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# Algebraic Quantum Field Theory, Perturbation Theory, and the Loop Expansion

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*Dedicated to the memory of Harry Lehmann*

**Abstract:** The perturbative treatment of quantum field theory is formulated within the framework of algebraic quantum field theory. We show that the algebra of interacting fields is additive, i.e. fully determined by its subalgebras associated to arbitrary small subregions of Minkowski space. We also give an algebraic formulation of the loop expansion by introducing a projective system $\mathcal{A}^{(n)}$ of observables "up to $n$ loops", where $\mathcal{A}^{(0)}$ is the Poisson algebra of the classical field theory. Finally we give a local algebraic formulation for two cases of the quantum action principle and compare it with the usual formulation in terms of Green's functions.

## 1. Introduction

Quantum field theory is a very successful frame for our present understanding of elementary particle physics. In the case of QED it led to fantastically precise predictions of experimentally measurable quantities; moreover the present standard model of elementary particle physics is of a similar structure and is also in good agreement with experiments. Unfortunately, it is not so clear what an interacting quantum field theory really is, expressed in meaningful mathematical terms. In particular, it is by no means evident how the local algebras of observables can be defined. A direct approach by methods of constructive field theory led to the paradoxical conjecture that QED does not exist; the situation seems to be better for Yang-Mills theories because of asymptotic freedom, but there the problem of big fields which can appear at large volumes poses at present unsurmountable problems [1,21].

In this paper we will take a pragmatic point of view: interacting quantum field theory certainly exists on the level of perturbation theory, and our confidence on quantum field theory relies mainly on the agreement of experimental data with results from low orders of perturbation theory. On the other hand, the general structure of algebraic quantum

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field theory (or "local quantum physics") coincides nicely with the qualitative features of elementary particle physics, therefore it seems to be worthwhile to revisit perturbation theory from the point of view of algebraic quantum field theory. This will, on the one hand, provide physically relevant examples for algebraic quantum field theory, and on the other hand, give new insight into the structure of perturbation theory. In particular, we will see that we can reach a complete separation of the infrared problem from the ultraviolet problem. This might be of relevance for Yang-Mills theory, and it is important for the construction of the theory on curved spacetimes [7].

The plan of the paper is as follows. We will start by describing the Stückelberg–Bogoliubov–Shirkov–Epstein–Glaser-version of perturbation theory [6,14,28,26,7]. This construction yields the local $S$-matrices $S(g)$ ($g \in \mathcal{D}(\mathbb{R}^4)$) as formal power series in $g$ (Sect. 2). The most important requirement which is used in this construction is the condition of causality (15) which is a functional equation for $g \to S(g)$. The results of Sects. 3 and 4 are to a large extent valid beyond perturbation theory. We only assume that we are given a family of unitary solutions of the condition of causality. In terms of these local $S$-matrices we will construct nets of local observable algebras for the interacting theory (Sect. 3). We will see that, as a consequence of causality, the interacting theory is completely determined if it is known for arbitrary small spacetime volumes (Sect. 4).

In Sect. 5 we algebraically quantize a free field by deforming the (classical) Poisson algebra. In a second step we generalize this quantization procedure to the perturbative interacting field. We end up with an algebraic formulation of the expansion in $\hbar$ of the interacting observables ("loop expansion").

In the last section we investigate two examples for the quantum action principle: the field equation and the variation of a parameter in the interaction. Usually this principle is formulated in terms of Green's functions [20,18,22], i.e. the vacuum expectation values of time ordered products of interacting fields. Here we give a local algebraic formulation, i.e. an operator identity for a localized interaction. In the case of the variation of a parameter in the interaction this requires the use of the retarded product of interacting fields, instead of only time ordered products (as in the formulation in terms of Green's functions).

For a local construction of observables and physical states in gauge theories we refer to [10,11,5]. There, perturbative positivity ("unitarity") is, by a local version of the Kugo-Ojima formalism [17], reduced to the validity of BRST symmetry [3].

## 2. Free Fields, Borchers' Class and Local $S$-Matrices

An algebra of observables corresponding to the Klein–Gordon equation

$$(\square + m^2)\varphi = 0 \tag{1}$$

can be defined as follows: Let $\Delta_{\mathrm{ret,av}}$ be the retarded, resp. advanced Green's functions of $(\square + m^2)$

$$(\square + m^2)\Delta_{\mathrm{ret,av}} = \delta, \qquad \mathrm{supp}\ \Delta_{\mathrm{ret,av}} \subset \bar{V}_{\pm}, \tag{2}$$

where $\bar{V}_{\pm}$ denotes the closed forward, resp. backward lightcone, and let $\Delta = \Delta_{\mathrm{ret}} - \Delta_{\mathrm{av}}$. The algebra of observables $\mathcal{A}$ is generated by smeared fields $\varphi(f),\ f \in \mathcal{D}(\mathbb{R}^4)$, which obey the following relations

$$f \mapsto \varphi(f) \text{ is linear,} \tag{3}$$
$$\varphi((\Box + m^2)f) = 0, \tag{4}$$
$$\varphi(f)^* = \varphi(\bar{f}), \tag{5}$$
$$[\varphi(f), \varphi(g)] = i < f, \Delta * g >, \tag{6}$$

where the star denotes convolution and $< f, g >= \int d^4x f(x)g(x)$. As a matter of fact, $\mathcal{A}$ (as a $*$-algebra with unit) is uniquely determined by these relations.

The Fock space representation $\pi$ of the free field is induced via the GNS-construction from the vacuum state $\omega_0$. Namely, let $\omega_0 : \mathcal{A} \to \mathbb{C}$ be the quasifree state given by the two-point function

$$\omega_0(\varphi(f)\varphi(g)) = i < f, \Delta_+ * g >, \tag{7}$$

where $\Delta_+$ is the positive frequency part of $\Delta$. Then the Fock space $\mathcal{H}$, the vector $\Omega$ representing the vacuum and the Fock representation are up to equivalence determined by the relation

$$(\Omega, \pi(A)\Omega) = \omega_0(A), \;\; A \in \mathcal{A}.$$

On $\mathcal{H}$, the field $\varphi$ (we will omit the representation symbol $\pi$) is an operator valued distribution, i.e. there is some dense subspace $\mathcal{D} \subset \mathcal{H}$ with

(i) $\varphi(f) \in \mathrm{End}(\mathcal{D})$
(ii) $f \mapsto \varphi(f)\Phi$ is continuous $\forall \Phi \in \mathcal{D}$.

There are other fields $A$ on $\mathcal{H}$, on the same domain, which are relatively local to $\varphi$,

$$[A(f), \varphi(g)] = 0 \qquad \text{if} \quad (x-y)^2 < 0 \quad \forall (x, y) \in (\mathrm{supp}\, f \times \mathrm{supp}\, g). \tag{8}$$

They form the so called Borchers class $\mathcal{B}$. In the case of the free field in 4 dimensions, $\mathcal{B}$ consists of Wick polynomials and their derivatives [13]. Fields from the Borchers class can be used to define local interactions,

$$H_I(t) = -\int d^3x \, g(t, \boldsymbol{x}) A(t, \boldsymbol{x}), \qquad g \in \mathcal{D}(\mathbb{R}^4), \tag{9}$$

(where the minus sign comes from the interpretation of $A$ as an interaction term in the Lagrangian) provided they can be restricted to spacelike surfaces. The corresponding time evolution operator from $-\tau$ to $\tau$, where $\tau > 0$ is so large that $\mathrm{supp}\, g \subset (-\tau, \tau) \times \mathrm{R}^3$, (the $S$-matrix) is formally given by the Dyson series

$$S(g) = \mathbf{1} + \sum_{n=1}^{\infty} \frac{i^n}{n!} \int dx_1 \dots dx_n \, T\big(A(x_1) \dots A(x_n)\big) g(x_1) \dots g(x_n). \tag{10}$$

with the time ordered products ("$T$-products") $T\big(\dots\big)$. It is difficult to derive (10) from (9) if the field $A$ cannot be restricted to spacelike surfaces. Unfortunately, this is almost always the case in four spacetime dimensions, the only exception being the field $\varphi$ itself and its derivatives. Therefore one defines the timeordered products of $n$ factors directly as multilinear (with respect to $C^\infty$-functions as coefficients) symmetric mappings from

$\mathcal{B}^n$ to operator valued distributions $T\big(A_1(x_1)\dots A_n(x_n)\big)$ on $\mathcal{D}$ such that they satisfy the factorization condition[1]

$$T\big(A(x_1)\dots A(x_n)\big) = T\big(A(x_1)\dots A(x_k)\big)T\big(A(x_{k+1})\dots A(x_n)\big) \tag{11}$$

if $\{x_{k+1},\dots,x_n\} \cap (\{x_1,\dots,x_k\} + \bar{V}_+) = \emptyset$. The $S$-matrix $S(g)$ is then, as a formal power series, by definition given by (10) . Since its zeroth order term is $\mathbf{1}$, it has an inverse in the sense of formal power series

$$S(g)^{-1} = \mathbf{1} + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int dx_1 \dots dx_n\, \bar{T}\big(A(x_1)\dots A(x_n)\big) g(x_1)\dots g(x_n), \tag{12}$$

where the "antichronological products" $\bar{T}(\dots)$ can be expressed in terms of the time ordered products

$$\bar{T}\big(A(x_1)\dots A(x_n)\big) \overset{\text{def}}{=} \sum_{P\in\mathcal{P}(\{1,\dots,n\})} (-1)^{|P|+n} \prod_{p\in P} T\big(A(x_i), i \in p\big). \tag{13}$$

(Here $\mathcal{P}(\{1,\dots,n\})$ is the set of all ordered partitions of $\{1,\dots,n\}$ and $|P|$ is the number of subsets in $P$.) The $\bar{T}$-products satisfy anticausal factorization

$$\bar{T}\big(A(x_1)\dots A(x_n)\big) = \bar{T}\big(A(x_{k+1})\dots A(x_n)\big)\bar{T}\big(A(x_1)\dots A(x_k)\big) \tag{14}$$

if $\{x_{k+1},\dots,x_n\} \cap (\{x_1,\dots,x_k\} + \bar{V}_+) = \emptyset$.

The crucial observation now (cf. [16]) is that $S(g)$ satisfies the remarkable functional equation

$$S(f+g+h) = S(f+g)S(g)^{-1}S(g+h), \tag{15}$$

$f, g, h \in \mathcal{D}(\mathbb{R}^4)$, whenever $(\operatorname{supp} f + \bar{V}_+) \cap \operatorname{supp} h = \emptyset$ (independent of $g$). Equivalent forms of this equation play an important role in [6] and [14]. For $g = 0$ this is just the functional equation for the time evolution and may be interpreted as the requirement of causality [6]. Actually, for formal power series $S(\cdot)$ of operator valued distributions, the $g = 0$ equation is equivalent to the seemingly stronger relation (15), because both are equivalent to condition (11) for the time ordered products. We call (15) the "condition of causality".

## 3. Interacting Local Nets

The arguments of this and the next section are to a large extent independent of perturbation theory. We start from the assumption that we are given a family of unitaries $S(f) \in \mathcal{A}$, $\forall f \in \mathcal{D}(\mathbb{R}^4, \mathcal{V})$ (i.e. $f$ has the form $f = \sum_i f_i(x)A_i$, $f_i \in \mathcal{D}(\mathbb{R}^4, \mathbb{R})$, $A_i \in \mathcal{V}$), where $\mathcal{V}$ is an abstract, finite dimensional, real vector space, interpreted as the space of possible interaction Lagrangians, and $\mathcal{A}$ is some unital $*$-algebra. In perturbation theory $\mathcal{V}$ is a real subspace of the Borchers' class. The unitaries $S(f)$ are required to satisfy the causality condition (15). We first observe that we obtain new solutions of (15) by introducing the relative $S$-matrices

$$S_g(f) \overset{\text{def}}{=} S(g)^{-1}S(g+f), \tag{16}$$

[1] Due to the symmetry and linearity of $T(\dots)$ it suffices to consider the case $A_1 = A_2 = \dots = A_n$.

where now $g$ is kept fixed and $S_g(f)$ is considered as a functional of $f$. In particular, the relative $S$-matrices satisfy local commutation relations

$$[S_g(h), S_g(f)] = 0 \qquad \text{if} \quad (x-y)^2 < 0 \quad \forall (x,y) \in \operatorname{supp} h \times \operatorname{supp} f. \tag{17}$$

Therefore their functional derivatives $A_g(x) = \frac{\delta}{\delta h(x)} S_g(hA)|_{h=0}$, $A \in \mathcal{V}$, $h \in \mathcal{D}(\mathbb{R}^4)$, provided they exist, are local fields (in the limit $g \to$ constant this is Bogoliubov's definition of interactig fields) [6].

We now introduce local algebras of observables by assigning to a region $\mathcal{O}$ of Minkowski space the $*$-algebra $\mathcal{A}_g(\mathcal{O})$ which is generated by $\{S_g(h)\ ,\ h \in \mathcal{D}(\mathcal{O}, \mathcal{V})\}$.

A remarkable consequence of relation (15) is that the structure of the algebra $\mathcal{A}_g(\mathcal{O})$ depends only locally on $g$ [16,7], namely, if $g \equiv g'$ in a neighbourhood of a causally closed region containing $\mathcal{O}$, then there exists a unitary $V \in \mathcal{A}$ such that

$$V S_g(h) V^{-1} = S_{g'}(h), \qquad \forall\, h \in \mathcal{D}(\mathcal{O}, \mathcal{V}). \tag{18}$$

Hence the system of local algebras of observables (according to the principles of algebraic quantum field theory this system ("the local net") contains the full physical content of a quantum field theory) is completely determined if one knows the relative $S$-matrices for test functions $g \in \mathcal{D}(\mathbb{R}^4, \mathcal{V})$.

The construction of the global algebra of observables for an interaction Lagrangian $\mathcal{L} \in \mathcal{V}$ may be performed explicitly (cf. [7]). Let $\Theta(\mathcal{O})$ be the set of all functions $\theta \in \mathcal{D}(\mathbb{R}^4)$ which are identically to 1 in a causally closed open neighbourhood of $\mathcal{O}$ and consider the bundle

$$\bigcup_{\theta \in \Theta(\mathcal{O})} \{\theta\} \times \mathcal{A}_{\theta\mathcal{L}}(\mathcal{O}). \tag{19}$$

Let $\mathcal{U}(\theta, \theta')$ be the set of all unitaries $V \in \mathcal{A}$ with

$$V S_{\theta\mathcal{L}}(h) = S_{\theta'\mathcal{L}}(h) V, \qquad \forall\, h \in \mathcal{D}(\mathcal{O}, \mathcal{V}). \tag{20}$$

Then $\mathcal{A}_{\mathcal{L}}(\mathcal{O})$ is defined as the algebra of covariantly constant sections, i.e.

$$\mathcal{A}_{\mathcal{L}}(\mathcal{O}) \ni A = (A_\theta)_{\theta \in \Theta(\mathcal{O})} \qquad (A_\theta \in \mathcal{A}_{\theta\mathcal{L}}(\mathcal{O})) \tag{21}$$

$$V A_\theta = A_{\theta'} V, \qquad \forall V \in \mathcal{U}(\theta, \theta'). \tag{22}$$

$\mathcal{A}_{\mathcal{L}}(\mathcal{O})$ contains in particular the elements $S_{\mathcal{L}}(h)$,

$$(S_{\mathcal{L}}(h))_\theta = S_{\theta\mathcal{L}}(h). \tag{23}$$

The construction of the local net is completed by fixing the embeddings $i_{21} : \mathcal{A}_{\mathcal{L}}(\mathcal{O}_1) \hookrightarrow \mathcal{A}_{\mathcal{L}}(\mathcal{O}_2)$ for $\mathcal{O}_1 \subset \mathcal{O}_2$. But these embeddings are inherited from the inclusions $\mathcal{A}_{\theta\mathcal{L}}(\mathcal{O}_1) \subset \mathcal{A}_{\theta\mathcal{L}}(\mathcal{O}_2)$ for $\theta \in \Theta(\mathcal{O}_2)$ by restricting the sections from $\Theta(\mathcal{O}_1)$ to $\Theta(\mathcal{O}_2)$. The embeddings evidently satisfy the compatibility relation $i_{12} \circ i_{23} = i_{13}$ for $\mathcal{O}_3 \subset \mathcal{O}_2 \subset \mathcal{O}_1$ and define thus an inductive system. Therefore, the global algebra can be defined as the inductive limit of local algebras

$$\mathcal{A}_{\mathcal{L}} \stackrel{\text{def}}{=} \cup_{\mathcal{O}} \mathcal{A}_{\mathcal{L}}(\mathcal{O}). \tag{24}$$

In perturbation theory, the unitaries $V \in \mathcal{U}(\theta, \theta')$ are themselves formal power series, therefore it makes no sense to say that two elements $A, B \in \mathcal{A}_{\mathcal{L}}(\mathcal{O})$ agree *in* $n^{\text{th}}$

order, but only that they agree *up to* $n^{\text{th}}$ order (because $(A_\theta - B_\theta) = \mathcal{O}(g^{n+1})$ implies $A_{\theta'} - B_{\theta'} = V^{-1}(A_\theta - B_\theta)V = \mathcal{O}(g^{n+1})$).

The time ordered products and hence the relative $S$-matrices $S_{\theta\mathcal{L}}(h)$ are chosen as to satisfy Poincaré covariance (see the normalization condition **N1** below), i.e. the unitary positive energy representation $U$ of the Poincaré group $\mathcal{P}_+^\uparrow$ under which the free field transforms satisfies

$$U(L)S_{\theta\mathcal{L}}(h)U(L)^{-1} = S_{\theta_L\mathcal{L}}(h_L),\\ \theta_L(x) := \theta(L^{-1}x),\ h_L(x) := D(L)h(L^{-1}x), \tag{25}$$

$\forall L \in \mathcal{P}_+^\uparrow$ provided $\mathcal{L}$ is a Lorentz scalar and $\mathcal{V}$ transforms under the finite dimensional representation $D$ of the Lorentz group. This enables us to define an automorphic action of the Poincaré group on the algebra of observables. Let for $A \in \mathcal{A}_\mathcal{L}(\mathcal{O}),\ \theta \in \Theta(L\mathcal{O})$

$$(\alpha_L(A))_\theta \stackrel{\text{def}}{=} U(L)A_{\theta_{L^{-1}}}U(L)^{-1}. \tag{26}$$

By inserting the definitions one finds that $\alpha_L(A)$ is again a covariantly constant section (22). So $\alpha_L$ is an automorphism of the net which realizes the Poincaré symmetry

$$\alpha_L\mathcal{A}_\mathcal{L}(\mathcal{O}) = \mathcal{A}_\mathcal{L}(L\mathcal{O}), \qquad \alpha_{L_1L_2} = \alpha_{L_1}\alpha_{L_2}. \tag{27}$$

For the purposes of perturbation theory, we have to enlarge the local algebras somewhat. In perturbation theory, the relative $S$-matrices are formal power series in two variables, and therefore the generators of the local algebras

$$S_\mathcal{L}(\lambda f) = \sum_{n=0}^\infty \frac{i^n\lambda^n}{n!}T_\mathcal{L}(f^{\otimes n}) \tag{28}$$

are formal power series with coefficients which are covariantly constant sections in the sense of (22). The first order terms in (28) are, according to Bogoliubov, the interacting local fields,

$$T_\mathcal{L}(hA) =: A_\mathcal{L}(h),\ A \in \mathcal{V},\ h \in \mathcal{D}(\mathbb{R}^4), \tag{29}$$

the higher order terms satisfy the causality condition (11) and may therefore be interpreted as time ordered products of interacting fields (cf. [14], Sect. 8.1).

Our enlarged local algebra $\mathcal{A}_\mathcal{L}(\mathcal{O})$ (we use the same symbol as before) now consists of all formal power series with coefficients from the algebra generated by all timeordered products $T_\mathcal{L}(f^{\otimes n})$ with $f \in \mathcal{D}(\mathcal{O}, \mathcal{V}),\ n \in \mathbb{N}_0$.

## 4. Consequences of Causality

Another consequence of the causality relation (15) is that the $S$-matrices $S(f)$ are *uniquely* fixed if they are known for test functions with arbitrarily small supports. Namely, by a repeated use of (15) we find that $S(\sum_{i=1}^n f_i)$ is a product of factors $S(\sum_{i\in K} f_i)^{\pm 1}$, where the sets $K \subset \{1, \dots, n\}$ have the property that for every pair $i, j \in K$ the causal closures of supp $f_i$ and supp $f_j$ overlap. Hence if the supports of all $f_i$ are contained in double cones of diameter $d$, the supports of $\sum_{i\in K} f_i$ fit into double cones of diameter

$2d$. As $d > 0$ can be chosen arbitrarily small and the relative $S$-matrices also satisfy (15), this implies additivity of the net,

$$\mathcal{A}_{\mathcal{L}}(\mathcal{O}) = \bigvee_{\alpha} \mathcal{A}_{\mathcal{L}}(\mathcal{O}_\alpha), \tag{30}$$

where $(\mathcal{O}_\alpha)$ is an arbitrary covering of $\mathcal{O}$ and where the symbol $\bigvee$ means the generated algebra.

One might also pose the *existence* question: Suppose we have a family of unitaries $S(f)$ for all $f$ with sufficiently small support which satisfy the causality condition (15) for $f, g, h \in \mathcal{D}(\mathcal{O}, \mathcal{V})$, diam$(\mathcal{O})$ sufficiently small, and local commutativity for arbitrary big separation

$$[S(f), S(g)] = 0 \quad \text{if} \quad \text{supp}\, f \text{ is spacelike to supp}\, g.$$

By repeated use of the causality (15) we can then define $S$-matrices for test functions with larger support. It is, however, not evident that these $S$-matrices are independent of the way of construction and that they satisfy the causality condition. (We found a consistent construction only in the simple case of one dimension: $x =$ time.) Fortunately, a general positive answer can be given in perturbation theory.

Let $S(f)$ be given for $f \in \mathcal{D}(\mathcal{O}, \mathcal{V})$ for all double cones with diam$(\mathcal{O}) < r$. The time ordered product of $n$ factors is the $n$-fold functional derivative of $S$ at $f = 0$. It is an operator valued distribution[2] $T_n$ defined on test functions of $n$ variables with support contained in $\mathcal{U}_n \stackrel{\text{def}}{=} \{(y_1, \dots, y_n) \in \mathbb{R}^{4n} \mid \max_{i<j} |y_i - y_j| < \frac{r}{2}\}$ and with values in $\mathcal{V}^{\otimes n}$. Especially we know $T_1(x)$ on $\mathbb{R}^4$. On this domain the time ordered products satisfy the factorization condition (11). In addition, local commutativity of the $S$-matrices implies

$$[T_n(x_1, \dots, x_n), T_m(y_1, \dots y_m)] = 0 \tag{31}$$

for $(x_i - y_j)^2 < 0 \quad \forall (i, j)$ and $(x_1, \dots x_n) \in \mathcal{U}_n,\ (y_1, \dots, y_m) \in \mathcal{U}_m$. By construction $T_n|_{\mathcal{U}_n}$ is symmetric with respect to permutations of the factors.

We now show that this input suffices to construct $T_n(x_1, \dots, x_n)$ on the whole $\mathbb{R}^{4n}$ by induction on $n$. We assume that the $T_k$'s were constructed for $k \le n-1$, that they fulfil causality (11) and

$$[T_m(x_1, \dots, x_m), T_k(y_1, \dots y_k)] = 0 \quad \text{for} \quad (x_1, \dots x_m) \in \mathcal{U}_m,\ k \le n-1 \tag{32}$$

($m$ arbitrary) and

$$[T_l(x_1, \dots, x_l), T_k(y_1, \dots y_k)] = 0 \quad \text{for} \quad l, k \le n-1, \tag{33}$$

if $(x_i - y_j)^2 < 0 \quad \forall (i, j)$ in the latter two equations. We can now proceed as in Sect. 4 of [7].[3]

[2] Here we change the notation for the time ordered products: let $f = \sum_i f_i(x) A_i,\ f_i \in \mathcal{D}(\mathbb{R}^4),\ A_i \in \mathcal{V}$. Instead of $\int dx_1 \dots dx_n \sum_{i_1 \dots i_n} T\big(A_{i_1}(x_1) \dots A_{i_n}(x_n)\big) f_{i_1}(x_1) \dots f_{i_n}(x_n)$ (10) we write $\int dx_1 \dots dx_n\, T_n(x_1, \dots, x_n) f(x_1) \dots f(x_n) \equiv T_n(f^{\otimes n})$.

[3] In contrast to the (inductive) Epstein–Glaser construction of $T_n(x_1, \dots, x_n)$ [14,7] the present construction is unique, normalization conditions (e.g. **N1**–**N4** in Sect. 5) are not needed, because the non-uniqueness of the Epstein–Glaser construction is located at the total diagonal $\Delta_n \equiv \{(x_1, \dots, x_n) \mid x_1 = \dots = x_n\}$. But here the time ordered products are given in the neighbourhood $\mathcal{U}_n$ of $\Delta_n$.

Let $\mathcal{J}$ denote the family of all non-empty proper subsets $I$ of the index set $\{1,\ldots,n\}$ and define the sets $\mathcal{C}_I \stackrel{\text{def}}{=} \{(x_1,\ldots,x_n) \in \mathbb{R}^{4n} \mid x_i \notin J^-(x_j),\ i \in I,\ j \in I^c\}$ for any $I \in \mathcal{J}$. Then

$$\bigcup_{I\in\mathcal{J}} \mathcal{C}_I \cup \mathcal{U}_n = \mathbb{R}^{4n}. \tag{34}$$

We use the short-hand notations

$$T^I(x_I) = T(\prod_{i\in I} A_i(x_i)), \qquad x_I = (x_i, i \in I). \tag{35}$$

On $\mathcal{D}(\mathcal{C}_I)$ we set

$$T_I(x) \stackrel{\text{def}}{=} T^I(x_I)T^{I^c}(x_{I^c}) \tag{36}$$

for any $I \in \mathcal{C}_I$. For $I_1, I_2 \in \mathcal{J},\ \mathcal{C}_{I_1} \cap C_{I_2} \neq \emptyset$ one easily verifies[4]

$$T_{I_1}|_{\mathcal{C}_{I_1}\cap\mathcal{C}_{I_2}} = T_{I_2}|_{\mathcal{C}_{I_1}\cap\mathcal{C}_{I_2}}. \tag{37}$$

Let now $\{f_I\}_{I\in\mathcal{J}} \cup \{f_0\}$ be a finite smooth partition of unity of $\mathbb{R}^{4n}$ subordinate to $\{\mathcal{C}_I\}_{I\in\mathcal{J}} \cup \mathcal{U}_n$: supp $f_I \subset \mathcal{C}_I$, supp $f_0 \subset \mathcal{U}_n$. Then we define

$$T_n(h) \stackrel{\text{def}}{=} T_n|_{\mathcal{U}_n}(f_0 h) + \sum_{I\in\mathcal{J}} T_I(f_I h), \qquad h \in \mathcal{D}(\mathbb{R}^{4n}, \mathcal{V}^{\otimes n}). \tag{38}$$

As in [7] one may prove that this definition is independent of the choice of $\{f_I\}_{I\in\mathcal{J}} \cup \{f_0\}$ and that $T_n$ is symmetric with respect to permutations of the factors and satisfies causality (11). Local commutativity (32) and (33) (with $n-1$ replaced by $n$) is verified by inserting the definition (38) and using the assumptions. By (10) we obtain from the $T$-products the corresponding $S$-matrix $S(g)$ for arbitrary large support of $g \in \mathcal{D}(\mathbb{R}^4, \mathcal{V})$, and $S(g)$ satisfies the functional equation (15).

## 5. Perturbative Quantization and Loop Expansion

Causal perturbation theory was traditionally formulated in terms of operator valued distributions on Fock space. It is therefore well suited for describing the deformation of the free field into an interacting field by turning on the interaction $g \in \mathcal{D}(\mathbb{R}^4, \mathcal{V})$. It is much less clear how an expansion in powers of $\hbar$ can be performed, describing the deformation of the classical field theory, mainly because the Fock space has no classical counter part.

Usually the expansion in powers of $\hbar$ is done in functional approaches to field theory by ordering Feynman graphs according to loop number. In this section we show that the algebraic description provides a natural formulation of the loop expansion, and we point out the connection to formal quantization theory.

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[4] In contrast to [7] the Wick expansion of the $T$-products is not used here, because local commutativity of the $T$-products is contained in the inductive assumption.

*5.1. Quantization of a free field and Wick products.* In quantization theory one associates to a given classical theory a quantum theory. One procedure is the deformation (or star-product) quantization [2]. This procedure starts from a Poisson algebra, i.e. a commutative and associative algebra together with a second product: a Poisson bracket, satisfying the Leibniz rule and the Jacobi identity; and to deform the product as a function of $\hbar$, such that[5] $a \times_\hbar b$ is a formal power series in $\hbar$, the associativity is maintained and

$$a \times_\hbar b \quad \stackrel{\hbar\to 0}{\longrightarrow} \quad ab, \qquad \frac{1}{\hbar}(a \times_\hbar b - b \times_\hbar a) \quad \stackrel{\hbar\to 0}{\longrightarrow} \quad \{a, b\}. \tag{39}$$

Actually this scheme can easily be realized in free field theory (cf. [9]). Basic functions are the evaluation functionals $\varphi_{\text{class}}(x)$, $(\Box + m^2)\varphi_{\text{class}} = 0$, with the Poisson bracket

$$\{\varphi_{\text{class}}(x), \varphi_{\text{class}}(y)\} = \Delta(x - y) \tag{40}$$

($\Delta$ is the commutator function (2)). Because of the singular character of $\Delta$ the fields should be smoothed out in order to belong to the Poisson algebra. Hence our fundamental classical observables are

$$\begin{aligned} \phi(t) = t_0 + \sum_{n=1}^{N} \int \varphi_{\text{class}}(x_1)\dots\varphi_{\text{class}}(x_n) t_n(x_1, \dots, x_n) dx_1 \dots dx_n, \\ t \equiv (t_0, t_1, \dots), \end{aligned} \tag{41}$$

where $t_0 \in \mathbb{C}$ arbitrary, $N < \infty$, $t_n$ is a suitable test "function" (we will admit also certain distributions) with compact support. The Klein Gordon equation shows up in the property: $A(t) = 0$ if $t_0 = 0$ and $t_n = (\Box_i + m^2)g_n$ for all $n > 0$, some $i = i(n)$ and some $g_n$ with compact support.

In the quantization procedure we identify $\varphi_{\text{class}}(x_1)\dots\varphi_{\text{class}}(x_n)$ with the normally ordered product (Wick product) $:\varphi(x_1)\dots\varphi(x_n):$ ($\varphi$ is the free quantum field ((3)–(6)). Wick's theorem may be interpreted as the definition of a $\hbar$-dependent associative product,

$$\begin{aligned} &: \prod_{i\in I} \varphi(x_i) : \times_\hbar \; : \prod_{j\in J} \varphi(x_j) : \\ &\quad = \sum_{K\subset I} \sum_{\alpha:K\to J \text{ injective}} \prod_{j\in K} i\hbar\Delta_+(x_j - x_{\alpha(j)}) : \prod_{l\in(I\setminus K)\cup(J\setminus\alpha(K))} \varphi(x_l) : \end{aligned} \tag{42}$$

in the linear space spanned by Wick products (the "Wick quantization").[6] To be precise we have to fix a suitable test function space (or better: test distribution space) in (41) which is small enough such that the product is well defined for all $\hbar$ and which contains the interesting cases occuring in perturbation theory, e.g. products of translation invariant distributions (particularly $\delta$-distributions of difference variables) with test functions of compact support should be allowed for $t_n$ as in Theorem 0 of Epstein and Glaser.

[5] The deformed product is called a $*$-product in deformation theory. In order to avoid confusion with the $*$-operation we denote the product by $\times_\hbar$.

[6] The observation that the Wick quantization is appropriate for the quantization of the free field goes back to Dito [9].

Let

$$\mathcal{W}_n \stackrel{\text{def}}{=} \{t \in \mathcal{D}'(\mathbb{R}^{4n})_{\text{symm}}\ ,\ \text{supp}\, t \text{ compact}, \\ \text{WF}(t)\ \cap\ (\mathbb{R}^{4n} \times \overline{V_+^n \cup V_-^n}) = \emptyset\} \tag{43}$$

(see the Appendix for a definition of the wave front set WF of a distribution). In [7] it was shown that Wick polynomials smeared with distributions $t \in \mathcal{W}_n$,

$$(\varphi^{\otimes n})(t) \stackrel{\text{def}}{=} \int :\varphi(x_1)\ldots\varphi(x_n): t(x_1,\ldots,x_n)\, dx_1 \ldots dx_n, \quad (\varphi^{\otimes 0}) \stackrel{\text{def}}{=} \mathbf{1}, \tag{44}$$

are densely defined operators on an invariant domain in Fock space. This includes in particular the Wick powers

$$:\varphi^n(f) := (\varphi^{\otimes n})(t),\ f \in \mathcal{D}(\mathbb{R}^4),\ t(x_1,\ldots,x_n) = f(x_1)\prod_{i=2}^{n}\delta(x_i - x_1) \tag{45}$$

The product of two such operators is given by

$$(\varphi^{\otimes n})(t) \times_\hbar (\varphi^{\otimes m})(s) = \sum_{k=0}^{\min\{n,m\}} \hbar^k (\varphi^{\otimes(n+m-2k)})(t \otimes_k s) \tag{46}$$

with the $k$-times contracted tensor product

$$(t \otimes_k s)(x_1,\ldots,x_{n+m-2k}) = \mathcal{S}\frac{n!m!i^k}{k!(n-k)!(m-k)!}\int dy_1\ldots dy_{2k}\Delta_+(y_1-y_2)\ldots \\ \Delta_+(y_{2k-1}-y_{2k})t(x_1,\ldots,x_{n-k},y_1,y_3,\ldots,y_{2k-1}) \\ s(x_{n-k+1},\ldots,x_{n+m-2k},y_2,y_4,\ldots,y_{2k}) \tag{47}$$

($\mathcal{S}$ means the symmetrization in $x_1,\ldots,x_{n+m-2k}$). The conditions on the wave front sets of $t$ and $s$ imply that the product $(t \otimes_k s)$ exists (see the Appendix) and is an element of $\mathcal{W}_{n+m-2k}$. The $*$-operation reduces to complex conjugation of the smearing function.

Let $\mathcal{W}_0 \stackrel{\text{def}}{=} \mathbb{C}$ and $\mathcal{W} \stackrel{\text{def}}{=} \bigoplus_{n=0}^{\infty}\mathcal{W}_n$. For $t \in \mathcal{W}$ let $t_n$ denote the component of $t$ in $\mathcal{W}_n$. The $*$-operation is defined by $(t^*)_n \stackrel{\text{def}}{=} (\bar{t}_n)$. Equation (46) can be thought of as the definition of an associative product on $\mathcal{W}$,

$$(t \times_\hbar s)_n = \sum_{m+l-2k=n} \hbar^k t_m \otimes_k s_l. \tag{48}$$

The Klein–Gordon equation defines an ideal $\mathcal{N}$ in $\mathcal{W}$ which is generated by $(\square + m^2)f,\ f \in \mathcal{D}(\mathbb{R}^4)$. Actually this ideal is independent of $\hbar$ (because a contraction with $(\square + m^2)f$ vanishes) and coincides with the kernel of $\phi$ defined in (41). Hence the product (48) is well defined on the quotient space $\bar{\mathcal{W}} = \mathcal{W}/\mathcal{N}$. For a given positive value of $\hbar$, $\bar{\mathcal{W}}$ is isomorphic to the algebra generated by Wick products $(\varphi^{\otimes n})(t),\ t \in \mathcal{W}_n$ (44). In the limit $\hbar \to 0$ we find

$$\lim_{\hbar\to 0}\phi(t) \times_\hbar \phi(s) = \lim_{\hbar\to 0}\phi(\sum_n \hbar^n t \otimes_n s) \\ = \phi(t \otimes_0 s) = \phi(t)\cdot\phi(s) \tag{49}$$

(we set $(t \otimes_k s)_n \stackrel{\text{def}}{=} \sum_{m+l=n} t_{m+k} \otimes_k s_{l+k}$, cf. (47)), with the classical product $\cdot$, and

$$\lim_{\hbar\to 0} \frac{1}{i\hbar}[\phi(t), \phi(s)]_\hbar = \phi(t \otimes_1 s - s \otimes_1 t) = \{\phi(t), \phi(s)\} \tag{50}$$

with the classical Poisson bracket. Thus $(\overline{\mathcal{W}}, \times_\hbar)$ provides a quantization of the given Poisson algebra of the classical free field $\varphi_{\text{class}}$ (40). We point out that we have formulated the algebraic structure of smeared Wick products without using the Fock space.

The Fock representation is recovered, via the GNS construction, from the vacuum state $\omega_0(t) = t_0$. It is faithful for $\hbar \neq 0$ but is one dimensional in the classical limit $\hbar = 0$. This illustrates the superiority of the algebraic point of view for a discussion of the classical limit.

*5.2. Normalization conditions and retarded products.* To study the perturbative quantization of interacting fields we need some technical tools which are given in this subsection.

The time ordered products are constructed by induction on the number $n$ of factors (which is also the order of the perturbation series (10)). In contrast to the inductive construction of the $T$-products in sect. 4, we do not know $T_n|_{\mathcal{U}_n}$ here. So causality (11) and symmetry determine the time ordered products uniquely (in terms of time ordered products of less factors) up to the total diagonal $\Delta_n = \{(x_1, \ldots, x_n) \in \mathbb{R}^{4n} | x_1 = x_2 = \cdots = x_n\}$. There is some freedom in the extension to $\Delta_n$. To restrict it we introduce the following additional defining conditions ("normalization conditions", formulated for a scalar field without derivative coupling, i.e. $\mathcal{L}$ is a Wick polynomial solely in $\phi$, it does not contain derivatives of $\phi$; for the generalization to derivative couplings see [5])

**N1** covariance with resp. to Poincaré transformations and possibly discrete symmetries, in particular
**N2** unitarity: $T(A_1(x_1)\ldots A_n(x_n))^* = \bar{T}(A_1^*(x_1)\ldots A_n^*(x_n))$,
**N3** $[T(A_1(x_1)\ldots A_n(x_n)), \phi(x)]$
$= i\hbar \sum_{k=1}^n T(A_1(x_1)\ldots \frac{\partial A_k}{\partial\phi}(x_k)\ldots A_n(x_n))\Delta(x_k - x)$,
**N4** $(\Box_x + m^2)T(A_1(x_1)\ldots A_n(x_n)\phi(x))$
$= -i\hbar \sum_{k=1}^n T(A_1(x_1)\ldots \frac{\partial A_k}{\partial\phi}(x_k)\ldots A_n(x_n))\delta(x_k - x)$,

where $[\phi(x), \phi(y)] = i\hbar\Delta(x - y)$. **N1** implies covariance of the arising theory, and **N2** provides a $*$-structure. **N3** gives the relation to time ordered products of sub Wick polynomials. Once these are known (in an inductive procedure), only a scalar distribution has to be fixed. Due to translation invariance the latter depends only on the relative coordinates. Hence, the extension of the (operator valued) $T$-product to $\Delta_n$ is reduced to the extension of a C-number distribution $t_0 \in \mathcal{D}'(\mathbb{R}^{4(n-1)} \setminus \{0\})$ to $t \in \mathcal{D}'(\mathbb{R}^{4(n-1)})$. (We call $t$ an extension of $t_0$ if $t(f) = t_0(f),\ \forall f \in \mathcal{D}(\mathbb{R}^{4(n-1)} \setminus \{0\})$). The singularity of $t_0(y)$ and $t(y)$ at $y = 0$ is classified in terms of Steinmann's scaling degree [27,7]

$$\text{sd}(t) \stackrel{\text{def}}{=} \inf\{\delta \in \mathbb{R}\,,\ \lim_{\lambda\to 0} \lambda^\delta t(\lambda x) = 0\}. \tag{51}$$

By definition $\text{sd}(t_0) \leq \text{sd}(t)$, and the possible extensions are restricted by requiring

$$\text{sd}(t_0) = \text{sd}(t). \tag{52}$$

Then the extension is unique for $\mathrm{sd}(t_0) < 4(n-1)$, and in the general case there remains the freedom to add derivatives of the $\delta$-distribution up to order $(\mathrm{sd}(t_0) - 4(n-1))$, i.e.

$$t(y) + \sum_{|a| \leq \mathrm{sd}(t_0)-4(n-1)} C_a \partial^a \delta(y) \tag{53}$$

is the general solution, where $t$ is a special extension [7, 24, 14], and the constants $C_a$ are restricted by **N1**, **N2**, **N4**, permutation symmetries and possibly further normalization conditions, e.g. the Ward identities for QED [10, 5]. For an interaction with mass dimension $\dim(\mathcal{L}) \leq 4$ the requirement (52) implies renormalizability by power counting, i.e. the number of indeterminate constants $C_a$ does not increase by going over to higher perturbative orders. In [10] it is shown that the normalization condition **N4** implies the field equation for the interacting field corresponding to the free field $\phi$ (see also (77) and Sect. 6.1 below).

We have defined the interacting fields as functional derivatives of relative $S$-matrices (29). Hence, to formulate the perturbation series of interacting fields we need the perturbative expansion of the relative $S$-matrices:

$$S_g(f) = \sum_{n,m} \frac{i^{n+m}}{n!m!} R_{n,m}(g^{\otimes n}; f^{\otimes m}), \tag{54}$$

where $g, f \in \mathcal{D}(\mathbb{R}^4, \mathcal{V})$. The coefficients are the so called retarded products ("$R$-products"). They can be expressed in terms of time ordered and anti-time ordered products by

$$R_{n,m}(g^{\otimes n}; f^{\otimes m}) = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \bar{T}_k(g^{\otimes k}) \times_\hbar T_{n-k+m}(g^{\otimes(n-k)} \otimes f^{\otimes m}). \tag{55}$$

They vanish if one of the first $n$ arguments is not in the past light cone of some of the last $m$ arguments ([14], Sect. 8.1),

$$\mathrm{supp}\ R_{n,m}\big(\ldots\big) \subset \{(y_1, \ldots y_n, x_1, \ldots, x_m)\ ,\ \{y_1, \ldots y_n\} \subset (\{x_1, \ldots, x_m\} + \bar{V}_-)\}. \tag{56}$$

In the remaining part of this subsection we show that the time ordered products can be defined in such a way that $R_{n,m}$ is of order $\hbar^n$. For this purpose we will introduce the connected part $(a_1 \times_\hbar \cdots \times_\hbar a_n)^c$ of $(a_1 \times_\hbar \cdots \times_\hbar a_n)$, where the $a_i$ are normally ordered products of free fields, and the connected part $T_n^c$ of the time ordered product $T_n$ (or "truncated time ordered product"). In both cases the connected part corresponds to the sum of connected diagrams, provided the vertices belonging to the same $a_i$ are identified. Besides the (deformed) product $\times_\hbar$ (42)

$$a \times_\hbar b = \sum_{n \geq 0} \hbar^n M_n(a, b), \tag{57}$$

where $a, b$ are normally ordered products of free fields, we have the classical product $a \cdot b = M_0(a, b)$, which is just the Wick product

$$:\prod_{i\in I} \varphi(x_i) : \cdot : \prod_{j\in J} \varphi(x_j) :=: \prod_{i\in I} \varphi(x_i) \prod_{j\in J} \varphi(x_j) : \tag{58}$$

and which is also associative and in addition commutative. Then we define $(a_1 \times_\hbar \cdots \times_\hbar a_n)^c$ recursively by

$$(a_1 \times_\hbar \cdots \times_\hbar a_n)^c \stackrel{\text{def}}{=} (a_1 \times_\hbar \cdots \times_\hbar a_n) - \sum_{|P|\geq 2} \prod_{J \in P} (a_{j_1} \times_\hbar \cdots \times_\hbar a_{j_{|J|}})^c, \tag{59}$$

where $\{j_1, \ldots, j_{|J|}\} = J,\ j_1 < \cdots < j_{|J|}$, the sum runs over all partitions $P$ of $\{1, \ldots, n\}$ in at least two subsets and $\prod$ means the classical product (58). $T_n^c$ is defined analogously

$$T_n^c(f_1 \otimes \cdots \otimes f_n) \stackrel{\text{def}}{=} T_n(f_1 \otimes \cdots \otimes f_n) - \sum_{|P|\geq 2} \prod_{p \in P} T_{|p|}^c(\otimes_{j\in p} f_j), \tag{60}$$

and similarly we introduce the connected antichronological product $\bar{T}_n^c \equiv (\bar{T}_n)^c$.

**Proposition 1.** *Let the normally ordered products of free fields $a_1, \ldots, a_n$ be of order $\mathcal{O}(\hbar^0)$. Then*

$$(a_1 \times_\hbar \cdots \times_\hbar a_n)^c = \mathcal{O}(\hbar^{n-1}). \tag{61}$$

*Proof.* We identify the vertices belonging to the same $a_i$ and apply Wick's theorem (42) to $a_1 \times_\hbar \cdots \times_\hbar a_n$. Each "contraction" (i.e. each factor $\Delta_+$) is accompanied by a factor $\hbar$. In the terms $\sim \hbar^0$ (i.e. without any contraction) $a_1, \ldots, a_n$ are completely disconnected, the number of connected components is $n$. By a contraction this number is reduced by 1 or 0. So to obtain a connected term we need at least $(n-1)$ contractions. Hence the connected terms are of order $\mathcal{O}(\hbar^{n-1})$. □

Let $\mathcal{B} \ni A_1, \ldots, A_n = \mathcal{O}(\hbar^0)$ and $x_i \neq x_j,\ \forall 1 \leq i < j \leq n$. Then there exists a permutation $\pi \in \mathcal{S}_n$ such that

$$T^c\big(A_1(x_1)\ldots A_n(x_n)\big) = (A_{\pi 1}(x_{\pi 1}) \times_\hbar \cdots \times_\hbar A_{\pi n}(x_{\pi n}))^c = \mathcal{O}(\hbar^{n-1}). \tag{62}$$

We want this estimate to hold true also for coinciding points

$$T^c\big(A_1(x_1)\ldots A_n(x_n)\big) = \mathcal{O}(\hbar^{n-1}) \qquad \text{on} \quad \mathcal{D}(\mathbb{R}^{4n}). \tag{63}$$

By the following argument this can indeed be satisfied by appropriate normalization of the time ordered products, i.e. (63) is an additional normalization condition, which is compatible with **N1**–**N4**. We proceed by induction on the number $n$ of factors. Let us assume that the $T^c$-products with less than $n$ factors fulfil (63) and that we are away from the total diagonal $\Delta_n$. Using causal factorization, (60) and the shorthand notation $T(J) := T(\prod_{j\in J} A_j(x_j)),\ J \subset \{1, \ldots, n\}$, we then know that there exists $I \subset \{1, \ldots, n\},\ I \neq \emptyset,\ I^c \neq \emptyset$, with

$$T\big(A_1(x_1)\ldots A_n(x_n)\big) = T(I) \times_\hbar T(I^c) = \sum_{r=1}^{|I|}\sum_{s=1}^{|I^c|} \sum_{I_1 \sqcup \cdots \sqcup I_r = I} \sum_{J_1 \sqcup \cdots \sqcup J_s = I^c} \sum_{k\geq 0} \hbar^k M_k\Big(T^c(I_1)\cdot \cdots \cdot T^c(I_r), T^c(J_1)\cdot \cdots \cdot T^c(J_s)\Big), \tag{64}$$

where $\sqcup$ means the disjoint union. We now pick out the connected diagrams. The term $k = 0$ on the r.h.s. has $(r + s)$ disconnected components. Analogously to Proposition 1 we conclude that it must hold $k \geq (r+s-1)$ for a connected diagram. Taking the validity of (63) for $T^c(I_l)$ and $T^c(J_m)$ into account, we obtain $\sum_{l=1}^{r}(|I_l|-1)+\sum_{m=1}^{s}(|J_m|-1)+(r+s-1)=n-1$ for the minimal order in $\hbar$ of a connected diagram. So the $\hbar$-power behaviour (62) holds true on $\mathcal{D}(\mathbb{R}^{4n}\setminus\Delta_n)$, and (63) is in fact a normalization condition.

Due to (60) $(T_n - T_n^c)$ is completely given by timeordered products of lower orders $< n$ and hence is known also on $\Delta_n$. The problem of extending $T_n$ to $\Delta_n$ concerns solely $T_n^c$. The normalization conditions **N1**–**N4** are equivalent to the same conditions for $T_n^c$ and $\bar{T}_n^c$ (i.e. $T_n$ and $\bar{T}_n$ everywhere replaced by $T_n^c$ and $\bar{T}_n^c$). Due to **N3**–**N4** it remains only the extension of $< \Omega, T^c(A_1 \dots A_n)\Omega >$, where all $A_j$ are different from free fields and $\Omega$ is the vacuum. It is obvious that this can be done in a way which maintains (63) and is in accordance with **N1**–**N2**.

We emphasize that the (ordinary) time ordered product $T_n$ does not satisfy (63) because of the presence of disconneted diagrams. On the other hand the connected antichronological product $\bar{T}_n^c$ fulfills the estimate (63), as may be seen by unitarity **N2**. We now turn to the retarded products (55):

**Proposition 2.** *Let $\mathcal{D}(\mathbb{R}^4,\mathcal{V}) \ni f_j, g_k = \mathcal{O}(\hbar^0)$. Then the following statements hold true:*

(i) *All diagrams which contribute to $R_{n,m}(f_1\otimes\cdots\otimes f_n; g_1\otimes\cdots\otimes g_m)$ have the property that each $f_j$-vertex is connected with at least one $g_k$-vertex.*
(ii) $R_{n,m}(f_1\otimes\cdots\otimes f_n; g_1\otimes\cdots\otimes g_m)=\mathcal{O}(\hbar^n)$.

*Proof.* (i) We work with the notation $R_{n,m}(Y;X)$, $Y\equiv\{y_1,\dots,y_n\}$, $X\equiv\{x_1,\dots,x_m\}$ (cf. [14]), and consider a subdiagram with vertices $J\subset Y$ which is not connected with the other vertices $(Y\setminus J)\cup X$. Because disconneted diagrams factorize with respect to the classical product (58), the corresponding contribution to $R_{n,m}(Y;X)$ (55) reads

$$\sum_{I\subset Y}(-1)^{|I|}\Big(\bar{T}(I\cap J^c)\bar{T}(I\cap J)\Big)\times_\hbar\Big(T(I^c\cap J)T(I^c\cap J^c,X)\Big). \tag{65}$$

However, this expression vanishes due to $\sum_{P\subset J}(-1)^{|P|}\bar{T}(P)\times_\hbar T(J\setminus P)=0$ (the latter equation is equivalent to (13), it is the perturbative version of $S^{-1}S=\mathbf{1}$). Hence for non-vanishing diagrams $J$ must be the empty set.

(ii) We express the $R$-product in terms of the connected $T$- and $\bar{T}$-products

$$\begin{aligned} &R_{n,m}(f_1\otimes\cdots\otimes f_n; g_1\otimes\cdots\otimes g_m)\\ &\quad=\sum_{I\subset\{1,\dots,n\}}(-1)^{|I|}\sum_{P\in\mathrm{Part}(I)}\;\sum_{Q\in\mathrm{Part}(I^c\sqcup\{1,\dots,m\})}\\ &\qquad\Big(\prod_{p\in P}\bar{T}^c_{|p|}(\otimes_{i\in p}f_i)\Big)\times_\hbar\Big(\prod_{q\in Q}T^c_{|q|}(\otimes_{i\in q}f_i\otimes\otimes_{j\in q}g_j)\Big), \end{aligned} \tag{66}$$

where again $\prod$ means the classical product (58) and $\sqcup$ stands again for the disjoint union. From (63) we know

$$\begin{aligned} &\prod_{p\in P}\bar{T}^c_{|p|}(\otimes_{i\in p}f_i)=\mathcal{O}(\hbar^{|I|-|P|}),\\ &\prod_{q\in Q}T^c_{|q|}(\otimes_{i\in q}f_i\otimes\otimes_{j\in q}g_j)=\mathcal{O}(\hbar^{|I^c|+m-|Q|}). \end{aligned} \tag{67}$$

From part (i) we conclude that the terms of lowest order (in $\hbar$) in

$$\Big(\prod_{p\in P}\bar{T}^c_{|p|}(\dots)\Big)\times_\hbar\Big(\prod_{q\in Q}T^c_{|q|}(\dots)\Big)=\sum_{n\geq 0}\hbar^n M_n\Big(\prod_{p\in P}\bar{T}^c_{|p|}(\dots),\prod_{q\in Q}T^c_{|q|}(\dots)\Big) \quad (68)$$

do not contribute. For simplicity we first consider the special case $m=1$. Then only connected diagrams contribute. Hence we obtain $n\geq |P|+|Q|-1$ similarly to the reasoning after (64). For arbitrary $m\geq 1$ the terms with minimal power in $\hbar$ correspond to diagrams which are maximally disconnected. According to part (i) these diagrams have $m$ disconnected components each component containing precisely one vertex $g_j$. Applying the $m=1$-argument to each of this components we get $n\geq |P|+|Q|-m$. Taking (67) into account it results the assertion: $(|I|-|P|)+(|I^c|+m-|Q|)+(|P|+|Q|-m)=n$. □

*5.3. Interacting fields.* We first describe the perturbative construction of the interacting classical field. Let $\mathcal{L}$ be a function of the field which serves as the interaction Lagrangian (for simplicity, we do not consider derivative couplings). We want to find a Poisson algebra generated by a solution of the field equation

$$(\square+m^2)\varphi_{\mathcal{L}}(x)=-\Big(\frac{\partial\mathcal{L}}{\partial\varphi}\Big)_{\mathcal{L}}(x), \quad (69)$$

with the initial conditions

$$\begin{aligned}&\{\varphi_{\mathcal{L}}(0,\mathbf{x}),\varphi_{\mathcal{L}}(0,\mathbf{y})\}=0=\{\dot{\varphi}_{\mathcal{L}}(0,\mathbf{x}),\dot{\varphi}_{\mathcal{L}}(0,\mathbf{y})\}\\&\{\varphi_{\mathcal{L}}(0,\mathbf{x}),\dot{\varphi}_{\mathcal{L}}(0,\mathbf{y})\}=\delta(\mathbf{x}-\mathbf{y}).\end{aligned} \quad (70)$$

We proceed in analogy to the construction of the interacting quantum field in Sect. 3 and construct in a first step solutions with localized interactions $\theta\mathcal{L}$ with $\theta\in\mathcal{D}(\mathbb{R}^4)$ which coincide at early times with the free field (hence the initial conditions (70) are trivially satisfied for sufficiently early times). They are given by a formal power series in the Poisson algebra of the free field

$$\begin{aligned}\varphi_{\theta\mathcal{L}}(x)=\sum_{n=0}^{\infty}\int_{y_1^0\leq y_2^0\leq\dots y_n^0\leq x^0}&dy_1dy_2\dots dy_n\,\theta(y_1)\dots\theta(y_n)\\&\{\mathcal{L}(y_1),\{\mathcal{L}(y_2),\dots\{\mathcal{L}(y_n),\varphi(x)\}\dots\}\}\end{aligned} \quad (71)$$

Analogous to the quantum case, the structure of the Poisson algebra associated to a causally closed region $\mathcal{O}$ does not depend on the behaviour of the interaction Lagrangian outside of $\mathcal{O}$, i.e. there is, for $\theta,\theta'\in\Theta(\mathcal{O})$ a canonical transformation $v$ with $v(\varphi_{\theta\mathcal{L}}(x))=\varphi_{\theta'\mathcal{L}}(x)$ for all $x\in\mathcal{O}$. The interacting field $\varphi_{\mathcal{L}}$ may then be defined as a covariantly constant section within a bundle of Poisson algebras.

Starting from the classical interacting field, one may try to define the quantized interacting field by replacing products of free classical fields by the normally ordered product of the corresponding free quantum fields (as in sect. 5.1) and the Poisson brackets in (71) by commutators

$$\{\cdot,\cdot\}\rightarrow\frac{1}{i\hbar}[\cdot,\cdot]_\hbar, \quad (72)$$

where the commutator refers to the quantized product $\times_\hbar$. Note that in general this replacement produces additional terms, e.g. the terms $k \geq 2$ in

$$\frac{1}{i\hbar}[:\varphi^n(x):, \quad :\varphi^m(y):]_\hbar = \sum_{k=1}^{\min\{n,m\}} (i\hbar)^{k-1} \frac{n!m!}{(n-k)!(m-k)!} \Big(\Delta_+(x-y)^k - \Delta_+(y-x)^k\Big) :\varphi^{(n-k)}(x)\varphi^{(m-k)}(y): \tag{73}$$

which correspond to loop diagrams. Due to the distributional character of the fields with respect to the quantized product the integral in (71), as it stands, is not well defined (there is an ambiguity for coinciding points due to the time ordering). But as we will see Bogoliubov's formula (29) for the interacting quantum field as a functional derivative of the relative $S$-matrix may be interpreted as a precise version of this integral.

From the factorization property (11), (14) of time ordered and anti-time ordered products, one gets the following recursion formula for the retarded products ((54), (55)): if supp $g$ is contained in the past and supp $f$, supp $h$ in the future of some Cauchy surface, we find

$$R_{n+1,m}(g \otimes h^{\otimes n}; f^{\otimes m}) = -[T_1(g), R_{n,m}(h^{\otimes n}; f^{\otimes m})]_\hbar, \tag{74}$$

where we used the fact that $\bar{T}_1 = T_1$. Hence, for $m = 1$ and $y_i \neq y_j \; \forall i \neq j$ the retarded product $R_{n,1}(y_1, \ldots, y_n; x)$ can be written in the form[7]

$$R\big(\mathcal{L}(y_1)\ldots\mathcal{L}(y_n);\varphi(x)\big) = (-1)^n \sum_{\pi \in \mathcal{S}_n} \Theta(x^0 - y^0_{\pi n})\Theta(y^0_{\pi n} - y^0_{\pi(n-1)}) \ldots \Theta(y^0_{\pi 2} - y^0_{\pi 1})[\mathcal{L}(y_{\pi 1}), [\mathcal{L}(y_{\pi 2}) \ldots [\mathcal{L}(y_{\pi n}), \varphi(x)]_\hbar \ldots]_\hbar]_\hbar. \tag{75}$$

(Due to the locality of the interaction $\mathcal{L}$ this is a Poincaré covariant expression.) This formula confirms part (ii) of Proposition 2 for non-coinciding $y_i$. Our main application of (75) is the study of the classical limit $\hbar \to 0$ of the quantized interacting field (29). Due to Proposition 2 (part (ii)) $R\big(\hbar^{-1}\mathcal{L}(y_1)\ldots\hbar^{-1}\mathcal{L}(y_n); \varphi(x)\big)$ contains no terms with negative powers of $\hbar$ and thus has a well-defined classical limit. We conclude that the quantized interacting field (29), (54)

$$\varphi_{\theta\mathcal{L}}(h) = \sum_{n=0}^{\infty} \frac{i^n}{n!\hbar^n} R_{n,1}((\theta\mathcal{L})^{\otimes n}; h\varphi), \qquad h \in \mathcal{D}(\mathbb{R}^4), \tag{76}$$

tends to the classical interacting field (71) in this limit. Note that the factor $\hbar^{-1}$ in the interaction Lagrangian is in accordance with the quantization rule (72), since in (75) there is for each factor $\mathcal{L}$ precisely one commutator. In $R_{n,1}((\theta\mathcal{L})^{\otimes n}; f\varphi)$ the above mentioned ambiguities for coinciding points in the iterated retarded commutators have been fixed by the definition of time ordered products as everywhere defined distributions.

The normalization condition **N4** implies an analogous equation for the retarded product $R_{n,1}$ (cf. [10]). The latter means that $\varphi_\mathcal{L}$ (76) satisfies the same field equation as the classical interacting field (69)

$$(\square + m^2)\varphi_\mathcal{L}(x) = -\Big(\frac{\partial\mathcal{L}}{\partial\varphi}\Big)_\mathcal{L}(x). \tag{77}$$

[7] The notation for the time ordered products introduced in Sect. 2 is used here for the retarded products.

Here $\left(\frac{\partial\mathcal{L}}{\partial\varphi}\right)_{\mathcal{L}}$ is not necessarily a polynomial in $\varphi_{\mathcal{L}}$ (the pointwise product of interacting fields is in general not defined).

We found that the relative $S$-matrices $S_{\hbar^{-1}\theta\mathcal{L}}(f)$ $(f \in \mathcal{D}(\mathbb{R}^4, \mathcal{V}))$, and hence all elements of the algebra $\mathcal{A}_{\hbar^{-1}\theta\mathcal{L}}$ are power series in $\hbar$. For the global algebras of covariantly constant sections we recall from [7] that the unitaries $V \in \mathcal{U}(\theta, \theta')$ can be chosen as relative $S$-matrices

$$V = S_{\hbar^{-1}\theta\mathcal{L}}(\hbar^{-1}\theta_{-}\mathcal{L})^{-1} \in \mathcal{U}(\theta, \theta'), \tag{78}$$

where $\theta_{-} \in \mathcal{D}(\mathbb{R}^4)$ depends in the following way on $(\theta - \theta')$: we split $\theta - \theta' = \theta_{+} + \theta_{-}$ with supp $\theta_{+} \cap (C(\mathcal{O}) + \bar{V}_{-}) = \emptyset$ and supp $\theta_{-} \cap (C(\mathcal{O}) + \bar{V}_{+}) = \emptyset$, (where $C(\mathcal{O})$ means the causally closed region containing $\mathcal{O}$ in which $\theta$ and $\theta'$ agree, cf. (18)). So $V$ is a formal Laurent series in $\hbar$, and the sections are no longer well defined power series. Replacing $\mathcal{A}$ and $\mathcal{A}(\mathcal{O})$ by $\bigvee_{n\in\mathbb{N}_0} \hbar^n\mathcal{A}$ and $\bigvee_{n\in\mathbb{N}_0} \hbar^n\mathcal{A}(\mathcal{O})$ (for the new algebras the same symbol $\mathcal{A}$ will be used again) we obtain modules over the ring of formal power series in $\hbar$ with complex coefficients. For the further construction the validity of part (iii) of the following Proposition is crucial:

**Proposition 3.** (i) *Let* $R_{n,m}(\ldots;\ldots) = \sum_{a=1}^{m} R_{n,m}^{(a)}(\ldots;\ldots)$, *where* $R_{n,m}^{(a)}(\ldots;\ldots)$ *is the sum of all diagrams with* $a$ *connected components. Then*

$$R_{n,m}^{(a)}((\hbar^{-1}\theta\mathcal{L})^{\otimes n}; (\hbar^{-1}\theta_{-}\mathcal{L})^{\otimes m}) = \mathcal{O}(\hbar^{-a}). \tag{79}$$

*(Note that the range of* $a$ *is restricted by part* (i) *of Proposition 2.) This estimate is of more general validity: instead of a retarded product we could have e.g. a multiple* $\times_{\hbar}$*-product, a time ordered or antichronological product and the factors may be quite arbitrary. It is only essential that each factor is of order* $\mathcal{O}(\hbar^{-1})$.

(ii) *Let* $A \in \mathcal{A}(\mathcal{O})$. *Then all diagrams which contribute to* $V \times_{\hbar} A \times_{\hbar} V^{-1}$, *(where* $V$ *is given by (78)) have the property that each vertex of* $V$ *and of* $V^{-1}$ *is connected with at least one vertex of* $A$. *(It may happen that a connected component of* $V$ *is not directly connected with* $A$, *but that it is connectecd with a connected component of* $V^{-1}$ *and the latter is connected with* $A$.*)*

(iii)

$$\mathcal{A}(\mathcal{O}) \ni A = \mathcal{O}(\hbar^n) \quad \Longrightarrow \quad V \times_{\hbar} A \times_{\hbar} V^{-1} = \mathcal{O}(\hbar^n). \tag{80}$$

*In particular if* $A$ *is the term of* $n$*-th order in* $\hbar$ *of an interacting field, then* $V \times_{\hbar} A \times_{\hbar} V^{-1}$ *is a power series in* $\hbar$ *in which the terms up to order* $\hbar^{n-1}$ *vanish.*

*Proof.* Part (i) is obtained essentially in the same way as Proposition 1. Part (iii) is a consequence of parts (i) and (ii), and the following observation: let us consider a diagram which contributes to $V \times_{\hbar} A \times_{\hbar} V^{-1}$ according to part (ii). If the subdiagrams belonging to $V$ and $V^{-1}$ have $r$ and $s$ connected components, then the whole diagram has at least $(r+s)$ contractions, which yield a factor $\hbar^{(r+s)}$.

It remains the proof of (ii): We use the same notations as in the proof of Proposition 2. Let $Y_1 \sqcup Y_2 = Y$, $X_1 \sqcup X_2 = X$. We now consider the sum of all diagrams contributing to $R(Y, X)$ in which the vertices $(Y_1, X_1)$ are not connected with the vertices $(Y_2, X_2)$.

Using (55) and the fact that disconnected diagrams factorize with respect to the classical product (58), this (partial) sum is equal to

$$\sum_{I\subset Y}(-1)^{|I\cap Y_1|}[\bar{T}(I\cap Y_1)\times_\hbar T(I^c\cap Y_1,X_1)]\cdot \\ (-1)^{|I\cap Y_2|}[\bar{T}(I\cap Y_2)\times_\hbar T(I^c\cap Y_2,X_2)] = R(Y_1,X_1)\cdot R(Y_2,X_2). \tag{81}$$

From $\mathbf{1}=VV^{-1}=VV^*$, (54) and (78) we know

$$\sum_{Y_1\sqcup Y_2=Y,\ X_1\sqcup X_2=X}(-1)^{(|Y_1|+|X_1|)}R^*(Y_1,X_1)\times_\hbar R(Y_2,X_2)=0 \tag{82}$$

for fixed $(Y,X)$, $Y\cup X\neq\emptyset$. Next we note

$$V\times_\hbar A\times_\hbar V^{-1}=\sum_{n,m}\frac{1}{n!m!}\int dy_1\dots dy_n dx_1\dots dx_m\,\theta(y_1)\dots \\ \theta(y_n)\theta_-(x_1)\dots\theta_-(x_m)\sum_{Y_1\sqcup Y_2=Y,\ X_1\sqcup X_2=X}(-i)^{(|Y_1|+|X_1|)} \\ \times\, i^{(|Y_2|+|X_2|)}R^*(Y_1,X_1)\times_\hbar A\times_\hbar R(Y_2,X_2), \tag{83}$$

where we have used the notations $Y\equiv\{y_1,\dots y_n\}$, $X\equiv\{x_1,\dots,x_n\}$. In the integrand of the latter expression we consider (for given $Y$ and $X$) fixed decompositions $Y=Y_3\sqcup Y_4$ and $X=X_3\sqcup X_4$, $Y_3\cup X_3\neq\emptyset$. Now we consider the (partial) sum of all diagrams in which the vertices $(Y_3,X_3)$ are not connected with $A$ and each of the vertices $(Y_4,X_4)$ is connected with $A$. Part (ii) is proved if we can show that this partial sum vanishes. This holds in fact true because $R^*$ and $R$ factorize according to (81), and due to the unitarity (82):

$$\sum_{Y_1\sqcup Y_2=Y,\ X_1\sqcup X_2=X}(-1)^{(|Y_1\cap Y_4|+|X_1\cap X_4|)}\big[R^*(Y_1\cap Y_4,X_1\cap X_4) \\ \times_\hbar A\times_\hbar R(Y_2\cap Y_4,X_2\cap X_4)\big] \\ (-1)^{(|Y_1\cap Y_3|+|X_1\cap X_3|)}\big[R^*(Y_1\cap Y_3,X_1\cap X_3) \\ \times_\hbar R(Y_2\cap Y_3,X_2\cap X_3)\big]=0. \qquad \square$$

Now we are ready to give an algebraic formulation of the expansion in $\hbar$. Let $I_n\overset{\text{def}}{=}\hbar^n\mathcal{A}_\mathcal{L}$. $I_n$ is an ideal in the global algebra $\mathcal{A}_\mathcal{L}$. We define

$$\mathcal{A}_\mathcal{L}^{(n)}\overset{\text{def}}{=}\frac{\mathcal{A}_\mathcal{L}}{I_{n+1}},\quad \mathcal{A}_\mathcal{L}^{(n)}(\mathcal{O})\overset{\text{def}}{=}\frac{\mathcal{A}_\mathcal{L}(\mathcal{O})}{I_{n+1}\cap\mathcal{A}_\mathcal{L}(\mathcal{O})}. \tag{84}$$

which means that we neglect all terms which are of order $\mathcal{O}(\hbar^{n+1})$. The embeddings $i_{21}:\mathcal{A}_\mathcal{L}(\mathcal{O}_1)\hookrightarrow\mathcal{A}_\mathcal{L}(\mathcal{O}_2)$ for $\mathcal{O}_1\subset\mathcal{O}_2$ induce embeddings $\mathcal{A}_\mathcal{L}^{(n)}(\mathcal{O}_1)\hookrightarrow\mathcal{A}_\mathcal{L}^{(n)}(\mathcal{O}_2)$. Thus we obtain a projective system of local nets $(\mathcal{A}_\mathcal{L}^{(n)}(\mathcal{O}))$ of algebras of quantum observables up to order $\hbar^{n+1}$.

Note that we may equip our algebras $\mathcal{A}_\mathcal{L}^{(n)}$ also with the Poisson bracket induced by $\frac{1}{i\hbar}[\cdot,\cdot]_\hbar$, because the ideals $I_n$ are also Poisson ideals with respect to these brackets. Then

$\mathcal{A}_{\mathcal{L}}^{(0)}$ becomes the local net of Poisson algebras of the classical field theory, whereas for $n \neq 0$ we obtain a net of noncommutative Poisson algebras.

The expansion in powers of $\hbar$ is usually called "loop expansion". This is due to the fact that the order in $\hbar$ of a certain Feynman diagram belonging to $R_{n,m}((\hbar^{-1}\theta\mathcal{L})^{\otimes n}; f_1 \otimes \cdots \otimes f_m),\ \mathcal{D}(\mathbb{R}^4, \mathcal{V}) \ni f_j = \mathcal{O}(\hbar^0)$, is equal to: (number of propagators (i.e. inner lines)) - $n$ = (number of loops) + $m$ - (number of connected components). In particular, using part (i) of Proposition 2, we find that for the interacting fields ($m = 1$) the order in $\hbar$ agrees with the number of loops.

## 6. Local Algebraic Formulation of the Quantum Action Principle

The method of algebraic renormalization (for an overview see [22]) relies on the so called "quantum action principle" (QAP), which is due to Lowenstein [20] and Lam [18]. This principle is a formula for the variation of (possibly connected or one-particle-irreducible) Green's functions (or of the corresponding generating functional) under

- a change of coordinates (e.g. one applies the differential operator of the field equation to the Green's functions),
- a variation of the fields (e.g. the BRST-transformation)
- a variation of a parameter. This may be a parameter in the Lagrangian or in the normalization conditions for the Green's functions.

These are different theorems with different proofs. The common statement is that the variation of the Green's functions is equal to the insertion of a local or spacetime integrated composite field operator (for details see [22]). In this section we study two simple cases of the QAP: the field equation and the variation of a parameter which appears only in the interaction Lagrangian.

The aim of this section is to formulate the QAP (in these two cases) for our local algebras of observables $\mathcal{A}_{\mathcal{L}}(\mathcal{O})$, i.e. we are looking for an *operator* identity which holds true independently of the adiabatic limit. Such an identity does not depend on the choice of a state, as it is the case for the Green's functions.

In a second step we compare our formula with the usual formulation of the QAP in terms of Green's functions. The latter are the vacuum expectation values in the adiabatic limit $g \to 1$.[8] We specialize to models for which the adiabatic limit is known to exist. This is the case for pure massive theories [14] and certain theories with (some) massless particles such as QED and $\lambda : \varphi^{2n} :$ -theories [4], provided the time ordered products are appropriately normalized.

*Remarks.* (1) From the usual QAP (in terms of Green's functions) one obtains an operator identity by means of the Lehmann–Symanzik–Zimmermann-reduction formalism [19]. Although the latter relies on the adiabatic limit an analogous conclusion from the Fock vacuum expectation values to arbitrary matrix elements is possible in our local construction: let $\mathcal{O}$ be an open double cone and let $x_1, \ldots, x_k \notin ((\bar{\mathcal{O}} \cup \{x_{k+l+1}, \ldots, x_n\}) + \bar{V}_-),\ x_{k+1}, \ldots, x_{k+l} \in \mathcal{O}$ and $x_{k+l+1}, \ldots, x_n \notin (\bar{\mathcal{O}} + \bar{V}_+)$. Using the causal factorization of time ordered products of interacting fields (28) we

[8] This limit is taken by scaling the test function $g$: let $g_0 \in \mathcal{D}(\mathbb{R}^4),\ g_0(0) = 1$; then one considers the limit $\epsilon \to 0\ (\epsilon > 0)$ of $g_\epsilon(x) \equiv g_0(\epsilon x)$. Uniqueness of the adiabatic limit means the independence of the particular choice of $g_0$.

obtain

$$\Big(\Omega, T_{\theta\mathcal{L}}\big(\varphi(x_1)\dots\varphi(x_n)\big)\Omega\Big) = \Big(T_{\theta\mathcal{L}}\big(\varphi(x_1)\dots\varphi(x_k)\big)^*\Omega,\\ T_{\theta\mathcal{L}}\big(\varphi(x_{k+1})\dots\varphi(x_{k+l})\big)T_{\theta\mathcal{L}}\Big(\varphi(x_{k+l+1})\dots\varphi(x_n)\big)\Omega\Big). \quad (85)$$

Now we choose $\theta \in \Theta(\mathcal{O})$ such that $\{x_1, \dots, x_k\} \cap (\text{supp } \theta + \bar{V}_-) = \emptyset$ and $\{x_{k+l+1}, \dots, x_n\} \cap (\text{supp } \theta + \bar{V}_+) = \emptyset$. Due to the retarded support (56) of the $R$-products we then know that $T_{\theta\mathcal{L}}\big(\varphi(x_{k+l+1})\dots\varphi(x_n)\big)$ agrees with the time ordered product $T_0\big(\varphi(x_{k+l+1})\dots\varphi(x_n)\big)$ of the corresponding free fields. By means of $S_{\theta\mathcal{L}}(f\varphi) = S(\theta\mathcal{L})^{-1}S(f\varphi)S(\theta\mathcal{L})$ for supp $f \cap (\text{supp } \theta + \bar{V}_-) = \emptyset$ we obtain

$$T_{\theta\mathcal{L}}\big(\varphi(x_1)\dots\varphi(x_k)\big)^* = S(\theta\mathcal{L})^{-1}T_0\big(\varphi(x_1)\dots\varphi(x_k)\big)^* S(\theta\mathcal{L}). \quad (86)$$

Our assertion follows now from the fact that the states $T_0\big(\varphi(x_{k+l+1})\dots\varphi(x_n)\big)\Omega$ generate a dense subspace of the Fock space and the same for the states $S(\theta\mathcal{L})^{-1}$ $T_0\big(\varphi(x_1)\dots\varphi(x_k)\big)^* S(\theta\mathcal{L})\Omega$. (For the validity of the latter statement it is important that $x_1, \dots, x_k$ can be arbitrarily spread over a Cauchy surface which is later than $(\bar{\mathcal{O}} \cup \{x_{k+l+1}, \dots, x_n\})$.)

(2) Recently Pinter [23] presented an alternative derivation of the QAP for the variation of a parameter in the Lagrangian also in the framework of causal perturbation theory. In contrast to our presentation Pinter's QAP is formulated for the $S$-matrix.

*6.1. Field equation.* The normalization condition **N4** implies

$$\begin{aligned}(\square_x + m^2)&R\big(\mathcal{L}(y_1)\dots\mathcal{L}(y_n); \phi(x)\phi(x_1)\dots\phi(x_m)\big)\\ &= -i\sum_{l=1}^{n}\delta(x-y_l)R\Big(\mathcal{L}(y_1)\dots\hat{l}\dots\mathcal{L}(y_n); \frac{\partial\mathcal{L}}{\partial\phi}(x)\phi(x_1)\dots\phi(x_m)\Big)\\ &\quad -i\sum_{j=1}^{m}\delta(x-x_j)R\big(\mathcal{L}(y_1)\dots\mathcal{L}(y_n); \phi(x_1)\dots\hat{j}\dots\phi(x_m)\big),\end{aligned} \quad (87)$$

where $\hat{l}$ and $\hat{j}$ means that the corresponding factor is omitted. This equation takes a simple form for the corresponding generating functionals (i.e. the relative $S$-matrices (16))

$$f(x)S_{g\mathcal{L}}(f\phi) = (\square_x + m^2)\frac{\delta}{i\delta f(x)}S_{g\mathcal{L}}(f\phi) - \frac{\delta}{i\delta\rho(x)}\Big|_{\rho=0} S_{g\mathcal{L}}\Big(f\phi + \rho g\frac{\partial\mathcal{L}}{\partial\phi}\Big). \quad (88)$$

To formulate this in terms of our local algebras of observables (cf. sect. 3) we set $g \equiv \theta \in \Theta(\mathcal{O})$ and for $x \in \mathcal{O}$ we can choose $\rho$ such that supp $\rho \subset \{y|\theta(y) = 1\}$. Then (88) turns into

$$(\square_x + m^2)\frac{\delta}{i\delta f(x)}S_{\mathcal{L}}(f\phi) = f(x)S_{\mathcal{L}}(f\phi) + \frac{\delta}{i\delta\rho(x)}\Big|_{\rho=0} S_{\mathcal{L}}\Big(f\phi + \rho\frac{\partial\mathcal{L}}{\partial\phi}\Big), \quad x \in \mathcal{O}. \quad (89)$$

This is the QAP (in the case of the field equation) for the local algebras of observables.

To compare with the usual form of the QAP we consider the generating functional $Z(f)$ for the Green's functions $< \Omega|T\big(\phi_{\mathcal{L}}(x_1)\ldots\phi_{\mathcal{L}}(x_m)\big)|\Omega >$ which is obtained from the relative $S$-matrices by

$$Z(f) = \lim_{g\to 1}(\Omega, S_{g\mathcal{L}}(f\phi)\Omega), \tag{90}$$

where $\Omega$ is the Fock vacuum [14]. So by taking the vacuum expectation value and the adiabatic limit of (88) we get

$$f(x)Z(f) = -\Delta(x)\cdot Z(f), \tag{91}$$

where $\Delta(x)$ is a insertion of UV-dimension[9] 3, coinciding with the classical field polynomial $\frac{\delta S}{\delta\phi(x)}$ in the classical approximation (where $S = \int d^4x\,[\frac{1}{2}(\partial_\mu\phi(x)\partial^\mu\phi(x) - m^2\phi^2(x)) + g(x)\mathcal{L}(x)]$ is the classical action). Equation (91) is the usual form of the QAP (cf. eqn. (3.20) in [22]). In the present case the local algebraic formulation (89) contains more information than the usual QAP (91).

*6.2. Variation of a parameter in the interaction.* In (54) we have defined retarded products of Wick polynomials, i.e. elements of the Borchers class. Analogously we now introduce retarded products $R_{\mathcal{L}}(g^{\otimes n}; f^{\otimes m})$ of interacting fields

$$S_{\mathcal{L}+g}(f) = S_{\mathcal{L}}(g)^{-1}S_{\mathcal{L}}(g+f) \stackrel{\text{def}}{=} \sum_{n,m=0}^{\infty}\frac{i^{n+m}}{n!m!}R_{\mathcal{L}}(g^{\otimes n}; f^{\otimes m}), \tag{92}$$

where $\mathcal{L}, g, f \in \mathcal{D}(\mathbb{R}^4, \mathcal{V})$. Obviously they can be expressed in terms of antichronological and time ordered products of interacting fields by exactly the same formula as in the case of Wick polynomials (55)

$$R_{\mathcal{L}}(g^{\otimes n}; f^{\otimes m}) = \sum_{k=0}^{n}(-1)^k\frac{n!}{k!(n-k)!}\bar{T}_{\mathcal{L}}(g^{\otimes k})T_{\mathcal{L}}(g^{\otimes(n-k)}\otimes f^{\otimes m}). \tag{93}$$

Thereby the antichronological product of interacting fields is defined analogously to the time ordered product (28), namely by

$$\bar{T}_{\mathcal{L}}(f^{\otimes m}) = \frac{d^m}{(-i)^m d\lambda^m}\Big|_{\lambda=0}S_{\mathcal{L}}(\lambda f)^{-1}, \tag{94}$$

and satisfies anticausal factorization (14) (which justifies the name). The support property (56) of the retarded products relies on the (anti)causal factorization of the $T$- and $\bar{T}$-products (11, 14), hence, the $R$-product of interacting fields ((92), (93)) has also retarded support (56).

Similarly to Lowenstein in [20], Sect. II.B, we consider an infinitesimal change of the interaction Lagrangian

$$\mathcal{L}_0 \to \mathcal{L}_0 + \epsilon\mathcal{L}_1, \tag{95}$$

[9] We assume that $\mathcal{L}$ has UV-dimension 4.

where $\mathcal{L}_0, \mathcal{L}_1 \in \mathcal{V}$ or $\mathcal{D}(\mathbb{R}^4, \mathcal{V})$. For the $m$-fold variation of the time ordered product of the interacting fields (28) we obtain

$$\frac{d^m}{d\epsilon^m}\Big|_{\epsilon=0} T_{\theta(\mathcal{L}_0+\epsilon\mathcal{L}_1)}(f^{\otimes l}) = \frac{\partial^m}{\partial\epsilon^m}\Big|_{\epsilon=0} \frac{\partial^l}{i^l\partial\lambda^l}\Big|_{\lambda=0} S_{\theta(\mathcal{L}_0+\epsilon\mathcal{L}_1)}(\lambda f) \\ = i^m R_{\theta\mathcal{L}_0}((\theta\mathcal{L}_1)^{\otimes m}; f^{\otimes l}). \tag{96}$$

To formulate this identity for our local algebras of observables we assume that $\mathcal{L}_1$ has compact support, i.e. $\mathcal{L}_1 \in \mathcal{D}(\mathbb{R}^4, \mathcal{V})$. We set

$$\Theta_0(\mathcal{O}) \stackrel{\text{def}}{=} \{\theta \in \Theta(\mathcal{O}) \quad | \quad \theta|_{\text{supp}\,\mathcal{L}_1} \equiv 1\}. \tag{97}$$

We consider the observables as covariantly constant sections in the bundle over $\Theta_0(\mathcal{O})$ (instead of $\Theta(\mathcal{O})$ as in sect. 3). Then we obtain

$$\frac{d^m}{d\epsilon^m}|_{\epsilon=0} T_{\mathcal{L}_0+\epsilon\mathcal{L}_1}(f^{\otimes l}) = i^m R_{\mathcal{L}_0}(\mathcal{L}_1^{\otimes m}; f^{\otimes l}). \tag{98}$$

This is the local algebraic formulation of the QAP for the variation of a parameter in the interaction.

We are now going to investigate the usual QAP by using Epstein and Glaser's definition of Green's functions (90). In (96) the $m$-fold variation of the parameter $\epsilon$ results in a *retarded* insertion of $(\theta\mathcal{L}_1)^{\otimes m}$. In the usual QAP $(\theta\mathcal{L}_1)^{\otimes m}$ is inserted into the *time ordered* product, i.e. one considers

$$i^m T_{\theta\mathcal{L}_0}((\theta\mathcal{L}_1)^{\otimes m} \otimes f^{\otimes l}) = \frac{\partial^m}{\partial\epsilon^m}\Big|_{\epsilon=0} \frac{\partial^l}{i^l\partial\lambda^l}\Big|_{\lambda=0} S_{\theta\mathcal{L}_0}(\theta\epsilon\mathcal{L}_1 + \lambda f). \tag{99}$$

Obviously (96) and (99) do not agree. However, let us assume that we are dealing with a purely massive theory and that $\mathcal{L}_0$ and $\mathcal{L}_1$ have UV-dimension $\dim(\mathcal{L}_j) = 4$. Or: if $\dim(\mathcal{L}_j) < 4$ we assume that $\mathcal{L}_j$ is treated in the extension to the total diagonal as if it would hold $\dim(\mathcal{L}_j) = 4$. Hence it may occur that the scaling degree increases in the extension to a certain amount: $\text{sd}(t_0) \leq \text{sd}(t) \leq 4n - b$ for a scalar theory without derivative couplings, where $b$ is the number of external legs (cf. (51)–(53)). (In the BPHZ framework one says that $\mathcal{L}_j$ is "oversubtracted with degree 4".) Then there exists a normalization of the time ordered products, which is compatible with the other normalization conditions **N1**–**N4** and (63), such that the Green's functions corresponding to (99) exist and agree, i.e. we assert

$$\frac{d^m}{d\epsilon^m}\Big|_{\epsilon=0} \lim_{\theta\to 1}\Big(\Omega, T_{\theta(\mathcal{L}_0+\epsilon\mathcal{L}_1)}(f^{\otimes l})\Omega\Big) = i^m \lim_{\theta\to 1}\Big(\Omega, T_{\theta\mathcal{L}_0}((\theta\mathcal{L}_1)^{\otimes m} \otimes f^{\otimes l})\Omega\Big) \tag{100}$$

for all $m, l \in \mathbb{N}_0$, which is equivalent to

$$\lim_{\theta\to 1}\Big(\Omega, S_{\theta(\mathcal{L}_0+\epsilon\mathcal{L}_1)}(\lambda f)\Omega\Big) = \lim_{\theta\to 1}\Big(\Omega, S_{\theta\mathcal{L}_0}(\theta\epsilon\mathcal{L}_1 + \lambda f)\Omega\Big). \tag{101}$$

(We assume that the derivatives $\frac{\partial^m}{\partial\epsilon^m}$ and $\frac{\partial^l}{\partial\lambda^l}$ commute with the adiabatic limit $\theta \to 1$. This seems to be satisfied for vacuum expectation values in pure massive theories as it is the case here [14].) This is the usual form of the QAP (in terms of Epstein and Glaser's

Green's functions) for the present case (cf. Eq. (2.6) of [20][10]). In contrast to the field equation, the QAP (100) does not hold for the operators before the adiabatic limit.

*Proof of (100).* For a better comparison with Lowenstein's formulation, we present a proof which makes the detour over the corresponding Gell–Mann Low expressions. First we comment on the equality of Epstein and Glaser's Green's functions with the Gell–Mann Low series

$$\lim_{\theta\to 1}(\Omega,\, S_{\theta\mathcal{L}}(f)\Omega) = \lim_{\theta\to 1}\frac{(\Omega,\, S(\theta\mathcal{L}+f)\Omega)}{(\Omega,\, S(\theta\mathcal{L})\Omega)}, \tag{102}$$

which is proved in the appendix of [12]. This can be understood in the following way: let $P_\Omega$ be the projector on the Fock vacuum $\Omega$ and $P_\Omega^\perp \overset{\text{def}}{=} 1-P_\Omega$. Using $S(\theta\mathcal{L})^* = S(\theta\mathcal{L})^{-1}$ we obtain

$$\begin{aligned}(\Omega,\, S_{\theta\mathcal{L}}(f)\Omega) &= (S(\theta\mathcal{L})\Omega,\, (P_\Omega + P_\Omega^\perp)S(\theta\mathcal{L}+f)\Omega)\\ &= \frac{(\Omega,\, S(\theta\mathcal{L}+f)\Omega)}{(\Omega,\, S(\theta\mathcal{L})\Omega)}\cdot |(\Omega,\, S(\theta\mathcal{L})\Omega)|^2\\ &\quad + (\Omega,\, S(\theta\mathcal{L})^{-1}P_\Omega^\perp S(\theta\mathcal{L}+f)\Omega)\end{aligned} \tag{103}$$

and

$$\begin{aligned}1 &= (\Omega,\, S(\theta\mathcal{L})^{-1}(P_\Omega + P_\Omega^\perp)S(\theta\mathcal{L})\Omega)\\ &= |(\Omega,\, S(\theta\mathcal{L})\Omega)|^2 + (\Omega,\, S(\theta\mathcal{L})^{-1}P_\Omega^\perp S(\theta\mathcal{L})\Omega).\end{aligned} \tag{104}$$

In $(\Omega,\, S(\theta\mathcal{L})^{-1}P_\Omega^\perp S(\theta\mathcal{L}+f)\Omega)$ there is at least one contraction between $S(\theta\mathcal{L})^{-1}$ and $S(\theta\mathcal{L}+f)$ (or: the terms without contraction are precisely $(\Omega,\, S(\theta\mathcal{L})^{-1}\Omega)(\Omega,\, S(\theta\mathcal{L}+f)\Omega)$). In the mentioned reference the support properties in momentum space of the contracted terms are analysed and in this way it is proved

$$\lim_{\theta\to 1}(\Omega,\, S(\theta\mathcal{L})^{-1}P_\Omega^\perp S(\theta\mathcal{L}+f)\Omega) = 0. \tag{105}$$

Inserting this into (103) and (with $f=0$) into (104) it results (102).

Because of (102) our assertion (101) is equivalent to

$$\lim_{\theta\to 1}\frac{(\Omega,\, S(\theta(\mathcal{L}_0+\epsilon\mathcal{L}_1)+\lambda f)\Omega)}{(\Omega,\, S(\theta(\mathcal{L}_0+\epsilon\mathcal{L}_1))\Omega)} = \lim_{\theta\to 1}\frac{(\Omega,\, S(\theta(\mathcal{L}_0+\epsilon\mathcal{L}_1)+\lambda f)\Omega)}{(\Omega,\, S(\theta\mathcal{L}_0)\Omega)}. \tag{106}$$

This is the QAP in terms of the Gell–Mann Low series. Obviously the nontrivial statement is

$$\lim_{\theta\to 1}\frac{(\Omega,\, S(\theta(\mathcal{L}_0+\epsilon\mathcal{L}_1))\Omega)}{(\Omega,\, S(\theta\mathcal{L}_0)\Omega)} = 1. \tag{107}$$

A possibility to ensure the validity of this equation is the above assumption (which has not been used so far) that $\mathcal{L}_0$ and $\mathcal{L}_1$ have mass dimension $\dim(\mathcal{L}_j)\le 4$ and are treated as

[10] Lowenstein works with Zimermanns definition of normal products of interacting fields: $N_\delta\{\prod_{j=1}^l \varphi_{i_j\,\mathcal{L}}(x)\}$, $\delta \ge d \equiv \sum_{j=1}^l d(\varphi_{i_j\,\mathcal{L}})$ [29]. For $\delta = d$ (i.e. without oversubtraction) $N_\delta\{\prod_{j=1}^l \varphi_{i_j\,\mathcal{L}}(x)\}$ agrees essentially with our $(:\prod_{j=1}^l \varphi_{i_j}(x):)_{g\mathcal{L}}$ (29). The difference is due to the adiabatic limit and the different ways of defining Green's functions (Zimmermann uses the Gell–Mann Low series, cf. (102), (106)).

dimension 4 vertices in the renormalization procedure. Due to this additional assumption and the requirements that the adiabatic limit exists and is unique, the normalization of the vacuum diagrams is uniquely fixed, and with this normalization the vacuum diagrams vanish in the adiabatic limit

$$\lim_{\theta\to 1}(\Omega, S(\theta\mathcal{L}_0)\Omega) = 1, \qquad \lim_{\theta\to 1}(\Omega, S(\theta(\mathcal{L}_0 + \epsilon\mathcal{L}_1))\Omega) = 1. \tag{108}$$

(For a proof see also the appendix of [12].) □

*Remarks.* (1) Without the assumption about $\mathcal{L}_0$ and $\mathcal{L}_1$ we find

$$\lim_{\theta\to 1}\Big(\Omega, S_{\theta(\mathcal{L}_0+\epsilon\mathcal{L}_1)}(\lambda f)\Omega\Big) = \lim_{\theta\to 1}\frac{\Big(\Omega, S_{\theta\mathcal{L}_0}(\theta\epsilon\mathcal{L}_1 + \lambda f)\Omega\Big)}{\Big(\Omega, S_{\theta\mathcal{L}_0}(\theta\epsilon\mathcal{L}_1)\Omega\Big)} \tag{109}$$

instead of (101), by using (102) only. This is a formulation of the QAP for general situations in which (107) does not hold.

(2) By means of the QAP (98) (or (100), or (109)) one can compute the change of the time ordered products of interacting fields (or of the Green's functions) under the variation of parameters $\lambda_1, \ldots, \lambda_s$ if the interaction Lagrangian has the form $\mathcal{L}(x) = \sum_i a_i(\lambda_1, \ldots, \lambda_s)\mathcal{L}_i(x)$, $\mathcal{L}_i \in \mathcal{V}$ resp. $\mathcal{D}(\mathbb{R}^4, \mathcal{V})$ (cf. Eqs. (2.7), (2.8)) of [20]). But only the interaction $\mathcal{L}$ may depend on the parameters and not the time ordering operator (i.e. the normalization conditions for the time ordered products).

## Appendix: Wavefront Sets and the Pointwise Product of Distributions

In this appendix we briefly recall the definition of the wavefront set of a distribution and mention a simple criterion for the existence of the pointwise product of distributions in terms of their wavefront sets. For a detailed treatment we refer to Hörmander [15], the application to quantum field theory on curved spacetimes can be found in [25, 8, 7].

Let $t \in \mathcal{D}'(\mathbb{R}^d)$ be singular at the point $x$ and let $f \in \mathcal{D}(\mathbb{R}^4)$ with $f(x) \neq 0$. Then $ft \in \mathcal{D}'(\mathbb{R}^d)$ is also singular at $x$ and $ft$ has compact support. Hence the Fourier transform $\widehat{ft}$ is a $\mathcal{C}^\infty$-function. In some directions $\widehat{ft}$ does not rapidly decay, because otherwise $ft$ would be infinitly differentiable at $x$. Thereby a function $g$ is called rapidly decaying in the direction $k \in \mathbb{R}^d \setminus \{0\}$, if there is an open cone $C$ with $k \in C$ and $\sup_{k'\in C} |k'|^N |g(k')| < \infty$ for all $N \in \mathbf{N}$.

**Definition.** *The wavefront set* $\mathrm{WF}(t)$ *of a distribution* $t \in \mathcal{D}'(\mathbb{R}^d)$ *is the set of all pairs* $(x, k) \in \mathbb{R}^d \times \mathbb{R}^d \setminus \{0\}$ *such that the Fourier transform* $\widehat{ft}$ *does not rapidly decay in the direction* $k$ *for all* $f \in \mathcal{D}(\mathbb{R}^d)$ *with* $f(x) \neq 0$.

For example the delta distribution satisfies $\widehat{f\delta}(k) = f(0)$, hence $\mathrm{WF}(\delta) = \{0\} \times \mathbb{R}^d \setminus \{0\}$. The wavefront set is a refinement of the singular support of $t$ (which is the complement of the largest open set where $t$ is smooth):

$$t \text{ is singular at } x \quad \Longleftrightarrow \quad \exists k \in \mathbb{R}^d \setminus \{0\} \text{ with } (x, k) \in \mathrm{WF}(t).$$

For the wavefront set of the two-point function one finds

$$\mathrm{WF}(\Delta_+) = \{(x, k) \mid x^2 = 0,\ k^2 = 0,\ x \| k,\ k_0 > 0\}. \tag{110}$$

Let $t$ and $s$ be two distributions which are singular at the same point $x$. We localize them by multiplying with $f \in \mathcal{D}(\mathbb{R}^d)$, where $f(x) \neq 0$. We assume that $(ft)$ and $(fs)$ have only one overlapping singularity, namely at $x$. In general the pointwise product $(ft)(y)(fs)(y)$ does not exist. Heuristically this can be seen by the divergence of the convolution integral $\int dk\, \widehat{(ft)}(p-k)\widehat{(fs)}(k)$. But this integral converges if $k_1+k_2 \neq 0$ for all $k_1, k_2$ with $(x, k_1) \in \mathrm{WF}(t)$ and $(x, k_2) \in \mathrm{WF}(s)$. This makes plausible the following theorem:

**Theorem.** *Let* $t, s \in \mathcal{D}'(\mathbb{R}^d)$ *with*

$$\{(x, k_1 + k_2) \mid (x, k_1) \in \mathrm{WF}(t) \wedge (x, k_2) \in \mathrm{WF}(s)\} \cap (\mathbb{R}^d \times \{0\}) = \emptyset. \qquad (111)$$

*Then the pointwise product* $(ts) \in \mathcal{D}'(\mathbb{R}^d)$ *exists.*

By means of this theorem one verifies the existence of the distributional products $(\varphi^{\otimes n})_\hbar(t)$ (44) and $(t \otimes_{k,\hbar} s)$ (47).

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**Note added in proof.** Renormalization can also be done entirely on the level of retarded products [1, 2, 3]. This leads to a direct proof that the interacting fields are power series in $\hbar$.

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