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A minimal stabilisation procedure for mixed finite element methods

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Summary. Stabilisation methods are often used to circumvent the difficulties associated with the stability of mixed finite element methods. Stabilisation however also means an excessive amount of dissipation or the loss of nice conservation properties. It would thus be desirable to reduce these disadvantages to a minimum. We present a general framework, not restricted to mixed methods, that permits to introduce a *minimal* stabilising term and hence a minimal perturbation with respect to the original problem. To do so, we rely on the fact that *some part of the problem* is stable and should not be modified. Sections 2 and 3 present the method in an abstract framework. Section 4 and 5 present two classes of stabilisations for the inf-sup condition in mixed problems. We present many examples, most arising from the discretisation of flow problems. Section 6 presents examples in which the stabilising terms is introduced to cure coercivity problems.

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1 Introduction

This paper will be devoted primarily to the stabilisation of mixed finite element methods. However, we shall introduce a general setting which might be applied to other situations.

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Let us thus consider, to fix ideas, the standard problem: find $(u, p) \in V \times Q$ such that,

$$\left\{\begin{array}{ll} a(u,v) + b(v,p) = \langle f, v\rangle & \forall v \in V, \\ b(u,q) = \langle g, q\rangle & \forall q \in Q, \end{array}\right. \tag{1.1}$$

where f and g are given elements in V' and Q' respectively. Throughout all the paper, we shall always assume that V and Q are Hilbert spaces and that $a(\ ,\ )$ and $b(\ ,\ )$ are continuous bilinear forms on $V \times V$ and $V \times Q$ respectively. Let then B denote the linear operator defined by

$$\langle Bv, q\rangle_{Q' \times Q} = b(v,q) \quad \forall v \in V, \forall q \in Q. \tag{1.2}$$

The kernel of B,

$$\ker B = \{v_0 \in V \mid b(v_0, q) = 0 \quad \forall q \in Q\} \tag{1.3}$$

will also play a fundamental role. For this problem, which has been the object of intensive studies, the classical theory (e.g. [8,9]) states that one gets a unique solution provided the following conditions hold:

– coercivity on the kernel of B, that is

$$\exists\, \alpha_0 > 0\ s.t. \quad a(v_0, v_0) \geq \alpha_0 \|v_0\|_V^2 \qquad \forall v_0 \in \ker B, \tag{1.4}$$

– inf-sup condition

$$\exists\, k_0 > 0\ s.t. \quad \sup_{v \neq 0} \frac{b(v,q)}{\|v\|_V} \geq k_0 \|q\|_Q \qquad \forall q \in Q. \tag{1.5}$$

Let us introduce a discrete problem: find $(u_h, p_h) \in V_h \times Q_h$, $V_h \subset V$, $Q_h \subset Q$, such that:

$$\left\{\begin{array}{ll} a(u_h,v_h) + b(v_h,p_h) = \langle f, v_h\rangle & \forall v_h \in V_h, \\ b(u_h,q_h) = \langle g, q_h\rangle & \forall q_h \in Q_h. \end{array}\right. \tag{1.6}$$

The bilinear form $b(\cdot,\cdot)$ now defines a discrete operator B_h from V_h into Q_h' and we must consider its kernel,

$$\ker B_h = \{v_{0h} \in V_0 \mid b(v_{0h}, q_h) = 0 \quad \forall q_h \in Q_h\}. \tag{1.7}$$

To get existence and uniqueness of the discrete problem, we must have conditions corresponding to (1.4) and (1.5), that is,

$$\exists\, \alpha_h > 0\ s.t. \quad a(v_{0h}, v_{0h}) \geq \alpha_h \|v_{0h}\|_V^2 \qquad \forall v_{0h} \in \ker B_h \tag{1.8}$$

$$\exists\, k_h > 0\ s.t. \quad \sup_{v_h \in V_h} \frac{b(v_h,q_h)}{\|v_h\|_V} \geq k_h \|q_h\|_Q. \tag{1.9}$$

To obtain error estimates, we must also assume the **stability conditions**:

(1.10) $$\alpha_h \geq \widetilde{\alpha}_0 > 0.$$

(1.11) $$k_h \geq \tilde{k}_0 > 0.$$

Problems may arise with both of these conditions. For (1.8) and (1.10) the trouble is that $\ker B_h$ is not, in general, a subspace of $\ker B$, so that (1.8) is not a consequence of (1.4) (unless coercivity hold for the whole space V.)

In the same way, an improper choice of the spaces V_h and Q_h can lead to k_h vanishing to 0 with h in (1.9). In many instances, conditions (1.10) and (1.11) impose contradictory requirements on the choice of the discrete spaces V_h and Q_h, and only quite special choices are admissible.

There are cases where these elaborate constructions are felt as inadequate. In some situations, for example, it happens that (1.1) is only a part of a larger problem, for which the choice of V_h and Q_h is not really free, and we are led to employ discrete spaces which are not suitable for (1.6).

Stabilisation methods, then, try to recover (1.8)-(1.11) through a modification of the variational formulation. This modification should obviously preserve consistency. Ideally, it should be as small as possible, restoring stability without introducing unwanted smoothing properties.

In this paper we shall describe a general framework for the study of stability issues. We shall also present a general technique that yields many examples of stabilised methods which can be analysed in this framework. The basic idea of the technique is that, in several cases, the discretisation at hand has some sort of "partial stability" (to fix ideas, we have a priori bounds for a certain *seminorm* of the solution, but not for the true norm.) Our technique consists then, somehow, in adding the minimum modification that allows to restore the full stability.

In the next section, we present and discuss the abstract framework in which we are going to set our examples. In Sect. 3 we present, always at the abstract level, a general stabilisation technique, with abstract stability theorems and error estimates. A first class of applications, together with several examples, will be discussed in Sect. 4, and a second class of applications, with several other examples, will be the object of Sect. 5. Roughly speaking, the two classes of applications will correspond to two different ways of stabilising problems of type (1.1) when the *inf-sup* condition (1.9),(1.11) does not hold: in the first class of stabilisations we assume that we have a stability result for a pair $V_h - \overline{Q}_h$, where $\overline{Q}_h \subset Q_h$, while in the second class we only assume a sort of *weak stability* that will be made precise later on. Applications to problems where the ellipticity in the kernel (1.8), (1.10) is needed are then considered in Sect. 6.

Other important general results on stabilisations for this type of problems can be found in [19,4,5,24] and the references therein. See also [9] for additional references.

2 An abstract framework

We consider here a very general problem. Let $\mathcal{W}$ be a Hilbert space, let A be in $\mathcal{L}(\mathcal{W},\mathcal{W}')$ (the space of linear continuous operators from $\mathcal{W}$ to $\mathcal{W}'$,) and let F be in $\mathcal{W}'$. We want to find $X \in \mathcal{W}$ such that,

$$(2.1) \qquad \langle AX, Y\rangle_{\mathcal{W}'\times\mathcal{W}} = \langle F, Y\rangle_{\mathcal{W}'\times\mathcal{W}} \qquad \forall Y \in \mathcal{W}.$$

From now on, we shall always assume that

$$(2.2) \qquad \langle AY, Y\rangle \geq 0 \qquad \forall Y \in \mathcal{W}.$$

The following result is an exercise in functional analysis, but, for the convenience of the readers, we sketch a proof.

Proposition 2.1. *If (2.2) holds, then the two following conditions are equivalent:*

i) A *is an isomorphism from* $\mathcal{W}$ *onto* $\mathcal{W}'$*.*
ii) $\exists \Phi \in \mathcal{L}(\mathcal{W},\mathcal{W})$ *and a positive real number* α_Φ *such that*

$$(2.3) \qquad \langle AY, \Phi(Y)\rangle_{\mathcal{W}'\times\mathcal{W}} \geq \alpha_\Phi \|Y\|^2_{\mathcal{W}} \quad \forall Y \in \mathcal{W}.$$

Proof of Proposition 2.1. Let J be the Riesz's operator from $\mathcal{W}'$ to $\mathcal{W}$. The implication $i) \implies ii)$ follows by taking $\Phi = JA$. To prove the converse implication we denote by Id the identity operator in $\mathcal{W}$, and we remark that, if (2.2) holds, then for every positive real number s we have, for all $Y \in \mathcal{W}$,

$$\langle (s\Phi + Id)^{\mathrm{t}} AY, Y\rangle_{\mathcal{W}'\times\mathcal{W}} = \langle AY, (s\Phi + Id)Y\rangle_{\mathcal{W}'\times\mathcal{W}} \geq s\,\alpha_\Phi \|Y\|^2_{\mathcal{W}}.$$

This easily implies that $(s\Phi + Id)^{\mathrm{t}}A$ is an isomorphism from $\mathcal{W}$ onto $\mathcal{W}'$. Since $s\Phi + Id$ is an isomorphism for s small enough, then $i)$ follows easily. □

Remark 2.1. If (2.2) is not satisfied, we always have $i) \implies ii)$ but the converse is false. This can be seen by considering in $L^2(]0,+\infty[)$ the mapping:

$$\begin{cases} (Au)(x) = u(x-1) \text{ for } x > 1 \\ (Au)(x) = 0 \text{ for } 0 < x \leq 1 \end{cases}$$

with $\Phi u := Au$. Clearly $ii)$ is satisfied, but $i)$ is not, as A is injective but not surjective. For an operator that does not satisfy (2.2), we would need two conditions instead of (2.3), that is: $\exists \Phi_1 \in \mathcal{L}(\mathcal{W},\mathcal{W}), \Phi_2 \in \mathcal{L}(\mathcal{W},\mathcal{W})$ such that, for all $Y \in \mathcal{W}$,

$$\begin{cases} \langle AY, \Phi_1(Y)\rangle_{\mathcal{W}'\times\mathcal{W}} \geq \alpha_1 \|Y\|^2_{\mathcal{W}}, \\ \langle \Phi_2(Y), A^{\mathrm{t}}Y\rangle_{\mathcal{W}\times\mathcal{W}'} \geq \alpha_2 \|Y\|^2_{\mathcal{W}}, \end{cases} \tag{2.4}$$

implying that A is both injective and surjective. □

Remark 2.2 (stability constant). It must be noted that the "stability constant" of Problem (2.1), that is the smallest constant C such that

$$\|X\| \leq C\|AX\| \qquad \forall X \in \mathcal{W},$$

is not $1/\alpha_\Phi$ (see (2.3)) but rather $\|\Phi\|/\alpha_\Phi$. □

As we are mostly interested in mixed problems, it might be worth showing that this abstract formalism contains the usual theory for Problem (1.1). Indeed, let $\mathcal{W} = V \times Q$, $X = (u,p)$, $Y = (v,q)$, and define

$$\begin{cases} \langle AX, Y\rangle = a(u,v) + b(v,p) - b(u,q), \\ \langle F, Y\rangle = \langle f, v\rangle_{V'\times V} - \langle g, q\rangle_{Q'\times Q}. \end{cases} \tag{2.5}$$

In this context, it is clear that (2.1) is just another way of writing (1.1). We suppose that $a(u,u) \geq 0$ for any $u \in V$, which clearly implies (2.2). We now want to get (2.3) from (1.4) and (1.5). We thus consider, for any given $(u,p) \in V \times Q$, two auxiliary problems, which have a unique solution if (1.4) and (1.5) hold:

– Find (u_1, p_1), solution of

$$\begin{cases} a(v,u_1) - b(v,p_1) = (u,v)_V \ \forall v \in V, \\ b(u_1,q) = 0 \qquad\qquad\qquad\quad \forall q \in Q. \end{cases} \tag{2.6}$$

– Find (u_2, p_2), solution of

$$\begin{cases} a(v,u_2) - b(v,p_2) = 0 \ \forall v \in V, \\ b(u_2,q) = (p,q)_Q \qquad \forall q \in Q. \end{cases} \tag{2.7}$$

We now set $\Phi(\{u,p\}) := \{(u_1+u_2),(p_1+p_2)\}$ and we have:

$$\begin{aligned} \langle A(X), \Phi(X)\rangle &= a(u, u_1+u_2) + b(u_1+u_2, p) - b(u, p_1+p_2) \\ &= \|u\|^2_V + \|p\|^2_Q. \end{aligned} \tag{2.8}$$

Remark 2.3. Problems (2.6) and (2.7) could, by linearity, be combined into one. We preferred to make more explicit the separate control of $\|u\|_V$ and $\|p\|_Q$. □

Let us now turn to the discretisation of (2.1). For a given sequence of subspaces $\mathcal{W}_h$ of $\mathcal{W}$ (usually of finite dimension) we consider, for each h, the discrete problem: find $X_h \in \mathcal{W}_h$ such that

$$\langle AX_h, Y_h\rangle = \langle F, Y_h\rangle \qquad \forall Y_h \in \mathcal{W}_h. \tag{2.9}$$

In general, for an arbitrary choice of $\mathcal{W}_h$, (2.9) will not be stable. In particular, we cannot ensure that there exists a sequence of linear operators $\Phi_h \in \mathcal{L}(\mathcal{W}_h, \mathcal{W}_h)$, uniformly bounded in h, such that, for some $\alpha_1 > 0$ independent of h,

$$\langle AY_h, \Phi_h(Y_h)\rangle \geq \alpha_1 \|Y_h\|^2_{\mathcal{W}} \qquad \forall Y_h \in \mathcal{W}_h. \tag{2.10}$$

We however suppose that stability holds for some semi-norm $[Y_h]_h$ on $\mathcal{W}_h$, that is we assume that we have two positive constants c_Φ and α_Φ and an operator $\Phi_h \in \mathcal{L}(\mathcal{W}_h, \mathcal{W}_h)$ such that

$$\|\Phi_h(Y_h)\| \leq c_\Phi \|Y_h\| \qquad \forall Y_h \in \mathcal{W}_h, \tag{2.11}$$

$$\langle AY_h, \Phi_h(Y_h)\rangle \geq \alpha_\Phi [Y_h]^2_h \qquad \forall Y_h \in \mathcal{W}_h, \tag{2.12}$$

which we can loosely state as “ some part of the problem is stable”.

What we shall try to do in the sequel is then to modify problem (2.9) in order to make it stable. We shall thus consider a stabilised problem of the type

$$\langle AX_h + R(X_h), Y_h\rangle = \langle F, Y_h\rangle \qquad \forall Y_h \in \mathcal{W}_h, \tag{2.13}$$

where $R(X_h)$ will be chosen in order to make (2.13) stable (in the sense of condition (2.10)) while **preserving consistency**. The following section will introduce a general mechanism for this construction.

Remark 2.4. In the case of the mixed problem (1.1), assuming for simplicity that $a(\cdot,\cdot)$ is V-elliptic, that is

$$\exists\, \alpha > 0 \; s.t. \quad a(v,v) \geq \alpha \|v\|^2_V \qquad \forall v \in V, \tag{2.14}$$

we can always have (2.12) by using the following semi-norm

$$[Y_h]^2_h = [(v_h, q_h)]^2 := \|v_h\|^2_V + [\![q_h]\!]^2_h \tag{2.15}$$

where

$$[\![q_h]\!]_h := \sup_{v_h \in V_h} \frac{b(v_h, q_h)}{\|v_h\|_V} \tag{2.16}$$

as it is shown in the following proposition.

Proposition 2.2. *Let A be of the form (2.5) and assume that (2.14) holds. Then, for every choice of subspaces V_h, Q_h we can find a linear operator $\Phi_h \in \mathcal{L}(\mathcal{W}_h, \mathcal{W}_h)$ such that (2.11) and (2.12) hold, with*

$$\alpha_\Phi = \frac{\alpha}{2}\, min\,(1, \frac{1}{\|a\|^2}), \tag{2.17}$$

$$c_\Phi = 1 + \frac{\alpha\|b\|}{\|a\|^2}, \tag{2.18}$$

and the semi-norm defined in (2.15) and (2.16).

Proof of Proposition 2.2. For a given $Y_h = (v_h, q_h)$, let $v_h^* \in V_h$ be such that

$$\frac{b(v_h^*, q_h)}{\|v_h^*\|_V} = \sup_{v_h \in V_h} \frac{b(v_h, q_h)}{\|v_h\|_V} =: [\![q_h]\!]_h \tag{2.19}$$

and

$$\|v_h^*\|_V = [\![q_h]\!]_h. \tag{2.20}$$

We now choose

$$\Phi_h(Y_h) = (v_h - \delta v_h^*, q_h), \tag{2.21}$$

with $\delta \in \mathbb{R}$ to be chosen later on. We have from (2.5) and (2.21):

$$\begin{aligned} \langle AY_h, \Phi_h(Y_h)\rangle &= a(v_h, v_h) - \delta a(v_h, v_h^*) \\ &\quad + b(v_h, q_h) - b(v_h, q_h) + \delta b(v_h^*, q_h) \\ &\geq \alpha\|v_h\|_V^2 - \delta\|a\|\|v_h\|_V\|v_h^*\|_V + \delta[\![q_h]\!]_h\|v_h^*\|_V \\ &= \alpha\|v_h\|_V^2 - \delta\|a\|\|v_h\|_V[\![q_h]\!]_h + \delta[\![q_h]\!]_h^2 \end{aligned} \tag{2.22}$$

having used (2.14), (2.19), and, in the last step, (2.20). It is now clear that, choosing $\delta = \alpha/\|a\|^2$, (2.22) implies

$$\langle AY_h, \Phi_h\, Y_h)\rangle \geq \frac{\alpha}{2}\|v_h\|_V^2 + \frac{\delta}{2}[\![q_h]\!]_h^2 \tag{2.23}$$

having used $2ab \leq a^2 + b^2$. Hence we have (2.12) with the constant α_Φ given by (2.17). On the other hand, (2.20) and the choice of δ imply (2.11) and (2.18) since

$$\|v_h - \delta v_h^*\| \leq \|v_h\| + \delta\|v_h^*\| = \|v_h\| + \delta[\![q_h]\!]_h \leq \|v_h\| + \delta\|B\|\,\|q_h\|_Q.$$

□

3 Abstract stabilisation and error estimates

We still consider the abstract setting of the previous section and our goal is to find approximate solutions of problem (2.1). We thus have a Hilbert space $\mathcal{W}$, and a sequence of approximation spaces $\mathcal{W}_h$. We suppose as in (2.12) that we have a " partial" stability result. More precisely, we make the following hypothesis:

H.1 *For every h there exists*

i) a semi-norm $[\,\cdot\,]_h$ *on* $\mathcal{W}$*,*

ii) an operator $\Phi_h \in \mathcal{L}(\mathcal{W}_h, \mathcal{W}_h)$*,*

iii) a constant c_Φ *such that*

(3.1) $$\|\Phi_h(Y_h)\| \leq c_\Phi \|Y_h\| \quad \forall Y_h \in \mathcal{W}_h.$$

iv) a constant $\alpha_\Phi > 0$ *such that*

(3.2) $$\langle AY_h, \Phi_h(Y_h)\rangle \geq \alpha_\Phi [Y_h]_h^2 \quad \forall Y_h \in \mathcal{W}_h.$$

□

We now want to modify the problem in order to stabilise it, and we assume that we find a bilinear form $R(X_h, Y_h)$ on $\mathcal{W}_h \times \mathcal{W}_h$ satisfying the following hypotheses.

H.2 *There exist a Hilbert space* $\mathcal{H}$*, and, for every* h*:*

i) an operator $G_h \in \mathcal{L}(\mathcal{W}_h, \mathcal{H})$*,*

ii) a constant $c_R > 0$ *such that*

(3.3) $$R(X_h, Y_h) \leq c_R \|X_h\|\,\|Y_h\| \qquad \forall X_h,\, Y_h \in \mathcal{W}_h,$$

iii) a constant $\alpha_R > 0$ *such that*

(3.4) $$R(Y_h, Y_h) \geq \alpha_R \|G_h Y_h\|_{\mathcal{H}}^2 \qquad \forall Y_h \in \mathcal{W}_h.$$

H.3 *With the notation of assumption H.2, there exist two positive constants* γ_2 *and* γ_3 *such that*

(3.5) $$[Y_h]_h^2 + \gamma_2 \|G_h Y_h\|_{\mathcal{H}}^2 \geq \gamma_3 \|Y_h\|_{\mathcal{W}}^2 \qquad \forall Y_h \in \mathcal{W}_h.$$

□

Remark 3.1. It is clear from (3.1) and (3.3) that, for every $Y_h \in \mathcal{W}_h$, we have

$$R(Y_h, \Phi_h(Y_h)) \leq c_R c_\Phi \|Y_h\|^2.$$

However, indicating by $c_{R\Phi}$ the best possible constant such that

(3.6) $$R(Y_h, \Phi_h(Y_h)) \leq c_{R\Phi} \|Y_h\|^2 \qquad \forall Y_h \in \mathcal{W}_h,$$

it might be possible that, in particular cases, $c_{R\Phi}$ is much smaller than $c_R c_\Phi$. Indeed, in some cases, $c_{R\Phi}$ could even be zero. In the following estimates, we shall therefore use the constant $c_{R\Phi}$ instead of the (always pessimistic)

$c_R c_\Phi$. Moreover, in several cases, the following additional property H.4 will hold. We shall see that, if this is the case, many technicalities could be avoided. □

H.4 *With the notation of assumptions H.1 and H.2 we have*

$$R(Y_h, \Phi_h(Y_h)) \geq 0 \qquad \forall Y_h \in \mathcal{W}_h. \tag{3.7}$$

□

We now consider, for some positive real number r, the regularized operator $\widetilde{A}$ defined as

$$\langle \widetilde{A} X_h, Y_h \rangle := \langle A X_h, Y_h \rangle + r R(X_h, Y_h) \qquad \forall Y_h, X_h \in \mathcal{W}_h, \tag{3.8}$$

and the corresponding regularised problem

$$\langle \widetilde{A} X_h, Y_h \rangle = \langle F, Y_h \rangle \qquad \forall Y_h \in \mathcal{W}_h. \tag{3.9}$$

We begin by proving the following lemma.

Lemma 3.1. *Assume that H.1, H.2 and H.4 hold. For every positive real numbers r and γ let $\widetilde{A}$ be defined as in (3.8) and $\widetilde{\Phi}_h$ be defined as*

$$\widetilde{\Phi}_h(Y_h) := Y_h + \gamma \Phi_h(Y_h) \quad \forall Y_h \in \mathcal{W}_h. \tag{3.10}$$

Then we have, for all $Y_h \in \mathcal{W}_h$,

$$\langle \widetilde{A} Y_h, \widetilde{\Phi}_h(Y_h) \rangle \geq \min(\alpha_R, \alpha_\Phi)(\gamma [Y_h]_h^2 + r \| G_h(V_h) \|_{\mathcal{H}}^2). \tag{3.11}$$

Proof of Lemma 3.1. From definitions (3.8), (3.10), and assumptions (3.2), (3.4), one immediately obtains for every positive γ and r :

$$\begin{aligned} &\langle \widetilde{A} Y_h, \widetilde{\Phi}_h(Y_h) \rangle = \langle A Y_h, \widetilde{\Phi}_h(Y_h) \rangle + r R(Y_h, \widetilde{\Phi}_h(Y_h) \\ &= \langle A Y_h, Y_h + \gamma \Phi_h(Y_h) \rangle + r R(Y_h, Y_h + \gamma \Phi_h(Y_h)) \\ &\geq \alpha_\Phi \gamma [Y_h]_h^2 + \alpha_R r \| G_h(Y_h) \|_{\mathcal{H}}^2 + r\gamma R(Y_h, \Phi_h(Y_h)), \end{aligned} \tag{3.12}$$

and the result follows easily from (3.7). □

It is clear that, if assumption H.3 is also verified, then (3.11) will give a stability result of type (2.10), where the explicit value of the constant α_1 can be easily deduced from the values of the other constants. On the other hand, the estimate (3.11) will be used in the sequel also in cases when some constant (r, mostly, and sometimes γ_2) might depend on h, so that it is convenient to leave it in its actual form.

In the applications that we are going to examine in the following sections, assumption H.4 will always be satisfied. However, for completeness, we present the following result, that can be used for the cases in which (3.7) does not hold.

Lemma 3.2. *Assume that H.1, H.2 and H.3 hold, and let* $\widetilde{A}$ *and* $\widetilde{\Phi}_h$ *be defined as in (3.8) and (3.10), respectively. Set now*

$$r_0 := \frac{\alpha_\Phi \gamma_3}{2c_{R\Phi}} \qquad \textit{and} \qquad \gamma_0 := \frac{\alpha_R \gamma_3}{2\gamma_2 c_{R\Phi}} \tag{3.13}$$

(or $+\infty$ *when* $c_{R\Phi} = 0$*.) Then, for all* $\gamma \leq \gamma_0$ *and for all* $r \leq r_0$ *we have*

$$\begin{aligned} \langle \widetilde{A}Y_h, \widetilde{\Phi}_h(Y_h)\rangle \geq \frac{1}{2}\min(\alpha_R, \alpha_\Phi) \\ \times(\gamma[Y_h]_h^2 + r\|G_h(V_h)\|_{\mathcal{H}}^2) \ \ \forall Y_h \in \mathcal{W}_h. \end{aligned} \tag{3.14}$$

Proof of Lemma 3.2. We restart as in (3.12), but using now (3.6) and assumption H.3:

$$\begin{aligned} &\langle \widetilde{A}Y_h, \widetilde{\Phi}_h(Y_h)\rangle = \langle AY_h, \widetilde{\Phi}_h(Y_h)\rangle + rR(Y_h, \widetilde{\Phi}_h(Y_h)) \\ &\geq \alpha_\Phi \gamma [Y_h]_h^2 + \alpha_R r\|G_h(Y_h)\|_{\mathcal{H}}^2 - r\gamma c_{R\Phi}\|Y_h\|^2. \end{aligned} \tag{3.15}$$

Using (3.5), the right-hand side of (3.15) is bounded below by

$$(\alpha_\Phi \gamma - r\gamma c_{R\Phi}/\gamma_3)[Y_h]_h^2 + (\alpha_R r - r\gamma\gamma_2 c_{R\Phi}/\gamma_3)\|G_h(Y_h)\|_{\mathcal{H}}^2 \tag{3.16}$$

If we choose now $r \leq r_0$ and $\gamma \leq \gamma_0$ then (3.14) follows immediately from (3.15), (3.16) and (3.13). □

Lemmata 3.1 and 3.2 will ensure stability for a wide class of stabilising procedures. We now consider the problem of error estimates. As we introduced sufficient conditions to ensure stability, the question will be to check consistency, and in particular the effect on consistency of the extra stabilising terms.

In order to retain a certain amount of generality, we shall make now some stability assumptions, that, in different particular cases, can be proved by means of the stability lemmata seen before. However, as we shall see, this part of Sect. 3 is presented in a way that makes it logically independent from the previous one. We make therefore the following assumptions.

H.5 *We have:*

i) a continuous problem

$$\langle AX, Y\rangle = \langle F, Y\rangle \qquad \forall Y \in \mathcal{W}, \tag{3.17}$$

that we assume to have a unique solution,

ii) a sequence of stabilised discrete problems

$$\langle \widetilde{A}X_h, Y_h\rangle = \langle F, Y_h\rangle \qquad \forall Y_h \in \mathcal{W}_h \tag{3.18}$$

where $\widetilde{A}$ *is still defined as in (3.8) for some* $r > 0$*,*

iii) two constants $\widetilde{c}_\Phi$ *and* $\widetilde{\alpha}_\Phi$*, and an operator* $\widetilde{\Phi}_h \in \mathcal{L}(\mathcal{W}_h, \mathcal{W}_h)$ *such that*

$$\|\widetilde{\Phi}_h(Y_h)\| \le \widetilde{c}_\Phi \|Y_h\| \quad \forall Y_h \in \mathcal{W}_h, \tag{3.19}$$

and

$$\langle \widetilde{A} Y_h, \widetilde{\Phi}_h(Y_h)\rangle \ge \widetilde{\alpha}_\Phi \|Y_h\|^2 \quad \forall Y_h \in \mathcal{W}_h. \tag{3.20}$$

We have then the following error bound.

Lemma 3.3. *Assume that (3.19) and (3.20) hold, and let X and X_h be the solutions of (3.17) and (3.18) respectively. For every $X_I \in \mathcal{W}_h$ we set*

$$\mathcal{R}(X_I) := \sup_{Y_h \in \mathcal{W}_h} \frac{R(X_I, Y_h)}{\|Y_h\|}, \tag{3.21}$$

and we have

$$\frac{\widetilde{\alpha}_\Phi}{\widetilde{c}_\Phi}\|X_I - X_h\| \le \|A\|\,\|X - X_I\| + r\mathcal{R}(X_I). \tag{3.22}$$

Proof of Lemma 3.3. Set $\delta X = X_I - X_h$, and $\widetilde{Y}_h = \widetilde{\Phi}_h(\delta X)$. From (3.19) we immediately have

$$\|\widetilde{Y}_h\| \le \widetilde{c}_\Phi\, \|\delta X\|. \tag{3.23}$$

On the other hand, using (3.20) and (3.8), adding and subtracting X, then using (3.17)-(3.18), and finally (3.21) we obtain:

$$\begin{aligned} &\widetilde{\alpha}_\Phi\, \|\delta X\|^2 \le \langle \widetilde{A}\delta X, \widetilde{Y}_h\rangle = \langle A(\delta X), \widetilde{Y}_h\rangle + rR(\delta X, \widetilde{Y}_h)\\ &= \langle A(X_I - X), \widetilde{Y}_h\rangle + \langle AX, \widetilde{Y}_h\rangle - \langle \widetilde{A}X_h, \widetilde{Y}_h\rangle + rR(X_I, \widetilde{Y}_h)\\ &= \langle A(X_I - X), \widetilde{Y}_h\rangle + rR(X_I, \widetilde{Y}_h)\\ &\le \|\widetilde{Y}_h\|\,(\|A\|\,\|X_I - X\| + r\mathcal{R}(X_I)) \end{aligned} \tag{3.24}$$

and (3.22) follows immediately using (3.23). □

Remark 3.2. In several applications, R will be chosen of the form

$$R(X_h, Y_h) = (G_h X_h, G_h Y_h)_{\mathcal{H}} \tag{3.25}$$

where G_h is the operator appearing in H.2. Moreover, the operator G_h will have a kernel, say $\overline{\mathcal{W}}_h$ (which in general will be a subspace of the " part of $\mathcal{W}_h$ controlled by A, before stabilisation".) In these cases, for every $\overline{X}_h \in \overline{\mathcal{W}}_h$, the second term in the right hand side of (3.22) can be estimated by

$$\begin{aligned} &\frac{R(X_I, Y_h)}{\|Y_h\|} = \frac{R(X_I - \overline{X}_h, Y_h)}{\|Y_h\|}\\ &\le c_R\|X_I - \overline{X}_h\| \le c_R(\|X_I - X\| + \|X - \overline{X}_h\|), \end{aligned} \tag{3.26}$$

so that, from (3.22) we have, in this case

$$\begin{aligned} &\frac{\widetilde{\alpha}_\Phi}{\widetilde{c}_\Phi}\|X_I - X_h\|\\ &\le \|A\|\,\|X - X_I\| + r\,c_R(\|X_I - X\| + \|X - \overline{X}_h\|). \end{aligned} \tag{3.27}$$

Clearly the choice (3.25) satisfies (3.4) with $\alpha_R = 1$. □

Remark 3.3. It is clear that, if all the constants appearing in H.1, H.2 and H.3 are independent of h, we can choose r and γ in Lemma 3.2 to be independent of h as well. Hence the assumption H.5 will also be satisfied with $\widetilde{c}_\Phi$ and $\widetilde{\alpha}_\Phi$ independent of h, and the combination of H.3, Lemma 3.2, and Lemma 3.3 (plus the obvious triangle inequality) will yield

$$\|X - X_h\| \leq C \left(\inf_{Y_h \in \mathcal{W}_h} \|X - Y_h\| + \inf_{\overline{Y}_h \in \overline{\mathcal{W}}_h} \|X - \overline{Y}_h\| \right). \tag{3.28}$$

with a constant C independent of h. If assumption H.4 holds as well, the choice of r can be done arbitrarily, for instance $r = 1$. □

A certain number of applications can be analysed with the instruments that we have developped so far, as indicated in the previous remark. However, there are cases in which it is convenient to use an r depending on h. In such cases, the previous analysis has to be readjusted, starting again from Lemma 3.1 and Lemma 3.2. In particular, we cannot expect to have a stability result of the type (3.20), but only the weaker one that comes from Lemma 3.1. Hence we have to modify H.5 as follows.

H.6 *We retain assumptions i) and ii) of H.5, and we change iii) into:*
iii bis) there exist two constants $\widetilde{c}_\Phi$ and α^*_Φ, independent of h and r, and a sequence of linear operators $\widetilde{\Phi}_h \in \mathcal{L}(\mathcal{W}_h, \mathcal{W}_h)$ such that (3.19) holds together with

$$\begin{aligned} &\langle \widetilde{A} Y_h, \widetilde{\Phi}_h(Y_h) \rangle \\ &\quad \geq \alpha^*_\Phi \left([Y_h]_h^2 + r \|G_h(Y_h)\|^2_{\mathcal{H}} \right) \quad \forall Y_h \in \mathcal{W}_h. \end{aligned} \tag{3.29}$$

□

It is clear that the above assumption will be satisfied by every stabilising method that satisfies the assumptions of Lemma 3.1 or the ones of Lemma 3.2, as it can be seen from (3.11) and (3.14). In this case we can prove the following more sophisticated and more useful error bound.

Lemma 3.4. *Let X and X_h be the solutions of (3.17) and (3.18) respectively. Assume that H.3 and H.6 hold. Then, for every $X_I \in \mathcal{W}_h$ we have*

$$\begin{aligned} &[X_I - X_h]_h^2 + r \|G_h(X_I - X_h)\|^2_{\mathcal{H}} \\ &\leq \left(\frac{\widetilde{c}_\Phi}{\alpha^*_\Phi} \right)^2 \frac{4(r + \gamma_2)}{r \gamma_3} \left(\|A\|^2 \|X_I - X\|^2 + r^2 (\mathcal{R}(X_I))^2 \right), \end{aligned} \tag{3.30}$$

where $\mathcal{R}(X_I)$ is still defined as in (3.21).

Proof of Lemma 3.4. We set $\delta X = X_I - X_h$. Arguing as in the proof of Lemma 3.3 we get, from (3.29), (3.8), (3.17), and (3.18)

$$\begin{aligned}\alpha^*_\Phi([\delta X]^2_h + r\|G_h\delta X\|^2) &\le \langle \widetilde{A}(\delta X), \widetilde{\Phi}(\delta X)\rangle \\ &\le \widetilde{c}_\Phi(\|A\|\,\|X_I - X\| + r\mathcal{R}(\mathcal{X}_\mathcal{I}))\|\delta X\|),\end{aligned} \tag{3.31}$$

and using (3.5) we immediately obtain

$$\begin{aligned}&[\delta X]^2_h + r\|G_h\delta X\|^2 \le \\ &\le \frac{\widetilde{c}_\Phi}{\alpha^*_\Phi}(\|A\|\,\|X_I - X\| + r\mathcal{R}(\mathcal{X}_\mathcal{I})) \\ &\quad\times(([\delta X]^2_h + \gamma_2\|G_h\delta X\|^2)/\gamma_3)^{1/2} \\ &\le \frac{\widetilde{c}_\Phi}{(\gamma_3)^{1/2}\alpha^*_\Phi}(\|A\|\,\|X_I - X\| + r\mathcal{R}(\mathcal{X}_\mathcal{I})) \\ &\quad\times([\delta X]_h + (\gamma_2)^{1/2}\|G_h\delta X\|).\end{aligned} \tag{3.32}$$

Then we apply the inequality $ab \le \frac{4}{3}a^2 + \frac{1}{3}b^2$ four times to the right-hand side, move four terms to the left and multiply the resulting equation by 3 to get (3.30). □

Remark 3.4. In applications, as we shall see, (3.30) will often be used with an r depending on h, while the other constants are independent of h. Still, we shall find cases in which the constant γ_2 in (3.5) can also be chosen to be dependent on h, and of the same order of magnitude of r. In these latter cases, (3.30) will provide an estimate of the type:

$$[\delta X]^2_h + r\|G_h(\delta X)\|^2_\mathcal{H} \le C\left(|X_I - X\|^2 + r^2(\mathcal{R}(X_I))^2\right), \tag{3.33}$$

with C independent of h, which, in its turn, can become

$$\begin{aligned}&[\delta X]^2_h + r\|G_h(\delta X)\|^2_\mathcal{H} \\ &\quad\le C\left((1 + r^2)\|X_I - X\|^2 + r^2\|X - \overline{X}_I)\|^2\right),\end{aligned} \tag{3.34}$$

using the bound (3.26) for $\mathcal{R}$. More generally, if there exists a constant κ independent of h such that $r \ge \kappa\gamma_2$, then we can apply H.3 to the left-hand side of (3.30) obtainig

$$\begin{aligned}&\min(1,\kappa)\,\|\delta X\|^2 \\ &\le \left(\frac{\widetilde{c}_\Phi}{\alpha^*_\Phi}\right)^2 \frac{4(1 + 1/\kappa)}{\gamma_3^2}\left(\|A\|^2\|X_I - X\|^2 + r^2(\mathcal{R}(X_I))^2\right).\end{aligned} \tag{3.35}$$

In other applications, r will depend on h but γ_2 will not. In these cases (3.30) will provide (for r "small") an estimate of the type

$$(3.36)\qquad \begin{aligned}&[\delta X]_h^2 + r\|G_h(\delta X)\|_{\mathcal{H}}^2 \\ &\le C\left(\frac{1}{r}\|X_I - X\|^2 + r\|X - \overline{X}_I)\|^2\right),\end{aligned}$$

that will then become

$$[\delta X]_h^2 + r\|G_h \delta X\|^2 \le C\left(\frac{1}{r}h^{s_1} \ + \ rh^{s_2}\right),$$

by usual interpolation estimates with, in general, $s_1 \ge s_2 \ge 0$. Then by taking $r = h^s$ we get

$$[\delta X]_h^2 + h^s\|G_h \delta X\|^2 \le C\left(h^{s_1 - s} + h^{s_2+s}\right)$$

with the optimal choice given by $s = (s_1 - s_2)/2$. □

4 A first class of applications

We shall start by considering a framework which is still abstract but deals with a subclass of problems (with similar features) containing many of the applications that will be discussed later on.

Suppose that we are in the context of mixed methods as in (1.1), and that we want to stabilise an inf-sup condition (1.5). Then we have

$$(4.1)\qquad \langle AX_h, Y_h\rangle = a(u_h, v_h) + b(v_h, p_h) - b(u_h, q_h).$$

We want to work on a choice of spaces $V_h \times Q_h$ for which we do not have stability, and we are aiming at using stabilised problems of the form

$$(4.2)\quad \begin{cases} a(u_h, v_h) + b(v_h, p_h) - b(u_h, q_h) + rR((u_h, p_h), (v_h, q_h)) \\ = \langle f, v_h\rangle - \langle g, q_h\rangle \quad \forall v_h \in V_h,\ \forall q_h \in Q_h. \end{cases}$$

for a suitable choice of R. This section will be dedicated to the stabilisation of problems of type (1.1) that satisfy the following assumptions.

A.0 *The bilinear forms $a(\ ,\ )$ and $b(\ ,\ )$ are continuous on $V \times V$ and $V \times Q$ respectively. Moreover $a(\ ,\ )$ is V-elliptic (see (2.14)) and $b(\ ,\ )$ satisfies the inf-sup condition (1.5) in $V \times Q$.*

A.1 *There exists a subspace $\overline{Q}_h \subset Q_h$ such that in $V_h \times \overline{Q}_h$ the problem is stable, that is*

$$(4.3)\ \exists\, \overline{k} > 0\ s.t. \quad \sup_{v_h \in V_h} \frac{b(v_h, \overline{q}_h)}{\|v_h\|_V} \ge \overline{k}\, \|\overline{q}_h\|_Q \qquad \forall \overline{q}_h \in \overline{Q}_h.$$

□

We note, incidentally, that the V-ellipticity assumption on $a(\, , \,)$ is not really relevant and only serves to simplify the presentation. Under the assumption A1 it is possible to explicitly build Φ_h and a semi-norm $[\![\cdot]\!]_h$ to apply our results, as we shall see in the follofing lemma.

Lemma 4.1. *Assumptions A.0 and A.1 imply H.1.*

Proof of Lemma 4.1. As we have stability in $V_h \times \overline{Q}_h$, we can solve, for any $p_h \in Q_h$, the problem: find $\overline{u}_h = \overline{u}_h(p_h) \in V_h$ and $\overline{\phi}_h = \overline{\phi}_h(p_h) \in \overline{Q}_h$ such that

$$\begin{cases} a(v_h, \overline{u}_h) - b(v_h, \overline{\phi}_h) = 0 & \forall v_h \in V_h, \\ b(\overline{u}_h, \bar{q}_h) = (p_h, \bar{q}_h)_Q = (\bar{p}_h, \bar{q}_h)_Q & \forall \bar{q}_h \in \overline{Q}_h. \end{cases} \tag{4.4}$$

where $\bar{p}_h = \overline{P}(p_h) =$ projection of p_h onto $\overline{Q}_h$. We define now

$$\Phi_h((u_h, p_h)) := (u_h + \alpha \overline{u}_h(p_h), p_h + \alpha \overline{\phi}_h(p_h)) \tag{4.5}$$

using, for instance, the same α as in (2.14). From (4.5) and (4.4) we have

$$\begin{aligned} &a(u_h, u_h + \alpha \overline{u}_h) + b(u_h + \alpha \overline{u}_h, p_h) - b(u_h, p_h + \alpha \overline{\phi}_h) \\ &\quad = a(u_h, u_h) + \alpha [a(u_h, \overline{u}_h) - b(u_h, \overline{\phi}_h)] + \alpha b(\overline{u}_h, \overline{\phi}_h)) \\ &\quad \geq \alpha (\|u_h\|^2 + \|\bar{p}_h\|^2). \end{aligned} \tag{4.6}$$

We then have that hypothesis H.1 is satisfied with the choice (4.5) for Φ_h and the seminorm

$$[(v, q)]_h := \|v\|_V^2 + \|\overline{P} q\|_Q^2. \tag{4.7}$$

□

We note in particular that, if A.0 and A.1 hold then the constants c_Φ, α_Φ in H.1 will be independent of h. On the other hand, hypotheses H.2 and H.3 will be easily fullfilled, with constants independent of h, if we take

$$G_h((v_h, q_h)) = q_h - \overline{P} q_h \tag{4.8}$$

with $\mathcal{H} = Q$, and, as in (3.25),

$$R((u_h, p_h), (v_h, q_h)) = \left(p_h - \overline{P} p_h, q_h - \overline{P} q_h \right)_Q . \tag{4.9}$$

It is however clear that the class of possible stabilisations is much wider, as shown in the following Lemma.

Lemma 4.2. *Let s_h be a linear operator from Q_h into itself, satisfying*

$$\begin{cases} s_h(\bar{q}_h) = 0 & \forall \bar{q}_h \in \overline{Q}_h, \\ \|s_h(q_h)\|_Q^2 \geq \alpha_S \|q_h - \overline{P} q_h\|_Q^2 & \forall q_h \in Q_h, \\ \|s_h(q_h)\|_Q \leq c_S \|q_h\|_Q & \forall q_h \in Q_h, \end{cases} \tag{4.10}$$

with constants α_S and c_S independent of h. We take now $\mathcal{H} = Q$ and

$$G_h((v_h, q_h)) := s_h(q_h), \tag{4.11}$$

$$R((u_h, p_h), (v_h, q_h)) := (s_h(p_h), s_h(q_h))_Q. \tag{4.12}$$

If assumptions A.0 and A.1 hold, then H.2 and H.3 will also hold, with constants independent of h. Moreover, if Φ_h is defined as in (4.5), then H.4 will also hold. Finally, H.5 will hold with $\widetilde{\Phi}_h$ defined as in Lemma 3.1.

Proof of Lemma 4.2. It is clear that (3.4) follows from (4.12) and (4.11) with $\alpha_R = 1$. It is also clear that (3.3) holds with constant $c_R = c_S^2$. Similarly (3.5) follows, with constants γ_2 and γ_3 independent of h, from (4.7) and (4.10)-(4.12). Finally, from (4.5) and (4.10)-(4.12) we have

$$\begin{aligned} & R((v_h, q_h), \Phi_h(v_h, q_h)) = (s_h(q_h), s_h(q_h + \alpha \overline{\phi}_h(q_h)))_Q \\ & = \|s_h(q_h)\|_Q^2 + \alpha (s(q_h), s(\overline{\phi}_h(q_h)))_Q = \|s_h(q_h)\|_Q^2 \geq 0. \end{aligned} \tag{4.13}$$

The validity of H.5 follows then directly from Lemma 3.1. □

We can now conclude with a general error estimate for this type of stabilisations.

Theorem 4.1. *Assume that A.0 and A.1 hold, and let (u, p) be the solution of Problem (1.1). Assume that in (4.2) R is defined through (4.11) and (4.12) using an s_h that satisfies (4.10). Then for every positive r Problem (4.2) has a unique solution (u_h, p_h), and there exist a consant C, independent of h, such that, for every $(u_I, p_I) \in V_h \times Q_h$ and for every $\overline{q}_h \in \overline{Q}_h$ we have*

$$\begin{aligned} & \|u_I - u_h\|_V^2 + \|\overline{P}(p_I - p_h)\|_Q^2 + r\|(Id - \overline{P})(p_I - p_h)\|_Q^2 \\ & \leq C\,(\frac{1+r}{r}) \left(\|u - u_I\|_V^2 + \|p - p_I\|_Q^2 + r^2 \|p_I - \overline{q}_h\|\right). \end{aligned} \tag{4.14}$$

Proof of Theorem 4.1. Under the above assumptions, we can immediately apply Lemma 3.4, that in our case gives, for every $(u_I, p_I) \in V_h \times Q_h$,

$$\begin{aligned} & \|u_I - u_h\|_V^2 + \|\overline{P}(p_I - p_h)\|_Q^2 + r\|(Id - \overline{P})(p_I - p_h)\|_Q^2 \\ & \leq C\,(\frac{1+r}{r}) \left(\|u - u_I\|_V^2 + \|p - p_I\|_Q^2 + r^2 (\mathcal{R}((u_I, p_I))^2\right) \end{aligned} \tag{4.15}$$

with a constant C independent of h. Using then (4.10)-(4.12) we have, for every $\overline{q}_h \in \overline{Q}_h$

$$\mathcal{R}((u_I, p_I)) = \sup_{q_h} \frac{(s_h(p_I), s_h(q_h))_Q}{\|q_h\|_Q}$$

$$= \sup_{q_h} \frac{(s_h(p_I - \overline{q}_h), s_h(q_h))_Q}{\|q_h\|_Q} \leq c_S^2 \|p_I - \overline{q}_h\|_Q, \tag{4.16}$$

which inserted in (4.15) gives the result. □

It is clear that traditional bounds for the error between the continuous solution and the discrete solution can be obtained from (4.14) by a suitable use of the triangle inequality, as we are going to do in the following corollaries. However, as we shall see, it will be convenient to split them in two cases: one in which r is bounded from below by a positive constant independent of h (but we allow it to be arbitrarily large), and the other in which r is bounded from above by a positive constant independent of h (but we allow it to go to zero for h going to zero). In order to simplify the exposition, we introduce first the following notation: given a Hilbert space W, an element $w \in W$ and a subspace $W_h \subset W$ we set:

$$E(w, W_h) := \inf_{w_h \in W_h} \|w - w_h\|_W. \tag{4.17}$$

The following notation will also be convenient: for $p \in Q$, for $\overline{Q}_h \subset Q_h \subset Q$ and r a positive real number, we set:

(4.18)

$$E_r(p, Q_h, \overline{Q}_h) := \left(\inf_{q_h \in Q_h} \inf_{\bar{q}_h \in \bar{Q}_h} \left(\|p - q_h\|_Q^2 + r^2 \|q_h - \bar{q}_h\|_Q^2 \right) \right)^{1/2}.$$

We have then the following two corollaries, whose proof follows immediately from Theorem 4.1.

Corollary 4.1. *In the same hypotheses of Theorem 4.1, assume that there exists an r_0 independent of h such that $r \geq r_0$. The there exist a constant C, independent of r and h, such that*

$$\begin{aligned} &\|u - u_h\|_V^2 + \|p - p_h\|_Q^2 + r\|p_h - \overline{P}p_h\|_Q^2 \\ &\leq C\,(\frac{1+r}{r}) \left(E^2(u, V_h) + E_r^2(p, Q_h, \overline{Q}_h) \right). \end{aligned} \tag{4.19}$$

Corollary 4.2. *In the same hypotheses of Theorem 4.1, assume that there exists an r_0 independent of h such that $r \leq r_0$. The there exist a constant C, independent of r and h, such that*

$$\begin{aligned} &\|u - u_h\|_V^2 + \|\overline{P}(p - p_h)\|_Q^2 + r\|(p - p_h) - \overline{P}(p - p_h)\|_Q^2 \\ &\leq C\,(\frac{1+r}{r}) \left(E^2(u, V_h) + E_r^2(p, Q_h, \overline{Q}_h) \right), \end{aligned} \tag{4.20}$$

Remark 4.1. As we have said, the result of Corollary 4.1 applies as well to the cases in which r is very large. In these cases, we remark that in (4.18) we can obviously choose $q_h \in \overline{Q}_h$, so that $E_r(p, Q_h, \overline{Q}_h) \leq E(p, \overline{Q}_h)$ and then from (4.19) we easily obtain

$$\text{(4.21)} \quad \|u - u_h\|_V^2 + \|p - p_h\|_Q^2 \leq C\left((E^2(u, V_h) + E^2(p, \overline{Q}_h)\right),$$

as we could have obtained directly from Lemma 3.3. In fact for r large the method is equivalent to penalising the *unstable part* of Q_h to actually obtain a solution in $\overline{Q}_h$. The *theoretical* interest of this choice seems questionable, as we could use directly a discretisation with V_h and $\overline{Q}_h$, that would be stable and provide essentially the same error bound. In practice however this choice could still be interesting for various reasons. For instance the choice of Q_h might be dictated by other equations that have to be solved toghether with (1.1), or by some optimistic hope of an improvement in the constants, providing better results for a fixed h.

Remark 4.2. In fact the most interesting case is covered by Corollary 4.2, and corresponds to use an r that goes to zero when h goes to zero, in the spirit of Remark 3.4. This becomes specially interesting when $E(p, \overline{Q}_h)$ is of a lower order than $E(u, V_h)$. In this case, we can add and subtract p in the expression of $E_r(p, Q_h, \overline{Q}_h)$ to obtain

(4.22)
$$\|p - q_h\|_Q^2 + r^2\|q_h - \bar{q}_h\|_Q^2 \leq (1 + 2r^2)\|p - q_h\|_Q^2 + 2r^2\|p - \bar{q}_h\|_Q^2$$

and then (4.20) easily becomes

$$\text{(4.23)} \quad \begin{aligned} &\|u - u_h\|_V^2 + \|\overline{P}(p - p_h)\|_Q^2 + r\|(p - p_h) - \overline{P}(p - p_h)\|_Q^2 \\ &\leq C\left(\frac{1}{r}(E^2(u, V_h) + E^2(p, Q_h)) + rE^2(p, \overline{Q}_h)\right). \end{aligned}$$

When $\overline{Q}_h$ provides a worse accuracy (with respect to V_h and Q_h,) so that the term $E^2(p, \overline{Q}_h)$ is bigger than the term $E^2(u, V_h) + E^2(p, Q_h)$, a small r can, somehow, compensate the difference (see Remark 3.4). Notice that, in this case, the theory can be applied with $\overline{Q}_h = \{0\}$ (*pure penalty* methods.) □

Example 4.1 Stabilisation of the $Q_1 - P_0$ element

The first case that we consider has been studied by Sylvester [23] for the Stokes problem. The goal is the stabilisation of the classical bilinear velocity–constant pressure (Q_1–P_0) approximation which notoriously suffers from stability problems ([21,9,18]). We thus consider the Stokes problem

(4.24)
$$\begin{cases} \int_\Omega \epsilon(\underline{u}) : \epsilon(\underline{v})\,\mathrm{dx} - \int_\Omega p\,\mathrm{div}\underline{v}\,\mathrm{dx} = \int_\Omega \underline{f} \cdot \underline{v}\,\mathrm{dx} & \forall \underline{v} \in (H_0^1(\Omega))^2 \\ \int_\Omega q\,\mathrm{div}\underline{u}\,\mathrm{dx} = 0 & \forall q \in L_0^2(\Omega), \end{cases}$$

where $L_0^2(\Omega)$ is the set of square integrable functions with null average.

Let $\mathcal{T}_h$ be a partition of Ω into rectangles (we restrict ourselves to this simplified setting, instead of the general isoparametric case, for the sake of a lighter presentation.) We now take for V_h the space of piecewise bilinear continuous functions, and for Q_h the space of piecewise constants:

(4.25)
$$\begin{cases} V_h = \{\underline{v}_h \in (H_0^1(\Omega))^2 \mid \underline{v}_{h|K} = \underline{a} + \underline{b}x + \underline{c}y + \underline{d}xy, \forall K \in \mathcal{T}_h\} \\ Q_h = \{q_h \in L_0^2(\Omega) \mid q_{h|K} = constant, \forall K \in \mathcal{T}_h\}\ . \end{cases}$$

On a rectangular mesh, it is well known that this approximation suffers from the *checkerboard spurious mode* on p_h: the kernel of the discrete gradient (in the space of piecewise constants) is two-dimensional and contains besides the expected global constants (which do not belong to Q_h) a second mode alternating values in a checkerboard pattern.

There also exist other unstable modes which emanate from local checkerboard patterns ([20]). Indeed, let us split $\mathcal{T}_h$ into 2×2 macroelements and on a macroelement M

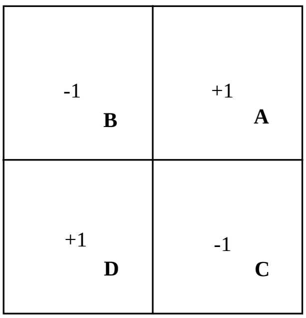

Fig. 4.1. Macroelement

let us define

$$CB_M = \begin{cases} 1 \text{ on } A, \\ -1 \text{ on } B, \\ -1 \text{ on } C, \\ 1 \text{ on } D \end{cases} \tag{4.26}$$

and

$$CB_h = \Big\{ q_h \mid \ q_{h|M} = \alpha CB_M \quad \forall M \Big\} . \tag{4.27}$$

It is easily seen (cfr. e.g.[21]) that, defining $\overline{Q}$ to be the orthogonal complement of CB_h in Q_h, the pair $V_h \times \overline{Q}_h$ gives a stable approximation which is equivalent (from the point view of degrees of freedom) to the $Q_2 - P_1$ piecewise quadratic–piecewise linear approximation. The above theory provides different possibilities for stabilising: we can take

$$R = R_1(p_h, q_h) = \big(p_h - \overline{P}p_h, q_h - \overline{P}q_h\big) \tag{4.28}$$

which corresponds to (4.9), or set, in each 2×2 macroelement, $s_h(q_h) = q_A + q_B - q_C - q_D$, and then use

$$R = R_2(p_h, q_h) = (s_h(p_h), s_h(q_h)) \tag{4.29}$$

which clearly satisfies (4.10). As we have seen in the previous section, both choices can be used with arbitrarily large r. It is clear that, for r large, the use of these stabilisations is equivalent to penalising the checkerboard mode and that the result is essentially the same as if one had used the stable approximation $V_h \times \overline{Q}_h$.

In Sylvester [23], one also uses

$$R(q_h, q_h) = ((q_B - q_A)^2 + (q_C - q_A)^2 + (q_D - q_B)^2 + (q_D - q_C)^2) \ .$$

For r large, this amounts to take

$$\overline{Q}_h = \{q_h | \, q_h = \text{constant on } M\} \, ,$$

that is the space of piecewise constants on macroelements, which is actually an “overstabilisation”.

Remark 4.3. An identical situation is met if we consider a triangular grid $\mathcal{T}_h$ which has been obtained from a coarser one, say $\widetilde{\mathcal{T}}_h$, by splitting as usual each triangle into four identical ones. Taking the space of piecewise linear continuous vectors on $\mathcal{T}_h$ for velocities and piecewise constants on $\widetilde{\mathcal{T}}_h$ for pressures we clearly have a stable pair. As above, this can be used to stabilise a $P1 - P0$ approximation on $\widetilde{\mathcal{T}}_h$, which by itself would be highly unstable. □

Example 4.2 Taylor-Hood approximation for Stokes

Another widely employed approximation for the Stokes problem is the Taylor -Hood P_2-P_1 element which uses, on triangles, a continuous approximation for pressure of degree *one* and a continuous approximation for velocity of degree *two*. This is apparently a drawback for many users who prefer the simplicity of the $P_2 - P_2$ equal-order interpolation. One could eventually think of using stabilisation as follows. Suppose that we use a piecewise quadratic approximation for the pressure.

Let us consider an edge at the interface of two triangles.Let A and B be the endpoints of this edge and C its midpoint. We can define

$$s_h(q_h) = q(A) - 2q(C) + q(B)$$

and

$$R((\underline{v}_h, q_h), (\underline{v}_h, q_h)) = \Sigma_{edges}(s_h(q_h))^2 \ .$$

Introducing this term with a large r obviously forces q_h to become linear on the edge, thus reducing the approximation to the Taylor-Hood approximation. It is easily seen that the theory applies and that, for r large, we get the usual $O(h^2)$ error estimates. We could also employ for both variables a piecewise linear approximation on macro-elements obtained by subdividing each triangle into four subtriangles. One can then use the same trick, forcing the pressure to be linear on each macro-element, obtaining in the limit the popular variant often called the $P1 - isoP2$ approximation, with the usual $O(h)$ error estimate. We will not develop further, as this procedure (for obtaining $P1 - isoP2$ as a limit of a penalty method on $P1 - P1$) has never been implemented to our knowledge.

Example 4.3 Penalty methods

We still consider the Stokes problem (4.24), and we employ for $V_h \times Q_h$ the unstable choice,

$$\text{(4.30)} \quad \begin{cases} V_h = \left\{ \underline{v}_h \in (H_0^1(\Omega))^2 \mid \underline{v}_{h|K} \in P_2(K), \quad \forall K \in \mathcal{T}_h \right\} \\ Q_h = \left\{ q_h \in L_0^2 \mid q_{h|K} \in P_1(K), \quad \forall K \in \mathcal{T}_h \right\} . \end{cases}$$

Note that (4.30) is a discontinuous pressure approximation as we impose no continuity requirement on Q_h at interfaces.This not a stable choice and the classical procedures to make it stable are

1. Use a larger V_h. The Crouzeix-Raviart element [13] is built along this option by adding cubic bubble functions to V_h .

2. Use a smaller Q_h. Taking as previously $\overline{Q}_h$ as the space of piecewise constant pressures yields a stable approximation, at the price of a loss of accuracy: this $P_2 - P_0$ approximation is only $O(h)$ instead of the $O(h^2)$ that one expects from the choice of V_h.

Stabilisation opens another avenue. The couple $V_h \times \overline{Q}_h$ is stable and we can define, as in Example 4.1, $R(p_h, q_h) = \big(p_h - \overline{P}p_h, q_h - \overline{P}q_h\big)$. The Stokes problem (4.24) becomes

(4.31)
$$\begin{cases} \int_\Omega \epsilon(\underline{u}_h) : \epsilon(\underline{v}_h)\,\mathrm{dx} - \int_\Omega p_h\, \mathrm{div}\underline{v}_h\,\mathrm{dx} = \int_\Omega \underline{f}\cdot\underline{v}_h\,\mathrm{dx} \;\forall\underline{v}_h \in V_h \\ \int_\Omega q_h\, \mathrm{div}\underline{u}_h\,\mathrm{dx} + r\left(p_h - \overline{P}p_h, q_h - \overline{P}q_h\right) = 0 \qquad \forall q_h \in Q_h. \end{cases}$$

This can also be written, after a few algebraic manipulations, as

$$(4.32)\quad \begin{cases} \int_\Omega \epsilon(\underline{u}_h) : \epsilon(\underline{v}_h)\,\mathrm{dx} - \int_\Omega \overline{p}_h\, \mathrm{div}\underline{v}_h\,\mathrm{dx} + \\ \qquad\qquad \dfrac{1}{r}\int_\Omega \mathrm{div}\underline{u}_h \mathrm{div}\underline{v}_h \mathrm{dx} = \int_\Omega \underline{f}\cdot\underline{v}_h\,\mathrm{dx} \;\forall\underline{v}_h \in V_h \\ \int_\Omega \overline{q}_h\, \mathrm{div}\underline{u}_h\,\mathrm{dx} = 0 \qquad \forall\overline{q}_h \in \overline{Q}_h, \end{cases}$$

where $\overline{p}_h$ now lies in $\overline{Q}_h$. This can be read as an augmented Lagrangian formulation for the constraint div $u_h = 0$. It can also be seen that, for r large, (4.32) reduces to the standard $P_2 - P_0$ approximation, as the "penalty" term (containig $1/r$) becomes negligible.

We can now apply the general results. For a fixed value of r, we get an $O(h)$ convergence rate as the consistency term

$$\mathcal{R} := \sup_{q_h} \frac{R(p_I, q_h)}{\|q_h\|} = \sup_{q_h} \frac{(p_I - \overline{P}p_I, q_h)}{\|q_h\|}$$

is obviouly only $O(h)$. However if we now employ the technique of Remark 4.2, taking $r = O(h)$ in (4.23) yields an $O(h^{3/2})$ estimate for velocities in H^1 and for the elementwise mean value of the pressure in L^2, as it has been pointed out in [7].

One can also see that, taking $\overline{Q}_h = \{0\}$, we obtain a *pure penalty* method. In this case our analysis provides the following result: if the space V_h yields an $O(h^k)$ approximation, taking $r = O(h^{k/2})$ we obtain globally an $O(h^{k/2})$ error estimate on velocities, regardless of the choice of approximation for Q_h.

5 A second class of applications

We consider now another general situation in which our abstract framework can be applied. We go back to a problem of the form (4.1) and we still make the assumption that $a(\cdot,\cdot)$ is elliptic on V. On the other hand, instead of assuming that we know a stable approximation $V_h \times \overline{Q}_h$, we make the following hypotheses.

A.2 *i) there exists a Hilbert space H with $V \subset H \equiv H' \subset V'$ and a function $\omega : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that*

$$\omega(h)\|v_h\|_V \leq \|v_h\|_H, \qquad \forall v_h \in V_h. \tag{5.1}$$

ii) if $B^{\mathrm{t}} : Q \longrightarrow V'$ is the linear operator associated with the bilinear form $b(v,q)$, we have

$$B^{\mathrm{t}}(Q_h) \subset H \tag{5.2}$$

iii) there exists a linear operator I from V into V_h and two positive constants σ and c_I, independent of h, such that

(5.3)

$$\|I(v) - v\|_H \leq \sigma\omega(h)\|v\|_V, \text{ and } \|I(v)\|_V \leq c_I\|v\|_V \quad \forall v \in V.$$

□

As an example, let us say that this assumption is verified when the pressure of Stokes problem is discretised by a space of continuous finite elements. Let us recall that from Proposition 2.2 we have a priori stability in the semi-norm

$$[(v_h, q_h)]_h^2 = \|v_h\|_V^2 + [\![q_h]\!]_h^2 \tag{5.4}$$

with

$$[\![q_h]\!]_h := \sup_{v_h} \frac{b(v_h, q_h)}{\|v_h\|_V} = \sup_{v_h} \frac{(v_h, B^{\mathrm{t}}q_h)_H}{\|v_h\|_V}. \tag{5.5}$$

We then have that assumption H.1 holds in our case, for the seminorm (5.4), with constants independent of h. Moreover it is obvious from (5.1) and (5.2) that we have

$$[\![q_h]\!]_h \geq \omega(h)\|P_{V_h}B^{\mathrm{t}}q_h\|_H \tag{5.6}$$

where P_{V_h} is the projection operator, in H, onto $V_h \subset V \subset H$.

The stability in the semi-norm (5.4) therefore implies also the stability in

$$[(v_h, q_h)]_*^2 = \|v_h\|^2 + \omega^2(h)\|P_{V_h}B^{\mathrm{t}}q_h\|_H^2. \tag{5.7}$$

and H.1 will hold, with constants independent of h, for the seminorm (5.7) as well. In agreement with the general procedure developed in Sect. 3, we can now take $\mathcal{H} = H$ with $G_h((v_h, q_h)) = B^{\mathrm{t}}q_h - P_{V_h}B^{\mathrm{t}}q_h$, and define:

$$R((u_h, p_h), (v_h, q_h)) = \left(B^{\mathrm{t}}p_h - P_{V_h}B^{\mathrm{t}}p_h, B^{\mathrm{t}}q_h - P_{V_h}B^{\mathrm{t}}q_h\right)_H. \tag{5.8}$$

It is clear that H.2 will hold with constants independent of h. Moreover it is easy to see, checking the expression of Φ_h in (2.21), that it leaves the second component invariant. Then from (5.8) we easily have that H.4 holds. We are left with H.3 which will be proved in the next two propositions using essentially the so-called Verfürth's trick [25].

Lemma 5.1. *Assume that A.0 and A.2 hold. Then*

$$c_I [\![q_h]\!]_h := c_I \sup_{v_h \in V_h} \frac{b(v_h, q_h)}{\|v_h\|_V} \geq k_0 \|q\|_Q - c_I \sigma \omega(h) \|B^{\mathrm{t}} q_h\|_H \; \forall q_h \in Q_h, \tag{5.9}$$

where k_0 is the inf-sup constant appearing in (1.5), $\omega(h)$ is given in (5.1), and σ, c_I are given in (5.3).

Proof of Lemma 5.1. We have from the inf-sup condition (1.5), and (5.3)

$$\begin{aligned} k_0 \|q_h\|_Q &\leq \sup_v \frac{b(v, q_h)}{\|v\|_V} = \sup_v \left(\frac{b(I(v), q_h)}{\|v\|_V} + \frac{b(v - I(v), q_h)}{\|v\|_V} \right) \\ &\leq c_I \sup_v \frac{b(I(v), q_h)}{\|I(v)\|_V} + \sup_v \frac{(v - I(v), B^{\mathrm{t}}(q_h))_H}{\|v\|_V} \\ &\leq c_I \sup_{v_h} \frac{b(v_h, q_h)}{\|v_h\|_V} + \sup_v \frac{\|v - I(v)\|_H \|B^{\mathrm{t}}(q_h)\|_H}{\|v\|_V} \\ &\leq c_I [\![q_h]\!]_h + \sigma \omega(h) \|B^{\mathrm{t}} q_h\|_H \quad \forall q_h \in Q_h. \end{aligned} \tag{5.10}$$

□

We can now easily get the following result.

Lemma 5.2. *Under the assumptions A.0 and A.2 there exists a constant $\widetilde{k}$, independent of h, such that*

$$[\![q_h]\!]_h^2 + \omega^2(h) \|B^{\mathrm{t}} q_h - P_{V_h} B^{\mathrm{t}} q_h\|_H^2 \geq \widetilde{k} \|q_h\|_Q^2 \quad \forall q_h \in Q_h. \tag{5.11}$$

Proof of Lemma 5.2. Indeed, from (5.6) one easily obtains

$$[\![q_h]\!]_h^2 + \omega^2(h) \|B^{\mathrm{t}} q_h - P_{V_h} B^{\mathrm{t}} q_h\|_H^2 \geq \omega^2(h) \|B^{\mathrm{t}} q_h\|_H^2, \quad \forall q_h \in Q_h, \tag{5.12}$$

and from (5.9)

$$2 c_I^2 [\![q_h]\!]_h^2 \geq k_0^2 \|q_h\|_Q^2 - 2 \sigma^2 \omega^2(h) \|B^{\mathrm{t}} q_h\|_H^2 \; . \tag{5.13}$$

Then (5.11) is obtained by summing (5.13) and (5.13) with appropriate constants. □

Lemma 5.2 implies that H.3 holds, with the above choices for $[\,\cdot\,]_h$ and G_h, with a constant γ_3 independent of h, and with $\gamma_2 = \omega^2(h)$. In a sense, we have a stability result that is *stronger* than necessary. However, if we look at the statement of Lemma 3.4, it is clear that a small γ_2 offers the possibility of using a small r without "paying the price". This is indeed what happens in the following convergence theorem.

Theorem 5.1. *Assume that A.2 holds, and let* (u, p) *be the solution of Problem (1.1). Assume that in (4.2)* R *is defined through (5.8). Then for every positive* r *Problem (4.2) has a unique solution* (u_h, p_h) *and there exists a constant* C*, independent of* h *and* r*, such that:*

(5.14)
$$\begin{aligned}&\|u-u_h\|_V^2+\|p-p_h\|_Q^2\\&\le C\,(\frac{\omega^2(h)+r}{r})\left(E^2(u,V_h)+(1+r^2)E^2(p,Q_h)+r^2E^2(B^{\mathrm{t}}p,V_h)\right)\end{aligned}$$

with the notation introduced in (4.17).

Proof of Theorem 5.1. The proof follows directly from Lemma 3.1, Lemma 3.4 the definition of R given in (5.8) and a few triangle inequalities. □

In certain cases, as we are going to see in the sequel, it might be more convenient to consider another subspace $\widetilde{V}_h$ of H and change the definition of R (5.8) into

$$R((u_h,p_h),(v_h,q_h)) = \left(B^{\mathrm{t}}p_h - P_{\widetilde{V}_h}B^{\mathrm{t}}p_h, B^{\mathrm{t}}q_h - P_{\widetilde{V}_h}B^{\mathrm{t}}q_h\right)_H. \tag{5.15}$$

This will be allowed, and it will still give optimal error estimates, provided that we have the following inequality.

A.3 *With the notation of aAssumption A.2, there exists a positive constant* $\widetilde{\beta}$*, independent of* h*, such that*

$$\|P_{V_h}B^{\mathrm{t}}q_h\|^2+\|B^{\mathrm{t}}q_h-P_{\widetilde{V}_h}B^{\mathrm{t}}q_h\|^2\ge\widetilde{\beta}\,\|B^{\mathrm{t}}q_h\|^2\quad\forall q_h\in Q_h. \tag{5.16}$$

□

Indeed, proceeding as in Lemma 5.2, and using (5.6) inequality (5.16) will imply

$$[\![q_h]\!]_h^2+\omega^2(h)\|B^{\mathrm{t}}q_h-P_{\widetilde{V}_h}B^{\mathrm{t}}q_h\|_H^2\ge\widetilde{k}\|q_h\|_Q^2\quad\forall q_h\in Q_h. \tag{5.17}$$

We summarise the above discussion in the following theorem.

Theorem 5.2. *Assume that A.2 and A.3 hold, and let* (u, p) *be the solution of Problem (1.1). Assume that in (4.2)* R *is defined through (5.15). Then* *for every positive r Problem (4.2) has a unique solution (u_h, p_h) and there exists a constant C independent of r and h such that:*

$$\|u-u_h\|_V^2+\|p-p_h\|_Q^2+r\|B^{\mathrm{t}}(p-p_h)-P_{\widetilde{V}_h}B^{\mathrm{t}}(p-p_h)\|_Q^2$$
$$\leq C\,(\frac{\omega^2(h)+r}{r})\Big(E^2(u,V_h)+(1+r^2)E^2(p,Q_h)+r^2E^2(B^{\mathrm{t}}p,\widetilde{V}_h)\Big)$$
(5.18)

with the notation introduced in (4.17).

Proof of Theorem 5.2. The proof follows the same lines as the one of Theorem 5.1. □

Example 5.1

We consider again Stokes problem as described in Example 4.1. However we now consider a piecewise linear approximation for both V_h and Q_h, a so-called equal interpolation case. On a *general* mesh, it is not possible, to our knowledge, to build (as in the previous section) a subspace $\overline{Q}_h$ yielding a stable approximation. We can however apply in a straightforward way the previous results with $H=(L^2(\Omega))^2$. We have here $\omega^2(h)=0(h^2)$. We can also write the bilinear form $b(\cdot,\cdot)$ in two ways,

$$b(\underline{v}_h,q_h)=-\int_\Omega \operatorname{div}\underline{v}_h\, q_h\mathrm{dx}=+\int_\Omega \underline{v}_h\cdot\underline{\nabla} q_h\mathrm{dx}. \tag{5.19}$$

The stabilised Stokes problem now reads

$$\begin{cases} a(\underline{u}_h,\underline{v}_h)+(\underline{v}_h,\underline{\nabla}\,p_h)=(\underline{f},\underline{v}_h) \quad \forall\underline{v}_h\in V_h,\\ (\underline{u}_h,\underline{\nabla}\,q_n)-r(\underline{\nabla}\,p_h-P_{V_h}\,\underline{\nabla}\,p_h,\underline{\nabla}\,q_h)=(g,q_h)\quad \forall q_h\in Q_h,\end{cases}$$
(5.20)

where we also introduced a possible right-hand side g in the second equation. The projection operator is not local and it is more convenient to write (5.20) in the form

$$\begin{cases} a(\underline{u}_n,\underline{v}_h)+(\underline{v}_h,\underline{\nabla}\,p_h)=(\underline{f},\underline{v}_h) & \forall\underline{v}_h\in V_h,\\ (\hat{\underline{u}}_h,\underline{\nabla}\,q_h)=r(\underline{\nabla}\,p_h,\underline{\nabla}\,q_h)+(g,q_h) & \forall q_h\in Q_h,\\ (\hat{\underline{u}}_h,\underline{v}_h)_H=(\underline{u}_h,\underline{v}_h)_H+(\underline{\nabla}\,p_h,\underline{v}_h)_H & \forall\underline{v}_h\in V_h,\end{cases} \tag{5.21}$$

where stability is seen to have been gained at the expense of a larger, non symmetric linear system. In practice, this can be solved by some iterative process.

To study convergence, we consider the estimate (5.14). It is clear that some potential trouble might lie in the last term of this inequality, that in our case is $r\|\underline{\nabla}\,p-P_{V_h}\underline{\nabla}\,p\|_H^2$. Indeed, the space V_h is made of functions vanishing on the boundary, while $\underline{\nabla}\, p$ does not. This induces a bad approximation near the boundary and it is easy to see that the term at hand is $O(h)$ for p regular enough. To get the correct order of convergence, we are thus led to use $r = O(h)$, which still is going to ensure convergence, as it will give $r \geq c_1\omega^2(h)$ asymptotically.

It must be recalled that the more classical stabilised problem (Brezzi-Pitkäranta [ ])

$$\begin{cases} a(\underline{u}_h, \underline{v}_h) + (\underline{v}_h, \underline{\nabla}\, p_h) = (\underline{f}, \underline{v}_h) & \forall v_h \in V_h, \\ (\underline{u}_h, \underline{\nabla}\, p_h,) + r(\underline{\nabla}\, p_h, \underline{\nabla}\, q_h) = 0 & \forall q_h \in Q_h, \end{cases} \tag{5.22}$$

requires $r = 0(h^2)$ in order to get the right order of approximation. This is also clear from our estimate if we take $P_{V_h} \equiv 0$. At the first sight, one might think that stabilising with $r = O(h^2)$ is somehow better than using $O(h)$, as the consistency error becomes smaller. However, numerical experiments show that the scheme (5.22) with $r = \delta\, h^2$ suffers from minor instabilities (oscillations of the pressure variable near to the boundary) when δ is too small, while for a larger δ a boundary layer will appear (corresponding to a Neumann boundary condition $r\partial p/\partial n = 0$.) The same is true for the scheme (5.20) if we take $r = \delta\, h$. On the other hand, very good results have been observed experimentally by Habashi et alii [6] if, instead of P_{V_h}, one uses the projection $P_{\widetilde{V}_h}$ on the space $\widetilde{V}_h$ in which boundary conditions are ignored. In particular, this choice eliminates the boundary layer effect, and allows to take a much bigger r (for instance $r = 1$) in order to suppress the oscillations. It is clear that with this choice we could recover the right order of convergence in the right-hand side of (5.14). We are then in the situation of Theorem 5.2, and we have to prove that inequality (5.16) of Assuption A3 holds. A result of this type is claimed in [12]. Since the proof there is rather complicated and might require some minor fixing, for convenience of the reader we report here another proof, limited to the case k=1. The proof follows, in essence, similar lines (macroelements, continuous dependence of the constant on the shape of the macroelement and so on) of the original one in [12], but has a simpler presentation.

Proposition 5.1. *Let Q_h and V_h be the space of piecewise linear pressures and velocities as above, and let $\widetilde{V}_h$ be the space of piecewise linear continuous vectors on $\mathcal{T}_h$ (without boundary conditions.) There exists a constant $\beta^* > 0$, independent of h, such that, for every $q_h \in Q_h$ and for every $\underline{w}_h \in \widetilde{V}_h$, there exists a $\underline{v}_h^0 \in V_h$ verifying*

$$\|\underline{v}_h^0\| \leq \|\underline{\nabla} q_h\| \tag{5.23}$$

and

$$(\underline{v}_h^0, \underline{\nabla} q_h) + \|\underline{\nabla} q_h - \underline{w}_h\|^2 \geq \beta^* \|\underline{\nabla} q_h\|^2 \tag{5.24}$$

where scalar products and norms are all in $L^2(\Omega)$.

Proof of Proposition 5.1. Let us consider first a macroelement K made by the collection of triangles having one vertex P of $\mathcal{T}_h$ in common. Split $q_h = q_0 + q_\ell$, where q_0 is such that $\underline{\nabla} q_0$ has zero mean value in K and q_ℓ is linear on K (hence $\underline{\nabla} q_\ell$ = constant in K.) It is clear that $(\underline{\nabla} q_0, \underline{\nabla} q_\ell)_K = 0$. We take now $\underline{v}_h^0$, piecewise linear, continuous, vanishing on the boundary of K and having value $\sqrt{6}\underline{\nabla} q_\ell$ at the internal vertex P. An easy computation shows that:

$$\|\underline{v}_h^0\|_{0,K} = \|\underline{\nabla} q_\ell\|_{0,K} \tag{5.25}$$

and

$$(\underline{v}_h^0, \underline{\nabla} q_\ell)_K = \sqrt{\frac{2}{3}}\, \|\underline{\nabla} q_\ell\|_{0,K}^2. \tag{5.26}$$

On the other hand, $\underline{\nabla} q_0$ belongs to a space (piecewise constant vectors on K, with continuous tangential components, and zero mean on K) whose intersection with piecewise linear continuous vectors on K is reduced to the zero vector. As we are in finite dimension, there exists a positive constant δ_K such that, for every $\underline{\nabla} q_0$ and for every $\underline{w}_h$

$$\|\underline{\nabla} q_0 - \underline{w}_h\|^2 \geq \delta_K \|\underline{\nabla} q_0\|_{0,K}^2. \tag{5.27}$$

As $\underline{\nabla} q_\ell$ is clearly continuous and piecewise linear, (5.27) easily implies that

$$\begin{aligned} &\|\underline{\nabla} q_h - \underline{w}_h\|^2 = \|\underline{\nabla} q_0 + \underline{\nabla} q_\ell - \underline{w}_h\|^2 \\ &= \|\underline{\nabla} q_0 - \tilde{\underline{w}}_h\|^2 \geq \delta_K \|\underline{\nabla} q_0\|_{0,K}^2, \end{aligned} \tag{5.28}$$

and a simple scaling argument shows immediately that δ_K is independent of the *size* of K (notice that (5.28) holds *for every* $\underline{w}_h$.)

Finally we explicitly point out that

$$(\underline{v}_h^0, \underline{\nabla} q_0)_K = \frac{\underline{v}_h^0(P)}{3} \int_K \underline{\nabla} q_0 \mathrm{d}x = 0, \tag{5.29}$$

where P is the only vertex internal to K. From (5.26)-(5.29) one then gets that, for every q_h and for every $\underline{w}_h$, there is a $\underline{v}_h^0$, piecewise linear, continuous, and vanishing on the boundary of K, such that (5.25) holds and

$$(\underline{v}_h^0, \underline{\nabla} q_h)_K + \|\underline{\nabla} q_h - \underline{w}_h\|_{0,K}^2 \geq \beta_K \|\underline{\nabla} q_h\|_{0,K}^2, \tag{5.30}$$

for some positive constant β_K independent of q_h and $\underline{w}_h$. The result (5.23), (5.24) follows then easily from (5.30) by typical instruments (continuity of β_K, splitting of Ω into macroelements such that each triangle belongs at most to three different macroelements, and so on.) □

With the aid of Proposition 5.1 we can now prove Assumption A.3.

Proposition 5.2. *Let Q_h, V_h and $\widetilde{V}_h$ be as in Proposition 5.1. Then there exists a constant $\widetilde{\beta} > 0$ such that*

$$\|P_{V_h}\underline{\nabla} q_h\|^2 + \|\underline{\nabla} q_h - P_{\widetilde{V}_h}\underline{\nabla} q_h\|^2 \geq \widetilde{\beta}\,\|\underline{\nabla} q_h\|^2 \qquad \forall q_h \in Q_h \tag{5.31}$$

where all the norms are in L^2.

Proof of Proposition 5.2. We start by observing that, for every $\underline{v}_h^0$ and q_h we have

$$\begin{aligned}(\underline{v}_h^0, \underline{\nabla} q_h) &= (\underline{v}_h^0, P_{V_h}\underline{\nabla} q_h) \leq \|\underline{v}_h^0\|\,\|P_{V_h}\underline{\nabla} q_h\| \\ &\leq \frac{\beta^*}{2}\|\underline{v}_h^0\|^2 + \frac{1}{2\beta^*}\|P_{V_h}\underline{\nabla} q_h\|^2\end{aligned} \tag{5.32}$$

where the last inequality clearly holds for every positive β^*, but we shall use it for the value of β^* given in (5.24). For every q_h we take now $\underline{v}_h^0$ as given by Proposition 5.1, and using (5.23) we have

$$(\underline{v}_h^0, \underline{\nabla} q_h) \leq \frac{\beta^*}{2}\|\underline{\nabla} q_h\|^2 + \frac{1}{2\beta^*}\|P_{V_h}\underline{\nabla} q_h\|^2, \tag{5.33}$$

that, inserted in (5.24) with $\underline{w}_h = P_{\widetilde{V}_h}\underline{\nabla} q_h$ gives

$$\frac{\beta^*}{2}\|\underline{\nabla} q_h\|^2 + \frac{1}{2\beta^*}\|P_{V_h}\underline{\nabla} q_h\|^2 + \|\underline{\nabla} q_h - P_{\widetilde{V}_h}\underline{\nabla} q_h\|^2 \geq \beta^*\,\|\underline{\nabla} q_h\|^2, \tag{5.34}$$

and (5.31) follows immediately. □

We can then apply Theorem 5.2, and see that, for $r = O(1)$, the stabilised Stokes problem:

$$\begin{cases} a(\underline{u}_h, \underline{v}_h) + (\underline{v}_h, \underline{\nabla}\, p_h) = (\underline{f}, \underline{v}_h) \quad \forall \underline{v}_h \in V_h, \\ (\underline{u}_h, \underline{\nabla}\, q_n) - r(\underline{\nabla}\, p_h - P_{\widetilde{V}_h}\, \underline{\nabla}\, p_h, \underline{\nabla}\, q_h) = (g, q_h) \quad \forall q_h \in Q_h, \end{cases} \tag{5.35}$$

is stable and optimally convergent when we take piecewise linear continuous velocities and pressure, and for $\widetilde{V}_h$ the space of piecewise linear continuous vectors without boundary conditions. □

Example 5.2

Let us consider a “ mixed formulation” of the Dirichet problem.

$$\begin{cases} (\underline{\sigma}, \underline{\tau}) + (\underline{\tau}, \underline{\nabla}\, \psi) = 0 \ \ \forall \underline{\tau} \in \Sigma, \\ (\underline{\tau}, \underline{\nabla}\, \varphi) = (f, \varphi) \qquad \forall \varphi \in \Psi, \end{cases} \tag{5.36}$$

having taken $\Sigma = (L^2(\Omega))^2 = H = \Sigma'$, $\Psi = H^1_0(\Omega)$. At the continuous level, this is nothing but a somewhat bizarre way of writing the standard formulation

$$\int_\Omega \underline{\nabla}\, \psi \cdot \underline{\nabla}\, \varphi \,\mathrm{dx} = \int_\Omega f\, \varphi \,\mathrm{dx} \quad \forall \varphi \in \Psi. \tag{5.37}$$

This equivalence however does not hold in general for discretised problems, unless Ψ_h and Σ_h are chosen in such a way that the space of gradients of Ψ_h is contained in Σ_h. Let us consider, as an example, a case in which this condition is violated: the so-called equal-order interpolation. We take

$$\begin{cases} \Psi_h = \{\varphi_h \in H^1_0(\Omega) \mid \varphi_{h|K} \in P_k(K)^2 \ \ \forall K \in \mathcal{T}_h\} \\ \Sigma_h = \{\underline{\tau}_h \in (H^1(\Omega))^2 \mid \underline{\tau}_{h|K} \in (P_k(K))^2 \ \ \forall K \in \mathcal{T}_h\} \end{cases} \tag{5.38}$$

and we look for $(\underline{\tau}_h, \varphi_h)$ in $\Sigma_h \times \Psi_h$ such that:

$$\begin{cases} (\underline{\sigma}_h, \underline{\tau}_h) + (\underline{\tau}_h, \underline{\nabla}\, \psi_h) = 0 \quad \forall \underline{\tau}_h \in \Sigma_h, \\ (\underline{\sigma}_h, \underline{\nabla}\, \varphi_h) = (f, \varphi_h) \qquad \forall \varphi_h \in \Psi_h. \end{cases} \tag{5.39}$$

This is not stable. Indeed one easily checks that we have

$$\sup_{\underline{\tau}_h} \frac{b(\underline{\tau}_h, \varphi_h)}{\|\underline{\tau}_h\|} =: [\![\varphi_h]\!]_h \geq \|P_{\Sigma_h} \underline{\nabla}\, \varphi_h\|_H \tag{5.40}$$

instead of

$$[\![\varphi_h]\!]_h \geq \|\underline{\nabla}\, \varphi_h\|_H, \tag{5.41}$$

which would ensure stability from Poincaré's inequality. Applying our procedure (with, clearly, $V = \Sigma$ and $Q = \Psi$,) we consider the stabilised problem,

$$\begin{cases} (\underline{\sigma}_h, \underline{\tau}_h) + (\underline{\tau}_h, \underline{\nabla}\, \psi_h) = 0, \ \ \forall \underline{\tau}_h \in \Sigma_h \\ (\sigma_h, \underline{\nabla}\, \varphi_h) + r(\underline{\nabla}\, \psi_h - P_{\Sigma_h} \underline{\nabla}\, \psi_h, \underline{\nabla}\, \varphi_h) = (f, \varphi_h) \ \forall \varphi_h \in \Psi_h. \end{cases} \tag{5.42}$$

Theorem 5.1 applies directly and we can get a convergence proof to the correct order in h. Notice that in this case there are no troubles with the boundary conditions, as we have them on Ψ and not on Σ.

Remark 5.1. The stabilised formulation (5.42) can be read as a convex combination of the standard discrete formulation

$$\int_\Omega \underline{\nabla}\, \psi_h \cdot \underline{\nabla}\, \varphi_h \,\mathrm{dx} = \int_\Omega f\, \varphi_h \,\mathrm{dx} \quad \forall \varphi_h \in \Psi_h. \tag{5.43}$$

and the mixed formulation (5.36). Indeed $P_{\Sigma_h} \underline{\nabla}\, \psi_h = \sigma_h$. □

Remark 5.2. Although we have followed the same general framework, there is a fundamental difference between Example 5.1 and Example 5.2 (beside the role of boundary conditions.) Indeed in this last case we have $V = \Sigma = H$ so that the constant $\omega(h) = 1$, while in Example 5.1 we had to employ the equivalence of norms in finite dimensional spaces. □

Example 5.3

We discuss now a "viscoelastic "problem. We consider a variant of Stokes problem, as a model problem for situations appearing in the finite element approximation of some viscoelastic flow problems. This example was, in fact, the first instance where the stabilisation technique developed in this paper was introduced. We refer to [14] and [15] for a more detailed presentation.

We take $V = (H_0^1(\Omega))^2, Q = L^2(\Omega)$ and $\Sigma = (L^2(\Omega))_s^2$, the space of symmetric square-integrable tensors, and we look for $(\underline{u}, p, \underline{\underline{\sigma}}) \in V \times Q \times \Sigma$ such that:

$$(5.44) \quad \begin{cases} (\underline{\underline{\sigma}}, \underline{\underline{\tau}}) + (G(\underline{\underline{\sigma}}), \underline{\underline{\tau}}) = \eta\,(\underline{\underline{\tau}}, \underline{\underline{\epsilon}}(\underline{u})) + (F(\underline{u}), \underline{\underline{\tau}}) \qquad \forall \underline{\underline{\tau}} \in \Sigma, \\ (div\ \underline{u}, q) = 0 \qquad \forall q \in Q, \\ (\underline{\underline{\sigma}}, \underline{\underline{\epsilon}}(\underline{v})) + (p, div\ \underline{v}) = (\underline{f}, \underline{v}) \qquad \forall \underline{v} \in V. \end{cases}$$

Here, η is a constant depending on the viscosity, and the functions $G(\cdot)$ and $F(\cdot)$ are representing rather complex terms which may vary from a model to another and can include Lie derivatives in convected models. They can be left undefined for our present purpose.
We now consider the discrete problem,

$$(5.45) \quad \begin{cases} (\underline{\underline{\sigma}}_h, \underline{\underline{\tau}}_h) + (G(\underline{\underline{\sigma}}_h), \underline{\underline{\tau}}_h) = \eta\,(\underline{\underline{\tau}}_h, \underline{\underline{\epsilon}}(\underline{u}_h)) + (F(\underline{u}_h), \underline{\underline{\tau}}_h) \quad \forall \underline{\underline{\tau}}_h \in \Sigma_h, \\ (div\ \underline{u}_h, q_h) = 0 \quad \forall q_h \in Q_h, \\ (\underline{\underline{\sigma}}_h, \underline{\underline{\epsilon}}(\underline{v}_h)) + (p_h, div\ \underline{v}_h) = (\underline{f}, \underline{v}_h) \qquad \forall \underline{v}_h \in V_h, \end{cases}$$

where V_h, Q_h and Σ_h are finite element subspaces of V, Q and Σ, respectively. Let us reduce this temporarily to a simple Stokes problem:

$$(5.46) \quad \begin{cases} (\underline{\underline{\sigma}}_h, \underline{\underline{\tau}}_h) = \eta\,(\underline{\underline{\tau}}_h, \underline{\underline{\epsilon}}(\underline{u}_h)) \quad \forall \underline{\underline{\tau}}_h \in \Sigma_h, \\ (div\ \underline{u}_h, q_h) = 0 \quad \forall q_h \in Q_h, \\ (\underline{\underline{\sigma}}_h, \underline{\underline{\epsilon}}(\underline{v}_h)) + (p_h, div\ \underline{v}_h) = (\underline{f}, \underline{v}_h) \qquad \forall \underline{v}_h \in V_h. \end{cases}$$

The first equation can now be read as:

$$(5.47) \qquad \underline{\underline{\sigma}}_h = P_{\Sigma_h}(\underline{\underline{\epsilon}}(\underline{u}_h))$$

and we can understand why we may have a stability problem, as $P_{\Sigma_h}(\underline{\underline{\epsilon}}(\underline{u}_h))$ is not strong enough to control $\underline{u}_h$ through a Korn's inequality, unless Σ_h is rich enough ([16]). Following the general procedure, we thus write, instead of (5.45), a stabilised form

(5.48)
$$\begin{cases} (\underline{\underline{\sigma}}_h, \underline{\underline{\tau}}_h) + (G(\underline{\underline{\sigma}}_h), \underline{\underline{\tau}}_h) = \eta\,(\underline{\underline{\tau}}_h, \underline{\underline{\epsilon}}(\underline{u}_h)) + (F(\underline{u}_h), \underline{\underline{\tau}}_h) \quad \forall \underline{\underline{\tau}}_h, \\ (div\; \underline{u}_h, q_h) = 0 \quad \forall q_h, \\ (\underline{\underline{\sigma}}_h, \underline{\underline{\epsilon}}(\underline{v}_h)) + r(\underline{\underline{\epsilon}}(\underline{u}_h) - P_{\Sigma_h}(\underline{\underline{\epsilon}}(\underline{u}_h)), \underline{\underline{\epsilon}}(\underline{v}_h)) \\ \qquad\qquad\qquad + (p_h, div\; \underline{v}_h) = (\underline{f}, \underline{v}_h) \quad \forall \underline{v}_h. \end{cases}$$

Applying the theory is again straightforward. In fact this is very close to the previous example but is much more relevant in applications, as it strongly widens the range of possible approximations of (5.44). Indeed, we may now use any reasonable approximation for Σ_h, the only constraint being to get the right order of precision. The price to pay is that the projection operator is most often not local, and that it has to be considered as an extra equation in the problem, which can also be written as

(5.49)
$$\begin{cases} (\underline{\underline{\sigma}}_h, \underline{\underline{\tau}}_h) + (G(\underline{\underline{\sigma}}_h), \underline{\underline{\tau}}_h) = \eta\,(\underline{\underline{\tau}}_h, \underline{\underline{\epsilon}}(\underline{u}_h)) + (F(\underline{u}_h), \underline{\underline{\tau}}_h) \quad \forall \underline{\underline{\tau}}_h, \\ (div\; \underline{u}_h, q_h) = 0 \quad \forall q_h, \\ (\underline{\underline{\sigma}}_h, \underline{\underline{\epsilon}}(\underline{v}_h)) + r(\underline{\underline{\epsilon}}(\underline{u}_h) - \underline{\underline{\widetilde{\sigma}}}_h, \underline{\underline{\epsilon}}(\underline{v}_h)) + (p_h, div\; \underline{v}_h) = (f, \underline{v}_h)\, \forall \underline{v}_h, \\ \underline{\underline{\widetilde{\sigma}}}_h = P_{\Sigma_h}(\underline{\underline{\epsilon}}(\underline{u}_h)). \end{cases}$$

We refer to [14] for details about implementation and numerical results.

6 Coercivity on the kernel of B

In all previous examples, we have used stabilisation to ensure an inf-sup condition. In many problems, e.g., plate problems, coercivity of the bilinear form $a(\cdot,\cdot)$ is an equally important issue and we can apply the same general framework to get stability when needed. Let us then suppose that, in problem (1.1), we have a bilinear form on $V \times V$ that is coercive only on $\ker B$. It is then natural to suppose that one has

(6.1)
$$a(v,v) + \|Bv\|^2 \geq \|v\|_V^2 \quad \forall v \in V.$$

The problem arises because, in general, $\ker B_h$ is not a subset of $\ker B$

Let us introduce then, instead of (1.6), a stabilised discrete problem: find $(u_h, p_h) \in V_h \times Q_h$, $V_h \subset V$, $Q_h \subset Q$, such that,

$$\begin{cases} a(u_h, v_h) + b(v_h, p_h) + r(Bu_h - B_h u_h, Bv_h)_Q = (f, v_h) & \forall v_h \in V_h, \\ b(u_h, q_h) = (g, q_h) & \forall q_h \in Q_h. \end{cases} \tag{6.2}$$

It is then obvious that we now have coercivity on the kernel of B_h. Here again we have employed the strategy of adding the minimum amount of stabilisation. In fact the stabilising term vanishes if $\ker B_h \subset \ker B$. We also notice that the stabilising term is in fact symmetric, as $B_h v_h$ is the projection of Bv_h on Q_h. As to error estimation, it is easy to obtain, using Lemma 3.3, the following result.

Proposition 6.1. *Let (u, p) be the solution of (1.6) and (u_h, p_h) the solution of the stabilised problem (6.2), with an r independent of h. Then there exists a constant C, independent of h, such that:*

$$\|u_h - u\|_V^2 + \|p_h - p\|_Q^2 \leq C\left(E^2(u, V_h) + E^2(p, Q_h) + E^2(Bu, Q_h)\right) \tag{6.3}$$

always with the notation (4.17). □

To fix ideas, let us consider a simple mixed formulation for the Dirichlet's problem: find $\underline{u} \in V = H(div, \Omega)$ and $p \in Q = L^2(\Omega)$ solution of

$$\begin{cases} (\underline{u}, \underline{v}) + (p, div\, \underline{v}) = 0, & \forall \underline{v} \in V \\ (div\, \underline{u}, q) = (g, q) & \forall q \in Q. \end{cases} \tag{6.4}$$

Here we have $B = div$ and this is a simple example in which the bilinear form $a(\cdot, \cdot)$ is coercive only on

$$\ker B = \{\underline{v}_0 \,|\; \underline{v}_0 \in H(div, \Omega),\, div\, \underline{v}_0 = 0\}\,.$$

Except for very special constructions (see e.g. [9] and the references therein) of the spaces V_h and Q_h, the discrete kernel $\ker B_h = \ker P_{Q_h} div$ is not a subset of $\ker B$. This is is the case, for instance, if one uses the Mini element of [3] to build V_h and Q_h. Let us recall it briefly: let $\mathcal{T}_h$ be a triangulation of Ω and let, for every $K \in \mathcal{T}_h$, b_K be the cubic bubble in K defined by $b_K = \lambda_1 \lambda_2 \lambda_3$. We set:

$$\begin{cases} V_h = \left\{\underline{v}_h \,|\, \underline{v}_h \in C^0(\Omega),\, \underline{v}_{h|K} \in (P_1(K) + \alpha_K b_K)^2 \quad \forall K \in \mathcal{T}_h\right\} \\ Q_h = \left\{q_h \,|\, q_h \in C^0(\Omega),\, q_{h|K} \in P_1(K) \quad \forall K \in \mathcal{T}_h\right\}. \end{cases} \tag{6.5}$$

It is classical that this approximation satisfies an inf-sup condition (in fact a stronger one than what we need here.)

To get a stabilised problem, we write

(6.6)
$$\begin{cases} (\underline{u}_h, \underline{v}_h) + (p_h, div\, \underline{v}_h) + r(div\underline{u}_h - P_{Q_h} div\underline{u}_h, div\underline{v}_h) = 0 \quad \forall \underline{v}_h \\ (div\, \underline{u}_h, q_h) = (g, q_h) \quad \forall q_h. \end{cases}$$

The projection operator is not, in general, local, and must be considered at the expenses of an extra equation. However it is easy to see that our general theory applies, and that the error estimate (6.3) yields the right order for the spaces at hand. In the particular case above, we can eliminate $P_{Q_h} div\underline{u}_h$ as the second equation of (6.6) states in fact that $P_{Q_h} div\underline{u}_h = P_{Q_h} g$. We can thus replace (6.6) by:

(6.7)
$$(\underline{u}_h, \underline{v}_h) + (p_h, div\, \underline{v}_h) + r(div\underline{u}_h - P_{Q_h} g, div\underline{v}_H) = 0 \quad \forall \underline{v}_h \in V_h$$
$$(div\, \underline{u}_h, q_h) = (g, q_h) \quad \forall q_h \in Q_h$$

This is very similar to the satbilisation introduced in [10].

This way of modifying the equations to bypass the coercivity problem proved to be fruitful also in the context of the approximation of Mindlin–Reissner plates ([1]) and shell problems ([22, 2]).

7 Conclusions

The various examples presented clearly show that the abstract theory developed here provides a unified framework for a wide class of applications, establishing links between apparently unrelated techniques. The theory also provides a general way of choosing the value of the stabilising parameter with respect to the mesh size and permits to obtain in some cases sharper error bounds.

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