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The Stability of Magnetic Vortices*

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Abstract: We study the linearized stability of n-vortex ($n \in \mathbb{Z}$) solutions of the magnetic Ginzburg–Landau (or Abelian Higgs) equations. We prove that the fundamental vortices ($n = \pm 1$) are stable for all values of the coupling constant, λ, and we prove that the higher-degree vortices ($|n| \geq 2$) are stable for $\lambda < 1$, and unstable for $\lambda > 1$. This resolves a long-standing conjecture (see, eg, [JT]).

1. Introduction

In this paper, we determine the stability of magnetic (or Abelian Higgs) vortices. These are certain critical points of the energy functional

$$\mathcal{E}(\psi, A) = \frac{1}{2} \int_{\mathbb{R}^2} \left\{ |\nabla_A \psi|^2 + (\nabla \times A)^2 + \frac{\lambda}{4}(|\psi|^2 - 1)^2 \right\} \tag{1}$$

for the fields

$$A : \mathbb{R}^2 \to \mathbb{R}^2 \quad \text{and} \quad \psi : \mathbb{R}^2 \to \mathbb{C}.$$

Here $\nabla_A = \nabla - iA$ is the covariant gradient, and $\lambda > 0$ is a coupling constant. For a vector, A, $\nabla \times A$ is the scalar $\partial_1 A_2 - \partial_2 A_1$, and for a scalar ξ, $\nabla \times \xi$ is the vector $(-\partial_2 \xi, \partial_1 \xi)$. Critical points of $\mathcal{E}(\psi, A)$ satisfy the *Ginzburg–Landau* (GL) equations

$$-\Delta_A \psi + \frac{\lambda}{2}(|\psi|^2 - 1)\psi = 0, \tag{2}$$

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$$\nabla \times \nabla \times A + Im(\bar{\psi} \nabla_A \psi) = 0, \tag{3}$$

where $\Delta_A = \nabla_A \cdot \nabla_A$.

Physically, the functional $\mathcal{E}(\psi, A)$ gives the difference in free energy between the superconducting and normal states near the transition temperature in the Ginzburg–Landau theory. A is the vector potential ($\nabla \times A$ is the induced magnetic field), and ψ is an *order parameter*. The modulus of ψ is interpreted as describing the local density of superconducting Cooper pairs of electrons.

The functional $\mathcal{E}(\psi, A)$ also gives the energy of a static configuration in the Yang-Mills-Higgs classical gauge theory on $\mathbb{R}^2$, with abelian gauge group $U(1)$. In this case A is a connection on the principal $U(1)$- bundle $\mathbb{R}^2 \times U(1)$, and ψ is the *Higgs field* (see [JT] for details).

A central feature of the functional $\mathcal{E}(\psi, A)$ (and the GL equations) is its infinite-dimensional symmetry group. Specifically, $\mathcal{E}(\psi, A)$ is invariant under $U(1)$ *gauge transformations,*

$$\psi \mapsto e^{i\gamma} \psi, \tag{4}$$

$$A \mapsto A + \nabla\gamma \tag{5}$$

for any smooth $\gamma : \mathbb{R}^2 \to \mathbb{R}$. In addition, $\mathcal{E}(\psi, A)$ is invariant under coordinate translations, and under the coordinate rotation transformation

$$\psi(x) \mapsto \psi(g^{-1}x) \qquad A(x) \mapsto gA(g^{-1}x) \tag{6}$$

for $g \in SO(2)$.

Finite energy field configurations satisfy

$$|\psi| \to 1 \qquad \text{as} \qquad |x| \to \infty \tag{7}$$

which leads to the definition of the *topological degree*, $\deg(\psi)$, of such a configuration:

$$\deg(\psi) = \deg\left(\frac{\psi}{|\psi|}\bigg|_{|x|=R} : \mathbb{S}^1 \to \mathbb{S}^1 \right)$$

(R sufficiently large). The degree is related to the phenomenon of *flux quantization*. Indeed, an application of Stokes' theorem shows that a finite-energy configuration satisfies

$$\deg(\psi) = \frac{1}{2\pi} \int_{\mathbb{R}^2} (\nabla \times A).$$

We study, in particular, "radially-symmetric" or "equivariant" fields of the form

$$\psi^{(n)}(x) = f_n(r) e^{in\theta}, \qquad A^{(n)}(x) = n \frac{a_n(r)}{r} \hat{x}^{\perp}, \tag{8}$$

where (r, θ) are polar coordinates on $\mathbb{R}^2$, $\hat{x}^{\perp} = \frac{1}{r}(-x_2, x_1)^t$, n is an integer, and

$$f_n, a_n : [0, \infty) \to \mathbb{R}.$$

It is easily checked that such configurations (if they satisfy (7)) have degree n. The existence of critical points of this form is well-known (see Sect. 2.1). They are called *n-vortices*.

Our main results concern the stability of these n-vortex solutions. Let

$$L^{(n)} = \text{Hess}\, \mathcal{E}(\psi^{(n)}, A^{(n)})$$

be the linearized operator for GL around the n-vortex, acting on the space

$$X = L^2(\mathbb{R}^2, \mathbb{C}) \oplus L^2(\mathbb{R}^2, \mathbb{R}^2).$$

The symmetry group of $\mathcal{E}(\psi, A)$ gives rise to an infinite-dimensional subspace of $\ker(L^{(n)}) \subset X$ (see Sect. 3.2), which we denote here by Z_{sym}. We say the n-vortex is (linearly) *stable* if for some $c > 0$,

$$L^{(n)}|_{Z_{\text{sym}}^{\perp}} \geq c,$$

and *unstable* if $L^{(n)}$ has a negative eigenvalue. The basic result of this paper is the following linearized stability statement:

Theorem 1. *1. (Stability of fundamental vortices)*
For all $\lambda > 0$, the ± 1-vortex is stable.
2. (Stability / instability of higher-degree vortices)
For $|n| \geq 2$, the n-vortex is

$$\begin{cases} \textit{stable} & \textit{for } \lambda < 1, \\ \textit{unstable} & \textit{for } \lambda > 1. \end{cases}$$

Theorem 1 is the basic ingredient in a proof of the nonlinear dynamical stability / instability of the n-vortex for certain dynamical versions of the GL equations. These include the GL gradient flow equations, and the Abelian Higgs (Lorentz-invariant) equations. These dynamical stability results are established in a separate work ([G2]). Other work on dynamics of magnetic vortices appears in [DS, S, S2].

The statement of Theorem 1 was conjectured in [JT] on the basis of numerical observations (see [JR]). Bogomolnyi ([B]) gave an argument for instability of vortices for $\lambda > 1$, $|n| \geq 2$. Our result rigorously establishes this property. The instability of higher-degree vortices for sufficiently large λ was established in [ABG]. The stability of vortices of Ginzburg–Landau equations without magnetic field was studied in [LL, M, OS1]. The stability of “monopole” solutions of a non-abelian generalization of (2-3) was studied in [AD] (see also [G1]).

The solutions of (2)–(3) are well-understood in the case of *critical coupling*, $\lambda = 1$. In this case, the *Bogomolnyi method* ([B]) gives a pair of first-order equations whose solutions are global minimizers of $\mathcal{E}(\psi, A)$ among fields of fixed degree (and hence solutions of the GL equations). Taubes ([T1, T2]) has shown that all solutions of GL with $\lambda = 1$ are solutions of these first-order equations, and that for a given degree n, the gauge-inequivalent solutions form a $2|n|$-parameter family. The $2|n|$ parameters describe the locations of the zeros of the scalar field. This is discussed in more detail in [JT] (see also [BGP]) and Sect. 6. We remark that for $\lambda = 1$, an n-vortex solution (8) corresponds to the case when all $|n|$ zeros of the scalar field lie at the origin.

The remainder of this paper is organized as follows. In Sect. 2 we describe in detail various properties of the n-vortex. In particular, we establish an important estimate on the n-vortex profiles which differentiates between the cases $\lambda < 1$ and $\lambda > 1$. In Sect. 3, we introduce the linearized operator, fix the gauge on the space of perturbations, and identify the zero-modes due to symmetry-breaking. Sections 4 through 7 comprise a proof of Theorem 1. A block-decomposition for the linearized operator is described in Sect. 4. This approach is similar to that used to study the stability of non-magnetic vortices in [OS1] and [G1]. In Sect. 5, we establish the positivity of certain blocks (those corresponding to the radially-symmetric variational problem, and those containing the translational zero-modes) for all λ, which completes the stability proof for the ± 1-vortices. The basic techniques are the characterization of symmetry-breaking in terms of zero-modes of the Hessian (or linearized operator), and a Perron-Frobenius type argument, based on a version of the maximum principle for systems (Proposition 6), which shows that the translational zero-modes correspond to the bottom of the spectrum of the linearized operator. A more careful analysis is needed for $|n| \geq 2$. This requires us to review some aspects of the critical case ($\lambda = 1$) in Sect. 6. The stability / instability proof for $|n| \geq 2$ is completed in Sect. 7. We use an extension of Bogomolnyi's instability argument, and another application of the Perron-Frobenius theory.

2. The *n*-Vortex

In this section we discuss the existence, and properties, of n-vortex solutions.

2.1. Vortex solutions. The existence of solutions of (GL) of the form (8) is well-known:

Theorem 2 (Vortex existence; [P,BC]). *For every integer n, and every $\lambda > 0$, there is a solution*

$$\psi^{(n)}(x) = f_n(r)e^{in\theta} \qquad A^{(n)}(x) = n\frac{a_n(r)}{r}\hat{x}^{\perp} \tag{9}$$

of the variational equations (2)–(3). In particular, the radial functions (f_n, a_n) minimize the radial energy functional

$$\mathcal{E}_r^{(n)}(f, a) = \frac{1}{2}\int_0^{\infty} \left\{ (f')^2 + n^2\frac{(1-a)^2 f^2}{r^2} + n^2\frac{(a')^2}{r^2} + \frac{\lambda}{4}(f^2-1)^2 \right\} rdr \tag{10}$$

(which is the full energy functional (1) restricted to fields of the form (8)) in the class

$$\{f, a : [0,\infty) \to \mathbb{R} \ | \ 1 - f \in H^1(rdr), \frac{a}{r} \in L^2_{loc}(rdr), \frac{a'}{r} \in L^2(rdr)\}.$$

The functions f_n, a_n are smooth, and have the following properties (for $n \neq 0$):

1. $0 < f_n < 1$, $0 < a_n < 1$ *on* $(0, \infty)$,
2. $f_n', a_n' > 0$,
3. $f_n \sim cr^n$, $a_n \sim dr^2$, *as* $r \to 0$ *(*$c > 0$ *and* $d > 0$ *are constants),*
4. $1 - f_n, 1 - a_n \to 0$ *as* $r \to \infty$*, with an exponential rate of decay.*

We call $(\psi^{(n)}, A^{(n)})$ an *n-vortex* (centred at the origin).

It follows immediately that the functions f_n and a_n satisfy the ODEs

$$-\Delta_r f_n + \frac{n^2(1-a_n)^2}{r^2}f_n + \frac{\lambda}{2}(f_n^2 - 1)f_n = 0 \tag{11}$$

and

$$-a_n'' + \frac{a_n'}{r} - f_n^2(1-a_n) = 0. \tag{12}$$

Remark 1. The n-vortex is known to be the unique solution of (GL) of the form (8) when $\lambda \geq 2n^2$ [ABGi]. In the appendix, we show that for $\lambda \geq 2n^2$, any such solution minimizes $\mathcal{E}_r^{(n)}$.

Remark 2. The functions f_n and a_n also depend on λ, but we suppress this dependence for ease of notation. When it will cause no confusion, we will also drop the subscript n.

Remark 3. The discrete symmetry $\psi \mapsto \bar{\psi}$, $A \mapsto -A$ of (GL) interchanges $(\psi^{(n)}, A^{(n)})$ and $(\psi^{(-n)}, A^{(-n)})$. Thus, we can assume $n \geq 0$.

2.2. An estimate on the vortex profiles. The following inequality, relating the exponentially decaying quantities f' and $1 - a$, plays a crucial role in the stability / instability proof.

Proposition 1. *We have*

$$\begin{cases} f'(r) > \frac{n(1-a(r))}{r} f(r) & \textit{for} \quad \lambda < 1 \\ f'(r) < \frac{n(1-a(r))}{r} f(r) & \textit{for} \quad \lambda > 1 \end{cases}. \tag{13}$$

Proof. Define $e(r) \equiv f'(r) - \frac{n(1-a(r))}{r} f(r)$. The properties listed in Theorem 2 imply that $e(r) \to 0$ as $r \to 0$ and as $r \to \infty$. Using the ODEs ((11)–(12)) we can derive the equation

$$(-\Delta_r + \alpha)e + \frac{e}{f}e' = (1-\lambda) f^2 f',$$

where

$$\alpha(r) = \frac{1 + n(1-a)}{r^2}(1 + \frac{rf'}{f}) + f^2 + \frac{na'}{r} > 0$$

and the result follows from the maximum principle. □

3. The Linearized Operator

In this section, we introduce the linearized operator (or Hessian) around the n-vortex, and identify its symmetry zero-modes.

3.1. Definition of the linearized operator. We work on the real Hilbert space

$$X = L^2(\mathbb{R}^2; \mathbb{C}) \oplus L^2(\mathbb{R}^2; \mathbb{R}^2)$$

with inner-product

$$< (\xi, B), (\eta, C) >_X = \int_{\mathbb{R}^2} \{Re(\bar{\xi}\eta) + B \cdot C\}.$$

We define the linearized operator, $L_{\psi,A}$ (= the Hessian of $\mathcal{E}(\psi, A)$) at a solution (ψ, A) of (2)–(3) through the quadratic form

$$\frac{\partial^2}{\partial\epsilon\partial\delta}\mathcal{E}(\psi + \epsilon\xi + \delta\eta, A + \epsilon B + \delta C)|_{\epsilon=\delta=0} = \langle(\eta, C), L_{\psi,A}(\xi, B)\rangle_X$$

for all (ξ, B), (η, C), $\in X$. The result is

$$L_{\psi,A}\begin{pmatrix} \xi \\ B \end{pmatrix} = \begin{pmatrix} [-\Delta_A + \frac{\lambda}{2}(2|\psi|^2 - 1)]\xi + \frac{\lambda}{2}\psi^2\bar{\xi} + i[2\nabla_A\psi + \psi\nabla] \cdot B \\ Im([\overline{\nabla_A\psi} - \bar{\psi}\nabla_A]\xi) + (-\Delta + \nabla\nabla + |\psi|^2) \cdot B \end{pmatrix}.$$

3.2. Symmetry zero-modes. We identify the part of the kernel of the operator

$$L^{(n)} \equiv L_{\psi^{(n)},A^{(n)}}$$

which is due to the symmetry group.

Proposition 2. *We have*

1.

$$L^{(n)} \begin{pmatrix} i\gamma\psi^{(n)} \\ \nabla\gamma \end{pmatrix} = 0 \tag{14}$$

for any $\gamma : \mathbb{R}^2 \to \mathbb{R}$.

2.

$$L^{(n)} \begin{pmatrix} \partial_j\psi^{(n)} \\ \partial_j A^{(n)} \end{pmatrix} = 0 \tag{15}$$

for $j = 1, 2$.

Proof. We use the basic result that the generator of a one-parameter group of symmetries of $\mathcal{E}(\psi, A)$, applied to the n-vortex, lies in the kernel of $L^{(n)}$. The vector in (14) is easily seen to be the generator of a one-parameter family of gauge transformations (4-5) applied to the n-vortex. Similarly, the vector in (15) is the generator of coordinate translations applied to the n-vortex. □

Remark 4. Applying the generator of the coordinate rotational symmetry (6) to the n-vortex gives us nothing new. This is covered by the gauge-symmetry case.

We define Z_{sym} to be the subspace of X spanned by the L^2 zero-modes described in Proposition 2. We recall that the n-vortex is called *stable* if there is a constant $c > 0$ such that

$$L^{(n)}|_{Z_{\text{sym}}^{\perp}} \geq c, \tag{16}$$

and *unstable* if $L^{(n)}$ has a negative eigenvalue.

3.3. Gauge fixing. In order to remove the infinite dimensional kernel of $L^{(n)}$ arising from gauge symmetry, we restrict the class of perturbations. Specifically, we restrict $L^{(n)}$ to the space of those perturbations $(\xi, B) \in X$ which are orthogonal to the L^2 gauge zero-modes (14). That is,

$$\left\langle \begin{pmatrix} i\gamma\psi^{(n)} \\ \nabla\gamma \end{pmatrix}, \begin{pmatrix} \xi \\ B \end{pmatrix} \right\rangle_X = 0$$

for all γ. Integration by parts gives the gauge condition

$$Im(\overline{\psi^{(n)}}\xi) = \nabla \cdot B. \tag{17}$$

As is done in [S], we consider a modified quadratic form $\tilde{L}^{(n)}$, defined by

$$< \alpha, \tilde{L}^{(n)}\alpha >=< \alpha, L^{(n)}\alpha > + \int (Im(\overline{\psi^{(n)}}\xi) - \nabla \cdot B)^2$$

for $\alpha = (\xi, B) \in X$. Clearly, $\tilde{L}^{(n)}$ agrees with $L^{(n)}$ on the subspace of X specified by the gauge condition (17). This modification has the important effect of shifting the essential spectrum away from zero (see (26)). A straightforward computation gives the following expression for $\tilde{L}^{(n)}$:

$$\tilde{L}^{(n)} \begin{pmatrix} \xi \\ B \end{pmatrix} = \begin{pmatrix} [-\Delta_A + \frac{\lambda}{2}(2|\psi|^2 - 1) + \frac{1}{2}|\psi|^2]\xi + \frac{1}{2}(\lambda - 1)\psi^2\bar{\xi} + 2i\nabla_A\psi \cdot B \\ 2Im[\overline{\nabla_A\psi}\xi] + [-\Delta + |\psi|^2]B \end{pmatrix}.$$

To establish Theorem 1, it suffices to prove that $\tilde{L}^{(n)} \geq c > 0$ on the subspace of X orthogonal to the translational zero-modes (15).

$\tilde{L}^{(n)}$ is a real-linear operator on X. It is convenient to identify $L^2(\mathbb{R}^2; \mathbb{R}^2)$ with $L^2(\mathbb{R}^2; \mathbb{C})$ through the correspondence

$$B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \leftrightarrow B^c \equiv B_1 - iB_2, \tag{18}$$

and then to complexify the space $X \mapsto \tilde{X} = [L^2(\mathbb{R}^2; \mathbb{C})]^4$ via

$$(\xi, B) \mapsto (\xi, \bar{\xi}, B^c, \bar{B}^c). \tag{19}$$

As a result, $\tilde{L}^{(n)}$ is replaced by the complex-linear operator

$$\tilde{\tilde{L}}^{(n)} = \text{diag}\,\{-\Delta_A, -\overline{\Delta_A}, -\Delta, -\Delta\} + V^{(n)},$$

where

$$V^{(n)} = \begin{pmatrix} \frac{\lambda}{2}(2|\psi|^2 - 1) + \frac{1}{2}|\psi|^2 & \frac{1}{2}(\lambda - 1)\psi^2 & -i(\partial_A^*\psi) & i(\partial_A\psi) \\ \frac{1}{2}(\lambda - 1)\bar{\psi}^2 & \frac{\lambda}{2}(2|\psi|^2 - 1) + \frac{1}{2}|\psi|^2 & -i(\overline{\partial_A\psi}) & i(\overline{\partial_A^*\psi}) \\ i(\overline{\partial_A^*\psi}) & i(\partial_A\psi) & |\psi|^2 & 0 \\ -i(\overline{\partial_A\psi}) & -i(\partial_A^*\psi) & 0 & |\psi|^2 \end{pmatrix}.$$

Here we have used the notation

$$\partial_A \equiv \partial_z - iA,$$

where $\partial_z = \partial_1 - i\partial_2$ (and the superscript c has been dropped from the complex function A obtained from the vector-field A via (18)).

The components of $V^{(n)}$ are bounded, and it follows from standard results ([RSII]) that $\tilde{\tilde{L}}^{(n)}$ is a self-adjoint operator on $\tilde{X}$, with domain

$$D(\tilde{\tilde{L}}^{(n)}) = [H^2(\mathbb{R}^2; \mathbb{C})]^4.$$

4. Block Decomposition

We write functions on $\mathbb{R}^2$ in polar coordinates. Precisely,

$$\tilde{X} = [L^2(\mathbb{R}^2;\mathbb{C})]^4 = [L^2_{rad} \otimes L^2(\mathbb{S}^1;\mathbb{C})]^4, \tag{20}$$

where $L^2_{\text{rad}} \equiv L^2(\mathbb{R}^+, rdr)$.

Let $\rho_n : U(1) \to \text{Aut}([L^2(\mathbb{S}^1;\mathbb{C})]^4)$ be the representation whose action is given by

$$\rho_n(e^{i\theta})(\xi, \eta, B, C)(x) = (e^{in\theta}\xi, e^{-in\theta}\eta, e^{-i\theta}B, e^{i\theta}C)(R_{-\theta}x),$$

where R_α is a counter-clockwise rotation in $\mathbb{R}^2$ through the angle α. It is easily checked that the linearized operator $\tilde{\tilde{L}}^{(n)}$ commutes with $\rho_n(g)$ for any $g \in U(1)$. It follows that $\tilde{\tilde{L}}^{(n)}$ leaves invariant the eigenspaces of $d\rho_n(s)$ for any $s \in i\mathbb{R} = \text{Lie}(U(1))$. The resulting block decomposition of $\tilde{\tilde{L}}^{(n)}$, which is described in this section, is essential to our analysis. In particular, the translational zero-modes each lie within a single subspace of this decomposition.

4.1. The decomposition of $L^{(n)}$. In what follows, we define, for convenience, $b(r) = \frac{n(1-a(r))}{r}$.

Proposition 3. *There is an orthogonal decomposition*

$$\tilde{X} = \bigoplus_{m\in\mathbb{Z}} (e^{i(m+n)\theta}L^2_{\text{rad}} \oplus e^{i(m-n)\theta}L^2_{\text{rad}} \oplus -ie^{i(m-1)\theta}L^2_{\text{rad}} \oplus ie^{i(m+1)\theta}L^2_{\text{rad}}), \tag{21}$$

under which the linearized operator around the vortex, $\tilde{\tilde{L}}^{(n)}$*, decomposes as*

$$\tilde{\tilde{L}}^{(n)} = \bigoplus_{m\in\mathbb{Z}} \hat{L}^{(n)}_m,$$

where

$$\hat{L}^{(n)}_m = -\Delta_r(Id) + \hat{V}^{(n)}_m \tag{22}$$

with

$$\hat{V}^{(n)}_m = \frac{1}{r^2}\text{diag}\,\{[m+n(1-a)]^2, [m-n(1-a)]^2, [m-1]^2, [m+1]^2\} + V'$$

and

$$V' = \begin{pmatrix} \frac{\lambda}{2}(2f^2-1)+\frac{1}{2}f^2 & \frac{1}{2}(\lambda-1)f^2 & f'-bf & -[f'+bf] \\ \frac{1}{2}(\lambda-1)f^2 & \frac{\lambda}{2}(2f^2-1)+\frac{1}{2}f^2 & -[f'+bf] & f'-bf \\ f'-bf & -[f'+bf] & f^2 & 0 \\ -[f'+bf] & f'-bf & 0 & f^2 \end{pmatrix}.$$

Proof. The decomposition (21) of $\tilde{X}$ follows from the usual Fourier decomposition of $L^2(\mathbb{S}^1; \mathbb{C})$, and the relation (20). An easy computation shows that $\tilde{\tilde{L}}^{(n)}$ preserves the space of vectors of the form

$$(\xi e^{i(m+n)\theta}, \eta e^{i(m-n)\theta}, -i\alpha e^{i(m-1)\theta}, i\beta e^{i(m+1)\theta}) \tag{23}$$

and that it acts on such vectors via (22). □

It follows that $\hat{L}_m^{(n)}$ is self-adjoint on $[L^2_{\mathrm{rad}}]^4$.

It will also be convenient to work with a rotated version of the operator $\hat{L}_m^{(n)}$,

$$L_m^{(n)} \equiv \begin{cases} R\hat{L}_m^{(n)}R^T & m \geq 0 \\ R'\hat{L}_m^{(n)}(R')^T & m < 0 \end{cases},$$

where

$$R = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \qquad R' = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

We have

$$L_m^{(n)} = -\Delta_r(Id) + V_m^{(n)}, \tag{24}$$

where

$$V_m^{(n)} = \begin{pmatrix} \frac{m^2}{r^2} + b^2 + \frac{\lambda}{2}(3f^2 - 1) & -2|m|\frac{b}{r} & -2bf & 0 \\ -2|m|\frac{b}{r} & \frac{m^2}{r^2} + b^2 + \frac{\lambda}{2}(f^2 - 1) + f^2 & 0 & -2f' \\ -2bf & 0 & \frac{m^2+1}{r^2} + f^2 & -2\frac{|m|}{r^2} \\ 0 & -2f' & -2\frac{|m|}{r^2} & \frac{m^2+1}{r^2} + f^2 \end{pmatrix}.$$

4.2. Properties of $L_m^{(n)}$.

Proposition 4. *We have the following:*

1.

$$L_m^{(n)} = L_{-m}^{(n)}. \tag{25}$$

2.

$$\sigma_{\mathrm{ess}}(L_m^{(n)}) = [\min(1, \lambda), \infty). \tag{26}$$

3. For $|n| = 1$ and $|m| \geq 2$,

$$L_m^{(n)} - L_1^{(n)} \geq 0 \tag{27}$$

with no zero-eigenvalue.

Proof. The first statement is obvious. The second statement follows in a standard way from the fact that

$$\lim_{r\to\infty} V_m^{(n)}(r) = \text{ diag }\{\lambda, 1, 1, 1\}.$$

To prove the third statement, we compute

$$\hat{L}_m^{(n)} - \hat{L}_1^{(n)} = \frac{m-1}{r^2} \text{ diag }\{m+1+2n(1-a),\ m+1-2n(1-a),\ m-1,\ m+3\}$$

which is non-negative, with no zero-eigenvalue for $m \geq 2$, $n = 1$. □

Remark 5. In light of (25), we can assume from now on that $m \geq 0$. This degeneracy is a result of the complexification (19) of the space of perturbations.

4.3. Translational zero-modes. The gauge fixing (Sect. 3.3) has eliminated the zero-modes arising from gauge symmetry. The translational zero-modes remain.

As written in (15), the translational zero-modes fail to satisfy the gauge condition (17). Further, they do not lie in L^2. A straightforward computation shows that if we adjust the vectors in (15) by gauge zero-modes given by (14) with $\gamma = -A_j$, $j = 1, 2$, we obtain

$$T_1 = \begin{pmatrix} (\nabla_A \psi)_1 \\ (\nabla \times A)e_2 \end{pmatrix}, \qquad T_2 = \begin{pmatrix} (\nabla_A \psi)_2 \\ -(\nabla \times A)e_1 \end{pmatrix},$$

where $e_1 = (1, 0)$ and $e_2 = (0, 1)$. T_1 and T_2 satisfy (17), and are zero-modes of the linearized operator. Note also that $T_{\pm 1}$ decay exponentially as $|x| \to \infty$, and hence lie in L^2.

It is easily checked that $T_1 \pm iT_2$ lie in the $m = \pm 1$ blocks for $\hat{L}_m^{(n)}$. After rotation by R, we have

$$L_{\pm 1}^{(n)} T = 0,$$

where

$$T = (f', bf, n\frac{a'}{r}, n\frac{a'}{r}).$$

5. Stability of the Fundamental Vortices

In this section we prove the first part of Theorem 1. Specifically, we show that for some $c > 0$, $L_m^{(\pm 1)} \geq c$ for $m \neq 1$, and $L_1^{(\pm 1)}|_{T^\perp} \geq c$. In light of the discussions in Sects. 3.3, 4.1, and 4.3, this will establish the stability of the ± 1-vortices.

5.1. Non-negativity of $L_0^{(n)}$ and radial minimization.

Proposition 5. *$L_0^{(n)} \geq 0$ for all λ.*

Proof. From the expression (24) we see that $L_0^{(n)}$ breaks up:

$$L_0^{(n)} = N_0 \oplus M_0 \tag{28}$$

(abusing notation slightly) where

$$M_0 = -\Delta_r(Id) + W_0$$

with

$$W_0 = \begin{pmatrix} b^2 + \frac{\lambda}{2}(3f_n^2 - 1) & -2bf \\ -2bf & \frac{1}{r^2} + f^2 \end{pmatrix}$$

and

$$N_0 = \begin{pmatrix} -\Delta_r + b^2 + \frac{\lambda}{2}(f^2 - 1) + f^2 & -2f' \\ -2f' & -\Delta_r + \frac{1}{r^2} + f^2 \end{pmatrix}.$$

An easy computation shows that M_0 is precisely the Hessian of the radial energy, $\mathrm{Hess}\mathcal{E}_r^{(n)}$ (see (10)). Since the n-vortex minimizes $\mathcal{E}_r^{(n)}$, we have $M_0 \geq 0$. It remains to show $N_0 \geq 0$. We establish the stronger result, $N_0 > 0$. Note that

$$N_0 = G_0^* G_0,$$

where

$$G_0 = \begin{pmatrix} \partial_r - f'/f & f \\ f & \partial_r + 1/r \end{pmatrix}.$$

In fact, G_0 has no zero-eigenvalue. To see this, we exploit some known results about the kernel of G_0 at $\lambda = 1$. In Sect. 6, we will show that at $\lambda = 1$, the full linearized operator is the square of a first-order differential operator, F: $\tilde{L}^{(n)}|_{\lambda=1} = F^* F$. The operator F was analyzed in [S], where it was shown to be Fredholm with index $2|n|$. The operator $F_0 \equiv G_0|_{\lambda=1}$ is F restricted to a particular invariant subspace. Thus F_0 is a Fredholm operator from its domain to L^2_{rad}. The kernels of F and F^* are known precisely, (see [S] and Sect. 6) and it follows that F_0 has index zero. Now, G_0 is a relatively compact perturbation of F_0 (due to the decay of the field components – see, again, [S]), and hence G_0 is also Fredholm with index zero. Finally, it is a simple matter to check that G_0^* has trivial kernel. If

$$G_0^* \begin{pmatrix} \xi \\ \beta \end{pmatrix} = 0$$

it follows that

$$(-\Delta_r + f^2)\beta = 0$$

and hence that $\beta = 0$, and so $\xi = 0$. The relation $N_0 > 0$ follows from this, and the fact that $\sigma_{\mathrm{ess}}(N_0) = [1, \infty)$. $\square$

5.2. A maximum principle argument. Removing the equality in Proposition 5 requires more work. First, we establish an extension of the maximum principle to systems (see, eg, [LM, PA] for related results). We will use this also in the proof that the translational zero-mode is the ground state of $L_1^{(n)}$ (Sect. 5.4).

Proposition 6. *Let L be a self-adjoint operator on $L^2(\mathbb{R}^n; \mathbb{R}^d)$ of the form*

$$L = -\Delta(Id) + V,$$

where V is a $d \times d$ matrix-multiplication operator with smooth entries. Suppose that $L \geq 0$ and that for $i \neq j$, $V_{ij}(x) \leq 0$ for all x. Further, suppose V is irreducible in the sense that for any splitting of the set $\{1, \dots, d\}$ into disjoint sets S_1 and S_2, there is an $i \in S_1$ and a $j \in S_2$ with $V_{ij}(x) < 0$ for all x. Finally, suppose that $L\xi = \eta \in L^2$ with $\eta \geq 0$ component-wise, and $\xi \not\equiv 0$. Then either

1. $\xi > 0$ or
2. $\eta \equiv 0$ and $\xi < 0$.

Proof. We write $\xi = \xi^+ - \xi^-$ with $\xi^+, \xi^- \geq 0$ component-wise, and compute

$$0 \ \leq \ < \xi^-, L\xi^- > \ = \ < \xi^-, L\xi^+ > - < \xi^-, L\xi > .$$

Since ξ_j^+ and ξ_j^- have disjoint support, we have

$$\text{r.h.s.} = \sum_{j \neq k} < \xi_j^-, V_{jk}\xi_k^+ > - < \xi^-, \eta > \ \leq 0.$$

Thus we have

1. $0 = \ < \xi^-, L\xi^- >$.
2. $0 = \ < \xi_j^-, V_{jk}\xi_k^+ >$ for all $j \neq k$.

Since $L \geq 0$, the first of these implies $L\xi^- = 0$ and hence $L\xi^+ = \eta$. So if $\eta \not\equiv 0$, then $\xi^+ \not\equiv 0$. If $\eta \equiv 0$ and $\xi^+ \equiv 0$, replace ξ with $-\xi$ in what follows. An application of the strong maximum principle (eg. [GT], Thm. 8.19) to each component of the equation

$$L\xi^+ = \eta$$

now allows us to conclude that for each k, either $\xi_k^+ > 0$ or $\xi_k^+ \equiv 0$. We know that for some k, $\xi_k^+ > 0$. Looking back at the second listed equation above, and using the irreducibility of V, we then see that $\xi_j^- \equiv 0$ for all j. Finally, we can easily rule out the possibility $\xi_k \equiv 0$ for some k, by looking back at the equation satisfied by ξ_k. Thus we have $\xi > 0$. □

5.3. Positivity of $L_0^{(n)}$. Now we apply Proposition 6 to show $M_0 > 0$. The trick here is to find a function ξ which satisfies $M_0\xi \geq 0$. This allows us to rule out the existence of a zero-eigenvector, which would be positive by Proposition 6. To obtain such a ξ, we differentiate the vortex with respect to the parameter λ. Specifically, differentiation of the Ginzburg–Landau equations with respect to λ results in

$$M_0\xi = \eta, \tag{29}$$

where

$$\xi = \begin{pmatrix} \partial_\lambda f \\ n\partial_\lambda a/r \end{pmatrix}$$

and

$$\eta = \begin{pmatrix} \frac{1}{2}(1-f^2)f \\ 0 \end{pmatrix} \geq 0.$$

We can now establish

Proposition 7. *For all* λ, $L_0^{(n)} \geq c > 0$.

Proof. We have already shown in the proof of Proposition 5, that $N_0 > 0$ and $M_0 \geq 0$. Hence, due to (28) and (26), it suffices to show that $Null(M_0) = \{0\}$. Suppose $M_0\zeta = 0$, $\zeta \not\equiv 0$. Proposition 6 then implies $\zeta > 0$ (or else take $-\zeta$). Now

$$0 = \, < M_0\zeta, \xi > \; = \; < \zeta, M_0\xi > \; = \; < \zeta, \eta > \; > \;\; 0$$

gives a contradiction. □

Remark 6. Proposition 6 applied to Eq. (29) also gives $\xi > 0$. That is, the vortex profiles increase monotonically with λ. This can be used to show that the rescaled vortex $(f_n(r/\sqrt{\lambda}), a_n(r/\sqrt{\lambda}))$ converges as $\lambda \to \infty$ to $(f^*, 0)$, where f^* is the (profile of) the n-vortex solution of the ordinary GL equation: $-\Delta_r f^* + n^2 f^*/r^2 + (f^{*2} - 1)f^* = 0$. This result was established by different means in [ABG].

5.4. Positivity of $L_1^{(\pm1)}$.

Proposition 8. $L_1^{(\pm1)} \geq 0$ *with non-degenerate zero-eigenvalue given by* T.

Proof. Let $\mu = inf spec L_1^{(\pm1)} \leq 0$, which is an eigenvalue by (26). Suppose $L_1^{(\pm1)} S = \mu S$. Applying Proposition 6 to $L_1^{(\pm1)} - \mu$ (note that $V_1^{(\pm1)}$ satisfies the irreducibility requirement) gives $S > 0$ (or $S < 0$). Further, μ is non-degenerate, as if μ were degenerate, we would have two strictly positive eigenfunctions which are orthogonal, an impossibility. Now if $\mu < 0$, we have $< S, T >= 0$, which is also impossible. Thus S is a multiple of T, and $\mu = 0$. □

5.5. Completion of stability proof for $n = \pm 1$. We are now in a position to complete the proof of the first statement of Theorem 1. By Proposition 7, $L_0^{(\pm1)} \geq c > 0$. By Proposition 8 and (26), $L_1^{(\pm1)}|_{T^\perp} \geq \tilde{c} > 0$. Finally, by (27), $L_m^{(\pm1)} \geq c' > 0$ for $|m| \geq 2$. It follows from Proposition 3 that $\tilde{L}^{(n)} \geq c > 0$ on the subspace of X orthogonal to the translational zero-modes. By the discussion of Sect. 3.3, this gives Theorem 1 for $n = \pm 1$. □

6. The Critical Case, $\lambda = 1$

In order to prove the remainder of Theorem 1, we exploit some results from the $\lambda = 1$ case.

6.1. The first-order equations. Following [B], we use an integration by parts to rewrite the energy (1) as

$$\mathcal{E}(\psi, A) = \frac{1}{2}\int_{\mathbb{R}^2} \Big\{|\partial_A^*\psi|^2 + \Big[\nabla \times A + \tfrac{1}{2}(|\psi|^2 - 1)\Big]^2 + \tfrac{1}{4}(\lambda - 1)(|\psi|^2 - 1)^2\Big\} + \pi \deg(\psi) \tag{30}$$

(recall, since we work in dimension two, $\nabla \times A$ is a scalar) where $\deg(\psi)$ is the topological degree of ψ, defined in the introduction. We assume, without loss of generality, that $deg(\psi) \geq 0$. Clearly, when $\lambda = 1$, a solution of the first-order equations

$$\partial_A^*\psi = 0, \tag{31}$$

$$\nabla \times A + \frac{1}{2}(|\psi|^2 - 1) = 0 \tag{32}$$

minimizes the energy within a fixed topological sector, $\deg(\psi) = n$, and hence solves GL. Note that we have identified the vector-field A with a complex field as in (18).

The n-vortices (9) are solutions of these equations (when $\lambda = 1$). Specifically,

$$n\frac{a'}{r} = \frac{1}{2}(1 - f^2) \tag{33}$$

and

$$f' = n\frac{(1-a)f}{r}. \tag{34}$$

In fact, it is shown in [T2] that for $\lambda = 1$, any solution of the variational equations solves the first- order equations (31)-(32).

Beginning from expression (30) for the energy, the variational equations (previously written as (2)-(3)) can be written as

$$\partial_A[\partial_A^*\psi] + \psi[\nabla \times A + \frac{1}{2}(|\psi|^2 - 1)] + \frac{1}{2}(\lambda - 1)(|\psi|^2 - 1)\psi = 0, \tag{35}$$

$$i\psi[\overline{\partial_A^*\psi}] - i\partial_{\bar{z}}[\nabla \times A + \frac{1}{2}(|\psi|^2 - 1)] = 0 \tag{36}$$

(here $\partial_A^* \equiv -\partial_{\bar{z}} + i\bar{A}$ is the adjoint of ∂_A).

6.2. First-order linearized operator. We show that the linearized operator at $\lambda = 1$ is the square of the linearized operator for the first-order equations.

Linearizing the first-order equations (31)–(32) about a solution, (ψ, A) (of the first-order equations) results in the following equations for the perturbation, $\alpha \equiv (\xi, B)$:

$$\partial_A^*\xi + i\psi\bar{B} = 0,$$

$$\nabla \times B + Re(\bar{\psi}\xi) = 0.$$

Now using $i\partial_{\bar{z}} B = \nabla \times B + i(\nabla \cdot B)$, and adding in the gauge condition (17), we can rewrite this as

$$F\alpha = 0, \tag{37}$$

where

$$F = \begin{pmatrix} \partial_A^* & i\psi(\bar{\ }) \\ \psi(\bar{\ }) & i\partial_z \end{pmatrix}.$$

If we linearize the full (second order) variational equations (in the form (35)-(36)) around (ψ, A), we obtain

$$\partial_A[\partial_A^*\xi + i\bar{B}\psi] + i\bar{B}[\partial_A^*\psi] + \psi[\nabla \times B + Re(\bar{\psi}\xi)]$$
$$+\xi[\nabla \times A + \tfrac{1}{2}(|\psi|^2 - 1)] + \tfrac{1}{2}(\lambda - 1)[(|\psi|^2 - 1)\xi + 2\psi Re(\bar{\psi}\xi)] = 0$$

and

$$i\bar{\psi}[\partial_A^*\xi + i\bar{B}\psi] + i\bar{\xi}[\partial_A^*\psi] - i\partial_{\bar{z}}[\nabla \times B + Re(\bar{\psi}\xi)] = 0.$$

Proposition 9. *When $\lambda = 1$, these linearized equations can also be written*

$$F^* F\alpha = 0.$$

Proof. This is a simple computation using the fact that the first-order equations (31–32) hold. □

This relation holds also on the level of the blocks. A straightforward computation gives

$$L_m^{(n)}|_{\lambda=1} = F_m^* F_m,$$

where

$$F_m = \begin{pmatrix} \partial_r - b & \frac{m}{r} & f & 0 \\ \frac{m}{r} & \partial_r - b & 0 & f \\ f & 0 & \partial_r + 1/r & -\frac{m}{r} \\ 0 & f & -\frac{m}{r} & \partial_r + 1/r \end{pmatrix}.$$

6.3. Zero-modes for $\lambda = 1$. It was predicted in [W] (and proved rigorously in [S]) that for $\lambda = 1$, the linearized operator around any degree-n solution of the first-order equations has a $2|n|$-dimensional kernel (modulo gauge transformations). This kernel arises because the Taubes solutions form a $2|n|$-parameter family, and all have the same energy. The zero-eigenvalues are identified in [B], and we describe them here. Let χ_m be the unique solution of

$$(-\Delta_r + \frac{m^2}{r^2} + f^2)\chi_m = 0$$

on $(0, \infty)$ with

$$\chi_m \sim r^{-m} \qquad \text{as} \qquad r \to 0$$

and

$$\chi_m \to 0 \qquad \text{as} \quad r \to \infty$$

for $m = 1, 2, \dots, n$. Then it is easy to check that when $\lambda = 1$,

$$F_m W_m = 0, \tag{38}$$

where

$$W_m = \begin{pmatrix} f\chi_m \\ f\chi_m \\ -(\chi_m' + m\chi_m/r) \\ -(\chi_m' + m\chi_m/r) \end{pmatrix}.$$

We remark that

$$\chi_1 = \frac{1-a}{r}$$

and it is easily verified that for $\lambda = 1$, $W_1 = \frac{1}{n}T$ gives the translational zero-modes.

7. The (In)stability Proof for $|n| \geq 2$

Here we complete the proof of Theorem 1.

The idea is to decompose $L_m^{(n)}$ into a sum of two terms, each of which has the same (translational) zero-mode (for $m = 1$) as $L_m^{(n)}$. One term is manifestly positive, and the other satisfies restrictions of Perron-Frobenius theory.

We begin by modifying F_m, and defining, for any λ,

$$\tilde{F}_m \equiv \begin{pmatrix} (\partial_r - \frac{f'}{f}) \cdot q & \frac{m}{r} & f & 0 \\ \frac{m}{r}q & \partial_r - \frac{f'}{f} & 0 & f \\ fq & 0 & \partial_r + 1/r & -\frac{m}{r} \\ 0 & f & -\frac{m}{r} & \partial_r + 1/r \end{pmatrix},$$

where we have defined

$$q(r) \equiv \frac{n(1-a)f}{rf'} \tag{39}$$

and $\partial_r \cdot q$ denotes an operator composition. By (34), we have $q \equiv 1$ for $\lambda = 1$. We also set, for $m = 1, \dots, n$,

$$\tilde{W}_m = \begin{pmatrix} q^{-1} f\chi_m \\ f\chi_m \\ -(\chi_m' + m\frac{\chi_m}{r}) \\ -(\chi_m' + m\frac{\chi_m}{r}) \end{pmatrix}.$$

Now $\tilde{W}_m$ has the following properties:

1. $\tilde{W}_1$ is the translational zero-mode $\frac{1}{n}T$ for all λ.

2. When $\lambda = 1$, $\tilde{W}_m = W_m$, $m = 1, \ldots, n$, give the $2|n|$ zero-modes (38) of the linearized operator.

These $\tilde{W}_m$ were chosen in [B] as candidates for directions of energy decrease (for $|m| \geq 2$) when $\lambda > 1$. Intuitively, we think of $\tilde{W}_m$ as a perturbation that tends to break the n-vortex into separate vortices of lower degree.

Now, $\tilde{F}_m$ was designed to have the following properties:

1. $\tilde{F}_m = F_m$ when $\lambda = 1$ (this is clear).
2. $\tilde{F}_m \tilde{W}_m = 0$ for all m and λ (this is easily checked).

A straightforward computation gives

$$L_m^{(n)} = \tilde{F}_m^* \tilde{F}_m + J M_m, \tag{40}$$

where $J = \text{diag}\{1, 0, 0, 0\}$ and

$$M_m = l_m - q l_m q + (\lambda - q^2) f^2$$

with

$$l_m = -\Delta_r + \frac{m^2}{r^2} + b^2 + \frac{\lambda}{2}(f^2 - 1).$$

By construction, when $m = 1$, the second term in the decomposition (40) must have a zero-mode corresponding to the original translational zero-mode. In fact, one can easily check that $M_1 f' = 0$.

Proposition 10. *For $|n| \geq 2$, M_1 has a non-degenerate zero-eigenvalue corresponding to f', and*

$$\begin{cases} M_1 \geq 0 & \lambda < 1 \\ M_1 \leq 0 & \lambda > 1 \end{cases}$$

on L^2_{rad}.

Proof. We recall inequality (13), which implies that for $\lambda < 1$, $q < 1$, and for $\lambda > 1$, $q > 1$. The operator M_1 is of the form

$$M_1 = (1 - q^2)(-\Delta_r) + \text{ first order } + \text{ multiplication}. \tag{41}$$

One can show that M_1 is bounded from below (resp. above) for $\lambda < 1$ (resp. $\lambda > 1$). We stick with the case $\lambda < 1$ for concreteness. Suppose $M_1 \eta = \mu \eta$ with $\mu = \text{infspec} M_1 \leq 0$. Applying the maximum principle (e.g. Proposition 6 for $d = 1$) to (41), we conclude that $\eta > 0$. If $\mu < 0$, we have $< \eta, f' >= 0$, a contradiction. Thus $\mu = 0$, and is non-degenerate by a similar argument. □

We also have

Lemma 1. *For $m \geq 2$, $M_m - M_1$ is non-negative for $\lambda < 1$, non-positive for $\lambda > 1$, and has no zero-eigenvalue.*

Proof. This follows from the equation

$$M_m - M_1 = (1 - q^2)\frac{m^2 - 1}{r^2}. \qquad \square$$

Completion of the proof of Theorem 1. Suppose now $\lambda < 1$. Since $\tilde{F}_m^* \tilde{F}_m$ is manifestly non-negative, and $M_m > M_1$ for $m \geq 2$, we have $L_m^{(n)} \geq 0$ for $m \geq 1$ (with only the translational 0-mode). Combined with (26) and Propositions 7 and 3, this gives stability of the n-vortex for $\lambda < 1$.

Now suppose $\lambda > 1$. By (40), Proposition 10 and Lemma 1, we have for $m = 2, \ldots n$,

$$< \tilde{W}_m, L_m^{(n)} \tilde{W}_m > \quad < \quad 0.$$

We remark that $\tilde{W}_m$ corresponds to an element of the un-complexified space X, and so $L^{(n)}$ has negative eigenvalues. This establishes the instability of the n-vortex for $|n| \geq 2$, $\lambda > 1$, and completes the proof of Theorem 1. □

8. Appendix: Vortex Solutions are Radial Minimizers

Proposition 11. *For $\lambda \geq 2n^2$, a solution of Eqs. (11)–(12) locally minimizes $\mathcal{E}_r^{(n)}$.*

Proof. It suffices then to show $M_0 = \text{Hess}\mathcal{E}_r^{(n)} > 0$ (see Sect. 5.1). We write $M_0 = L_0 + Z_0$, where

$$L_0 = \text{diag}\{l, -\Delta_r\}$$

with $l = -\Delta_r + b^2 + \frac{\lambda}{2}(f^2 - 1)$ and

$$Z_0 = \begin{pmatrix} 2\lambda f^2 & -2bf \\ -2bf & \frac{1}{r^2} + f^2 \end{pmatrix}.$$

We note that $lf = 0$ (one of the GL equations). It follows from the fact that $f > 0$ and a Perron-Frobenius type argument (see [OS1]) that $l \geq 0$ with no zero-eigenvalue. It suffices to show $Z_0 \geq 0$. Clearly $tr(Z_0) > 0$, and

$$\det(Z_0) = 2\lambda f^4 + \frac{2f^2}{r^2}[\lambda - 2n^2(1-a)^2]$$

is strictly positive for $\lambda \geq 2n^2$. □

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References

[ABG] Almeida, L., Bethuel, F., Guo, Y.: A remark on the instability of symmetric vortices with large coupling constant. Commun. Pure Appl. Math. **50**, 1295–1300 (1997)

[ABGi] Alama, S., Bronsard, L., Giorgi T.: Uniqueness of symmetric vortex solutions in the Ginzburg–Landau model of superconductivity. Preprint (1998)

[AD] Androulakis, G., Dostoglou, S.: On the stability of monopole solutions. Nonlinearity **11**, 377–408 (1998)

[BC] Berger, M.S., Chen, Y.Y.: Symmetric vortices for the nonlinear Ginzburg–Landau equations of superconductivity, and the nonlinear desingularization phenomenon. J. Funct. Anal. **82**, 259–295 (1989)

[B] Bogomol'nyi, E.B.: The stability of classical solutions. Yad. Fiz. **24**, 861–870 (1976)

[BGP] Boutet de Monvel–Berthier, A., Georgescu, V., Purice, R.: A boundary value problem related to the Ginzburg–Landau model. Commun. Math. Phys. **142**, 1–23 (1991)
[DS] Demoulini, S., Stuart, D.: Gradient flow of the superconducting Ginzburg–Landau functional on the plane. Commun. Anal. Geom. **5**, no.1, 121–198 (1997)
[GT] Gilbarg, D., Trudinger, N.S.: Elliptic Partial Differential Equations of Second Order. Berlin: Springer-Verlag, 1977
[G1] Gustafson, S.: Symmetric solutions of Ginzburg–Landau equations in all dimensions. Intern. Math. Res. Notices **No. 16**, 807–816 (1997)
[G2] Gustafson, S.: Dynamic stability of magnetic vortices. In preparation.
[JT] Jaffe, A., Taubes, C.: Vortices and Monopoles. Boston: Birkhauser, 1980.
[JR] Jacobs, L., Rebbi, C.: Interaction of superconducting vortices. Phys. Rev. **B19**, 4486–4494 (1979)
[LL] Lieb, E.H., Loss, M.: Symmetry of the Ginzburg–Landau Minimizer in a Disc. Math. Res. Lett. **1**, 701–715 (1994)
[LM] Lopez-Gomez, J., Molina-Meyer, M.: The maximum principle for cooperative weakly coupled elliptic systems and some applications. Diff. Int. Eqns. **7**, no. 2, 383–398 (1994)
[M] Mironescu, P.: On the stability of radial solutions of the Ginzburg–Landau equation. J. Funct. Anal. **130**, 334–344 (1995)
[OS1] Ovchinnikov, Y., Sigal, I.M.: Ginzburg–Landau equation I: Static vortices. In: Partial Differential Equations and their Applications, Greiner et. al., eds. Providence, RI: AMS, 1997, pp. 199–220
[P] Plohr, B.: The existence, regularity, and behaviour at infinity of isotropic solutions of classical gauge field theories. Princeton thesis
[PA] Pao, C.V.: Nonlinear elliptic systems in unbounded domains. Nonlinear Analysis: Theory, Methods, and Applications **22**, No. 11, 1391–1407 (1994)
[RSII] Reed, M., Simon, B.: Methods of Modern Mathematical Physics, Vol II: Fourier Analysis, Self-Adjointness. New York: Academic Press, 1975
[RSIV] Reed, M., Simon, B.: Methods of Modern Mathematical Physics, Vol IV: Analysis of Operators. New York: Academic Press, 1978
[S] Stuart, D.: Dynamics of Abelian Higgs vortices in the near Bogomolny regime. Commun. Math. Phys. **159**, 51–91 (1994)
[S2] Stuart, D.: Periodic solutions of the Abelian Higgs model and rigid rotation of vortices. GAFA **9**, 568–595 (1999)
[T1] Taubes, C.: Arbitrary n-vortex solutions to the first order Ginzburg–Landau equations. Commun. Math. Phys. **72**, 277–292 (1980)
[T2] Taubes, C.: On the equivalence of the first and second order equations for gauge theories. Commun. Math. Phys. **75**, 207–227 (1980)
[W] Weinberg, E.: Multivortex solutions of the Ginzburg–Landau equations. Phys. Rev. D **19**, 3008–3012 (1979)

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