{h}\right).$$

As $0 < b < 1$ then if $|h|$ is sufficiently big and h is negative, the function $F(h)$ is also negative. On the other hand as in the proof of Corollary 5 we have

$$F(h) = \frac{\alpha(0)x_2 + \beta(0)y_2 + \gamma(0)}{(h-1/6)\ln(1/6-h)} + o\left(\frac{1}{(h-1/6)\ln(1/6-h)}\right)$$

$$= \frac{b}{(h-1/6)\ln(1/6-h)} + o\left(\frac{1}{(h-1/6)\ln(1/6-h)}\right)$$

which shows that if $|h - 1/6|$ is sufficiently small, but $h - 1/6$ is negative, the function $F(h)$ is positive. Thus the number of the zeros of the function $F(h)$ (and hence of $\frac{d^2}{dh^2} I_{b10}(h)$) on the interval $(-\infty, 1/6)$ is odd, which completes the proof of Theorem 4.2. □

5. The geometry of the centroid curve

Denote by $\{H \le h\}$ the interior of the oval $\delta_1(h)$ of the curve Γ_h, where $h \in [0, 1/6]$. For $h = 0$ it reduces to a point and for $h = 1/6$ it is a closed loop containing the saddle point $(1, 0)$ of the vector field X_H. Recall that the *centroid point* $(\xi(h), \eta(h))$ of $\{H \le h\}$ has coordinates

$$(\xi(h), \eta(h)) = \left(\frac{\iint_{\{H\le h\}} x\,dx \wedge dy}{\iint_{\{H\le h\}} dx \wedge dy}, \frac{\iint_{\{H\le h\}} y\,dx \wedge dy}{\iint_{\{H\le h\}} dx \wedge dy}\right),\quad h \in \left[0, \frac{1}{6}\right]$$

and consider the centroid curve

$$L_{ab} = \{(\xi(h), \eta(h)) : h \in [0, 1/6]\}$$

associated to the polynomial

$$H(x, y) = H(x, y; a, b) = \frac{1}{2}(x^2 + y^2) - \frac{1}{3}x^3 + axy^2 + \frac{1}{3}by^3 .$$

The importance of the curve L_{ab} lies in the fact that it is an affine invariant of $H(x, y)$, containing a complete information on the number and the multiplicity of the zeros of the Abelian integral

$$I_{\alpha\beta\gamma}(h) = \iint_{\{H\le h\}} (\alpha\,x + \beta\,y + \gamma)\,dx \wedge dy$$

on the interval $(0, 1/6]$. Indeed, $I_{\alpha\beta\gamma}(h)$ vanishes at $h \in [0, \frac{1}{6}]$ if and only if the affine line

$$\{(x, y) \in \mathbb{R}^2 : \alpha x + \beta y + \gamma = 0\} \tag{36}$$

intersects L_{ab} at the point $(\xi(h), \eta(h))$. Note also that we have always $I_{\alpha\beta\gamma}(0) = 0$, and $\frac{d}{dh} I_{\alpha\beta\gamma}(0) = 0$ if and only if the line (36) intersects L_{ab} at the point

$$(\xi(0), \eta(0)) = \lim_{h \to 0^+} (\xi(h), \eta(h)).$$

In the sequel it will be useful to consider the curve $L^*_{ab} \subset (\mathbb{RP}^2)^*$ dual to the centroid curve $L_{ab} \subset \mathbb{R}^2$. L^*_{ab} is the set of tangent lines to L_{ab}. Let $[\alpha : \beta : \gamma]$ be homogeneous coordinates in $(\mathbb{RP}^2)^*$. Then $[\alpha : \beta : \gamma] \in L^*_{ab}$ if and only if the line (36) is tangent to L_{ab}. It is also clear that the sections $\mathbf{B}^{ab}$ of the bifurcation set $\mathbf{B}$ live in $(\mathbb{RP}^2)^*$ too. For example the projective line $l_i^{ab} \subset (\mathbb{RP}^2)^*$ is just the set of affine lines in $\mathbb{R}^2$ containing the critical point (x_i, y_i). The study of the position of the dual centroid curve L^*_{ab} with respect to the bifurcation set $\mathbf{B}^{ab}$ in $(\mathbb{RP}^2)^*$ will be one of the main ingredients in the proof of the main result of the present paper.

5.1. The regularity of the centroid curve

Theorem 5.1. *The curve L_{ab} is smooth.*

On its hand the above result will follow from the following stronger

Theorem 5.2. *There exist real constants r, s such that*

$$r\,\xi'(h) + s\,\eta'(h) \neq 0, \ \forall h \in [0, 1/6)\ . \tag{37}$$

Corollary 7 (see Proposition 7.1. in [22]). *When running the centroid curve L_{ab}, the tangential vector rotates within an angle less than π.*

Proof of Theorem 5.1 assuming Theorem 5.2. As $\xi'(h)$ and $\eta'(h)$ can not vanish simultaneously, then L is locally smooth, except eventually at $(\xi(1/6), \eta(1/6))$. The smoothness of L at this point is equivalent to the claim that the direction of the tangent vector $(\xi'(h), \eta'(h))$ tends to a given direction as h tends to $1/6$. In fact it tends to the direction of the line through the centroid point $(\xi(1/6), \eta(1/6))$ and the saddle point $(1, 0)$ ([19, Theorem 3.1.(ii), Fig. 4]). It remains to show that L_{ab} has no points of self-intersection. Indeed, if $(\xi(h'), \eta(h')) = (\xi(h''), \eta(h''))$ for some $h', h'' \in [0, 1/6]$, then the function $r\,\xi(h) + s\,\eta(h)$ takes the same values at h', h'', and hence its derivative vanishes for some $\tilde{h} \in (h', h'')$ in contradiction with Corollary 7. $\square$

Proof of Theorem 5.2. Following [19, Sect. 4], we note that it is enough to find constants $r, s \in \mathbb{R}$, such that for every $k \in \mathbb{R}$, the number of the zeros of $r\,X'(h) + s\,Y'(h) - k\,M'(h)$ in the complex domain $\mathcal{D}$ is less or equal to one. For then the function $r\,X(h) + s\,Y(h) - k\,M(h)$ will have at most two zeros on $[0, 1/6)$ one of them being $h = 0$, which shows that $(r\,X(h) + s\,Y(h))/M(h) - k$ has at most one simple zero on $[0, 1/6)$ (we used that $X(0) = Y(0) = M(0) = 0$, but $M'(0) \neq 0$). This would imply that $(r\,X(h) + s\,Y(h))/M(h)$ is a strictly monotone function on $[0, 1/6)$, and hence (37).

From now on we shall suppose that the vector field X_H has two centers and two saddle points. The case when X_H has one center and one saddle is simpler (because the interval $(1/6, \infty)$ does not contain critical values of $H(x, y)$) and will not be considered in detail. The case when X_H has one center and three saddles is studied in [19]. The differential one-forms $\nabla\omega_X$ and $\nabla\omega_Y$ on $\bar{\Gamma}_h$ are of third kind and have three simple poles at $\infty_1, \infty_2, \infty_3$. Moreover their residues are constant in h. Let γ_i, be cycles on Γ_h represented by small loops around ∞_i, $i = 1, 2, 3$. The cycles γ_i are homologous to zero in $H_1(\bar{\Gamma}_h, \mathbb{Z})$ and

$$2\pi\sqrt{-1}\,Res_{\infty_i}\omega = \int_{\gamma_i}\omega$$

for every meromorphic one-form ω on $\bar{\Gamma}_h$. For $h \in \mathbb{R}$ the curve Γ_h has an obvious real structure. The corresponding anti-holomorphic involution (the complex conjugation) induces involutions of $H_1(\Gamma_h, \mathbb{Z})$ and $H^1_{DR}(\Gamma_h, \mathbb{C})$. Without loss of generality we may suppose that

$$\bar{\infty}_1 = \infty_2,\ \bar{\infty}_2 = \infty_1,\ \bar{\infty}_3 = \infty_3$$

and hence

$$\bar{\gamma}_1 = \gamma_2,\ \bar{\gamma}_2 = \gamma_1,\ \bar{\gamma}_3 = -\gamma_3\ .$$

It follows that for every real meromorphic one-form on $\bar{\Gamma}_h$, $h \in \mathbb{R}$ holds

$$Res_{\infty_3}\omega \in \mathbb{R},\ \overline{Res_{\infty_1}\omega} = Res_{\infty_2}\omega\ .$$

The residues of the real one-forms $\nabla\omega_X$ and $\nabla\omega_Y$ are however constant in h and therefore the above holds even for $h \notin \mathbb{R}$. We conclude that there exist real constants r and s (unique up to multiplication by a non-zero constant) such that

$$Res_{\infty_3}(r\,\nabla\omega_X + s\,\nabla\omega_Y) = 0\,. \tag{38}$$

We note for a further use that also

$$Res_{\infty_1}(r\,\nabla\omega_X + s\,\nabla\omega_Y) + Res_{\infty_2}(r\,\nabla\omega_X + s\,\nabla\omega_Y) = 0$$

and hence

$$\sqrt{-1}\,Res_{\infty_1}(r\,\nabla\omega_X + s\,\nabla\omega_Y) \in \mathbb{R}\ . \tag{39}$$

Denote

$$F(h) = (rX'(h) + sY'(h) - kM'(h))/M'(h)\,.$$

We shall use the argument principle to show that the holomorphic function $F(h)$ has at most one zero in the complex domain $\mathcal{D}$. We shall copy the proof of Theorem 4.1, as well the notations introduced there, except in the computation of $Im(F(h))$ along $(1/6, \infty)$. In a neighborhood of $h = 1/6$ we have

$$F(h) = (r\,x_2 + s\,y_2) - k + O\left(\frac{1}{\log\left(h - \frac{1}{6}\right)}\right)$$

where $(x_2, y_2) = (1, 0)$ and hence the increase of the argument of $F(h)$ along the circle $\{h : |h - 1/6| = r, h \neq 1/6 + r\}$ is close to zero. In particular the imaginary part of $F(h)$ restricted to this circle vanishes exactly once (at $h = 1/6 - r$).

In a neighborhood of $h = \infty$ we have

$$F(h) = h^{\frac{1}{3}}\left(c + O(h^{-\frac{1}{3}})\right)$$

where c is a suitable real constant (possibly equal to zero). Therefore the increase of the argument of $F(h)$ along the circle $\{h \in \mathbb{C} : |h| = R, h \neq R\}$ is close to $2\pi/3$ (or less if $c = 0$).

Consider now the imaginary part of F along the intervals $[h_2+r, h_3-r]$, $[h_3 + r, h_4 - r]$, $[h_4 + r, \infty]$. We have

$$Im(F(h)) = \frac{W_{\delta_1^+\delta_1^-}(r\,\nabla\omega_X + s\,\nabla\omega_Y, \nabla\omega_M)}{2\int_{\delta_1^+(h)}\nabla\omega_M\int_{\delta_1^-(h)}\nabla\omega_M}\,.$$

The denominator does not vanish, and to compute the numerator above we shall use the reciprocity law for Abelian integrals of third and first kind. Namely, let δ', δ'' be two smooth loops on $\bar{\Gamma}_h$ which are transversal and represent cycles with intersection index one. Let $\Pi = \bar{\Gamma}_h \backslash \{\delta' \cup \delta''\}$ be the period parallelogram of $\bar{\Gamma}_h$. Let $P, P_0 \in \Pi$ be fixed points. Integrating the meromorphic one-form

$$(r\,\nabla\omega_X + s\,\nabla\omega_Y)\int_{P_0}^{P}\nabla\omega_M$$

along the border of Π we obtain the following reciprocity law for differential forms of first and third kind

$$\begin{aligned}&\left(\int_{\delta'} r\,\nabla\omega_X + s\,\nabla\omega_Y\right)\int_{\delta''}\nabla\omega_M - \left(\int_{\delta''} r\,\nabla\omega_X + s\,\nabla\omega_Y\right)\int_{\delta'}\nabla\omega_M\\ &= 2\pi i\sum_{P\in\Pi} Res_P(r\,\nabla\omega_X + s\,\nabla\omega_Y)\int_{P_0}^{P}\nabla\omega_M\end{aligned}$$

(see [17] for details). If we put $P_0 = \infty_1$, and taking into account that the intersection index of δ_1^+ and δ_1^- is non-zero, the above formula becomes

$$\begin{aligned} &c\, W_{\delta_1^+\delta_1^-}(r\,\nabla\omega_X + s\,\nabla\omega_Y, \nabla\omega_M) \\ &= 2\pi\, i\, Res_{\infty_2}(r\,\nabla\omega_X + s\,\nabla\omega_Y) \int_{\infty_1}^{\infty_2} \nabla\omega_M \end{aligned} \tag{40}$$

where c is a non-zero integer. The residue $Res_{\infty_2}(r\,\nabla\omega_X + s\,\nabla\omega_Y)$ can not vanish. Indeed, otherwise (38) would imply that $r\,\nabla\omega_X + s\,\nabla\omega_Y$ is holomorphic on $\bar{\Gamma}_h$. But $\nabla\omega_M$ is holomorphic too in contradiction with Lemma 1. Further $z = \int_{P_0}^{P} \nabla\omega_M$ is an uniformizing parameter on the compact elliptic curve $\bar{\Gamma}_h$, and hence

$$\int_{\infty_1}^{\infty_2} \nabla\omega_M \neq 0$$

(because $\infty_1 \neq \infty_2$ on $\bar{\Gamma}_h$).

Consider finally the behavior of $F(h)$ in a neighborhood of $h_i \in (1/6, \infty)$, $i = 3, 4$. We have

$$F(h) = (r\,x_i + s\,y_i) - k + O\left(\frac{1}{\log\left(h - \frac{1}{6}\right)}\right)$$

where (x_i, y_i) is the critical point associated to h_i, and hence the increase of the argument of $F(h)$ along the circle $\{h : |h - h_i| = r\}$ is close to zero. We shall show that, moreover, the imaginary part of $F^+(h)$ takes the same sign at $h_i - r$ and at $h_i + r$. This would imply that the embedded semi-circle

$$F\left(\left\{h = h_i + r\,e^{i\varphi} : 0 \le \varphi \le \pi\right\}\right)$$

has a zero intersection index with the line $\{Im(h) = 0\}$ and hence (summing up the preceding information) we get that $F(h)$ has at most one zero in the complex domain $\tilde{\mathcal{D}}$ (and hence in $\mathcal{D}$).

Consider first the case $h = h_3$. The sign of $Im(F(h))$ at $h_3 \pm r$ is the same as the sign of

$$W_{\delta_1^+\delta_1^-}(r\,\nabla\omega_X + s\,\nabla\omega_Y, \nabla\omega_M)$$

where

$$\bar{\delta}_1^+(h) = \delta_1^-(h),\ \ h = h_3 \pm r\ .$$

On the other hand, using the Picard-Lefschetz formula and the Dynkin diagram shown on Fig. 3, we obtain

$$\delta_1^-(h_3 - r) = \delta_1^+(h_3 - r) - < \delta_1^+, \delta_2 > \delta_2(h_3 - r)$$

and

$$\delta_1^-(h_3 + r) = \delta_1^+(h_3 + r) - < \delta_1^+, \delta_2 > \delta_2(h_3 - r) - < \delta_1^+, \delta_3 > \delta_3(h_3 + r)\ .$$

The cycle $\delta_2 - \delta_3$ has zero intersection index with δ_i, $\forall i$ and hence it is homologous to zero on $\bar{\Gamma}_h$. It is also an "imaginary" cycle on $[h_2+r, h_4-r]$ and hence for a suitable non-zero integer c we have $\delta_2 - \delta_3 = c\,\gamma_3$ which implies

$$\int_{\delta_2(h)} r\,\nabla\omega_X + s\,\nabla\omega_Y \equiv \int_{\delta_3(h)} r\,\nabla\omega_X + s\,\nabla\omega_Y \ .$$

Thus the sign of $Im(F(h))$ at $h_3 - r$ is the same as the sign of

$$-W_{\delta_1^+\delta_2}(r\,\nabla\omega_X + s\,\nabla\omega_Y, \nabla\omega_M)$$

and the sign of $Im(F(h))$ at $h_3 + r$ is the same as the sign of

$$-2\,W_{\delta_1^+\delta_2}(r\,\nabla\omega_X + s\,\nabla\omega_Y, \nabla\omega_M) \ .$$

The Picard-Lefschetz formula shows that the last Wronskian is a holomorphic function in a neighborhood of h_3. It remains to check that it does not vanish at h_3. For this purpose we take the limit $h \to h_3$ in (40). We already noted that

$$\int_{\infty_1}^{\infty_2} \nabla\omega_M \neq 0$$

for regular values of h. This hold true also for $h = h_3$. Indeed, the path of integration connecting ∞_1 to ∞_2 on $\bar{\Gamma}_h$ represents a "relative" cycle which has a zero intersection index with the cycle δ_3 (because the Wronskian under consideration is single-valued in a neighborhood of h_3). The desingularization

$$\bar{\Gamma}_{h_3} \to \Gamma_{h_3}$$

of the singular irreducible affine curve Γ_{h_3} is the Riemann sphere $\mathbb{CP}^1$. Denote by $(x_3^\pm, y_3^\pm) \in \bar{\Gamma}_{h_3} = \mathbb{CP}^1$ the two pre-images of $(x_3, y_3) \in \Gamma_{h_3}$. If we suppose further that $z(x_3^+, y_3^+) = 0$ and $z(x_3^-, y_3^-) = \infty$, where z is the uniformizing parameter on $\mathbb{CP}^1$, then the differential one-form $\nabla\omega_M$ is a multiple of dz/z. It follows that $\int_{\infty_1}^{\infty_2} \nabla\omega_M$ is a multiple of

$$\ln\left(\frac{z(\infty_2)}{z(\infty_1)}\right)$$

and hence it is not zero (as $\infty_1 \neq \infty_2$ on $\bar{\Gamma}_{h_3}$).

The sign of $Im(F^+(h))$ at $h = h_4 \pm r$ can be studied in a similar way. Namely, this sign is the same as the sign of

$$W_{\delta_1^+\delta_1^-}(r\,\nabla\omega_X + s\,\nabla\omega_Y, \nabla\omega_M)$$

where

$$\bar{\delta}_1^+(h) = \delta_1^-(h), \ h = h_4 \pm r \ .$$

On the other hand, using the Picard-Lefschetz formula and the Dynkin diagram shown on Fig. 3, we obtain

$$\delta_1^-(h_4-r) = \delta_1^+(h_4-r) - <\delta_1^+,\delta_2> \delta_2^+(h_4-r) - <\delta_1^+,\delta_3> \delta_3^+(h_4-r)$$

and

$$\begin{aligned}\delta_1^-(h_4+r) = \delta_1^+(h_4+r) &- <\delta_1^+,\delta_2> \delta_2^+(h_4+r) - <\delta_1^+,\delta_3> \delta_3^+(h_4+r)\\ &+ <\delta_1^+,\delta_4> \delta_4(h_4+r)\ .\end{aligned}$$

The Wronskian

$$W_{\delta_1^+\delta_4}(r\,\nabla\omega_X + s\,\nabla\omega_Y, \nabla\omega_M)$$

is a holomorphic function in a neighborhood of $h=h_4$. On the other hand

$$W_{\delta_1\delta_2}(r\,\nabla\omega_X + s\,\nabla\omega_Y, \nabla\omega_M) = W_{\delta_1\delta_3}(r\,\nabla\omega_X + s\,\nabla\omega_Y, \nabla\omega_M)\ .$$

Finally (40) and the Picard-Lefschetz formula show that in a neighborhood of $h=h_4$ holds

$$\begin{aligned}&2\,\pi\,i\,c\,Res_{\infty_2}(r\,\nabla\omega_X + s\,\nabla\omega_Y)\ln(h-h_4)\int_{\delta_4(h_4)}\nabla\omega_M + O(1)\\ &= W_{\delta_1\delta_2}(r\,\nabla\omega_X + s\,\nabla\omega_Y, \nabla\omega_M)\end{aligned} \tag{41}$$

where c is a non-zero integer. Indeed it is easy to see that (contrary to the preceding case) the Wronskian (41) is not single valued in a neighborhood of $h=h_4$. In particular the path of integration connecting ∞_1 to ∞_2 on $\bar{\Gamma}_h$ represents a "relative" cycle which has a non-zero intersection index with the cycle δ_4. The cycle $\delta_4(h)$ is real for $h\in\mathbb{R}$ in a neighborhood of h_4, and hence

$$\int_{\delta_4(h_4)}\nabla\omega_M$$

is a real non-zero constant. The residue

$$Res_{\infty_2}(r\,\nabla\omega_X + s\,\nabla\omega_Y)$$

is a non-zero imaginary constant (Lemma 3 and (39)) which shows that the real part of the Wronskian (41) is $c\,\ln(|h-h_4|)$ for suitable non-zero real constant c. This implies that the sign of $Im(F^+(h))$ at $h=h_4+r$ and $h=h_4-r$ is the same (provided that r is sufficiently small). This completes the proof of Theorem 5.2. □

5.2. The convexity of the centroid curve

Theorem 5.3. *Each line intersects at most twice (counting the multiplicity) the centroid curve L_{ab}.*

For $(a, b) \in \Omega_1^0$ this is the main result of [19]. We consider below the remaining cases $(a, b) \in \Omega_2^0 \cup \Omega_3^0$. At the point $(\xi(1/6), \eta(1/6))$ the claim of Theorem 5.3 means the following. In a neighborhood of $h = 1/6$ holds

$$\alpha\, X(h) + \beta\, Y(h) + \gamma\, M(h) = \sum_{k=0}^{\infty} c_{2k+1}\, h^k + c_{2k+2}\, h^{k+1}\, \log(1/6 - h)\,. \tag{42}$$

We shall say that the line

$$l = \{(x, y) \in \mathbb{R}^2 : \alpha\, x + \beta\, y + \gamma = 0\}$$

intersects L_{ab} at $(\xi(1/6), \eta(1/6))$ with multiplicity k (or has an order of tangency k), if

$$c_1 = c_2 = \ldots = c_k = 0,\, c_{k+1} \neq 0.$$

In the proof of Theorem 5.3 we shall need the following

Lemma 7. *The sign of the curvature of L_{ab} is one and the same near the points $(\xi(0), \eta(0))$ and $(\xi(1/6), \eta(1/6))$.*

Denote by t_c and t_s the lines tangent to L_{ab} at $C = (\xi(0), \eta(0))$, $Z = (\xi(1/6), \eta(1/6))$ and let m be the line passing through both C and Z. From Corollary 7 we see that t_c and t_s do intersect, and from the above Lemma, that their intersection point lies on the tangent t_c (t_s) in the same direction with respect to C (Z) as the part of L_{ab} near C (Z) (see Fig. 5). This implies

Corollary 8. *If the curve L_{ab} is convex then it is entirely contained in the triangle formed by t_c, t_s, m as shown on Fig. 5.*

Proof of Lemma 7. Denote

$$\tilde{\kappa}(h; a, b) = det \begin{pmatrix} X & Y & M \\ X' & Y' & M' \\ X'' & Y'' & M'' \end{pmatrix}$$

and let $\Omega_{23}^0 \subset \mathbb{R}^2$ be the connected open domain

$$\Omega_{23}^0 = \Omega_2^0 \cup \Omega_3^0 \cup \left\{ (a, b) \in \mathbb{R}^2 : b^2 - 4\, a^3 = 0,\; 0 < a < \frac{1}{2},\; 0 < b < \frac{\sqrt{2}}{2} \right\}.$$

If $\kappa(h; a, b)$ denotes the curvature of the smooth plane curve L_{ab}, then an elementary computation shows that

$$\kappa(h; a, b) = \tilde{\kappa}(h; a, b)[(\xi'(h))^2 + (\eta'(h))^2]^{-3/2}(M(h))^{-3}$$

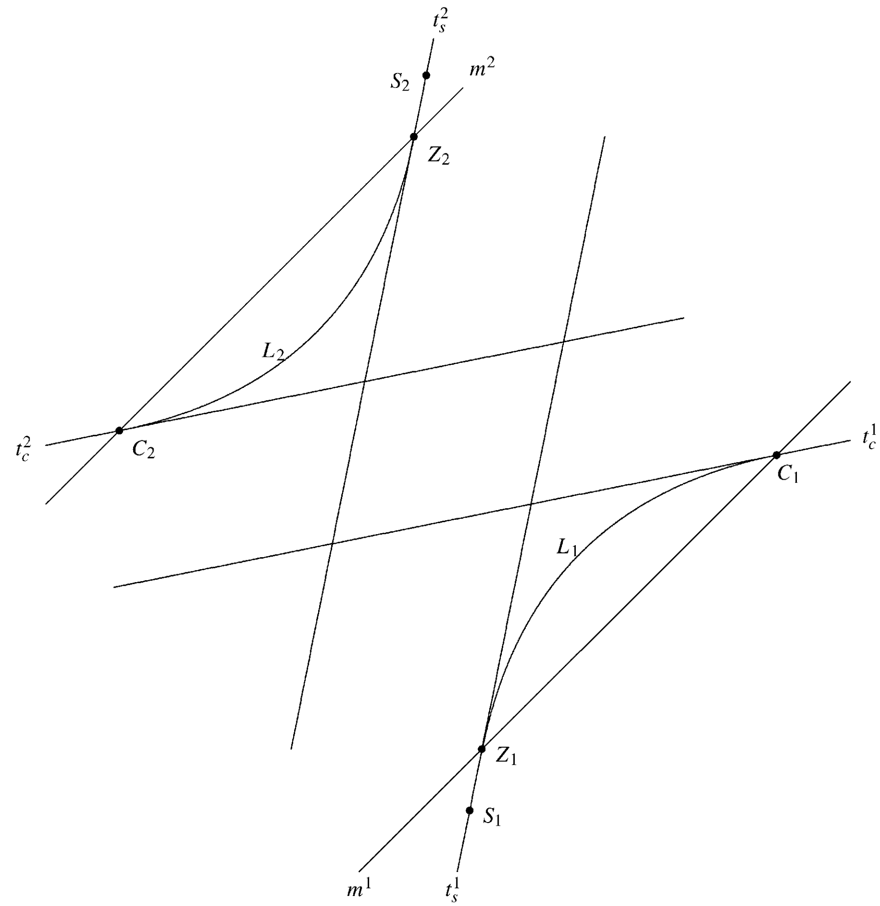


Fig. 5

where for every $h \in [0, 1/6)$, $(a, b) \in \Omega^0_{12}$, the function

$$[(\xi'(h))^2 + (\eta'(h))^2]^{-3/2}(M(h))^{-3}$$

is analytic and non-vanishing. It follows from [41] that $\tilde{\kappa}(0; a, b) = 0$ if and only if $I_{\alpha\beta\gamma} \equiv 0$. For every fixed $(a, b) \in \Omega^0_{12}$ the functions $X(h)$, $Y(h)$, $M(h)$ are linearly independent over $\mathbb{R}$ [15, Proposition 5] and hence

$$\tilde{\kappa}(0; a, b) \neq 0, \ \forall (a, b) \in \Omega^0_{12}.$$

In a neighborhood of $h = 1/6$ we have

$$X(h) = c_1^X + c_2^X (h - 1/6) \log(1/6 - h) + c_3^X (h - 1/6) + o(h - 1/6)$$

$$Y(h) = c_1^Y + c_2^Y (h - 1/6) \log(1/6 - h) + c_3^Y (h - 1/6) + o(h - 1/6)$$

$$M(h) = c_1^M + c_2^M (h - 1/6) \log(1/6 - h) + c_3^M (h - 1/6) + o(h - 1/6)$$

and hence

$$\tilde{\kappa}(h; a, b) = \frac{1}{1/6 - h} \left(D(a, b) + o(1/6 - h) \right)$$

where

$$D(a, b) = det \begin{pmatrix} c_1^X & c_1^Y & c_1^M \\ c_2^X & c_2^Y & c_2^M \\ c_3^X & c_3^Y & c_3^M \end{pmatrix} .$$

It follows from [20] that $D(a, b) = 0$ if and only if $I_{\alpha\beta\gamma} \equiv 0$ and as above we conclude that

$$D(a, b) \neq 0, \ \forall (a, b) \in \Omega_{12}^0 .$$

The Picard-Lefschetz formula implies that c_i^X, c_i^Y, c_i^M are analytic functions in $(a, b) \in \Omega_{12}^0$ and hence $D(a, b)$ is analytic too. Therefore it is enough to compare the signs of the curvature $\kappa(h; a, b)$ of L_{ab} at $h = 0$ and for $h \sim 1/6$ for at least one centroid curve L_{ab}, $(a, b) \in \Omega_{12}^0$. Indeed, this is already proved in the so called centro-symmetrical case, $a = 0, b \in (0, 1)$ [22]. □

Recall that in the case when the non-perturbed vector field X_H has three saddles and one center Theorem 5.3 is proved in [22] (see also the Introduction).

Proof of Theorem 5.3 in the case when X_H has one center and one saddle. It follows from [19] and Theorem 4.1. For convenience of the reader we recall the proof. First we note that the centroid curve is entirely contained (except its ends C and Z) in the triangle formed by the lines t_c, t_s and m. Indeed, otherwise L_{ab} would intersect one if these lines at four points (counting the multiplicity). This implies the existence of an Abelian integral $I_{\alpha\beta\gamma}(h)$ which has four zeros on the interval $(0, 1/6)$. As $I_{\alpha\beta\gamma}(0) = 0$, then the second derivative of $I_{\alpha\beta\gamma}(h)$ has at least three zeros which contradicts to Theorem 4.1. More generally, the same reasonings show that every line l intersects L_{ab} in at most three points. Suppose further that there exists a line l which intersects L_{ab} in *exactly* three points. Using a suitable continuous deformation of l we may always suppose that l is one of the lines t_c, t_s or m, but still intersects L_{ab} in exactly three points. This contradicts to the claim above that the centroid curve is entirely contained (except its ends C and Z) in the triangle formed by the lines t_c, t_s and m. □

Proof of Theorem 5.3 in the case when X_H has two centers and two saddles. By definition the ends of L_{ab}^* are the tangents t_c, t_s. From (32) we get $t_c = [\alpha(0), \beta(0), \gamma(0)] \in \Delta^{ab}$. The saddle point (x_2, y_2) lies on the line t_s [19] so $t_s \in l_2^{ab}$. We begin by establishing the following

Lemma 8. *If $(a, b) \in \Omega_2^0$ and the claim of Theorem 5.3 holds true then*

$$L_{ab}^* \cap l_3^{ab} = \emptyset, \ L_{ab}^* \cap l_4^{ab} = \emptyset, \ L_{ab}^* \cap l_2^{ab} = t_s .$$

Proof. The convexity of L_{ab} and Corollary 8 show that the only tangent of L_{ab} containing the point (x_2, y_2) is t_s and hence $L^*_{ab} \cap l_2^{ab} = t_s$. To find the position of L^*_{ab} with respect to l_3^{ab}, l_4^{ab} we shall use a deformation argument. As in [22] let us denote by L_i, $i = 1, 2$ the centroid curves of X_H and let C_i, S_i be the corresponding center and saddle points. Denote further by t_c^i the tangent to L_i at C_i of X_H , and by t_s^i, the tangent to L_i at the centroid point Z_i of the loop area. Let R_i denotes the sector formed by t_s^i, t_c^i, where the centroid curve L_i near Z_i, C_i lies and let R_i^{op} be the opposite sector. In the centro-symmetrical case $a = 0$, $0 < b < 1$ the mutual position of the sets we just defined is determined in [22] (Fig. 8) and it is shown on Fig. 5. More precisely, every line intersects at most twice the centroid curves L_i (counting the multiplicity) and hence $L_i \subset R_i$ (Corollary 8). The central symmetry implies that the tangent t_c^1 is parallel to t_c^2, and the tangent t_s^1 is parallel to t_s^2. It is proved that the centroid curve L_1 is contained in R_2^{op}, and L_2 is contained in R_1^{op}. In particular $C_2 \subset R_1^{op}$ and hence $L^*_{0b} \cap l_4^{ab} = \emptyset$. Finally the saddle point S_i lies on the tangent t_s^i, in the direction with respect to the the centroid point Z_i, opposite to the part of L_i near Z_i [19, Theorem 3.1]. This implies that $S_2 = (x_3, y_3) \in\in R_1^{op}$ and hence $L^*_{0b} \cap l_3^{ab} = \emptyset$.

To study the general case, $a \neq 0$, consider a continuous deformation $s \to (a(s), b(s))$ of the centroid curve $L_{a(0),b(0)}$, where

$$a(0) = 0,\ b(0) \in (0, 1),\ (a(s), b(s)) \in \Omega_2^0,\ s \in [0, 1]\ .$$

It is enough to check that the critical points (x_3, y_3) and (x_4, y_4) remain contained in the sector R_1^{op} (which also depends on a, b), or equivalently

$$(x_3, y_3) \notin t_s^1, t_c^1,\quad (x_4, y_4) \notin t_s^1, t_c^1\ .$$

The tangent t_s^1 contains the saddle (x_2, y_2) and intersects the period annulus of (x_1, y_1) (as it contains the centroid point Z_1). Therefore t_s^1 intersects a continuous orbit of X_H and hence it is tangent to X_H at some point contained in the period annulus. As X_H is a *quadratic* vector field, then t_s^1 does not contain any other equilibrium point of X_H, so $(x_3, y_3) \notin t_s^1$, $(x_4, y_4) \notin t_s^1$

The equation of the tangent t_c^1 is $b\,x + (a - 1)\,y = 0$ (33) and an easy computation (see (28) and the explications after) shows that

$$b\,x_i + (a - 1)\,y_i \neq 0,\ i = 2, 3, 4.$$

This completes the proof of Lemma 8. □

Suppose that the deformation of the centroid curve $L_{a(0),b(0)}$

$$s \to (a(s), b(s)),\ a(0) = 0,\ b(0) \in (0, 1),\ (a(s), b(s)) \in \Omega_2^0,\ s \in [0, 1]$$

is such that $L_{a(s),b(s)}$ is convex for $s \in [0, 1)$ and it is not convex for $s = 1$. The curvature of $L_{a(1)b(1)}$ can not vanish at its ends (Lemma 7) and hence it vanishes at some internal point $(\xi(h'), \eta(h'))$, $h' \in (0, 1/6)$. The curvature has moreover a double zero at h' and hence the Abelian integral $I_{\alpha\beta\gamma}(h)$ associated to the line $\{\alpha x + \beta y + \gamma = 0\}$ tangent to $L_{a(1)b(1)}$ at $(\xi(h'), \eta(h'))$ has a zero at h' of order four. Thus $\frac{d^2}{dh^2} I_{\alpha\beta\gamma}(h)$ has at least three zeros (counting the multiplicity) on the open interval $(0, 1/6)$. This implies that there exists $s_0 \in (0, 1)$ such that the Abelian integral $\frac{d^2}{dh^2} I_{\alpha\beta\gamma}(h)$ associated to the line tangent to $L_{a(s_0)b(s_0)}$ at $\xi(h'), \eta(h')$ has at least three zeros in the *complex domain* $\mathcal{D} = \mathbb{C}\backslash[1/6, \infty)$. We shall show that this is impossible.

Lemma 8 (after an obvious modification of the proof) shows that for every $s \in [0, s_0]$ holds

$$L^*_{a(s)b(s)} \cap l_3^{a(s)b(s)} = \emptyset,\ L^*_{a(s)b(s)} \cap l_4^{a(s)b(s)} = \emptyset,\ L^*_{a(s)b(s)} \cap l_2^{a(s)b(s)} = t_s\ .$$

The set $\mathbf{B}^{a(s)b(s)}$ is, however, an union of lines $l_i^{a(s)b(s)}$, $l_\infty^{a(s)b(s)}$ and one segment $\Delta^{a(s)b(s)}$ (Theorem 3.1). This implies that if $L^*_{a(s)b(s)}$ intersects an open connected component of

$$(\mathbb{RP}^2)^* \backslash \mathbf{B}^{a(s)b(s)}$$

then this component is either $\mathcal{U}$ or $\mathcal{U}'$, or $\mathcal{U}''$, where $\mathcal{U}$ was defined in Theorem 4.2, and $\mathcal{U}, \mathcal{U}', \mathcal{U}''$ are shown on Fig. 4. Note that $\mathcal{U}, \mathcal{U}', \mathcal{U}''$ are the only connected components containing the point $[\alpha(\infty) : \beta(\infty) : \gamma(\infty)]$. If $[\alpha : \beta : \gamma] \in \mathcal{U}$ the Abelian integral $\frac{d^2}{dh^2} I_{\alpha\beta\gamma}(h)$ has exactly one zero in the complex domain $\mathcal{D}$ and if $[\alpha : \beta : \gamma]$ belongs to $\mathcal{U}'$ or $\mathcal{U}''$ then $\frac{d^2}{dh^2} I_{\alpha\beta\gamma}(h)$ has at most two zeros in the complex domain $\mathcal{D}$ (Corollary 4 and Proposition 2). This contradicts to the claim that the Abelian integral $\frac{d^2}{dh^2} I_{\alpha\beta\gamma}(h)$ associated to the line tangent to $L_{a(s_0)b(s_0)}$ at $\xi(h'), \eta(h')$ has at least three zeros in the *complex domain* $\mathcal{D} = \mathbb{C}\backslash[1/6, \infty)$. We conclude that the centroid curve $L_{a(1)b(1)}$ is convex, which on its hand implies that for every $(a, b) \in \Omega_2^0$ the centroid curve L_{ab} is convex.

Suppose that there exists a line which intersects L_{ab} in three points. The convexity of L_{ab} implies that at least two of them are distinct and hence there exist $h', h'' \in (0, 1/6)$ such that the tangent vectors to L_{ab} at h' and h'' are co-linear. Using once again the convexity of L_{ab} we see that when h is running the interval $[h', h'']$ the tangent vector to L_{ab} at $(\xi(h), \eta(h))$ rotates within an angle π in contradiction to Corollary 7. This completes the proof of Theorem 5.3. $\square$

6. Proof of Theorem 1

The proof of Theorem 1 is based on Theorem 5.3 and on results of Roussarie [35,36]. Let X_0 be an analytic vector field with a center at the origin 0. This means that there is an open neighborhood V of O which is an union of O and periodic orbits of X_0. The maximal open neighborhood of O with this property is called *the (open) period annulus* Π *of* O. The *closed period annulus* $\bar{\Pi}$ is the closure of the open one. Consider an analytic unfolding X_λ, $\lambda \in \mathbb{R}^\Lambda$, O of X_0. Our first task is to define the notion of *cyclicity* of the period annulus of X_0 with respect to X_λ. The usual definition of cyclicity of limit periodic sets which uses the Hausdorf metric (e.g. [35]) is not convenient here. Indeed, the period annulus is an union of limit periodic sets.

Definition 3. *Let* X_λ *be a germ of a family of smooth plane vector fields,* $\lambda \in (\mathbb{R}^\Lambda, 0)$, *and let* $K \subset \mathbb{R}^2$ *be a compact invariant set of* X_0. *We say that the pair* (K, X_λ) *has cyclicity* $N = Cycl(K, X_\lambda)$ *if* N *is the smallest integer having the property: there exists* $\varepsilon_0 > 0$ *and a neighborhood* V_K *of* K, *such that for every* λ, $|\lambda| < \varepsilon_0$, *the vector field* X_λ *has no more than* N *limit cycles contained in* V_K. *If* $\tilde{K}$ *is an invariant set of* X_0 *(possibly non-compact), then the cyclicity of the pair* $(\tilde{K}, X_\lambda)$ *is*

$$Cycl(\tilde{K}, X_\lambda) = sup\{Cycl(K, X_\lambda) : K \subset \tilde{K},\ K \text{ is a compact}\}.$$

In the case when K is a limit periodic set our definition coincides with the usual one [35].

From now on we put $X_0 = X_H$, where X_H is the quadratic Hamiltonian vector field satisfying the genericity conditions imposed in Theorem 1. Consider the unfolding

$$X_\lambda = X_{H^\lambda} + \left(\lambda_1\, x\, y + \lambda_2\, \frac{y^2}{2} + \lambda_3\, y\right) \frac{\partial}{\partial y}$$

where X_{H^λ} is the Hamiltonian vector field associated to

$$H^\lambda(x, y) = H(x, y) + \sum_{1 \le i+j \le 3} \lambda_{ij} x^i y^j,\ \ H^0(x, y) = H(x, y),$$

$$\lambda = (\lambda_{10}, \lambda_{01}, ..., \lambda_{33}, \lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^{12}\ .$$

Every quadratic unfolding of X_0 is induced by X_λ, that is to say X_λ is versal among all quadratic unfoldings of X_0. According to a Theorem of Il'yashenko [25] X_λ has a center at the origin if and only if $\lambda_1 = \lambda_2 = \lambda_3 = 0$. This suggest to consider a "blow up" defined by the change of parameters

$$\lambda_1 = \varepsilon\, \alpha, \lambda_2 = \varepsilon\, \beta, \lambda_3 = \varepsilon\, \gamma$$

where

$$(\alpha, \beta, \gamma) \in S^2 = \{(\alpha, \beta, \gamma) : \alpha^2 + \beta^2 + \gamma^2 = 1\}\ .$$

We obtain a family of plane vector fields $X_{(\lambda_{ij}, \varepsilon, \alpha, \beta, \gamma)}$ defined in a neighborhood of the sphere $0 \times S^2 \subset \mathbb{R}^{10} \times S^2$. The next step is to localize this family along $0 \times S^2$.

Namely, for every fixed point $(0, \alpha^0, \beta^0, \gamma^0)$ consider a local diffeomorphism

$$\mu : \mathbb{R}^{12}, 0 \to \mathbb{R}^{10} \times S^2, (0, \alpha^0, \beta^0, \gamma^0) .$$

The localized vector field

$$X_\mu = X_{(\lambda_{ij}(\mu), \varepsilon(\mu), \alpha(\mu), \beta(\mu), \gamma(\mu))}$$

is an unfolding of the Hamiltonian vector field $X_0 = X_H$.

Theorem 6.1.

$$Cycl(\bar{\Pi}, X_\mu) \leq 2 .$$

Proof of Theorem 1 assuming Theorem 6.1. The compactness of the embedded sphere $0 \times S^2$ implies that there is a neighborhood $U \subset \mathbb{R}^{10} \times S^2$ of $0 \times S^2$ and a neighborhood $V \subset \mathbb{R}^2$ of the closed period annulus $\bar{\Pi}$, such that for every $(\lambda_{ij}, \varepsilon, \alpha, \beta, \gamma) \in U$ the vector field $X_{(\lambda_{ij}, \varepsilon, \alpha, \beta, \gamma)}$ has at most two limit cycles in V. On its turn this shows that

$$Cycl(\bar{\Pi}, X_\lambda) \leq 2 . \tag{43}$$

□

It can be shown that each limit cycle of a quadratic vector field surrounds exactly one equilibrium point which is of focus type [38]. This implies that every limit periodic set of $X_0 = X_H$ on the Poincaré compactification of the plane, is contained in a closed period annulus (we used that $H(x, y)$ is generic). If X_H has only one period annulus Π, then Theorem 1 is proved.

Suppose that X_H has two period annuli Π_1 and Π_2 with centroid curves L_1 and L_2. It is shown in [21, Theorem 3.5] that

If both centroid curves are strictly convex then any line can intersect their union at most in two points (counting the multiplicities).

It is easy to deduce from this (generalizing Theorem 6.1) that

$$Cycl(\bar{\Pi}_1, X_\lambda) + Cycl(\bar{\Pi}_2, X_\lambda) \leq 2 .$$

We prefer to use, however, the following recent result due to Zegeling and Kooij [40].

The only possible limit cycle distributions in quadratic systems with four real singular points are $(1, 1)$ *and* $(n, 0)$ *where n is an integer (examples with* $n = 0, 1, 2, 3$ *are well known).*

This combined to (43) completes the proof of Theorem 1. □

Proof of Theorem 6.1. We shall prove first that $Cycl(\Pi, X_\mu) \leq 2$. Without loss of generality we suppose that $\mu_1 = \varepsilon$. The vector field has a center if and only if $\mu_1 = 0$. We may also suppose that $X_\mu(0) = 0$.

Consider the displacement function

$$\delta_\mu(h) = P_\mu(h) - h$$

where $P_\mu(h)$ is the first return map associated to X_μ and to the arc $l = \{(x,0) : x \in [0,1]\}$ parameterized by $h = H(x,y)|_l$. Let $K \subset \Pi$ be a compact invariant set of X_H. As $\mu_1 = 0$ implies $\delta_\mu(h) \equiv 0$, then

$$\delta_\mu(h) = \mu_1 F(h,\mu)$$

where $F(h,0) = I_{\alpha\beta\gamma}(h)$ is the Pontryagin function defined in the Introduction (the proof is the same as in [33]). Let $\tilde{h} < 1/6$. For every fixed μ in a sufficiently small neighborhood of $0 \in \mathbb{R}^{12}$ the function $\delta_\mu(h)$ has an analytic continuation in a neighborhood of $[0,\tilde{h}]$ and hence in a complex neighborhood $U \subset \mathbb{C}$ of $[0,\tilde{h}]$. On the other hand $F(h,\mu)$ depends also analytically on μ. By Rouché's theorem $F(h,\mu)$ and $F(h,0)$ have the same number of zeros in h (counted with multiplicity) in U. We may always suppose that the zeros of $F(h,0)$ in U are contained in $[0,\tilde{h}]$. Their number equals to the number of intersection points of the line

$$l = \{(x,y) \in \mathbb{R}^2 : \alpha x + \beta y + \gamma = 0\}$$

with the piece of centroid curve $\{L(h) : h \in [0,\tilde{h}]\}$ plus one (because $F(h,\mu)$ always vanishes at $h = 0$).

Similarly, according to [35, Theorem 25(i)], the cyclicity of the homoclinic loop $\partial\,\Pi$ is bounded by the number of the vanishing coefficients in the expansion (42) of $F(h,0)$ near $h = 1/6$. This number also equals to the multiplicity of the intersection of the line L above with the centroid curve L at the endpoint $(\xi(1/6), \eta(1/6))$. Theorem 5.3 implies that

$$Cycl(\Pi, X_\mu) + Cycl(\partial\,\Pi, X_\mu) \leq 2$$

and hence $Cycl(\bar{\Pi}, X_\mu) \leq 2$. □

7. Concluding remarks

The systematic study of *global* phase portraits of plane polynomial vector fields has been initiated by Poincaré [31,32]. As most fundamental problems he recognized the problem of finding the limit cycles, and the problem of distinguishing between a center and a focus of such fields. The question about *the maximal number and position of the limit cycles of a plane polynomial vector field of degree n* was explicitly asked later by Hilbert in the second part of his 16th problem [18]. Recall that the first part of this problem asks for a (projective) classification of the ovals of a real plane algebraic curve

$$\{(x,y) \in \mathbb{R}^2 : H(x,y) = 0\} \tag{44}$$

where $H(x,y)$ is an arbitrary real polynomial of degree n. The analogy between limit cycles (which are transcendental curves in general) and ovals remains, however, very incomplete. Thus, according to Harnack's theorem, the number of the connected components of a smooth real plane projective curve is less or equal to $p_g + 1$ where $p_g = (n-1)(n-2)/2$ is the genus of (44) (the bound is exact). The analogue of Harnack's theorem for differential equations would be to find an *exact* upper bound $H(n)$ for the number of limit cycles of any plane polynomial vector field of degree at most n. Hilbert suggested that the the problem "may be attacked by the method of continuous variation of the coefficients". The latter was successfully used in the study of ovals of algebraic curves, and is actually known as the Hilbert-Rohn method.

To establish even the finiteness of the Hilbert numbers $H(n)$ turned out to be beyond the reach of the 20th century mathematics. The deepest known result in this direction is the following

Theorem. *The number of limit cycles of a* given *plane polynomial vector field is finite.*

For a long time this result was attributed to Dulac [8] which contained in fact an important gap. Complete proofs were obtained more recently by Ecalle [9] and Il'yashenko [26, 28].

The study of bifurcations of limit cycles (as suggested by Hilbert) leads naturally to the notions of *limit periodic set* and *cyclicity* of such sets with respect to a given family of vector fields [35]. The limit periodic sets can be classified according to their co-dimension in the space of all polynomial (or analytic) plane vector fields. If a polynomial vector field has a limit periodic set of infinite co-dimension, then it has a period annulus which is an infinite union of periodic orbits, each of them being a limit periodic set. Therefore in this case we have to study rather the cyclicity of the whole period annulus, than the cyclicity of an individual limit periodic set. An important example is the following. Consider a real polynomial $H(x, y)$ of degree $n + 1$ which is generic in the sense that it has n^2 distinct critical values. We shall also suppose that the differential equation $dH = 0$ has a center which is placed at the origin. Consider the perturbed Hamiltonian system

$$d\,H + \varepsilon\,\omega = 0, \quad \omega = P\,dx + Q\,dy$$

where $P = P(x, y)$, $Q = Q(x, y)$ are real polynomials of degree n, and ε is a "small" real parameter. Denote by $\gamma(h) \subset \{H = h\}$ the continuous family of ovals of $\{H = h\}$ which tend to the origin as $h \to 0$.

Theorem ([25, Il'yashenko]). *Either $d\,\omega \equiv 0$ (in which case the perturbed Hamiltonian system $d\,H + \varepsilon\,\omega = 0$ is Hamiltonian), or the Pontryagin function*

$$I(h) = \int_{\gamma(h)} \omega \tag{45}$$

does not vanish identically.

We explained in the Introduction, that in the above situation limit cycles bifurcate from ovals $\gamma(h)$, such that $I(h) = 0$. Therefore the cyclicity of an open period annulus is bounded by the maximal number of zeros which an Abelian integral $I(h)$ on a suitable interval can have (this holds true even for the closed period annulus [34]). Based on this Arnold [4, p. 313] formulated a weakened (or rather infinitesimal) version of the 16th Hilbert problem, which asks for the maximal number of zeros of Abelian integrals of the form (45) (see also [5]). It should be stressed, however, that if the polynomial $H(x, y)$ is not generic, or the degree of the polynomial one-form ω is strictly greater than $deg(H) - 1$, the problem of finding the limit cycles of $dH + \varepsilon\omega = 0$ is *not equivalent* to a problem on the zeros of Abelian integrals. The reason is that $I(h) \equiv 0$ does not imply in general that the return map is equal to the identity map. The higher order Pontryagin functions [10] have to be computed in this case and they are not always Abelian integrals (see Iliev [24] for examples). On the contrary, when the conditions of the Il'yashenko's theorem are satisfied, then the problem of finding the limit cycles becomes a problem in algebraic geometry. Because of its importance we shall formulate this *infinitesimal 16th Hilbert problem* in detail.

Let X_λ, $\lambda \in \mathbb{R}^\Lambda$, be the space of all plane vector fields of degree n. Suppose that

$$X_0 = X_H = H_y \frac{\partial}{\partial x} - H_x \frac{\partial}{\partial y}$$

where $H(x, y)$ is a real polynomial of degree $n+1$. Let $Z(n, H) = Cycl(\bar{\Pi}, X_\lambda)$ be the maximal cyclicity which a closed period annulus $\bar{\Pi}$ of X_H can have with respect to X_λ (of course X_0 can have several period annuli). We shall say that a real polynomial $H(x, y)$ of degree $n+1$ is *generic* if it has n^2 distinct critical values in a complex domain. The infinitesimal 16th Hilbert problem is then

Find the numbers

$$Z(n) = sup\{Z(n, H) : deg\, H \leq n+1, \quad H(x, y) \text{ is generic}\}.$$

As the dimension of the vector space of Abelian integrals

$$\mathcal{A}_n = \{I(h) : I(h) = \int_{\gamma(h)} \omega, \qquad (46)$$
$$\omega = P(x, y)\, dx + Q(x, y)\, dy, \; deg(P), deg(Q) \leq n\}$$

equals to $n(n+1)/2$ [25], then

$$Z(n) \geq \frac{n(n+1)}{2} - 1$$

and by a theorem of Varchenko [39] and Khovanskii [29]

$$Z(n) < \infty\,.$$

The main result of the present paper (Theorem 1) says that

$$Z(2) = 2$$

which suggests that in general

$$Z(n) = \frac{n(n+1)}{2} - 1 . \tag{47}$$

This is equivalent to say that the space of Abelian integrals (46) is a Chebishev space (i.e. the number of the zeros of a function which belongs to the space is less than its dimension).

Consider now the following vector space

$$\mathcal{A}'_n = \left\{ \frac{d}{dh} I(h) : I(h) = \int_{\gamma(h)} \omega, \right. \tag{48}$$

$$\left. \omega = P(x,y)\,dx + Q(x,y)\,dy,\ deg(P), deg(Q) \le n \right\} .$$

We have dim $\mathcal{A}_n$ = dim $\mathcal{A}'_n$ and it can be shown that $\mathcal{A}'_2$ is a Chebishev space (we just repeat the proofs of the present paper given for the space $\mathcal{A}_2$). As $I(h) \in \mathcal{A}_n$ implies that $I(0) = 0$ then the Chebishev property of $\mathcal{A}'_n$ implies the Chebishev property of $\mathcal{A}_n$. There is one more reason, however, which makes the space $\mathcal{A}'_n$ more natural than $\mathcal{A}_n$. The space $\mathcal{A}'_n$ is intimately related to the Jacobian variety of a singular curve Γ_{sing} constructed canonically from the plane affine curve $\Gamma_h = \{(x,y) \in \mathbb{C}^2 : H(x,y) = h\}$. Namely, let $\bar{\Gamma}_h$ be the compactified and normalized curve Γ_h. Consider the divisor

$$D_\infty = \sum_{i=1}^{n+1} \infty_i = \bar{\Gamma}_h \backslash \Gamma_h .$$

The singularized curve Γ_{sing} is a compact singular curve with topological space $\Gamma_h \cup \infty$ (we identify all points $\infty_i \in \bar{\Gamma}_h$ "at infinity" into a single point ∞). The structure sheaf of Γ_{sing} is the sheaf of functions f which are regular on the affine curve $\bar{\Gamma}_h$ and take the same value at ∞_i, $f(\infty_i) = f(\infty_j)$ (see Serre [37]). Let $\Omega^1(D_\infty)$ be the sheaf of meromorphic differentials on $\bar{\Gamma}_h$ which are regular over Γ_h and have at most a simple pole at $D_\infty = \sum_{i=1}^{n+1} \infty_i$. An easy exercise (which generalizes Lemma 3, and is proved in the same way) shows that the vector space of covariant derivatives of polynomial one-forms of a given degree

$$Span_{\mathbb{C}}\{\nabla\omega : \omega = P(x,y)\,dx + Q(x,y)\,dy,\ deg(P), deg(Q) \le n\}$$

coincides with $H^0(\bar{\Gamma}_h, \Omega^1(D_\infty))$. The generalized Jacobian [37] $J(\Gamma_{sing})$ of the singular curve Γ_{sing} is

$$J(\Gamma_{sing}) = H^0\big(\bar{\Gamma}_h, \Omega^1(D_\infty)\big)^* / H_1(\Gamma_h, \mathbb{Z}) .$$

Therefore the Abelian integrals

$$\frac{d}{dh}I(h) = \int_{\gamma(h)} \nabla\omega \in \mathcal{A}_n'$$

are coefficients of the period matrix of Γ_{sing}. We expect that in the case $n > 2$ the generalized Jacobian $J(\Gamma_{sing})$ will play the role of the elliptic curve Γ_h (isomorphic to its Jacobian $J(\Gamma_h)$) in the case $n = 2$. As the arithmetic genus of Γ_{sing} is $p_a = n(n+1)/2$ then (47) becomes

$$Z(n) = p_a - 1$$

which could be considered as a (partial) analogue of the Harnack's theorem.

References

1. N. A'Campo, Le groupe de monodromie du déploiment des singularités isolées de courbes planes I, Math. Ann. **213** (1975) 1–32
2. V.I. Arnold, S.M. Gusein-Zade, A.N. Varchenko, Singularities of Differentiable Maps, vols. 1 and 2, Monographs in mathematics, Birkhäuser, Boston, 1985 and 1988
3. V.I. Arnold, Yu.S. Il'yashenko, Ordinary Differential Equations, in Dynamical Systems I, Enciclopaedia of Math. Sci., vol.1, Springer, Berlin, 1988
4. V.I. Arnold, Geometrical Methods in the Theory of Ordinary Differential Equations, Springer, New York, 1988
5. V.I. Arnold, Some unsolved problems in the theory of differential equations and mathematical physics, Russian Math. Surveys **44** (1989), no. 4, 157–171
6. V.I. Arnold, Ten problems in Theory of singularities and its applications, 1–8, Adv. Soviet Math., **1**, Amer. Math. Soc., Providence, RI, 1990
7. W.A. Coppel, A survey of quadratic systems. J. Differential Equations **2** (1966) 293–304
8. H. Dulac, Sur les cycles limites, Bull. Soc. Math. France, **51** (1923) 45–188
9. J. Ecalle, Introduction aux fonctions analysables et preuve constructive de la conjecture de Dulac, Hermann, Paris 1992
10. J.P. Françoise, Successive derivatives of a first return map, application to the study of a quadratic vector field, Erg. Theory and Dyn. Systems **16** (1996) 87–96
11. L. Gavrilov, Le 16ème problème de Hilbert infinitésimal dans le cas quadratique, preprint 164, Laboratoire Emile Picard, University of Toulouse III, September 1999
12. L. Gavrilov, E. Horozov, Limit cycles of perturbations of quadratic vector fields, J. Math. Pures Appl. **72** (1993) 213–238
13. L. Gavrilov, Isochronicity of plane polynomial Hamiltonian systems, Nonlinearity **10** (1997) 433–448
14. L. Gavrilov, Petrov modules and zeros of Abelian integrals, Bull. Sci. math. **122** (1998) 571–584
15. L. Gavrilov, Abelian integrals related to Morse polynomials and perturbation of plane Hamiltonian vector fields, Ann. Inst. Fourier **49**, 2 (1999) 611–652
16. L. Gavrilov, I.D. Iliev, Second order analysis in polynomially perturbed reversible quadratic Hamiltonian systems, Erg. Theory and Dyn. Systems, to appear in 2000
17. P.A. Griffiths, J. Harris, Principles of Algebraic Geometry, John Wiley and Sons, 1978
18. D. Hilbert, Mathematical problems. Lecture delivered before the International Congress of Mathematicians at Paris in 1900. Proc. Symp. Pure Math. **28** (1976) 1–34
19. E. Horozov, I.D. Iliev, On the number of limit cycles in perturbations of quadratic Hamiltonian systems, Proc. London Math. Soc. **69** (1994) 198–224
20. E. Horozov, I.D. Iliev, On saddle-loop bifurcations of limit cycles in perturbations of quadratic Hamiltonian systems, J. Diff. Eqns. **113** (1994) 84–105

21. E. Horozov, I.D. Iliev, Hilbert-Arnold problem for cubic Hamiltonians and limit cycles, in: Proc. Fourth Intern. Coll. Diff. Equations, VSP Intern. Sci. Publ. Utrecht (1994) 115–124
22. E. Horozov, I. Iliev, Perturbations of quadratic Hamiltonian systems with symmetry, Ann. Inst. Henri Poincaré **13** (1996) 17–56
23. S.M. Husein-Zade, Dynkin digrams of singularities of functions of two variables, Functional Anal. Appl. **8** (1974) 10–13, 295-300
24. I.D. Iliev, Perturbations of quadratic centers, Bull. Sci. math. **22** (1998) 107–161
25. Yu. Il'yashenko, Appearance of limit cycles in perturbation of the equation $\frac{dw}{dz} = -\frac{R_z}{R_w}$ where $R(z, w)$ is a polynomial, USSR Math. Sbornik **78** (1969) 360–373 (in russian)
26. Yu. Il'yashenko, Dulac's memoir "On Limit Cycles" and related problems of the local theory of differential equations, Russian Math. Surveys **40** (1985) 1–49
27. Yu. Il'yashenko, S. Yakovenko, Double exponential estimate for the number of zeros of complete Abelian integrals, Invent. math. **121** (1995) 613–650
28. Yu. Il'yashenko, Finitness theorems for limit cycles, Transl. Math. Monographs **94**, AMS, Providence, RI, 1991
29. A.G. Khovanskiĭ, Real analytic manifolds with the property of finiteness, and complex abelian integrals. Functional Anal. Appl. **18** (1984) no. 2, 119–127
30. G.S. Petrov, Elliptic integrals and their nonoscillation, Funct. Anal. Appl. **20** (1986) 37–40
31. H. Poincaré, Mémoire sur les courbes définies par une équation différentielle, J. de Mathématiques, 3e série **7** (1881) 375–422, **8** (1882) 251–296
32. H. Poincaré, Sur les courbes définies par les équations différentielles, J. de Mathématiques, 4e série **1** (1885) 167–244
33. L.S. Pontryagin, On dynamic systems close to Hamiltonian systems, Zh. Eksp. Teor. Fiz. **4** (1934) 234–238, in russian
34. R. Roussarie, On the number of limit cycles which appear by perturbation of separatrix loop of planar vector fields, Bol. Soc. Bras. Math. (2), **17** (1986) 67–101
35. R. Roussarie, Bifurcations of Planar Vector Fields and Hilbert's sixteenth Problem, Progress in Mathematics vol. 164, Birkhauser, 1998
36. R. Roussarie, Cyclicité fini des lacets et des points cuspidaux, Nonlinearity **2** (1988) 73–117
37. J.-P. Serre, Groupes algébriques et corps de classes, Hermann, Paris, 1959
38. Ye Yankian Theory of limit cycles, Transl. of Mathematical Monographs **66**, AMS, 1986
39. A.N. Varchenko, Estimate of the number of zeros of Abelian integrals depending on a parameters and limit cycles, Funct. Analysis and Appl. **18** (1984) 98–108
40. A. Zegeling, R.E. Kooij, The distribution of limit cycles in quadratic systems with four finite singularities, J. Diff. Equations **151** (1999) 373–385
41. H. Żołądek, Quadratic systems with center and their perturbations, J. Diff. Equations **109** (1994) 223–273