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Convergence of a relaxation scheme for hyperbolic systems of conservation laws*

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Summary. This paper concerns the study of a relaxation scheme for $N \times N$ hyperbolic systems of conservation laws. In particular, with the compensated compactness techniques, we prove a rigorous result of convergence of the approximate solutions toward an entropy solution of the equilibrium system, as the relaxation time and the mesh size tend to zero.

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1 Introduction

The aim of this paper is to prove a rigorous result of convergence for a numerical scheme for the following $N \times N$ systems of conservation laws

$$u_t + f(u)_x = 0,$$

which is based on the well-known semilinear relaxation approximation of the form

$$\begin{cases} u_t + v_x = 0 \\ v_t + a^2 u_x = \frac{1}{\varepsilon}\left(f(u) - v\right). \end{cases} \tag{1.1}$$

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This kind of numerical schemes was firstly proposed in [5], where, in particular, the authors consider a first order upwind scheme and a second order MUSCL scheme, together with a second order TVD Runge-Kutta splitting scheme. The only rigorous prove of convergence of such schemes is done in the scalar case, namely, when the equilibrium system is a scalar conservation law, by proving the consistency and the monotonicity of the relaxing scheme. Another rigorous proof of convergence for slightly different first order and second order schemes can be found in [1], again in the scalar case.

The big advantage in considering this kind of schemes lies essentially on the special semilinear structure of the approximating system, which allows to solve it numerically without introducing Riemann solvers. Indeed, due to the presence of linear characteristic fields, the decomposition in characteristic variables for system (1.1) is trivial and it is possible to apply standard upwind schemes without solving Riemann problems, while the stiff relaxation term can be handled by introducing appropriate implicit time discretization.

In this contest, we are able to prove the convergence of a first order, upwind implicit relaxation scheme, when the limit is an $N \times N$ system, which satisfies to certain conditions. More precisely, we restrict ourselves to those equilibrium systems for which Serre in [11] proved that the relaxation approximation given in (1.1) converges as $\varepsilon \downarrow 0$.

For other theoretical results concerning hyperbolic systems with relaxation, we mention the book of Whitham [13], where for the first time this kind of models was taken into account, and the papers [8, 2, 6, 7] in the quasilinear case, and [3, 10] for the semilinear 2×2 case.

The remaining part of this paper is organized as follows. In the next section we describe our numerical scheme and we prove its stability in L^∞, by using mainly the result of [11] concerning the existence of positively compact invariant domains for systems of the form (1.1). Moreover, we prove also the basic L^2 estimates which are crucial in the control of the relaxation process and hence in the proof of the convergence of our approximate solution. In this point, we use the particular extension procedure considered again in [11], which allows to build global strictly convex and dissipative entropies for (1.1).

The last section is devoted to the proof of the convergence of the numerical solution to (1.1) toward an entropy solution of the equilibrium system

$$u_t + f(u)_x = 0. \tag{1.2}$$

To show this result, we will use the compensated compactness techniques. In particular, we will assume that (1.2) has enough entropy-entropy flux pairs (η, q) (see, for instance, the systems considered in [4]) and we will prove that the entropy production

$$\eta_t + q_x$$

lies in a relatively compact set of the space H^{-1}_{loc}, as the relaxation time ε and the mesh size of the approximate solutions tend to zero. Therefore, the convergence of our approximating sequence will follows from the arguments of [4].

We conclude this section by collecting the main properties of the system (1.1) proved in [11].

Proposition 1.1 *Let $\mathcal{K}$ be a convex characteristic set, namely, its boundary $\partial\mathcal{K}$ is stable under the differential $\nabla f(u)$. Let us assume the following subcharacteristic condition*

$$a > \max_{u\in\mathcal{K}}\{\rho(\nabla f(u))\}. \tag{1.3}$$

Then the following properties hold:

1. *The images $\mathcal{K}_\pm$ of $\mathcal{K}$ under the applications $h_\pm : u \longmapsto u \pm \frac{1}{a}f(u)$ are convex. Moreover, the maps $h_\pm : \mathcal{K} \longrightarrow \mathcal{K}_\pm$ are diffeomorphisms and*

$$\mathcal{K} = \frac{1}{2}\left(\mathcal{K}_+ + \mathcal{K}_-\right).$$

2. *The set*

$$D^a_{\mathcal{K}} = \left\{(u,v) : u + \frac{1}{a}v \in \mathcal{K}_+ \text{ and } u - \frac{1}{a}v \in \mathcal{K}_-\right\}$$

is a positively invariant set for the system (1.1).

Proposition 1.2 *Let $\mathcal{K}$ be a convex characteristic set and assume the subcharacteristic condition (1.3) holds. Moreover, let $(\eta,q) : \mathcal{K} \longrightarrow \mathbb{R}^2$ be an entropy-entropy flux pair for the equilibrium system (1.2). Then there exists an unique entropy-entropy flux pair $(E,Q) : D^a_{\mathcal{K}} \longrightarrow \mathbb{R}^2$ which coincides with (η,q) along the equilibrium curve $\{(u,v) : v = f(u)\}$. In particular, the pair (E,Q) is defined by*

$$E(u,v) = e_+\left(u + \frac{1}{a}v\right) + e_-\left(u - \frac{1}{a}v\right);$$
$$Q(u,v) = ae_+\left(u + \frac{1}{a}v\right) - ae_-\left(u - \frac{1}{a}v\right),$$

for any $(u,v) \in D^a_{\mathcal{K}}$ and the functions $e_\pm$ are uniquely defined on $\mathcal{K}_\pm$ by the following relations

$$e_+(h_+(u)) = \frac{1}{2}\left(\eta(u) + \frac{1}{a}q(u)\right) \quad e_-(h_-(u)) = \frac{1}{2}\left(\eta(u) - \frac{1}{a}q(u)\right),$$

for any $u \in \mathcal{K}$. Moreover,

- *if* η *is convex and nonnegative, then* E *is convex, nonnegative and* dissipative, *namely*

$$\nabla_v E(u,v)\cdot(f(u)-v)\le 0, \quad \textit{for any } (u,v)\in D^a_{\mathcal{K}};$$

- *if* $\nabla^2\eta > 0$ *on* $\mathcal{K}$*, then there exists a constant* $\alpha>0$ *such that*

$$\nabla_v E(u,v)\cdot(f(u)-v)\le -\alpha\|f(u)-v\|^2, \quad \textit{for any } (u,v)\in D^a_{\mathcal{K}}$$

  and $e_\pm$ *are strictly convex functions on* $\mathcal{K}_\pm$ *(in particular,* $E(u,v)$ *is strictly convex on* $D^a_{\mathcal{K}}$*).*

2 A priori estimates for the relaxation scheme

In this section we prove some a priori estimates which are the starting point to investigate the relaxation approximation of the $N\times N$ equilibrium system (1.2). More precisely, we will prove the stability in L^∞ of our numerical solution of the semilinear system (1.1), together with the L^2 control of the relaxation term $\frac{1}{\varepsilon}(f(u)-v)$, uniformly with respect to the relaxation time ε and the mesh sizes.

Let us consider the following one-step conservative scheme for the system (1.1)

(2.1)
$$\begin{cases} \frac{1}{k}\left[u_j^{n+1}-u_j^n\right]+\frac{1}{h}\left[v^n_{j+\frac12}-v^n_{j-\frac12}\right]=0\\ \frac{1}{k}\left[v_j^{n+1}-v_j^n\right]+\frac{a^2}{h}\left[u^n_{j+\frac12}-u^n_{j-\frac12}\right]=\frac{1}{\varepsilon}\left[f(u_j^{n+1})-v_j^{n+1}\right].\end{cases}$$

This scheme is obtained with a discretization of (1.1) in both the space and time variables and it is implicit in the source term, since, in general, it is possible to establish better stability properties for such kind of schemes without, in particular, mutual hypotheses on the relaxation parameter ε and the time and space mesh sizes k and h. Due to the semilinear structure of the relaxation system (1.1) and due to the particular structure of its entropy-entropy flux pair established in Proposition 1.2, it is convenient to introduce the following characteristic variables

$$w=u+\frac{1}{a}v \qquad\qquad z=u-\frac{1}{a}v.$$

Therefore, the scheme (2.1) can be rewritten in the following way

(2.2)
$$\begin{cases} \frac{1}{k}\left[w_j^{n+1}-w_j^n\right]+\frac{a}{h}\left[w^n_{j+\frac12}-w^n_{j-\frac12}\right]=\frac{1}{a\varepsilon}\left[f(u_j^{n+1})-v_j^{n+1}\right]\\ \frac{1}{k}\left[z_j^{n+1}-z_j^n\right]-\frac{a}{h}\left[z^n_{j+\frac12}-z^n_{j-\frac12}\right]=-\frac{1}{a\varepsilon}\left[f(u_j^{n+1})-v_j^{n+1}\right].\end{cases}$$

The point values $w^n_{j\pm\frac{1}{2}}$ and $z^n_{j\pm\frac{1}{2}}$ are defined by a (first order) upwind scheme as follows

$$\begin{aligned} w^n_{j+\frac{1}{2}} &= w^n_j \qquad & z^n_{j+\frac{1}{2}} &= z^n_{j+1} \\ w^n_{j-\frac{1}{2}} &= w^n_{j-1} \qquad & z^n_{j-\frac{1}{2}} &= z^n_j . \end{aligned}$$

Hence our upwind scheme in the Riemann variables is given by

$$\text{(2.3)} \quad \begin{cases} \frac{1}{k}\left[w^{n+1}_j - w^n_j\right] + \frac{a}{h}\left[w^n_j - w^n_{j-1}\right] = \frac{1}{a\varepsilon}\left[f(u^{n+1}_j) - v^{n+1}_j\right] \\ \frac{1}{k}\left[z^{n+1}_j - z^n_j\right] - \frac{a}{h}\left[z^n_{j+1} - z^n_j\right] = -\frac{1}{a\varepsilon}\left[f(u^{n+1}_j) - v^{n+1}_j\right]. \end{cases}$$

Moreover, in the original variables, we get the following relations

$$\begin{aligned} u^n_{j+\frac{1}{2}} &= \frac{1}{2}\left(u^n_j + u^n_{j+1}\right) - \frac{1}{2a}\left(v^n_{j+1} - v^n_j\right) \\ v^n_{j+\frac{1}{2}} &= \frac{1}{2}\left(v^n_j + v^n_{j+1}\right) - \frac{a}{2}\left(u^n_{j+1} - u^n_j\right) \\ u^n_{j-\frac{1}{2}} &= \frac{1}{2}\left(u^n_{j-1} + u^n_j\right) - \frac{1}{2a}\left(v^n_j - v^n_{j-1}\right) \\ v^n_{j-\frac{1}{2}} &= \frac{1}{2}\left(v^n_{j-1} + v^n_j\right) - \frac{a}{2}\left(u^n_j - u^n_{j-1}\right), \end{aligned}$$

and the scheme (2.1) becomes

$$\text{(2.4)} \quad \begin{cases} \frac{1}{k}\left[u^{n+1}_j - u^n_j\right] + \frac{1}{2h}\left[v^n_{j+1} - v^n_{j-1}\right] \\ \quad -\frac{a}{2h}\left[u^n_{j+1} - 2u^n_j + u^n_{j-1}\right] = 0 \\ \frac{1}{k}\left[v^{n+1}_j - v^n_j\right] + \frac{a^2}{2h}\left[u^n_{j+1} - u^n_{j-1}\right] \\ \quad -\frac{a}{2h}\left[v^n_{j+1} - 2v^n_j + v^n_{j-1}\right] = \frac{1}{\varepsilon}\left[f(u^{n+1}_j) - v^{n+1}_j\right]. \end{cases}$$

As usual, the first time steps u^0_j and v^0_j (and, equivalently, w^0_j and z^0_j) are defined as the mean values of the initial condition $u_0(x)$ and $v_0(x)$ over the j-th cell $I_j = \left[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}\right]$

(2.5)

$$u^0_j = \frac{1}{h}\int_{I_j} u_0(x)dx \qquad\qquad v^0_j = \frac{1}{h}\int_{I_j} v_0(x)dx$$

$$w^0_j = \frac{1}{h}\int_{I_j} w_0(x)dx = u^0_j + \frac{1}{a}v^0_j \qquad z^0_j = \frac{1}{h}\int_{I_j} z_0(x)dx = u^0_j - \frac{1}{a}v^0_j.$$

Moreover, we will assume the classical CFL condition for the schemes (2.3) and (2.4), namely

$$\text{(2.6)} \qquad a\lambda = a\frac{k}{h} \leq 1.$$

In the following lemma we establish the L^∞ stability for the schemes (2.3) and (2.4).

Lemma 2.1 *Let $\mathcal{K}$ be a convex characteristic set and assume $(u_0, v_0) \in D^a_{\mathcal{K}}$ (for instance, $u_0 \in \mathcal{K}$ and $v_0 = f(u_0)$). Moreover, assume the subcharacteristic condition (1.3) and the CFL condition (2.6) hold. Then*

$$(u_j^n, v_j^n) \in D^a_{\mathcal{K}}, \quad \textit{for any } n, j. \tag{2.7}$$

Proof. Due to the definition of $D^a_{\mathcal{K}}$, in order to prove (2.7), it suffices to prove

$$w_j^n \in \mathcal{K}_+ \text{ and } z_j^n \in \mathcal{K}_- \quad \text{for any } n, j.$$

From (2.5) and from the convexity of $\mathcal{K}_\pm$, it follows $w_j^0 \in \mathcal{K}_+$ and $z_j^0 \in \mathcal{K}_-$ for any j. Moreover, let us assume by induction

$$w_j^n \in \mathcal{K}_+ \text{ and } z_j^n \in \mathcal{K}_- \quad \text{for any } j.$$

Thus, Proposition 1.1 yields $u_j^n \in \mathcal{K}$ for any j. Hence, from the first line of (2.4) we get

$$u_j^{n+1} = \left(1 - \frac{ak}{h}\right) u_j^n + \frac{ak}{h}\left[\frac{1}{2}\left(u_{j+1}^n - \frac{1}{a}v_{j+1}^n + u_{j-1}^n + \frac{1}{a}v_{j-1}^n\right)\right]. \tag{2.8}$$

Now,

$$\begin{aligned}&\frac{1}{2}\left(u_{j+1}^n - \frac{1}{a}v_{j+1}^n + u_{j-1}^n + \frac{1}{a}v_{j-1}^n\right)\\ &\quad = \frac{1}{2}\left(z_{j+1}^n + w_{j-1}^n\right) \in \frac{1}{2}\left(\mathcal{K}_- + \mathcal{K}_+\right) = \mathcal{K},\end{aligned}$$

by using again Proposition 1.1. Therefore, from (2.8), the induction assumption, the convexity of $\mathcal{K}$ and the CFL condition (2.6), it follows

$$u_j^{n+1} \in \mathcal{K}. \tag{2.9}$$

With the aid of the function $h_\pm$ defined in Proposition 1.1, we can rewrite the right-hand-side of (2.3) as follows

$$\begin{aligned}\frac{1}{a\varepsilon}\left[f(u_j^{n+1}) - v_j^{n+1}\right] &= \frac{1}{\varepsilon}h_+(u_j^{n+1}) - \frac{1}{\varepsilon}w_j^{n+1}\\ -\frac{1}{a\varepsilon}\left[f(u_j^{n+1}) - v_j^{n+1}\right] &= \frac{1}{\varepsilon}h_-(u_j^{n+1}) - \frac{1}{\varepsilon}z_j^{n+1}.\end{aligned}$$

Thus, (2.3) yields

$$w_j^{n+1} = \left(1 + \frac{k}{\varepsilon}\right)^{-1} \left[\left(1 - \frac{ak}{h}\right) w_j^n + \frac{ak}{h} w_{j-i}^n + \frac{k}{\varepsilon} h_+(u_j^{n+1})\right]$$
$$z_j^{n+1} = \left(1 + \frac{k}{\varepsilon}\right)^{-1} \left[\left(1 - \frac{ak}{h}\right) z_j^n + \frac{ak}{h} z_{j+i}^n + \frac{k}{\varepsilon} h_-(u_j^{n+1})\right].$$

Since $u_j^{n+1} \in \mathcal{K}$ for any j, by definition we have $h_\pm(u_j^{n+1}) \in \mathcal{K}_\pm$ for any j. Hence, the above relation, together with the CFL condition (2.6), the convexity of $\mathcal{K}_\pm$ and the induction assumption, implies

$$(w_j^{n+1}, z_j^{n+1}) \in \mathcal{K}_+ \times \mathcal{K}_-,$$

which concludes the proof. □

We conclude the section by proving a discrete L^2_{loc} bound for the relaxation term $\frac{1}{\varepsilon}(f(u) - v)$. To perform this task, we will reduce ourselves to the case of a scheme verifying the following stronger CFL condition

$$a\lambda = \frac{ak}{h} = 1. \tag{2.10}$$

This restriction will allows us to get an extra control of the relaxation term, besides the usual L^2 control, which can be proved even under the weaker condition (2.6), which is necessary to carry out our argument in the study of the entropy production.

Lemma 2.2 *Let $\mathcal{K}$ be a convex characteristic set and assume $(u_0, v_0) \in D^a_{\mathcal{K}}$. Moreover, assume the subcharacteristic condition (1.3) and the CFL condition (2.10) hold. Finally, let (η, q) be a C^2 strictly convex entropy-entropy flux pair of the relaxed system (1.2) with $\eta \geq 0$. Then, for any N, H,*

$$\frac{hk}{\varepsilon} \sum_{n=0}^{N-1} \sum_{j=-H}^{H} \|f(u_j^{n+1}) - v_j^{n+1}\|^2 + \frac{hk^2}{\varepsilon^2} \sum_{n=0}^{N-1} \sum_{j=-H}^{H} \|f(u_j^{n+1}) - v_j^{n+1}\|^2 \leq C, \tag{2.11}$$

where the positive constant C depends only on $T = Nk$, $L = Hh$ and $D^a_{\mathcal{K}}$, and it is independent from ε, k and h.

Proof. Let $(E(u,v), Q(u,v))$ be the extension of $(\eta(u), q(u))$ given by Proposition 1.2 and let us denote with E_j^n the value of the function E in the point (u_j^n, v_j^n). Then

$$\begin{aligned}
&\frac{1}{k}\left[E_j^{n+1}-E_j^n\right]+\frac{1}{h}\left[Q^n_{j+\frac12}-Q^n_{j-\frac12}\right]\\
&\quad=\frac{1}{k}\left[e_+(w_j^{n+1})+e_-(z_j^{n+1})-e_+(w_j^n)-e_-(z_j^n)\right]\\
&\quad\;+\frac{a}{h}\left[e_+(w_j^n)-e_-(z_{j+1}^n)-\left(e_+(w_{j-1}^n)-e_-(z_j^n)\right)\right]\\
&\quad=\frac{1}{k}\left[e_+(w_j^{n+1})-e_+(w_{j-1}^n)\right]+\frac{1}{k}\left[e_-(z_j^{n+1})-e_-(z_{j+1}^n)\right]\\
&\quad=\frac{1}{a\varepsilon}\Big[\nabla e_+(w_j^{n+1})\cdot\left(f(u_j^{n+1})-v_j^{n+1}\right)\\
&\quad\;-\nabla e_-(z_j^{n+1})\cdot\left(f(u_j^{n+1})-v_j^{n+1}\right)\Big]\\
&\quad\;-\frac{k}{2a^2\varepsilon^2}\nabla^2(e_+(\widetilde{w})+e_-(\widetilde{z}))\\
&\quad\;\times\left[f(u_j^{n+1})-v_j^{n+1};f(u_j^{n+1})-v_j^{n+1}\right],
\end{aligned}\tag{2.12}$$

where with $\nabla^2 F[\xi;\xi]$ we denote the quadratic form in ξ associated with the hessian matrix of F. In (2.12) we used the CFL condition (2.10), the upwind reconstruction for the point values of the flux Q and the following relations, which come from (2.3) with the particular choice induced by (2.10)

$$w_j^{n+1}-w_{j-1}^n=-(z_j^{n+1}-z_{j+1}^n)=\frac{k}{a\varepsilon}\left[f(u_j^{n+1})-v_j^{n+1}\right].$$

Since η is strictly convex, Proposition 1.2 yields

$$\begin{aligned}
&\tfrac{1}{a\varepsilon}\Big[\nabla e_+(w_j^{n+1})\cdot\left(f(u_j^{n+1})-v_j^{n+1}\right)\\
&\qquad-\nabla e_-(z_j^{n+1})\cdot\left(f(u_j^{n+1})-v_j^{n+1}\right)\Big]\\
&\quad=\frac{1}{\varepsilon}\nabla_v E(u_j^{n+1},v_j^{n+1})\cdot\left(f(u_j^{n+1})-v_j^{n+1}\right)\\
&\quad\le-\frac{\alpha}{\varepsilon}\|f(u_j^{n+1})-v_j^{n+1}\|^2
\end{aligned}\tag{2.13}$$

and

$$\begin{aligned}
&\frac{k}{2a^2\varepsilon^2}\nabla^2(e_+(\widetilde{w})+e_-(\widetilde{z}))\left[f(u_j^{n+1})-v_j^{n+1};f(u_j^{n+1})-v_j^{n+1}\right]\\
&\qquad\ge\beta\frac{k}{\varepsilon^2}\|f(u_j^{n+1})-v_j^{n+1}\|^2,
\end{aligned}\tag{2.14}$$

for some constants α and β independent from ε, k and h. By using (2.13) and (2.14) in (2.12) we get

$$\begin{aligned}&\frac{1}{k}\left[E_j^{n+1}-E_j^n\right]+\frac{a}{h}\left[e_+(w_j^n)-e_-(z_{j+1}^n)-\left(e_+(w_{j-1}^n)-e_-(z_j^n)\right)\right]\\ &\leq -\frac{\alpha}{\varepsilon}\|f(u_j^{n+1})-v_j^{n+1}\|^2-\beta\frac{k}{\varepsilon^2}\|f(u_j^{n+1})-v_j^{n+1}\|^2,\end{aligned} \tag{2.15}$$

where the constants α and β are independent from ε, k and h. At this point, we sum in (2.15) in j and n to obtain

$$\begin{aligned}&\frac{k}{\varepsilon}\sum_{n=0}^{N-1}\sum_{j=-H}^{H}\|f(u_j^{n+1})-v_j^{n+1}\|^2+\frac{k^2}{\varepsilon^2}\sum_{n=0}^{N-1}\sum_{j=-H}^{H}\|f(u_j^{n+1})-v_j^{n+1}\|^2\\ &\leq C\left(D_{\mathcal{K}}^a\right)\left\{-\frac{ak}{h}\sum_{n=0}^{N-1}\left[e_+(w_H^n)-e_+(w_{-H-1}^n)-e_-(z_{H+1}^n)+e_-(z_{-H}^n)\right]\right.\\ &\qquad\left.+\sum_{j=-H}^{H}E_j^0\right\},\end{aligned} \tag{2.16}$$

because of $E \geq 0$. The L^∞ stability of the relaxation scheme implies

$$\begin{aligned}&\frac{k}{h}\left|\sum_{n=0}^{N-1}\left[e_+(w_H^n)-e_+(w_{-H-1}^n)-e_-(z_{H+1}^n)+e_-(z_{-H}^n)\right]\right|\\ &\qquad\leq\frac{1}{h}C\left(D_{\mathcal{K}}^a,T=Nk\right),\end{aligned} \tag{2.17}$$

while the Jensen inequality and the convexity of $e_\pm$ yields

$$\begin{aligned}\sum_{j=-H}^{H}E_j^0&=\sum_{j=-H}^{H}\left[e_+\left(\frac{1}{h}\int_{I_j}w_0(x)dx\right)+e_-\left(\frac{1}{h}\int_{I_j}z_0(x)dx\right)\right]\\ &\leq\frac{1}{h}\sum_{j=-H}^{H}\int_{I_j}\left[e_+(w_0(x))+e_-(z_0(x))\right]dx\\ &=\frac{1}{h}\int_{-L}^{L}E(u_0(x),v_0(x))dx\leq\frac{1}{h}C\left(D_{\mathcal{K}}^a,L\right).\end{aligned} \tag{2.18}$$

Finally, using (2.17) and (2.18) in (2.16) we obtain (2.11). □

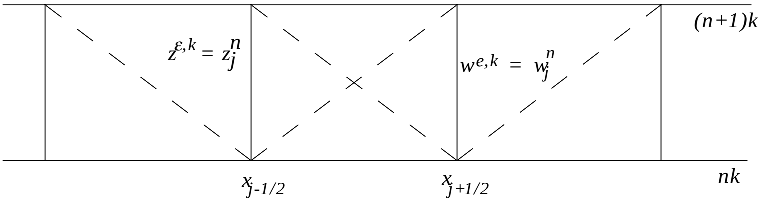

Fig. 1. Reconstruction of the approximate solution

3 The convergence of the numerical solution

In this section we prove the convergence of our approximate solution, based on the numerical scheme (2.4) with the CFL condition (2.10). Therefore we will take advantage of the results of Lemma 2.1 and Lemma 2.2 to control the relaxation process and to show that the entropy production $\eta_t + q_x$ for the equilibrium system (1.2) belongs to a compact set of H^{-1}_{loc}. Then we will conclude, thanks to the compensated compactness techniques [4, 11].

Given the discrete values u_j^n and v_j^n (or, equivalently, w_j^n and z_j^n), we reconstruct a piecewise approximate solution $(u^{\varepsilon,k}, v^{\varepsilon,k})$ in such a way that the characteristic variables w and z are constant along the characteristic directions $x - at$ and $x + at$ respectively, in any strip of the form $(n, (n + 1)k) \times \mathbb{R}$. Therefore, the approximate solution verifies the homogeneous system associated to (1.2) in any strip $(n, (n + 1)k) \times \mathbb{R}$. More precisely, we define the approximating sequence as follows (see also Fig. 1)

$$w^{\varepsilon,k} \equiv w_j^n \text{ for } t \in (nk, (n+1)k) \text{ and } x_{j-\frac{1}{2}} - at < x < x_{j+\frac{1}{2}} - at$$
$$z^{\varepsilon,k} \equiv z_j^n \text{ for } t \in (nk, (n+1)k) \text{ and } x_{j-\frac{1}{2}} + at < x < x_{j+\frac{1}{2}} + at.$$

It is clear from Figure 1 the big advantage we take in considering the CFL condition (2.10), namely $\frac{1}{a} = \frac{k}{h}$, in order to perform our reconstruction: the slope of the characteristic lines is equal to the ratio of the mesh sizes.

Remark 3.1 From the condition (2.11) of Lemma 2.2 we have the following uniform control for the approximating sequence $(u^{\varepsilon,k}, v^{\varepsilon,k})$

$$\begin{aligned} &\frac{1}{\varepsilon}\int_0^T\int_{-L}^L |f(u^{\varepsilon,k}) - v^{\varepsilon,k}|^2 dxdt + \frac{k}{\varepsilon^2}\int_0^T\int_{-L}^L |f(u^{\varepsilon,k}) \\ &\qquad - v^{\varepsilon,k}|^2 dxdt \le C\left(D^a_{\mathcal{K}}, T, L\right), \end{aligned} \tag{3.1}$$

for any T, L. In particular,

$$f(u^{\varepsilon,k}) - v^{\varepsilon,k} \longrightarrow 0, \quad \text{in } L^2_{loc},$$

as $\varepsilon \downarrow 0$, uniformly in k. □

Now we can prove the convergence of the approximate numerical solution we have constructed.

Theorem 3.2 *Let $\mathcal{K}$ be a convex characteristic set and assume $(u_0, v_0) \in D^a_{\mathcal{K}}$. Assume the subcharacteristic condition (1.3) and the CFL condition (2.10) hold and assume the relaxed system (1.2) is endowed with a C^2 strictly convex entropy-entropy flux pair (η^*, q^*) with $\eta^* \geq 0$. Finally, assume that the only probability measures with support in $\mathcal{K}$ satisfying the Tartar identity [12] for any entropy-entropy flux pair are Dirac masses. Then, extracting if necessary a subsequence,*

$$u^{\varepsilon,k} \longrightarrow u \text{ in } L^p_{loc} \text{ as } \varepsilon, k \downarrow 0, \quad \text{for any } p < +\infty \tag{3.2}$$

and u is a weak entropy solution of the equilibrium system

$$u_t + f(u)_x = 0. \tag{3.3}$$

Proof. Due to our hypothesis on the Tartar identity, in order to prove (3.2), we have to show that the entropy production verifies

$$\eta(u^{\varepsilon,k})_t + q(u^{\varepsilon,k})_x \in \text{comp } H^{-1}_{loc}, \tag{3.4}$$

for any entropy-entropy flux pair of (3.3) and

$$\eta(u^{\varepsilon,k})_t + q(u^{\varepsilon,k})_x \leq 0, \tag{3.5}$$

if, in addition, η is convex. Indeed, the L^∞ stability established in Lemma 2.1 implies, extracting if necessary a subsequence, the following weak convergence

$$u^{\varepsilon,k} \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty.$$

Once we have proved (3.4), applying the div-curl lemma [12], we obtain the commutative relation of Tartar for any entropy-entropy flux pair and hence the strong convergence of $u^{\varepsilon,k}$ toward u, because, from our hypothesis, we can say that the Young measure associated with this sequence reduces to a Dirac mass. Therefore, in view of the L^2_{loc} convergence of $f(u^{\varepsilon,k}) - v^{\varepsilon,k}$ toward zero established in Remark 3.1, we have in particular that u is a weak solution of (3.3) and, by passing into the limit in (3.5), we recover the entropy inequality for this solution.

Let (η, q) be an entropy-entropy flux pair for (3.3) and let (E, Q) be its extension. Therefore, proceeding as in [2, 6, 7, 11]

$$\begin{aligned} \eta(u^{\varepsilon,k})_t + q(u^{\varepsilon,k})_x &= \Big(E(u^{\varepsilon,k}, f(u^{\varepsilon,k})) - E(u^{\varepsilon,k}, v^{\varepsilon,k})\Big)_t \\ &\quad + \Big(Q(u^{\varepsilon,k}, f(u^{\varepsilon,k})) - Q(u^{\varepsilon,k}, v^{\varepsilon,k})\Big)_x \\ &\quad + E(u^{\varepsilon,k}, v^{\varepsilon,k})_t + Q(u^{\varepsilon,k}, v^{\varepsilon,k})_x \\ &= J_1^{\varepsilon,k} + J_2^{\varepsilon,k} + J_3^{\varepsilon,k}. \end{aligned} \tag{3.6}$$

From Remark 3.1, we know that $J_1^{\varepsilon,k}$ and $J_2^{\varepsilon,k}$ tend to zero in H^{-1}_{loc}, as $\varepsilon \downarrow 0$, uniformly in k. Therefore it remains to control the entropy production related to the extended entropy-entropy flux pair (E, Q). We will prove that this production is bounded in the space of measures $\mathcal{M}$ (and it is nonnegative if η is convex) and we will conclude applying the Murat lemma [9]. In any strip of the form $(nk, (n+1)k) \times \mathbb{R}$, since our approximating solutions have discontinuities which travel with characteristic speeds a and $-a$, the entropy production $E(u^{\varepsilon,k}, v^{\varepsilon,k})_t + Q(u^{\varepsilon,k}, v^{\varepsilon,k})_x$ is zero. More precisely, in any strip of this form, we have

$$(\partial_t + a\partial_x)\, w^{\varepsilon,k} = 0 \qquad\qquad (\partial_t - a\partial_x)\, z^{\varepsilon,k} = 0,$$

which yields the following relations

$$(\partial_t + a\partial_x)\, e_+(w^{\varepsilon,k}) = 0 \qquad\qquad (\partial_t - a\partial_x)\, e_-(z^{\varepsilon,k}) = 0,$$

for any $(t, x) \in (nk, (n+1)k) \times \mathbb{R}$, namely $E(u^{\varepsilon,k}, v^{\varepsilon,k})_t + Q(u^{\varepsilon,k}, v^{\varepsilon,k})_x = 0$ for any $(t, x) \in (nk, (n+1)k) \times \mathbb{R}$. Therefore the entropy production is due uniquely to the concentrations along the horizontal lines nk. Hence, for any test function Φ, we have

$$\begin{aligned}\mathcal{E}(\Phi) &= \int\int (E\Phi_t + Q\Phi_x)\, dxdt = \sum_n \int_{nk}^{(n+1)k} \int (E\Phi_t + Q\Phi_x)\, dxdt \\ &= \sum_n \int \Phi(t_{n+1}, x) \Big(E(u^{\varepsilon,k}(t_{n+1} - 0, x), v^{\varepsilon,k}(t_{n+1} - 0, x)) \\ &\quad - E(u^{\varepsilon,k}(t_{n+1} + 0, x), v^{\varepsilon,k}(t_{n+1} + 0, x)) \Big)\, dx,\end{aligned}$$

with the notation $t_n = nk$. Our particular choice for the reconstruction of the piecewise approximate solutions (Fig. 1) yields

$$\begin{aligned} E(u^{\varepsilon,k}(t_{n+1} - 0, x), v^{\varepsilon,k}(t_{n+1} - 0, x)) &= e_+(w_{j-1}^n) + e_-(z_{j+1}^n) \\ E(u^{\varepsilon,k}(t_{n+1} + 0, x), v^{\varepsilon,k}(t_{n+1} + 0, x)) &= e_+(w_j^{n+1}) + e_-(z_j^{n+1}),\end{aligned}$$

for any $x \in I_j$. Thus, proceeding as in the proof of Lemma 2.2, it follows

$$\begin{aligned}\mathcal{E}(\Phi) &= \sum_n \sum_j \int_{I_j} \Big(e_+(w_{j-1}^n) - e_+(w_j^{n+1}) + e_-(z_{j+1}^n) - e_-(z_j^{n+1}) \Big) \\ &\quad \times \Phi(t_{n+1}, x) dx \\ &= \sum_n \sum_j \int_{I_j} \Big\{ \frac{k^2}{2a^2\varepsilon^2} \nabla^2 (e_+(\widetilde{w}) + e_-(\widetilde{z})) \\ &\quad \times \Big[f(u_j^{n+1}) - v_j^{n+1}; f(u_j^{n+1}) - v_j^{n+1} \Big] \\ &\quad - \frac{K}{\varepsilon} \nabla_v E(u_j^{n+1}, v_j^{n+1}) \cdot \Big(f(u_j^{n+1}) - v_j^{n+1} \Big) \Big\} \Phi(t_{n+1}, x) dx. \qquad (3.7)\end{aligned}$$

From Proposition 1.2, we get $\nabla_v E(u, f(u)) = 0$ and in particular $\|\nabla_v E(u, v)\| \leq C\|f(u) - v\|$ for any $(u, v) \in D^a_{\mathcal{K}}$. Therefore, (2.11) and (3.7) imply

$$\begin{aligned} |\mathcal{E}(\Phi)| &\leq C\,(D^a_{\mathcal{K}})\,\|\Phi\|_{C_0} \\ &\quad \times \sum_{n=0}^{N-1} \sum_{j=-H}^{H} \left(\frac{hk}{\varepsilon} \|f(u_j^{n+1}) - v_j^{n+1}\|^2 + \frac{hk^2}{\varepsilon^2} \|f(u_j^{n+1}) - v_j^{n+1}\|^2 \right) \\ &\quad \leq C\,(D^a_{\mathcal{K}}, L, T)\,\|\Phi\|_{C_0} \end{aligned}$$

that is, the boundedness of $\mathcal{E}$ in the space of measures $\mathcal{M}$. Therefore, we have proved the strong convergence of the sequence $u^{\varepsilon,k}$ toward a weak solution u of the equilibrium system (3.3). Moreover, if η is convex, in view of Proposition 1.2, from (3.7) we get

$$\mathcal{E}(\Phi) \geq 0,$$

for any $\Phi \geq 0$. Hence, by passing into the limit in (3.6), we recover the entropy inequality for the limit function u. □

Remark 3.3 From the proof of Theorem 3.2, it follows also the convergence of the numerical solution $u^{\varepsilon,k}$ of (1.1) toward a numerical solution u^k of (1.2), letting $\varepsilon \downarrow 0$ and keeping k fixed. Due to the CFL condition (2.10), u^k is a solution of the Lax-Friedrichs scheme for (1.2), which is the limit, as $\varepsilon \downarrow 0$, of our relaxation scheme [5]. □

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