Numer. Math. 89: 561–590 (2001)
Digital Object Identifier (DOI) 10.1007/s002110100264

Numerische Mathematik

Semi-implicit finite volume scheme for solving nonlinear diffusion equations in image processing

Karol Mikula[1], **Narisoa Ramarosy**[2]

[1] Department of Mathematics, Slovak University of Technology, Radlinskeho 11, 81368 Bratislava, Slovakia; e-mail: mikula@vox.svf.stuba.sk
[2] Laboratoire d'Analyse Numérique et E.D.P, Université Paris Sud, Orsay, France

Received January 7, 1999 / Revised version received May 31, 2000 /
Published online March 20, 2001 –

Summary. We propose and prove a convergence of the semi-implicit finite volume approximation scheme for the numerical solution of the modified (in the sense of Catté, Lions, Morel and Coll) Perona–Malik nonlinear image selective smoothing equation (called *anisotropic diffusion* in the image processing). The proof is based on L_2 a-priori estimates and Kolmogorov's compactness theorem. The implementation aspects and computational results are discussed.

Mathematics Subject Classification (1991): 35K55, 65P05

1. Introduction

In this paper we study the convergence of the semi-implicit finite volume scheme for the following nonlinear initial-boundary value problem

$$\partial_t u - \nabla.(g(|\nabla G_\sigma * u|)\nabla u) = f(u_0 - u) \quad \text{in } Q_T \equiv I \times \Omega, \tag{1.1}$$

$$\partial_\nu u = 0 \quad \text{on } I \times \partial\Omega, \tag{1.2}$$

$$u(0, \cdot) = u_0 \quad \text{in } \Omega, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^d$ is a rectangular domain, $I = [0, T]$ is a scaling interval, and

$$g \text{ is a decreasing function, } g(0) = 1, 0 < g(s) \to 0 \text{ for } s \to \infty, g(\sqrt{t}) \text{ is smooth,} \tag{1.4}$$

Correspondence to: K. Mikula

(1.5) $G_\sigma \in C^\infty(\mathbb{R}^d)$ is a smoothing kernel with $\int_{\mathbb{R}^d} G_\sigma(x)dx = 1$

and $G_\sigma(x) \to \delta_x$ for $\sigma \to 0$, δ_x the Dirac measure at point x,

(1.6) f is a Lipschitz continuous, nondecreasing function, $f(0) = 0$,

(1.7) $u_0 \in L^2(\Omega)$.

We assume that

(1.8) supp $G_\sigma(x) \subset B_\sigma(0)$ ($B_\sigma(0)$ is a ball centered at 0 with radius σ)

and by the term $\nabla G_\sigma * u$ in (1.1) we mean $\int_{\mathbb{R}^d} \nabla G_\sigma(x-\xi)\tilde{u}(\xi,t)d\xi$, where $\tilde{u}$ is an extension of u given by periodic reflexion through the boundary of Ω in the region $\Omega_\sigma = \Omega \cup \bigcup_{x\in\partial\Omega} B_\sigma(x)$ and by 0 in $\mathbb{R}^d - \Omega_\sigma$.

In the image processing, (1.1)–(1.3) arises in the nonlinear data filtration, edge detection and image enhancement and restoration [14], [5]. The initial condition $u_0(x)$ represents the greylevel intensity function of the image which we want to process. The solution $u(t,x)$ of (1.1)–(1.3) represents the family of *scaled* (filtered, smoothed) versions of $u_0(x)$; t is understood as an abstract parameter called *scale*. In general, the processing of u_0 by evolutionary PDE like (1.1) is called *image multiscale analysis* [1,2,11, 15] and, in a sense, it represents an embedding of the initial image to the so called *nonlinear scale space*. In our case, (1.1)–(1.3) represent a slight modification of the well-known Perona–Malik equation called also *anisotropic diffusion* in computer vision community. It selectively diffuses an image in the regions where the signal is of constant mean in spite of those regions where the signal changes its tendency. This diffusion process is governed by the shape of the function g and its dependence on ∇u which is in a sense an edge indicator [14]. We note that in original Perona–Malik formulation ∇u stands in the place of $\nabla G_\sigma * u$ in (1.1). However, if the product $g(s)s$ is decreasing, the Perona–Malik equation can behave locally like the backward heat equation, which is an ill–posed problem. So, for g's used in practice ($g(s) = 1/(1+s^2)$, $g(s) = e^{-s}$) both existence and uniqueness of a solution cannot be obtained. One way how to preveal that mathematical disadvantage has been proposed by Catté, Lions, Morel and Coll in [5]. They have introduced the convolution with the Gaussian kernel G_σ into the decision process for the value of the diffusion coefficient. This slight modification (for σ small, the models are close and in a sense $\nabla G_\sigma * u \to \nabla u$ for $\sigma \to 0$) allowed them to prove the existence and uniqueness of the weak solution for the modified model and to keep all practical advantages of original formulation. Moreover, the usage of the *Gaussian gradient* makes the process more stable in the presence of noise. It has made explicit a *presmoothing* included implicitly in numerical realizations of Perona–Malik equation, too. Due to homogeneous Neumann boundary conditions the solution tends to a constant function with time evolution, provided $f \equiv 0$. By means of f on the right–hand side of (1.1), the solution u is forced to be close to u_0, which can weaken the influence of the *stopping time* T . In [12], $f(s) \equiv s$ is proposed.

Several approaches for the numerical solution of (1.1)–(1.3) have been suggested and studied regarding stability of the schemes, convergence to a weak solution of continuous problem and efficiency of implementations. There are used finite difference approximations (see e.g. [16] as well as methods based on finite element technique allowing adaptivity by coarsening of the discrete computational grid [8], [3]. The convergence of the semi-implicit scheme combined with finite elements in space to the unique weak solution of (1.1)–(1.3) has been proven in [8]. The finite difference schemes for (1.1)–(1.3), explicit, semi-implicit or implicit, have been studied only due to the stability properties on given spatial discrete grids [16].

In this paper we derive approximation scheme, corresponding to (1.1)–(1.3), using the so called finite volume method [13], [6]. In finite volume method, the discrete approximations are piecewise constant per control volumes corresponding to pixel/voxel structure of the discrete image, so such approach is the most natural in image processing. The nonlinearity of the equation is treated from the previous discrete scale step, thus our scheme is semi-implicit and leads to a solution of sparse linear systems in each discrete scale step of the algorithm. That can be done in efficient way using preconditioned iterative solvers. Moreover, the scheme allows to derive L_2 a-priori estimates for fully discrete solutions which force us to use Kolmogorov's compactness theorem in order to prove the convergence of the approximations to the unique weak solution of (1.1)–(1.3).

The organization of the paper is as follows. In Sect. 2 we present the fully discrete semi-implicit finite volume scheme and in Sect. 3 we prove its convergence to the unique weak solution of the problem. Finally, in Sect. 4 we discuss some computational results.

2. The finite volume scheme

Let τ_h be a mesh of Ω. The elements of τ_h are the so called control volumes. For every pair $(p,q) \in \tau_h^2$ with $p \neq q$, we denote their common interface by e_{pq}, i.e. $e_{pq} = \overline{p} \cap \overline{q}$ which is supposed to be included in a hyperplane of $\mathbb{R}^d$ not intersecting either p or q. Let $m\,(e_{pq})$ denote the measure of e_{pq}, and n_{pq} the unit vector normal to e_{pq} oriented from p to q. We denote by $\mathcal{E}$ the set of pairs of adjacent control volumes, defined by $\mathcal{E} = \left\{(p,q) \in \tau_h^2, p \neq q,\ m\,(e_{pq}) \neq 0\right\}$. We also use the notation $N(p) = \{q, (p,q) \in \mathcal{E}\}$. We assume that there exists family of points $(x_p)_{p \in \tau_h}$, $x_p \in p$ for every $p \in \tau_h$, such that for every $(p,q) \in \mathcal{E}$,

$\frac{x_q - x_p}{|x_q - x_p|} = n_{pq}$. Let $\delta(p)$ denote the diameter of the control volume p, $m(p)$ the measure in $\mathbb{R}^d$, of the control volume p, ∂p its boundary and let

$$h = \max_{p \in \tau_h} \delta(p).$$

We denote by $d_{pq} = |x_q - x_p|$ the Euclidean distance between x_p and x_q and by x_{pq} a point of e_{pq} intersecting the segment $\overline{x_p x_q}$. Finally we define $T_{pq} = \frac{m(e_{pq})}{d_{pq}}$.

Now, we are ready to write the **semi-implicit finite volume scheme** for solving regularized Perona-Malik problem (1.1)-(1.3):

Let $0 = t_0 \leq t_1 \leq ... \leq t_{N_{\max}} = T$ *denote the time discretization with* $t_n = t_{n-1} + k$, *where* k *is the time step. For* $n = 0, ..., N_{\max} - 1$ *we look for* $\overline{u}_p^{n+1}$, $p \in \tau_h$ *satisfying*

$$\frac{\overline{u}_p^{n+1} - \overline{u}_p^n}{k} m(p) = \sum_{q \in N(p)} g_{pq}^{\sigma,n}(\overline{u}_{h,k}) T_{pq} \left(\overline{u}_q^{n+1} - \overline{u}_p^{n+1}\right) \tag{2.1}$$
$$+ f\left(\overline{u}_p^0 - \overline{u}_p^n\right) m(p)$$

starting with

$$\overline{u}_p^0 = \frac{1}{m(p)} \int_p u_0(x)\, dx,\ p \in \tau_h, \tag{2.2}$$

where

$$g_{pq}^{\sigma,n}(\overline{u}_{h,k}) = g\left(|\nabla G_\sigma * \tilde{u}_{h,k}(x_{pq}, t_n)|\right) \tag{2.3}$$

and $\tilde{u}_{h,k}$ *is an extension of the piecewise constant function* $\overline{u}_{h,k}$ *defined as follows*

$$\overline{u}_{h,k}(x,t) = \sum_{n=0}^{N_{\max}} \sum_{p \in \tau_h} \overline{u}_p^n \chi_{\{x \in p\}} \chi_{\{t_{n-1} < t \leq t_n\}} \tag{2.4}$$

with the function $\chi_{\{A\}} = \begin{cases} 1 \text{ if } A \text{ is true} \\ 0 \text{ elsewhere.} \end{cases}$ *The extension is realized by periodic reflexion through the boundary of* Ω *in the region* Ω_σ *and by* 0 *in* $\mathbb{R}^d - \Omega_\sigma$.

Remark 2.1. The function $\overline{u}_{h,k}$, constructed using discrete values given by the scheme (2.1), is considered as the approximation of the solution of (1.1)-(1.3) and its convergence to a weak solution of (1.1)-(1.3), as $h \to 0$, $k \to 0$, will be studied in Sect. 3.

Remark 2.2. The scheme (2.1) is deduced from initial-boundary value problem (1.1)-(1.3) after integration of (1.1) over $p \in \tau_h$ and over $[t_n, t_{n+1}]$:

$$\int_{t_n}^{t_{n+1}} \int_p \partial_t u dx dt = \int_{t_n}^{t_{n+1}} \int_{\partial p} g(|\nabla G_\sigma * u|) \nabla u . \overrightarrow{n} dx dt + \int_{t_n}^{t_{n+1}} \int_p f(u_0 - u) dx dt \tag{2.5}$$

and taking into account the homogeneous Neumann boundary conditions (1.2).

Remark 2.3. The gradient of the convolution term in (2.1), i.e.

$$\nabla G_\sigma * \tilde{u}_{h,k}(x_{pq}, t_n) = \left(\frac{\partial}{\partial x_i} (G_\sigma * \tilde{u}_{h,k}(x_{pq}, t_n)) \right)_{i=1,\dots,d},$$

where x_i are space variables, is computed using the convolution derivative property

$$\frac{\partial}{\partial x_i} (G_\sigma * \tilde{u}_{h,k}(x_{pq}, t_n)) = \frac{\partial G_\sigma}{\partial x_i} * \tilde{u}_{h,k}(x_{pq}, t_n).$$

Then we have

$$\begin{aligned} \frac{\partial G_\sigma}{\partial x_i} * \tilde{u}_{h,k}(x_{pq}, t_n) &= \int_{\mathbb{R}^d} \frac{\partial G_\sigma}{\partial x_i}(x_{pq} - s) \tilde{u}_{h,k}(s, t_n) ds \\ &= \sum_r \overline{u}_r^n \int_r \frac{\partial G_\sigma}{\partial x_i}(x_{pq} - s) ds \end{aligned} \tag{2.6}$$

and thus

$$\nabla G_\sigma * \tilde{u}_{h,k}(x_{pq}, t_n) = \sum_r \overline{u}_r^n \int_r \nabla G_\sigma (x_{pq} - s) ds \tag{2.7}$$

where the sum is evaluated on control volumes $r \in \tau_h$ and on control volumes contained in the reflexion of τ_h through boundary of Ω which are arround x_{pq}. Hereby, the sum is restricted to control volumes intersecting $B_\sigma(x_{pq})$, the ball centered at x_{pq} with radius σ.

Theorem 2.1. (Existence and uniqueness of the discrete solution) *There exists unique solution $\overline{u}_{h,k}$ given by the scheme (2.1).*

Proof. We can prove the existence and uniqueness of $\overline{u}_{h,k}$ once we prove it for each $\overline{u}_p^n$, $p \in \tau_h$, $0 \le n \le N_{\max}$. We use an induction argument for that purpose. First $\overline{u}_p^0$ is given for each $p \in \tau_h$. Next, we suppose that $\overline{u}_p^n, p \in \tau_h$ is known. By (2.1) we have, for each $p \in \tau_h$, that

$$\left(\frac{m\,(p)}{k} + \sum_{q \in N(p)} g_{pq}^{\sigma,n}(\overline{u}_{h,k}) T_{pq} \right) \overline{u}_p^{n+1} - \sum_{q \in N(p)} g_{pq}^{\sigma,n}(\overline{u}_{h,k}) T_{pq} \overline{u}_q^{n+1} =$$

$$= \frac{m\,(p)}{k} \overline{u}_p^n + f\left(\overline{u}_p^0 - \overline{u}_p^n \right) m\,(p)\,. \tag{2.8}$$

let us define

$$\mathcal{P} = card\,(\tau_h)$$

and function $\alpha : \tau_h \to \mathbf{N} \cap [1, \mathcal{P}]$ which numbers each volume p, i.e. $p \longmapsto \alpha\,(p)$. Then we can construct $\mathbb{R}^{\mathcal{P}\times\mathcal{P}}$-matrix $A = (A_{ij})_{\mathcal{P}\times\mathcal{P}}$ coming from (2.8) for which

$$A_{\alpha(p)\alpha(p)} = \frac{m\,(p)}{k} + \sum_{q \in N(p)} g_{pq}^{\sigma,n}(\overline{u}_{h,k}) T_{pq},$$

$$A_{\alpha(p)\alpha(q)} = -g_{pq}^{\sigma,n}(\overline{u}_{h,k}) T_{pq}\ , \quad q \in N\,(p)$$

and otherwise

$$A_{ij} = 0.$$

Let $\mathbb{R}^{\mathcal{P}}$-vector $\overline{U}^{n+1}$ correspond to the discrete solution, i.e.

$$\left(\overline{U}^{n+1} \right)_{\alpha(p)} = \overline{u}_p^{n+1}$$

and set $\mathbb{R}^{\mathcal{P}}$-vector $\overline{F}^{n+1}$ to

$$\left(\overline{F}^{n+1} \right)_{\alpha(p)} = \frac{m\,(p)}{k} \overline{u}_p^n + f\left(\overline{u}_p^0 - \overline{u}_p^n \right) m\,(p)\,.$$

Using these definitions we can rewrite (2.8) into linear system in the matrix form

$$A\overline{U}^{n+1} = \overline{F}^{n+1}. \tag{2.9}$$

Since the matrix A is symmetric and strictly diagonally dominant, there exists unique $\mathbb{R}^{\mathcal{P}}$ vector $\overline{U}^{n+1}$ satisfying (2.9) which in turn implies the existence and uniqueness of $\overline{u}_p^{n+1}, p \in \tau_h$.

Definition 2.2. *A weak solution of the regularized Perona-Malik problem (1.1)-(1.3) is a function* $u \in L^2\left(0, T; H^1(\Omega)\right)$ *satisfying the identity*

(2.10)
$$\int_0^T \int_\Omega u \frac{\partial \varphi}{\partial t}(x,t)\, dx dt + \int_\Omega u_0(x)\, \varphi(x,0)\, dx -$$
$$- \int_0^T \int_\Omega g(|\nabla G_\sigma * u|) \nabla u \nabla \varphi dx dt + \int_0^T \int_\Omega f(u_0 - u) \varphi(x,t)\, dx dt = 0$$

for all $\varphi \in \Phi$*, where the function space*

$$\Phi = \left\{ \varphi \in L^2(0,T;H^1(\Omega)), \frac{\partial \varphi}{\partial t} \in L^2(0,T;H^1(\Omega)'), \varphi(.,T) = 0 \right\}$$
(2.11)

and $\left(H^1(\Omega)\right)'$ *denotes the dual space of* $H^1(\Omega)$ *.*

Remark 2.4. In [5] Catté, Lions, Morel and Coll proved that there exists unique solution of (1.1)-(1.3) (with $f \equiv 0$) in the distributional sense which is also the classical solution of the problem at the same time. Their result can be simply adopted for Lipschitz continuous right hand side f. To get existence they used Schauder's fixed point theorem with iterations in entire parabolic equation. In the next section, we will find such solution in a computationally natural and efficient way using semi-implicit finite volume scheme.

3. Convergence of the scheme to the weak solution

3.1. $L^2(\Omega)$ *- a priori estimates*

Lemma 3.1. *The scheme* (2.1) *leads to the following estimates:*

There exists a positive constant C *such that*

(i) $\max\limits_{0 \le n \le N_{\max}} \sum\limits_{p \in \tau_h} \left(\overline{u}_p^n\right)^2 m(p) \le C$

(ii) $\sum\limits_{n=0}^{N_{\max}} k \sum\limits_{(p,q) \in \mathcal{E}} \frac{\left(\overline{u}_p^n - \overline{u}_q^n\right)^2}{d_{pq}} m\left(e_{pq}\right) \le C$

(iii) $\sum\limits_{n=0}^{N_{\max}-1} \sum\limits_{p \in \tau_h} \left(\overline{u}_p^{n+1} - \overline{u}_p^n\right)^2 m(p) \le C$

hold for every k *sufficiently small with a constant* C *which does not depend on* h, k*.*

Proof. Let us consider n such that $0 \leq n < N_{\max}$. We multiply the scheme (2.1) by $\overline{u}_p^{n+1} k$ to obtain

$$\left(\overline{u}_p^{n+1} - \overline{u}_p^{n}\right) \overline{u}_p^{n+1} m\left(p\right) = k \sum_{q \in N(p)} \left[g_{pq}^{\sigma,n}(\overline{u}_{h,k}) T_{pq} \left(\overline{u}_q^{n+1} - \overline{u}_p^{n+1}\right) \overline{u}_p^{n+1}\right] + kf\left(\overline{u}_p^{0} - \overline{u}_p^{n}\right) \overline{u}_p^{n+1} m\left(p\right). \tag{3.1}$$

Using the property $\left(a-b\right)a = \frac{1}{2}a^2 - \frac{1}{2}b^2 + \frac{1}{2}\left(a-b\right)^2$ on the left hand side of (3.1) and after summing over $p \in \tau_h$, we have that

$$\begin{aligned} \frac{1}{2} \sum_{p \in \tau_h} \left(\overline{u}_p^{n+1}\right)^2 m(p) - \frac{1}{2} \sum_{p \in \tau_h} \left(\overline{u}_p^{n}\right)^2 m(p) & \\ + \frac{1}{2} \sum_{p \in \tau_h} \left(\overline{u}_p^{n+1} - \overline{u}_p^{n}\right)^2 m(p) & \\ = k \sum_{p \in \tau_h} \sum_{q \in N(p)} \left[g_{pq}^{\sigma,n}(\overline{u}_{h,k}) T_{pq} \left(\overline{u}_q^{n+1} - \overline{u}_p^{n+1}\right) \overline{u}_p^{n+1}\right] & \\ + k \sum_{p \in \tau_h} f\left(\overline{u}_p^{0} - \overline{u}_p^{n}\right) \overline{u}_p^{n+1} m\left(p\right). & \end{aligned} \tag{3.2}$$

Since $g_{pq}^{\sigma,n}(\overline{u}_{h,k}) T_{pq} = g_{qp}^{\sigma,n}(\overline{u}_{h,k}) T_{qp}$ we can rearrange the summation of the first term of the right hand side of (3.2) to obtain

$$\sum_{p \in \tau_h} \sum_{q \in N(p)} \left[g_{pq}^{\sigma,n}(\overline{u}_{h,k}) T_{pq} \left(\overline{u}_q^{n+1} - \overline{u}_p^{n+1}\right) \overline{u}_p^{n+1}\right] = \tag{3.3}$$

$$= -\frac{1}{2} \sum_{(p,q) \in \mathcal{E}} \left[g_{pq}^{\sigma,n}(\overline{u}_{h,k}) T_{pq} \left(\overline{u}_p^{n+1} - \overline{u}_q^{n+1}\right)^2\right].$$

Applying (3.3) in (3.2) and after summing over $n = 0, ..., m-1 < N_{\max}$, we have

$$\frac{1}{2} \sum_{p \in \tau_h} \left(\overline{u}_p^{m}\right)^2 m(p) + \frac{1}{2} \sum_{n=0}^{m-1} \sum_{p \in \tau_h} \left(\overline{u}_p^{n+1} - \overline{u}_p^{n}\right)^2 m(p)$$

$$+ \frac{1}{2} \sum_{n=0}^{m-1} k \sum_{(p,q) \in \mathcal{E}} \left[g_{pq}^{\sigma,n}(\overline{u}_{h,k}) T_{pq} \left(\overline{u}_p^{n+1} - \overline{u}_q^{n+1}\right)^2\right]$$

$$= \frac{1}{2} \sum_{p \in \tau_h} \left(\overline{u}_p^{0}\right)^2 m(p) + \frac{1}{2} \sum_{n=0}^{m-1} k \sum_{p \in \tau_h} f\left(\overline{u}_p^{0} - \overline{u}_p^{n}\right) \overline{u}_p^{n+1} m\left(p\right).$$

Then we use Young's inequality and the Lipschitz continuity of f (by $K_f > 0$ we denote the Lipschitz constant of f) in the right hand side to obtain

$$\frac{1}{2}\sum_{p\in\tau_h}\left(\overline{u}_p^0\right)^2 m(p) + \sum_{n=0}^{m-1} k \sum_{p\in\tau_h} f\left(\overline{u}_p^0 - \overline{u}_p^n\right)\overline{u}_p^{n+1} m\left(p\right) \leq$$

$$\leq \frac{1}{2}\sum_{p\in\tau_h}\left(\overline{u}_p^0\right)^2 m(p) + \frac{K_f}{2}\sum_{n=0}^{m-1} k \sum_{p\in\tau_h}\left(\overline{u}_p^0\right)^2 m\left(p\right)$$

$$+\frac{K_f}{2}\sum_{n=0}^{m-1} k \sum_{p\in\tau_h}\left(\overline{u}_p^n\right)^2 m\left(p\right) + \frac{1}{2}\sum_{n=0}^{m-1} k \sum_{p\in\tau_h}\left(\overline{u}_p^{n+1}\right)^2 m\left(p\right).$$

Since $u_0 \in L^2\left(\Omega\right)$, there exists $C_1 > 0$ such that

$$\frac{1}{2}\sum_{p\in\tau_h}\left(\overline{u}_p^0\right)^2 m(p) \leq C_1, \tag{3.4}$$

and one can show that there exists a positive constant C_2 such that

$$\frac{1}{2}\sum_{p\in\tau_h}\left(\overline{u}_p^m\right)^2 m(p) + \frac{1}{2}\sum_{n=0}^{m-1}\sum_{p\in\tau_h}\left(\overline{u}_p^{n+1} - \overline{u}_p^n\right)^2 m(p)$$

$$+\frac{1}{2}\sum_{n=0}^{m-1} k \sum_{(p,q)\in\mathcal{E}}\left[g_{pq}^{\sigma,n}(\overline{u}_{h,k})T_{pq}\left(\overline{u}_p^{n+1} - \overline{u}_q^{n+1}\right)^2\right] \tag{3.5}$$

$$\leq C_1 + C_2\sum_{n=0}^{m} k \sum_{p\in\tau_h}\left(\overline{u}_p^n\right)^2 m\left(p\right).$$

Now, we can apply the discrete Gronwall lemma to state the result *(i)* of the lemma, i.e. there exists a positive constant C_3 such that

$$\sum_{p\in\tau_h}\left(\overline{u}_p^m\right)^2 m(p) \leq C_3 \text{ for all } 0 \leq m \leq N_{\max} \tag{3.6}$$

hold for every k sufficiently small where the constant C_3 does not depend on h, k. We get also

$$\left|\frac{\partial}{\partial x_i}G_\sigma * \tilde{u}_{h,k}(x_{pq}, t_n)\right| \leq \int_{\mathbb{R}^d}\left|\frac{\partial}{\partial x_i}G_\sigma(x_{pq} - \xi)\tilde{u}(\xi, t_n)\right| d\xi \leq \tag{3.7}$$

$$\leq \frac{1}{2}\int_{\mathbb{R}^d}\left|\frac{\partial}{\partial x_i}G_\sigma(x_{pq} - \xi)\right|^2 d\xi + \frac{1}{2}\int_{\mathbb{R}^d}|\tilde{u}(\xi, t_n)|^2 d\xi \leq C_\sigma$$

$$+C_4\int_{\Omega_\sigma}|\tilde{u}(\xi, t_n)|^2 d\xi \leq C_\sigma + C_4\sum_{p\in\tau_h}\left(\overline{u}_p^n\right)^2 m\left(p\right) \leq C_5$$

It comes from (3.7) that $|\nabla G_\sigma * \tilde{u}_{h,k}(x_{pq}, t_n)| < \infty$, which in turn implies that there exists a positive constant α such that

$$g_{pq}^{\sigma,n}(\overline{u}_{h,k}) > \alpha > 0. \tag{3.8}$$

Using (3.8) and (3.6) in (3.5) , one can deduce assertions *(ii)* and *(iii)* of the lemma.

3.2. Space and time translate estimates

In order to show relative compactness in $L^2(Q_T)$ of $(\overline{u}_{h,k})_{h,k}$ verifying (2.1) - (2.4), we need to establish the estimates of differences in space and in time for the set of discrete solutions.

Lemma 3.2. (Space translate estimate) *For all vector $\xi \in \mathbb{R}^d$, there exists a positive constant C such that*

$$\int\limits_{\Omega_\xi \times (0,T)} (\overline{u}_{h,k}(x+\xi,t) - \overline{u}_{h,k}(x,t))^2 \, dxdt \le C |\xi| (|\xi| + 2h) \tag{3.9}$$

where $\Omega_\xi = \{x \in \Omega, [x, x+\xi] \in \Omega\}$.

Proof. Let $\xi \in \mathbb{R}^d$ be a given vector. For all $(p,q) \in \mathcal{E}$, we denote by ξ_{pq} the following value $\xi_{pq} = \frac{\xi}{|\xi|}.n_{pq}$. For all $x \in \Omega_\xi$, we denote by $E(x,p,q)$ the function defined as follows

$$E(x,p,q) = \begin{cases} 1 \text{ if the segment } [x, x+\xi] \text{ intersects} \\ \quad e_{pq}, p \text{ and } q; \text{ and } \xi_{pq} > 0 \\ 0 \text{ otherwise.} \end{cases}$$

For any $t \in (0,T)$ there exists $n \in \mathbf{N}$ such that $(n-1)k < t \le nk$. Then for almost all $x \in \Omega_\xi$ we can see that

$$\overline{u}_{h,k}(x+\xi,t) - \overline{u}_{h,k}(x,t) = \overline{u}^n_{p(x+\xi)} - \overline{u}^n_{p(x)} = \sum_{(p,q)\in\mathcal{E}} E(x,p,q) \left(\overline{u}^n_q - \overline{u}^n_p\right) \tag{3.10}$$

where $p(x)$ is the volume $p \in \tau_h$ where $x \in p$. We introduce the term $\sqrt{\xi_{pq} d_{pq}}$ in (3.10) by multiplying and dividing by it the right hand side. Using the Cauchy-Schwartz inequality we obtain

$$(\overline{u}_{h,k}(x+\xi,t) - \overline{u}_{h,k}(x,t))^2 \le \tag{3.11}$$

$$\le \left(\sum_{(p,q)\in\mathcal{E}} E(x,p,q)\, \xi_{pq} d_{pq} \right) \left(\sum_{(p,q)\in\mathcal{E}} E(x,p,q) \frac{\left(\overline{u}^n_q - \overline{u}^n_p\right)^2}{\xi_{pq} d_{pq}} \right).$$

Using the fact that $\xi_{pq} d_{pq} = \frac{\xi}{|\xi|} n_{pq} d_{pq} = \frac{\xi}{|\xi|} (x_q - x_p)$, we have that

$$\sum_{(p,q)\in\mathcal{E}} E(x,p,q)\,\xi_{pq} d_{pq} = \frac{\xi}{|\xi|}\left(x_{p(x+\xi)} - x_{p(x)}\right). \tag{3.12}$$

In order to bound the difference between the two volume center points in the right hand side of (3.12), we add and subtract $x+\xi$ to obtain

$$\left|x_{p(x+\xi)} - x_{p(x)}\right| = \left|\left(x_{p(x+\xi)} - (x+\xi)\right) - \left(x_{p(x)} - x\right) + \xi\right| \le$$

$$\le \left|x_{p(x+\xi)} - (x+\xi)\right| + \left|x_{p(x)} - x\right| + |\xi| \le 2h + |\xi|$$

since $(x+\xi) \in p(x+\xi)$ and $x \in p(x)$. This result implies that

$$\sum_{(p,q)\in\mathcal{E}} E(x,p,q)\,\xi_{pq} d_{pq} \le 2h + |\xi|. \tag{3.13}$$

Now, we integrate the relation (3.11) on $\Omega_\xi \times (0,T)$ and use (3.13) to obtain

$$\int_{\Omega_\xi\times(0,T)} \left(\overline{u}_{h,k}(x+\xi,t) - \overline{u}_{h,k}(x,t)\right)^2 dxdt$$

$$\le (2h+|\xi|)\sum_{n=0}^{N_{\max}} k \sum_{(p,q)\in\mathcal{E}} \frac{\left(\overline{u}_q^n - \overline{u}_p^n\right)^2}{\xi_{pq} d_{pq}} \int_{\Omega_\xi} E(x,p,q)\,dx \tag{3.14}$$

since $\overline{u}_{h,k}$ is piecewise constant for each interval $(nk,(n+1)k)$. By the geometrical argument given in [7], we have that

$$\int_{\Omega_\xi} E(x,p,q)\,dx \le m(e_{pq})\,|\xi|\,\xi_{pq}$$

and applying this result in (3.14) we obtain

$$\int_{\Omega_\xi\times(0,T)} \left(\overline{u}_{h,k}(x+\xi,t) - \overline{u}_{h,k}(x,t)\right)^2 dxdt$$

$$\le (2h+|\xi|)\,|\xi| \sum_{n=0}^{N_{\max}} k \sum_{(p,q)\in\mathcal{E}} \frac{\left(\overline{u}_q^n - \overline{u}_p^n\right)^2}{d_{pq}} m(e_{pq}) \;. \tag{3.15}$$

Finally, using the discrete a priori estimate *(ii)* of Lemma 3.1 we end the proof.

Lemma 3.3. (Time translate estimate) *There exists a positive constant* C *such that*

$$\int\limits_{\Omega\times(0,T-s)} \left(\overline{u}_{h,k}\left(x,t+s\right)-\overline{u}_{h,k}\left(x,t\right)\right)^2 dxdt \leq Cs$$

for all $s \in (0,T)$ *.*

Proof. Let $s \in (0,T)$ be a given number. Let us define the following functions of time t

$$A(t) = \int\limits_{\Omega} \left(\overline{u}_{h,k}\left(x,t+s\right)-\overline{u}_{h,k}\left(x,t\right)\right)^2 dxdt,$$

$$n_t = \left\lceil \frac{t}{k} \right\rceil \text{ and } n_{t+s} = \left\lceil \frac{t+s}{k} \right\rceil,$$

where $\lceil . \rceil$ means the upper integer part of positive real number. Since $\overline{u}_{h,k}$ is piecewise constant function we have that

$$A(t) = \sum_{p\in\tau_h} \left(\overline{u}_p^{n_{t+s}} - \overline{u}_p^{n_t}\right)^2 m\left(p\right)$$

which can be written as

$$A(t) = \sum_{p\in\tau_h} \left(\overline{u}_p^{n_{t+s}} - \overline{u}_p^{n_t}\right) \sum_{t\leq nk<t+s} \left(\overline{u}_p^{n+1} - \overline{u}_p^{n}\right) m\left(p\right). \tag{3.16}$$

We use the approximation scheme (2.1) in (3.16) to have

$$A(t) = \sum_{t\leq nk<t+s} k \sum_{p\in\tau_h} \Bigg(\left(\overline{u}_p^{n_{t+s}} - \overline{u}_p^{n_t}\right) \times \\ \times \sum_{q\in N(p)} \left[g_{pq}^{\sigma,n}\left(\overline{u}_{h,k}\right) T_{pq} \left(\overline{u}_q^{n+1} - \overline{u}_p^{n+1}\right)\right] \Bigg) + \\ + \sum_{t\leq nk<t+s} k \sum_{p\in\tau_h} \left(\left(\overline{u}_p^{n_{t+s}} - \overline{u}_p^{n_t}\right) f\left(\overline{u}_p^0 - \overline{u}_p^n\right) m\left(p\right)\right)$$

which after rearranging of the sum concerning the control volume variable leads to the relation

$$A(t) = \sum_{t\leq nk<t+s} \frac{k}{2} \sum_{(p,q)\in\mathcal{E}} \Bigg(\left(\overline{u}_p^{n_{t+s}} - \overline{u}_p^{n_t} - \overline{u}_q^{n_{t+s}} + \overline{u}_q^{n_t}\right) \times \\ \times g_{pq}^{\sigma,n}\left(\overline{u}_{h,k}\right) T_{pq} \left(\overline{u}_q^{n+1} - \overline{u}_p^{n+1}\right) \Bigg) + \\ + \sum_{t\leq nk<t+s} k \sum_{p\in\tau_h} \left(\left(\overline{u}_p^{n_{t+s}} - \overline{u}_p^{n_t}\right) f\left(\overline{u}_p^0 - \overline{u}_p^n\right) m\left(p\right)\right) \tag{3.17}$$

Applying Young's inequality in terms of the previous expresion and using the relation $\frac{1}{2}\left(a+b\right)^2 \leq a^2 + b^2$ yield

$$A(t) \leq \frac{1}{2}A_0(t) + \frac{1}{2}A_1(t) + \frac{1}{4}A_2(t) + A_3(t) \tag{3.18}$$

where

$$A_0(t) = \sum_{t \leq nk < t+s} k \sum_{(p,q)\in\mathcal{E}} T_{pq}\left(\overline{u}_q^{n_t} - \overline{u}_p^{n_t}\right)^2, \tag{3.19}$$

$$A_1(t) = \sum_{t \leq nk < t+s} k \sum_{(p,q)\in\mathcal{E}} T_{pq}\left(\overline{u}_q^{n_{t+s}} - \overline{u}_p^{n_{t+s}}\right)^2, \tag{3.20}$$

$$A_2(t) = \sum_{t \leq nk < t+s} k \sum_{(p,q)\in\mathcal{E}} \left(g_{pq}^{\sigma,n}\left(\overline{u}_{h,k}\right)\right)^2 T_{pq}\left(\overline{u}_q^{n+1} - \overline{u}_p^{n+1}\right)^2, \tag{3.21}$$

$$A_3(t) = \sum_{t \leq nk < t+s} k \sum_{p\in\tau_h} \left(\left(\overline{u}_p^{n_{t+s}} - \overline{u}_p^{n_t}\right) f\left(\overline{u}_p^0 - \overline{u}_p^n\right) m\left(p\right)\right). \tag{3.22}$$

Now, we integrate (3.18) according to the time variable to obtain

$$\int_0^{T-s} A(t)dt \leq \frac{1}{2}\int_0^{T-s} A_0(t)dt + \frac{1}{2}\int_0^{T-s} A_1(t)dt + + \frac{1}{4}\int_0^{T-s} A_2(t)dt + \int_0^{T-s} A_3(t)dt. \tag{3.23}$$

Next step is to give a bound for each term on the right hand side of (3.23). We begin with A_0 term.

$$\int_0^{T-s} A_0(t)dt = \int_0^{T-s} \sum_{t \leq nk < t+s} k \sum_{(p,q)\in\mathcal{E}} T_{pq}\left(\overline{u}_q^{n_t} - \overline{u}_p^{n_t}\right)^2 dt =$$

$$= \int_0^{T-s} k \sum_{(p,q)\in\mathcal{E}} T_{pq}\left(\overline{u}_q^{n_t} - \overline{u}_p^{n_t}\right)^2 \sum_{n\in\mathbf{N}} \chi_{\{t \leq nk < t+s\}} dt.$$

It is clear that

$$\chi_{\{t \leq nk < t+s\}} = \chi_{\{nk-s < t \leq nk\}}. \tag{3.24}$$

Then we split the integration over $(0, T-s)$ into a sum of time step intervals to have that

$$\int_0^{T-s} A_0(t)dt \le \sum_{n_t=0}^{N_{\max}-1} k \sum_{(p,q)\in\mathcal{E}} T_{pq} \left(\overline{u}_q^{n_t} - \overline{u}_p^{n_t}\right)^2 \times$$
$$\times \int_{n_t k}^{(n_t+1)k} \sum_{n\in\mathbf{N}} \chi_{\{nk-s<t\le nk\}} dt$$

since n_t depends on t and the integrated function is positive. But we have that

$$J = \int_{n_t k}^{(n_t+1)k} \sum_{n\in\mathbf{N}} \chi_{\{nk-s<t\le nk\}} dt = \sum_{n\in\mathbf{N}} \int_{n_t k}^{(n_t+1)k} \chi_{\{nk-s<t\le nk\}} dt \tag{3.25}$$

and if we change the variable to $w = t - nk + s$, we have that

$$J = \sum_{n\in\mathbf{N}} \int_{n_t k - nk + s}^{(n_t+1)k - nk + s} \chi_{\{0<w\le s\}} dw = s \tag{3.26}$$

Then it yields that

$$\int_0^{T-s} A_0(t)dt \le s \sum_{n_t=0}^{N_{\max}} k \sum_{(p,q)\in\mathcal{E}} T_{pq} \left(\overline{u}_q^{n_t} - \overline{u}_p^{n_t}\right)^2 . \tag{3.27}$$

Applying the estimate *(ii)* of Lemma 3.1 in (3.27) gives

$$\int_0^{T-s} A_0(t)dt \le Cs. \tag{3.28}$$

Similarly, only changing n_t into n_{t+s}, one can show that

$$\int_0^{T-s} A_1(t)dt \le Cs. \tag{3.29}$$

Using the definition (3.21) we have that

$$\int_0^{T-s} A_2(t)dt = \int_0^{T-s} \sum_{n\in\mathbf{N}} k \sum_{(p,q)\in\mathcal{E}} \left(g_{pq}^{\sigma,n}\left(\overline{u}_{h,k}\right)\right)^2 \times$$
$$\times T_{pq} \left(\overline{u}_q^{n+1} - \overline{u}_p^{n+1}\right)^2 \chi_{\{t\le nk<t+s\}} dt.$$

Since t varies over $(0, T-s)$, we can restrict the summation only to $n = 0, ..., N_{\max} - 1$ and applying (3.24) we have

$$\int_0^{T-s} A_2(t)dt \le \tag{3.30}$$

$$\sum_{n=0}^{N_{\max}-1} k \sum_{(p,q)\in\mathcal{E}} \left(g_{pq}^{\sigma,n}\left(\overline{u}_{h,k}\right)\right)^2 T_{pq}\left(\overline{u}_q^{n+1} - \overline{u}_p^{n+1}\right)^2 \int_0^{T-s} \chi_{\{nk-s<t\le nk\}} dt.$$

However, one can show that for all $0 \le n \le N_{\max} - 1$ holds

$$\int_0^{T-s} \chi_{\{nk-s<t\le nk\}} dt = \min\left(T-s, nk\right) - \max\left(0, nk-s\right) \le s. \tag{3.31}$$

Applying the estimate *(ii)* of Lemma 3.1 and (3.31) in (3.30) yield

$$\int_0^{T-s} A_2(t)dt \le Cs. \tag{3.32}$$

For the term $A_3(t)$ we use the assumptions for f stated in (1.6). Using Young's inequality and the relation $\frac{1}{2}\left(a-b\right)^2 \le a^2 + b^2$ in the Definition (3.22) we have

$$A_3(t) \le \sum_{t\le nk<t+s} k \sum_{p\in\tau_h} \Bigg(2\left(\overline{u}_p^{n_{t+s}}\right)^2 + \\ +2\left(\overline{u}_p^{n_t}\right)^2 + \left(f\left(\overline{u}_p^0 - \overline{u}_p^n\right)\right)^2 \Bigg) m\left(p\right). \tag{3.33}$$

Since f is Lipschitz continuous and $f\left(0\right) = 0$ we can deduce

$$A_3(t) \le 2 \sum_{t\le nk<t+s} k \sum_{p\in\tau_h} \Bigg(\left(\overline{u}_p^{n_{t+s}}\right)^2 + \left(\overline{u}_p^{n_t}\right)^2 + \\ + \left(K_f \overline{u}_p^0\right)^2 + \left(K_f \overline{u}_p^n\right)^2 \Bigg) m\left(p\right). \tag{3.34}$$

Now we integrate it over $(0, T-s)$ in order to obtain

$$\int_0^{T-s} A_3(t)dt \le 2B_1 + 2B_2 + 2K_f B_3 + 2K_f B_4 \tag{3.35}$$

where B_i, $i = 1, ..., 4$, correspond to

$$B_1 = \int_0^{T-s} \sum_{t \le nk < t+s} k \sum_{p \in \tau_h} \left(\overline{u}_p^{n_{t+s}}\right)^2 m(p)\, dt, \tag{3.36}$$

$$B_2 = \int_0^{T-s} \sum_{t \le nk < t+s} k \sum_{p \in \tau_h} \left(\overline{u}_p^{n_t}\right)^2 m(p)\, dt, \tag{3.37}$$

$$B_3 = \int_0^{T-s} \sum_{t \le nk < t+s} k \sum_{p \in \tau_h} \left(\overline{u}_p^{0}\right)^2 m(p)\, dt, \tag{3.38}$$

$$B_4 = \int_0^{T-s} \sum_{t \le nk < t+s} k \sum_{p \in \tau_h} \left(\overline{u}_p^{n}\right)^2 m(p)\, dt. \tag{3.39}$$

We use the same argument as in the estimate of $A_0(t)$ to state that

$$B_1 \le \sum_{n_{t+s}=0}^{N_{\max}-1} k \sum_{p \in \tau_h} \left(\overline{u}_p^{n_{t+s}}\right)^2 m(p) \int_{n_{t+s}k}^{(n_{t+s}+1)k} \sum_{n \in \mathbf{N}} \chi_{\{nk-s<t\le nk\}} dt.$$

The identities (3.25), (3.26) and the estimate *(i)* of Lemma 3.1 imply that

$$B_1 \le CTs \tag{3.40}$$

and, similarly,

$$B_2 \le CTs. \tag{3.41}$$

In order to give an upper bound of B_3, one can use the estimates (3.4) and (3.24) to have

$$B_3 \le C_1 \sum_{n=0}^{N_{\max}-1} k \int_0^{T-s} \chi_{\{nk-s<t\le nk\}} dt \tag{3.42}$$

which together with (3.31) implies that

$$B_3 \le C_1 Ts. \tag{3.43}$$

In order to estimate the last term B_4, we use that

$$B_4 \le \sum_{n=0}^{N_{\max}-1} k \sum_{p \in \tau_h} \left(\overline{u}_p^{n}\right)^2 m(p) \int_0^{T-s} \chi_{\{nk-s<t\le nk\}} dt \tag{3.44}$$

which together with the estimate *(i)* of Lemma 3.1 and (3.31) leads to

$$B_4 \le CTs. \tag{3.45}$$

Thus, using (3.40), (3.41), (3.43) and (3.45), we can deduce that

$$\int_0^{T-s} A_3(t)dt \le 2T\left(2C + K_f\left(C_1 + C\right)\right)s. \tag{3.46}$$

Finally, applying (3.28), (3.29), (3.32) and (3.46) in (3.23) we have proved the lemma.

Let us define the set

$$\mathcal{E}_{ext} = \left\{\kappa,\ \text{such that there exists } p \in \tau_h,\ \kappa \subset \partial p \cap \partial\Omega\right\},$$

and let

$$\overline{u}_\kappa := \overline{u}_p \text{ where } p \in \tau_h,\ \kappa \subset \partial p \cap \partial\Omega.$$

The following lemma represents the so called **trace inequality** given in [6]:

Lemma 3.4. *Let Ω be an open bounded polygonal connected subset of $\mathbb{R}^d$. Let $\overline{\gamma}\left(\overline{u}_{h,k}\right)$ be defined by $\overline{\gamma}\left(\overline{u}_{h,k}\right) = \overline{u}_\kappa$ a.e. for the $(d-1)-$Lebesgue measure on $\kappa \in \mathcal{E}_{ext}$. Then there exists positive C, depending only on Ω, such that*

$$\left\|\overline{\gamma}\left(\overline{u}_{h,k}\right)\right\|_{L^2(\partial\Omega)} \le C\left(\left|\overline{u}_{h,k}\right|_{1,\tau_h} + \left\|\overline{u}_{h,k}\right\|_{L^2(\Omega)}\right)$$

where

$$\left|\overline{u}_{h,k}\right|_{1,\tau_h} = \left(\sum_{(p,q)\in\mathcal{E}} \frac{\left(\overline{u}_q^n - \overline{u}_p^n\right)^2}{d_{pq}} m\left(e_{pq}\right)\right)^{\frac{1}{2}}.$$

Lemma 3.5. (Convergence of $\overline{u}_{h,k}$) *There exists $u \in L^2\left(Q_T\right)$ such that for some subsequence of $\overline{u}_{h,k}$*

$$\overline{u}_{h,k} \to u \text{ in } L^2\left(Q_T\right)$$

as $h, k \to 0$.

Proof. From the estimate *(i)* of Lemma 3.1. we have that $\left\|\overline{u}_{h,k}\right\|_{L^2(Q_T)} \le C$ and we have proved the time and space translate estimates given in Lemmas 3.2 and 3.3. In order to use Kolmogorov's compactness criterion [4], Theorem IV.25), it will be sufficient to prove that

$$K = \int_{\Omega\times(0,T)} \left(\overline{u}_{h,k}\left(x+\xi,t\right) - \overline{u}_{h,k}\left(x,t\right)\right)^2 dxdt \le C\left|\xi\right|. \tag{3.47}$$

In fact, we can write

$$\overline{u}_{h,k}(x+\xi,t)-\overline{u}_{h,k}(x,t)=$$

$$=\sum_{(p,q)\in\mathcal{E}}E(x,p,q)\left(\overline{u}_q^n-\overline{u}_p^n\right)+\sum_{\kappa\in\mathcal{E}_{ext}}\chi\left([x,x+\xi]\cap\kappa\right)\overline{u}_\kappa^n$$

and thus

$$\left(\overline{u}_{h,k}(x+\xi,t)-\overline{u}_{h,k}(x,t)\right)^2\leq$$

$$\leq 2\left(\sum_{(p,q)\in\mathcal{E}}E(x,p,q)\,\xi_{pq}d_{pq}\right)\left(\sum_{(p,q)\in\mathcal{E}}E(x,p,q)\,\frac{\left(\overline{u}_q^n-\overline{u}_p^n\right)^2}{\xi_{pq}d_{pq}}\right)$$

$$+2\sum_{\kappa\in\mathcal{E}_{ext}}\chi\left([x,x+\xi]\cap\kappa\right)\left(\overline{u}_\kappa^n\right)^2.$$

Using the same technique as in the proof of Lemma 3.2 one obtains that

$$K\leq(2h+|\xi|)\,|\xi|\,C+2\sum_{n=0}^{N_{\max}}k\int_{\Omega}\sum_{\kappa\in\mathcal{E}_{ext}}\chi\left([x,x+\xi]\cap\kappa\right)\left(\overline{u}_\kappa^n\right)^2dxdt$$

from where

$$K\leq(2h+|\xi|)\,|\xi|\,C+2\,|\xi|\sum_{n=0}^{N_{\max}}k\sum_{\kappa\in\mathcal{E}_{ext}}\left(\overline{u}_\kappa^n\right)^2m(\kappa)$$

which in turn gives

$$K\leq(2h+|\xi|)\,|\xi|\,C+2\,|\xi|\sum_{n=0}^{N_{\max}}k\left\|\overline{\gamma}\left(\overline{u}_{h,k}\right)\right\|_{L^2(\partial\Omega)}.$$

Now, using Lemma 3.4 we have

$$K\leq(2h+|\xi|)\,|\xi|\,C+2C\,|\xi|\sum_{n=0}^{N_{\max}}k\left(\left|\overline{u}_{h,k}\right|_{1,\tau_h}+\left\|\overline{u}_{h,k}\right\|_{L^2(\Omega)}\right)$$

Then, the a-priori estimates of Lemma 3.1 give us that there exists $C>0$ such that (3.47) holds true. Then $\overline{u}_{h,k}$ is relatively compact in $L^2(Q_T)$. This implies that there is a subsequence of $\overline{u}_{h,k}$ converging to a limit u in $L^2(Q_T)$.

3.3. Convergence of the discrete solution to the weak solution

In this section we consider the subsequence $\overline{u}_{h_m,k_m}$ of $\overline{u}_{h,k}$ that converges to u when $h_m, k_m \to 0$ (see Lemma 3.5). Next step is to prove that u is the weak solution of (1.1)-(1.3). For the sake of simplicity, we still call this subsequence $\overline{u}_{h,k}$. First let us define the set of functions

$$\Psi = \left\{\varphi \in C^{2,1}\left(\overline{\Omega} \times [0,T]\right), \nabla\varphi.\overrightarrow{n} = 0 \textit{ on } \partial\Omega \times (0,T), \varphi(.,T) = 0\right\}$$

which is dense in the set Φ defined in (2.11).

Let $\varphi \in \Psi$ be given. In order to have a discrete analogy of the weak solution identity (2.10), we multiply the scheme (2.1) by $\varphi(x_p,t_n)\,k$. Then we sum the resulting identity over all $p \in \tau_h$ and $n = 0, ..., N_{\max} - 1$. It yields

$$\begin{aligned} &\sum_{n=0}^{N_{\max}-1} k \sum_{p\in\tau_h} \frac{\left(\overline{u}_p^{n+1} - \overline{u}_p^n\right)}{k} \varphi(x_p,t_n)\, m(p) \\ = &\sum_{n=0}^{N_{\max}-1} k \sum_{p\in\tau_h} \varphi(x_p,t_n) \sum_{q\in N(p)} \left[g_{pq}^{\sigma,n}(\overline{u}_{h,k}) T_{pq}\left(\overline{u}_q^{n+1} - \overline{u}_p^{n+1}\right)\right] \\ &+ \sum_{n=0}^{N_{\max}-1} k \sum_{p\in\tau_h} \varphi(x_p,t_n)\, f\left(\overline{u}_p^0 - \overline{u}_p^n\right) m(p). \end{aligned} \tag{3.48}$$

Next we make a discrete integration by part of each term of the relation (3.48) in order to approach the weak solution form. Then we transfom the term on the left hand side by rearranging the summation over n, i.e., by putting the time difference in φ instead of in $\overline{u}$. We also take into account the fact that $\varphi(x_p,T) = 0$ for all $p \in \tau_h$. We obtain

$$\begin{aligned} &\sum_{n=0}^{N_{\max}-1} k \sum_{p\in\tau_h} \frac{\left(\overline{u}_p^{n+1} - \overline{u}_p^n\right)}{k} \varphi(x_p,t_n)\, m(p) \\ &= -\sum_{n=1}^{N_{\max}} k \sum_{p\in\tau_h} \overline{u}_p^n \frac{\varphi(x_p,t_n) - \varphi(x_p,t_{n-1})}{k} m(p) \\ &\quad - \sum_{p\in\tau_h} \overline{u}_p^0 \varphi(x_p,0)\, m(p). \end{aligned} \tag{3.49}$$

For the first term of the right hand side of (3.48), we gather the sum over $p \in \tau_h$ and over $q \in N(p)$ to have

$$\sum_{n=0}^{N_{\max}-1} k \sum_{p\in\tau_h} \varphi(x_p,t_n) \sum_{q\in N(p)} \left[g_{pq}^{\sigma,n}(\overline{u}_{h,k})\, T_{pq}\left(\overline{u}_q^{n+1} - \overline{u}_p^{n+1}\right)\right]$$

$$= -\frac{1}{2} \sum_{n=0}^{N_{\max}-1} k \sum_{(p,q)\in\mathcal{E}} g_{pq}^{\sigma,n}(\overline{u}_{h,k}) T_{pq} \left(\overline{u}_q^{n+1} - \overline{u}_p^{n+1}\right) \times$$

$$\times \left(\varphi(x_q,t_n) - \varphi(x_p,t_n)\right) \tag{3.50}$$

since $g_{pq}^{\sigma,n}(\overline{u}_{h,k}) T_{pq} = g_{qp}^{\sigma,n}(\overline{u}_{h,k}) T_{qp}$. Then we can write the scheme in its discrete weak form analogous to the identity (2.10), i.e.

$$\sum_{n=1}^{N_{\max}} k \sum_{p\in\tau_h} \overline{u}_p^n \frac{\varphi(x_p,t_n) - \varphi(x_p,t_{n-1})}{k} m(p) + \sum_{p\in\tau_h} \overline{u}_p^0 \varphi(x_p,0) m(p) -$$

$$-\frac{1}{2} \sum_{n=0}^{N_{\max}-1} k \sum_{(p,q)\in\mathcal{E}} g_{pq}^{\sigma,n}(\overline{u}_{h,k}) T_{pq} \left(\overline{u}_q^{n+1} - \overline{u}_p^{n+1}\right) \times$$

$$\times \left(\varphi(x_q,t_n) - \varphi(x_p,t_n)\right) \tag{3.51}$$

$$+ \sum_{n=0}^{N_{\max}-1} k \sum_{p\in\tau_h} \varphi(x_p,t_n) f\left(\overline{u}_p^0 - \overline{u}_p^n\right) m(p) = 0.$$

In the sequel, we prove the convergence of each term of (3.51) to its continuous analogy in (2.10) for all test functions $\varphi \in \Psi$.

Lemma 3.6. *We have that*

$$\sum_{n=1}^{N_{\max}} k \sum_{p\in\tau_h} \overline{u}_p^n \frac{\varphi(x_p,t_n) - \varphi(x_p,t_{n-1})}{k} m(p) \to \int_0^T \int_\Omega u \frac{\partial\varphi}{\partial t}(x,t)\, dxdt$$

as $h,k \to 0$ *for all* $\varphi \in \Psi$.

Proof. Let $\varphi \in \Psi$ be given. Then we define the difference of discrete and continuous terms of lemma by

$$T_1 = \sum_{n=1}^{N_{\max}} \sum_{p\in\tau_h} \left[\overline{u}_p^n \left(\varphi(x_p,t_n) - \varphi(x_p,t_{n-1})\right) m(p) - \right.$$

$$\left. - \int_{t_{n-1}}^{t_n} \int_p u \frac{\partial\varphi}{\partial t}(x,t)\, dxdt \right].$$

We add and substract $\overline{u}_p^n \int_{t_{n-1}}^{t_n} \int_p \frac{\partial\varphi}{\partial t}(x,t)\, dxdt$ in the summation to obtain

$$|T_1| \le \left| \sum_{n=1}^{N_{\max}} \sum_{p\in\tau_h} \overline{u}_p^n \int_{t_{n-1}}^{t_n} \int_p \left(\frac{\partial\varphi}{\partial t}(x_p,t) - \frac{\partial\varphi}{\partial t}(x,t) \right) dxdt \right| + \tag{3.52}$$

$$+\left|\int_0^T\int_\Omega(\overline{u}_{h,k}-u)\frac{\partial\varphi}{\partial t}(x,t)\,dxdt\right|.$$

To estimate the first term of the right hand side of (3.52) we apply the Cauchy-Schwartz inequality in order to separate $\overline{u}_p^n$ and to have its discrete $L^2(Q_T)$ norm. Then using the fact that

$$\left(\frac{\partial\varphi}{\partial t}(x_p,t)-\frac{\partial\varphi}{\partial t}(x,t)\right)\leq hM$$

since $\varphi\in C^{2,1}\left(\overline{\Omega}\times[0,T]\right)$, we can deduce that this term tends to 0 as $h,k\to 0$. From Lemma 3.5, which gives a strong convergence, $\overline{u}_{h,k}$ to u *in* $L^2(Q_T)$, we have the same assertion also for the second term on the right hand side of (3.52). Thus we proved that $|T_1|\to 0$ when $h,k\to 0$.

Lemma 3.7. *For a given u_0 and for $\overline{u}_p^0$ as defined in (2.2) we have that*

$$\sum_{p\in\tau_h}\overline{u}_p^0\varphi(x_p,0)\,m(p)\to\int_\Omega u_0(x)\,\varphi(x,0)\,dx$$

as $h,k\to 0$ for all $\varphi\in\Psi$.

Proof. Using (2.2) we have that

$$\sum_{p\in\tau_h}\overline{u}_p^0\varphi(x_p,0)\,m(p)-\int_\Omega u_0(x)\,\varphi(x,0)\,dx=$$
$$=\sum_{p\in\tau_h}\int_p(\varphi(x_p,0)-\varphi(x,0))\,u_0(x)\,dx$$

Since $\varphi\in C^{2,1}\left(\overline{\Omega}\times[0,T]\right)$, there exists a positive constant M_2 such that

$$|\varphi(x_p,0)-\varphi(x,0)|\leq hM_2$$

and one can deduce the assertion of the lemma.

Using the definition of Ψ, we apply the Green formula to obtain

$$\int_\Omega g(|\nabla G_\sigma*u|)\nabla u\nabla\varphi dx=-\int_\Omega div\left(g(|\nabla G_\sigma*u|)\nabla\varphi\right)\,udx, \tag{3.53}$$

which will be used in the sequel for the convergence proof.

Lemma 3.8. *We have that*

$$\frac{1}{2}\sum_{n=0}^{N_{\max}-1} k \sum_{(p,q)\in\mathcal{E}} g_{pq}^{\sigma,n}\left(\overline{u}_{h,k}\right) T_{pq}\left(\overline{u}_q^{n+1}-\overline{u}_p^{n+1}\right) \times$$

$$\times\left(\varphi\left(x_q,t_n\right)-\varphi\left(x_p,t_n\right)\right) \rightarrow \int_0^T\int_\Omega div\left(g(|\nabla G_\sigma * u|)\nabla\varphi\right) u dx dt$$

as $h,k\rightarrow 0$ *for all* $\varphi\in\Psi$.

Proof. We consider that

$$\frac{1}{2}\sum_{n=0}^{N_{\max}-1} k \sum_{(p,q)\in\mathcal{E}} g_{pq}^{\sigma,n}\left(\overline{u}_{h,k}\right) T_{pq}\left(\overline{u}_q^{n+1}-\overline{u}_p^{n+1}\right)\left(\varphi\left(x_q,t_n\right)-\varphi\left(x_p,t_n\right)\right)$$

$$-\int_0^T\int_\Omega div\left(g(|\nabla G_\sigma * u|)\nabla\varphi\right) u dx dt = \sum_{i=1}^{5} R_i$$

where $R_i, i=1,...,5$ comes from the splitting of the left hand side difference into several parts adding and substracting some extra terms and which will be defined and estimated in the sequel. First, we define

$$R_1=\frac{1}{2}\sum_{n=0}^{N_{\max}-1} k \sum_{(p,q)\in\mathcal{E}}\left(\overline{u}_q^{n+1}-\overline{u}_p^{n+1}\right) g_{pq}^{\sigma,n}\left(\overline{u}_{h,k}\right) R_{pq}^n m\left(e_{pq}\right), \tag{3.54}$$

where R_{pq}^n represents the difference between discrete and continuous normal derivative evaluated on (x_{pq},t_n), i.e.

$$R_{pq}^n=\left(\frac{\varphi\left(x_q,t_n\right)-\varphi\left(x_p,t_n\right)}{d_{pq}}\right)-\nabla\varphi\left(x_{pq},t_n\right) n_{pq}. \tag{3.55}$$

Since $\varphi\in C^{2,1}\left(\overline{\Omega}\times[0,T]\right)$, one can show that there exists a constant $M_3>0$ such that

$$\left|R_{pq}^n\right|\leq hM_3\ . \tag{3.56}$$

Then, using (1.4) we have

$$|R_1|\leq h\frac{M_3}{2}\sum_{n=0}^{N_{\max}-1} k \sum_{(p,q)\in\mathcal{E}}\left|\overline{u}_q^{n+1}-\overline{u}_p^{n+1}\right| m\left(e_{pq}\right). \tag{3.57}$$

We multiply and divide the right hand side of (3.57) by $\sqrt{d_{pq}}$ and apply Cauchy-Schwartz inequality to obtain

$$|R_1| \leq h\frac{M_3}{2}\left(\sum_{n=0}^{N_{\max}-1} k \sum_{(p,q)\in\mathcal{E}} T_{pq}\left(\overline{u}_q^{n+1}-\overline{u}_p^{n+1}\right)^2\right)^{\frac{1}{2}} \times$$

$$\times\left(\sum_{n=0}^{N_{\max}-1} k \sum_{(p,q)\in\mathcal{E}} d_{pq} m\left(e_{pq}\right)\right)^{\frac{1}{2}}. \tag{3.58}$$

It comes from geometrical arguments that there exists a positive constant M_4 such that

$$\sum_{(p,q)\in\mathcal{E}} d_{pq} m\left(e_{pq}\right) \leq M_4\left|\Omega\right|. \tag{3.59}$$

The estimate *(ii)* of Lemma 3.1 combined with (3.59) implies that

$$|R_1| \leq h\frac{M_3\left(M_4\left|\Omega\right| TC\right)^{\frac{1}{2}}}{2}$$

and one can conclude that

$$|R_1| \to 0 \text{ as } h, k \to 0. \tag{3.60}$$

Next, we set

$$R_2 = \frac{1}{2}\sum_{n=0}^{N_{\max}-1}\sum_{(p,q)\in\mathcal{E}}\left(\overline{u}_q^{n+1}-\overline{u}_p^{n+1}\right) g_{pq}^{\sigma,n}\left(\overline{u}_{h,k}\right)\int_{t_n}^{t_{n+1}}\int_{e_{pq}} \overline{R}_{pq}^n dx dt \tag{3.61}$$

where

$$\overline{R}_{pq}^n = \left(\nabla\varphi\left(x_{pq}, t_n\right) - \nabla\varphi\left(x, t\right)\right) n_{pq}\,. \tag{3.62}$$

Thanks to the regularity of φ, one can show that for any $x \in e_{pq}$ holds

$$\left|\overline{R}_{pq}^n\right| \leq \left(h+k\right) M_5$$

with a positive constant M_5 depending only on φ. We apply this result to replace $\overline{R}_{pq}^n$ in (3.61) and by the same argument as in estimating of the term R_1 we derive that

$$|R_2| \to 0 \text{ as } h, k \to 0. \tag{3.63}$$

Now, we denote

$$G_{pq}^n = g(|\nabla G_\sigma * \tilde{u}_{h,k}\left(x_{pq}, t_n\right)|) - g(|\nabla G_\sigma * \tilde{u}_{h,k}\left(x, t\right)|) \tag{3.64}$$

and define the third term

$$(3.65)\quad R_3 = \frac{1}{2} \sum_{n=0}^{N_{\max}-1} \sum_{(p,q)\in\mathcal{E}} \left(\overline{u}_q^{n+1} - \overline{u}_p^{n+1}\right) \int_{t_n}^{t_{n+1}} \int_{e_{pq}} G_{pq}^n \nabla\varphi\, n_{pq} dx dt\ .$$

To prove the convergence of R_3 to 0, first we bound G_{pq}^n. For that purpose, we use the fact that g is Lipschitz continuous. Let L_g be the Lipschitz constant of g, i.e., for any positive real numbers ζ_1 and ζ_2 hold

$$(3.66)\qquad |g(\zeta_1) - g(\zeta_2)| \le L_g |\zeta_1 - \zeta_2|\, .$$

Then we have

$$\left|G_{pq}^n\right| \le L_g \left|\, |\nabla G_\sigma * \tilde{u}_{h,k}(x_{pq}, t_n)| - |\nabla G_\sigma * \tilde{u}_{h,k}(x,t)|\, \right|$$

and one can use the triangular inequality for the Euclidean norm to obtain

$$\left|G_{pq}^n\right| \le L_g \left|\nabla G_\sigma * \tilde{u}_{h,k}(x_{pq}, t_n) - \nabla G_\sigma * \tilde{u}_{h,k}(x,t)\right|.$$

Using the form of the convolution as given in (2.7) and as $t \in (t_n, t_{n+1})$, one can show that for any $x \in e_{pq}$ holds

$$\left|G_{pq}^n\right| \le L_g \sum_r \left|\overline{u}_p^n\right| \int_r \left|\nabla G_\sigma(x_{pq} - s) - \nabla G_\sigma(x - s)\right| ds$$

where the sum is evaluated only on control volumes $r \in \tau_h$ and r in reflexion of τ_h through the boundary of Ω intersecting $B_\sigma(x_{pq})$, the ball centered at x_{pq} with radius σ.

Thanks to the hypotheses on G_σ, which is in $C^\infty(\mathbb{R}^d)$, the Cauchy-Schwarz inequality and the estimate *(i)* of Lemma 3.1 we obtain

$$\left|G_{pq}^n\right| \le h M_6$$

with a positive constant M_6. Since $\nabla\varphi$ is a continuous function, $S = \sup_{Q_T} |\nabla\varphi| < \infty$, we have that

$$|R_3| \le h \frac{M_6 S}{2} \sum_{n=0}^{N_{\max}-1} k \sum_{(p,q)\in\mathcal{E}} \left|\overline{u}_q^{n+1} - \overline{u}_p^{n+1}\right| m(e_{pq})$$

which together with Cauchy-Schwarz inequality and (3.59) leads to the desired result

$$(3.67)\qquad |R_3| \to 0 \text{ as } h, k \to 0.$$

Let the fourth term be given as

$$(3.68)\qquad R_4 \equiv \frac{1}{2} \sum_{n=0}^{N_{\max}-1} \sum_{(p,q)\in\mathcal{E}} \left(\overline{u}_q^{n+1} - \overline{u}_p^{n+1}\right)$$

$$\int_{t_n}^{t_{n+1}} \int_{e_{pq}} \left(g(|\nabla G_\sigma * \tilde{u}_{h,k}|) - g(|\nabla G_\sigma * u|) \right) \nabla \varphi n_{pq} dx dt \; .$$

The Lipschitz continuity of g and the triangular inequality for Euclidean norm give that

$$|g(|\nabla G_\sigma * \tilde{u}_{h,k}|) - g(|\nabla G_\sigma * u|)| \leq L_g \left| \nabla G_\sigma * (\tilde{u}_{h,k} - u) \right| \tag{3.69}$$

Using Cauchy-Schwartz inequality in the convolution term in (3.69) and the definition of extension lead to the following result

$$\begin{aligned} &|g(|\nabla G_\sigma * \tilde{u}_{h,k}|) - g(|\nabla G_\sigma * u|)| \leq \\ &\leq C \left(\int_{R^d} |\nabla G_\sigma (x-s)|^2 \, ds \right)^{\frac{1}{2}} \|\overline{u}_{h,k} - u\|_{L^2(Q_T)} \, . \end{aligned} \tag{3.70}$$

Since G_σ is C^∞, and G_σ has a compact support, one can show that there exists a constant $M_7 > 0$ such that

$$\left(\int_{R^d} |\nabla G_\sigma (x-s)|^2 \, ds \right)^{\frac{1}{2}} \leq M_7.$$

Then, we use the same technique as for the estimate of R_1 to see that there exists a positive constant M_8 such that

$$|R_4| \leq M_8 \|\overline{u}_{h,k} - u\|_{L^2(Q_T)} \tag{3.71}$$

and since $\overline{u}_{h,k}$ converges to u strongly in $L^2 (Q_T)$, we deduce that

$$|R_4| \to 0 \text{ as } h, k \to 0. \tag{3.72}$$

The last term is defined by

$$R_5 = \int_0^T \int_\Omega div \left(g(|\nabla G_\sigma * u|) \nabla \varphi \right) (u - \overline{u}_{h,k}) \, dx. \tag{3.73}$$

The form of the first term of (3.73) is slightly different from its equivalent in (3.68) but it can be justified by the fact that

$$\begin{aligned} &\int_\Omega div \left(g(|\nabla G_\sigma * u|) \nabla \varphi \right) \overline{u}_{h,k} (x,t) \, dx \\ &= \sum_{p \in \tau_h} \overline{u}_p^{n+1} \int_p div \left(g(|\nabla G_\sigma * u|) \nabla \varphi \right) dx \end{aligned} \tag{3.74}$$

by the definition of $\overline{u}_{h,k}$ and by the value of t which belongs to the interval $[t_n, t_{n+1}]$. Applying the Green formula in (3.74) implies that

$$\sum_{p\in\tau_h} \overline{u}_p^{n+1} \int\limits_p div\left(g(|\nabla G_\sigma * u|)\nabla\varphi\right)dx$$

$$= \sum_{p\in\tau_h} \overline{u}_p^{n+1} \sum_{q\in N(p)} \int\limits_{e_{pq}} g(|\nabla G_\sigma * u|)\nabla\varphi\, n_{pq} dx.$$

Since $g(|\nabla G_\sigma * u|)\nabla\varphi n_{pq} = -g(|\nabla G_\sigma * u|)\nabla\varphi n_{qp}$, we finally obtain

$$\int\limits_\Omega div\left(g(|\nabla G_\sigma * u|)\nabla\varphi\right)\overline{u}_{h,k}(x,t)\,dx$$

$$= -\frac{1}{2}\sum_{(p,q)\in\mathcal{E}} \left(\overline{u}_q^{n+1} - \overline{u}_p^{n+1}\right)\int\limits_{e_{pq}} g(|\nabla G_\sigma * u|)\nabla\varphi n_{pq} dx. \tag{3.75}$$

By the hypotheses, $g\left(\sqrt{\cdot}\right) \in C^\infty\left(\mathbb{R}^+\right)$, and convolution property state that $\nabla G_\sigma * u \in C^\infty\left(\mathbb{R}^d\right)$ since $G_\sigma \in C^\infty(\mathbb{R})$. Then $div\left(g(|\nabla G_\sigma * u|)\nabla\varphi\right) \in L^2(Q_T)$ and it comes from strong convergence of $\overline{u}_{h,k}$ to u that

$$|R_5| \to 0 \text{ as } h, k \to 0. \tag{3.76}$$

Finally, according to (3.60) , (3.63) , (3.67) , (3.72) and (3.76) ,we can conclude that

$$\frac{1}{2}\sum_{n=0}^{N_{\max}-1} k \sum_{(p,q)\in\mathcal{E}} g_{pq}^{\sigma,n}\left(\overline{u}_{h,k}\right) T_{pq}\left(\overline{u}_q^{n+1} - \overline{u}_p^{n+1}\right) \times$$

$$\times\left(\varphi(x_q, t_n) - \varphi(x_p, t_n)\right) \to \int\limits_0^T \int\limits_\Omega div\left(g(|\nabla G_\sigma * u|)\nabla\varphi\right) u dx dt$$

in $L^2(Q_T)$. This ends the proof of Lemma.

Lemma 3.9. *We have that*

$$\sum_{n=0}^{N_{\max}-1} k \sum_{p\in\tau_h} \varphi(x_p, t_n)\, f\left(\overline{u}_p^0 - \overline{u}_p^n\right) m(p) \to \int\limits_0^T \int\limits_\Omega f(u_0 - u)\varphi(x,t)\,dx dt$$

as $h, k \to 0$ *for all* $\varphi \in \Psi$.

Proof. The difference between the sequence and its desired limit can be written as

$$\sum_{n=0}^{N_{\max}-1} k \sum_{p\in\tau_h} \varphi\left(x_p, t_n\right) f\left(\overline{u}_p^0 - \overline{u}_p^n\right) m\left(p\right) \\ - \int_0^T \int_\Omega f(u_0 - u)\varphi\left(x,t\right) dxdt = N_1 + N_2$$

where

$$N_1 = \sum_{n=0}^{N_{\max}-1} \sum_{p\in\tau_h} \varphi\left(x_p, t_n\right) \int_{t_n}^{t_{n+1}} \int_p \left[f\left(\overline{u}_p^0 - \overline{u}_p^n\right) - f(u_0 - u)\right] dxdt, \tag{3.77}$$

$$N_2 = \sum_{n=0}^{N_{\max}-1} \sum_{p\in\tau_h} \int_{t_n}^{t_{n+1}} \int_p f(u_0 - u)\left[\varphi\left(x_p, t_n\right) - \varphi\left(x,t\right)\right] dxdt. \tag{3.78}$$

Our purpose is to prove that these two quantities tend to 0 as $h, k \to 0$. Due to the Lipschitz continuity of f we have

$$|N_1| \leq K_f \sum_{n=0}^{N_{\max}-1} \sum_{p\in\tau_h} |\varphi\left(x_p, t_n\right)| \int_{t_n}^{t_{n+1}} \int_p \left|\left(\overline{u}_p^0 - u_0\right) - \left(\overline{u}_p^n - u\right)\right| dxdt \tag{3.79}$$

and applying Cauchy-Schwartz inequality and the relation $\frac{1}{2}\left(a-b\right)^2 \leq a^2 + b^2$ we obtain

$$N_1 \leq K_f \left(\sum_{n=0}^{N_{\max}-1} k \sum_{p\in\tau_h} \left|\varphi\left(x_p, t_n\right)\right|^2 m(p)\right)^{\frac{1}{2}} \\ \left(\int_0^T \int_\Omega \left(\left(\overline{u}_{h,k}\left(x,0\right) - u_0\right)^2 + \left(\overline{u}_{h,k} - u\right)^2\right) dxdt\right)^{\frac{1}{2}} \tag{3.80}$$

from where we have

$$|N_1| \to 0 \text{ as } h, k \to 0. \tag{3.81}$$

Thanks to the regularity of φ, one can show the existence of a positive constant M_9 such that

$$\left|\varphi\left(x_p, t_n\right) - \varphi\left(x,t\right)\right| \leq \left(h+k\right) M_9.$$

Then it comes that

$$|N_2| \leq (h+k)\, M_9 K_f \int\limits_{Q_T} |u_0 - u| dxdt$$

which in turn implies that

(3.82) $$|N_2| \to 0 \text{ as } h,k \to 0.$$

Thus the lemma is proved.

Theorem 3.10. (Convergence to the weak solution) *The sequence* $\overline{u}_{h,k}$ *converges strongly in* $L^2(Q_T)$ *to the unique weak solution* u *defined in* (2.10) as $h,k \to 0$.

Proof. Using Lemma 3.5 and Lemma 3.2, we know from [6] that the limit u of the sequence $\overline{u}_{h,k}$ is in space $L^2(0,T;H^1(\Omega))$. Then we can use Green's theorem in the result of Lemma 3.8 and together with Lemmas 3.6 - 3.9 we can deduce that u satisfies the weak identity for all test functions $\varphi \in \Psi$. But Ψ is dense in Φ which implies the convergence result. The uniqueness of the weak solution is given in [5] (see Remark 2.4) and so not only subsequence but the sequence $\overline{u}_{h,k}$ itself converges to u.

4. Numerical experiments

In this section we present numerical experiments obtained by the scheme (2.1). We have choosen

$$g(s) = \frac{1}{1+Ks^2}$$

with a constant $K > 0$ and the convolution is realized with kernel

$$G_\sigma(x) = \frac{1}{Z} e^{\frac{|x|^2}{|x|^2-\sigma^2}},$$

where the constant Z is choosen so that G_σ has unit mass. In order to compute the diffusion coefficient $g_{pq}^{\sigma,n}(\overline{u}_{h,k})$ in (2.1) we use concept described in Remark 2.3. The terms $\int_r \nabla G_\sigma (x_{pq} - s)\, ds$ in (2.7) are computed using computer algebra system e.g. Mathematica. For any given σ they can be precomputed in advance. The sparse linear systems corresponding to (2.1) can be solved by any efficient linear solver.

In Fig. 1 we present embedding of the initial image (noisy corrupted four-petal shape, 256 x 256 pixels) into the so called *nonlinear scale space*

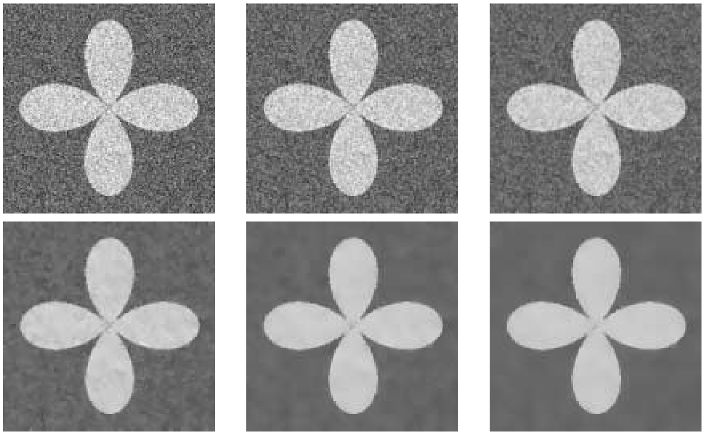

Fig. 1.

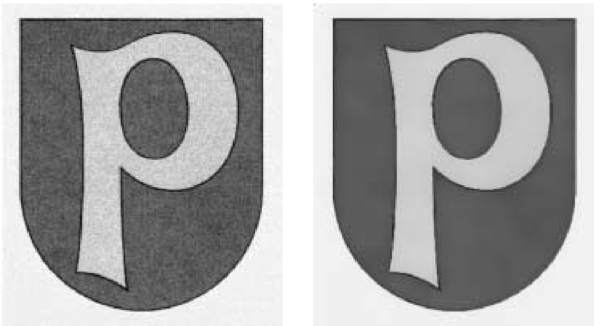

Fig. 2.

given by (1.1)-(1.3). We present the initial image and its processing in the scales $t = 10k, 20k, 30k, 40k, 50k$. We see the simplification (denoising) of the image together with preserving of important edges in the sequence of discrete scale steps. In Fig. 2, the scanned image (coat-of-arms from a book with not paper nor colours of good quality) is processed. We present scanned original (left) and processed image (right) after 40 discrete scale steps. In both experiments $k = h$, $\sigma = \frac{k}{10}$, $K = 20$ and h is a pixel size.

References

1. Alvarez, L., Guichard, F., Lions, P.L., Morel, J.M. (1993): Axioms and Fundamental Equations of Image Processing. Archive for Rat. Mech. Anal. **123**, 200–257
2. Alvarez, L., Morel, J.M. (1994): Formalization and computational aspects of image analysis. Acta Numerica, 1–59

3. Bänsch, E., Mikula, K. (1997): A coarsening finite element strategy in image selective smoothing. Computing and Visualization in Science **1**, 53–61
4. Brezis, H. (1992): Analyse fonctionelle, Theorie et applications. Masson
5. Catté, F., Lions, P.L., Morel, J.M., Coll, T. (1992): Image selective smoothing and edge detection by nonlinear diffusion. SIAM J. Numer. Anal. **129**, 182–193
6. Eymard, R., Gallouet, T., Herbin, R.: Finite Volume Methods. to appear
7. Eymard, R., Gutnic, M., Hilhorst, D. (1998): The finite volume scheme for the Richards equation. Preprint University Paris-Sud
8. Kačur, J., Mikula, K. (1995): Solution of nonlinear diffusion appearing in image smoothing and edge detection. Appl. Numerical Math. **17**, 47–59
9. Kačur, J., Mikula, K. (2001): Slow and fast diffusion effects in image processing. Computing and Visualization in Science **3**, 185–195 (2001)
10. Kačur, J., Mikula, K. (1997): Slowed anisotropic diffusion. In: Romeny, B.t.H, Florack, L., Koenderink, J., Viergever, M. (eds) Proceedings of Scale-Space Theory in Computer Vision, pp. 357-360. Lecture Notes in Computer Science 1252, Springer, Berlin
11. Lions, P.L. (1994): Axiomatic derivation of image processing models. M^3AS **4**, 467-475
12. Nordström, K.N. (1989): Biased anisotropic diffusion-A unified approach to edge detection. Preprint, Department of Electrical Engeneering and Computer Sciences, University of California, Berkeley
13. Patankar, S. (1980): Numerical heat transfer and fluid flow. Hemisphere Publ. Comp.
14. Perona, P., Malik, J.(1987): Scale space and edge detection using anisotropic diffusion. In Proc. IEEE Computer Society Workshop on Computer Vision
15. Romeny, B.M.t.H. (ed.) (1994): Geometry driven diffusion in computer vision. Kluwer, Dordrecht
16. Weickert, J., Romeny, B.M.t.H., Viergever, M.A. (1998): Efficient and reliable schemes for nonlinear diffusion filtering. IEEE Trans. Image Processing **7**, 398–410