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Generalized p-FEM in homogenization

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Summary. A new finite element method for elliptic problems with locally periodic microstructure of length $\varepsilon > 0$ is developed and analyzed. It is shown that the method converges, as $\varepsilon \to 0$, to the solution of the homogenized problem with optimal order in ε and exponentially in the number of degrees of freedom independent of $\varepsilon > 0$. The computational work of the method is bounded independently of ε. Numerical experiments demonstrate the feasibility and confirm the theoretical results.

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1 Introduction

Numerous problems in engineering and the sciences involve media with small-scale features, such as a large number of rivets, stiffeners, fibers etc. In many cases *scale resolution,* i.e., the discretization of the small-scale problem features with finite elements, is not feasible, even with advanced hardware. The derivation of macroscopic models as the small scales tend to zero by averaging or homogenization is by now well understood and established for periodic structures (see, e.g. [3, 4, 12]). The averaged equations have smooth coefficients and are therefore well-suited for Finite-Element discretization. The small-scale features of the solution, however, are lost in this process. Recovery of such features requires the computation of so-called correctors which are as difficult to compute as the original problem. Moreover, the averaged equations are obtained as leading term in asymptotic

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expansions of the solution as the scale-length $\varepsilon \to 0$. In practice, however, $\varepsilon > 0$ is given and fixed and the asymptotic limit may be a poor description of the phenomena of interest. Since asymptotic expansions generally do not converge, the inclusion of higher order terms at fixed $\varepsilon > 0$ into the homogenization process will not improve the solution, in general. In addition, the homogenization is basically related to a global periodic pattern of the microstructure.

Some researchers have therefore avoided the use of homogenization techniques. For example, finite element multigrid and multiscale techniques have been developed for the resolution of the small scales (see, e.g. [7]). Such schemes are successful in rather general situations, in particular in the absence of periodicity. However, they require scale-resolution, i.e., with linear elements in dimension d at least $O(\varepsilon^{-d})$ degrees of freedom. The multigrid techniques constitute an optimal order process for the solution of the resulting system of equations, but cannot overcome the requirement of scale resolution. If the scales are resolved, these approaches yield algebraic convergence rates.

In the present paper, we develop a new p-FE approach for the numerical solution of homogenization problems. Its main features are as follows:

a) under the assumption of *locally periodic structure*, the scale can be resolved with computational work which is bounded independently of ε,

b) for piecewise analytic input data, the method will converge exponentially, independent of the length scale ε, in particular also at fixed, positive ε.

c) as $\varepsilon \to 0$, the numerical solution converges to the homogenized limit with an optimal rate in ε.

d) the approach applies to general elliptic systems with locally periodic microstructure.

A related algorithm has been used successfully in large scale computations [1].

For the sake of illustration, the approach will be developed and analyzed here for the classical elliptic problem

$$-\nabla \cdot \left(a\left(\frac{x}{\varepsilon}\right)\nabla u\right) + a_0\left(\frac{x}{\varepsilon}\right)u = f \quad \text{in } \Omega, \tag{1.1}$$

$$Bu = 0 \quad \text{on } \partial\Omega. \tag{1.2}$$

Here Ω is a bounded, connected subset of $\mathbb{R}^d$ with boundary $\partial\Omega$ and boundary operator B which may be either the trace operator or the conormal derivative operator. The problem is assumed strongly elliptic, i.e. $a(\xi), a_0(\xi)$ are positive.

The Finite Element Method (FEM) for (1.1), (1.2) reads: find $u_{\mathrm{N}}^{\varepsilon} \in \mathcal{V}_{\varepsilon}^{\mathrm{N}}$ such that

$$a(u^{\varepsilon}, v) = \int_{\Omega} \left\{ a\left(\frac{x}{\varepsilon}\right) \nabla u \cdot \nabla v + a_0\left(\frac{x}{\varepsilon}\right) uv \right\} dx = (f, v) \quad \forall v \in \mathcal{V}_{\varepsilon}^{\mathrm{N}} \tag{1.3}$$

where $\mathcal{V}_{\varepsilon}^{\mathrm{N}} \subset H^1(\Omega)$ is a subspace of dimension N which carries the essential boundary conditions (if any). The FE-solution $u_{\mathrm{N}}^{\varepsilon}$ is optimal in the energy norm $\|\cdot\|_{\mathrm{E}}$

$$\forall v \in \mathcal{V}_{\varepsilon}^{\mathrm{N}} : \quad \|u^{\varepsilon} - u_{\mathrm{N}}^{\varepsilon}\|_{\mathrm{E}} \leq \|u^{\varepsilon} - v\|_{\mathrm{E}} \tag{1.4}$$

and the performance of the FEM (1.3) depends strongly on the design of the subspace $\mathcal{V}_{\varepsilon}^{\mathrm{N}}$.

The basic idea of our approach is the design of special, ε-dependent subspaces $\mathcal{V}_{\varepsilon}^{\mathrm{N}}$ which resolve the microscale with a number of degrees of freedom independent of ε and which give exponential convergence in N if the right hand side f of the problem is analytic. To this end, we must assume the coefficients $a(\xi), a_0(\xi)$ in (1.1) to be 1-periodic. The subspaces $\mathcal{V}_{\varepsilon}^{\mathrm{N}}$ will be built by analyzing the Fourier-Bochner representation from [10, 11] of the solution of (1.1) on the unbounded domain $\mathbb{R}^d$. We show that asymptotic expansion of the kernel with respect to ε about $\varepsilon = 0$ reproduces the classical homogenization approach – thus the method is at least as good as that approach. We obtain subspaces $\mathcal{V}_{\varepsilon}^{\mathrm{N}}$ by sampling the Fourier-Bochner kernel for fixed $\varepsilon > 0$ in the frequency domain. We prove that if the sampling points are properly selected, this yields function systems with exponential convergence independent of $\varepsilon > 0$. We calculate the ε-dependent shape functions by solving a parameter-dependent unit-cell problem with the hp-FEM. Finally, we address the calculation of stiffness matrices for our ε-dependent shape functions. We show that these matrices can be generated with work independent of ε. In order to present the ideas in the simplest setting, we concentrate here on the case $d = 1$ and globally periodic problems. We hasten to add, however, that all proofs apply verbatim in dimensions $d > 1$ [8]. Likewise, the assumption on global periodicity of the coefficients a, a_0 is not restrictive – if the coefficients are only patch-wise periodic, we may resort to the partition of unity method (PUM) and use $\mathcal{V}_{\varepsilon}^{\mathrm{N}}$ simply as local approximation spaces in the PUM (see [2] for more on the theory and applications of the PUM). Finally, we remark that the algorithms developed here have shown good results also in the non-periodic setting, see e.g., [1], even though the theoretical results do not apply there.

The outline of this paper is as follows. In Sect. 2 we present the homogenization problem on the unbounded domain and introduce the kernel $\phi(y, \varepsilon, t)$ together with its properties. In Sect. 3 we show how the classical homogenization result $\varepsilon \to 0$ can be obtained with our approach and derive also the new spectral homogenization result. Exponential and spectral convergence results are established. Section 4 addresses the computational aspects of the kernel and of the stiffness matrices if the ε-dependent shape functions are used in a p-version FEM. Computational examples in full agreement with the theory conclude the paper.

2 The homogenization problem on $\mathbb{R}$

2.1 Variational setting and representation formula

Based on (1.1), consider the following elliptic, second order equation

$$-\frac{d}{dx}\left(a\left(\frac{x}{\varepsilon}\right)\frac{du^\varepsilon}{dx}(x)\right)+a_0\left(\frac{x}{\varepsilon}\right)u^\varepsilon(x)=f(x) \tag{2.1}$$

on $\mathbb{R}$, in which $a(\cdot)$ and $a_0(\cdot)$ are $L^\infty(\mathbb{R})$, 2π-periodic functions, $\varepsilon > 0$ is a real parameter and $f \in L^2(\mathbb{R})$. It is also assumed that there exist positive constants $\gamma, \gamma_1 > 0$ such that

$$0 < \gamma \le a(\xi), a_0(\xi) \le \gamma_1, \quad \text{for a.e. } \xi \in \mathbb{R}. \tag{2.2}$$

Then, it is shown in [10], [11] that (2.1) admits a unique solution u^ε with the following representation:

$$u^\varepsilon(x)=\frac{1}{\sqrt{2\pi}}\overset{(B)}{\int_{\mathbb{R}}}\hat{f}(t)\mathrm{e}^{\mathrm{i}xt}\phi\left(\frac{x}{\varepsilon},\varepsilon,t\right)dt. \tag{2.3}$$

Here, $\hat{f}$ represents the Fourier transform of f and the integral is understood as Fourier-Bochner integral of Banach-space valued functions. The kernel $\phi(\cdot,\varepsilon,t)$ is the 2π-periodic weak solution of the so-called *unit-cell problem*:

$$\begin{aligned}-\frac{1}{\varepsilon^2}\frac{d}{dy}\left(a(y)\frac{d}{dy}\left(\phi(y,\varepsilon,t)\mathrm{e}^{\mathrm{i}\varepsilon ty}\right)\right) \quad &+a_0(y)\phi(y,\varepsilon,t)\mathrm{e}^{\mathrm{i}\varepsilon ty}\\ &=\mathrm{e}^{\mathrm{i}\varepsilon ty},\ y\in Q,\end{aligned} \tag{2.4}$$

where $Q := \{y : |y| < \pi\}$ denotes the fundamental period. To characterize precisely the notion of solution of (2.1), we introduce the following weighted Sobolev spaces on $\mathbb{R}$:

Definition 2.1 For $j = 0, 1$ and for any $\nu \in \mathbb{R}$ the weighted Sobolev spaces $H^j_\nu(\mathbb{R})$ are defined to be

$$H^j_\nu(\mathbb{R})=\overline{C_0^\infty(\mathbb{R};\mathbb{C})}^{\|\cdot\|_{j,\nu}}, \tag{2.5}$$

where

$$\|u\|_{j,\nu}^2 = \int_{\mathbb{R}} \left(\sum_{l=0}^{j} \left| \frac{d^l u}{dx^l}(x) \right|^2 \right) \mathrm{e}^{2\nu|x|}\, dx. \tag{2.6}$$

Let us associate with (2.1) the sesquilinear form $\Psi(\varepsilon)[\cdot,\cdot] : H^1_{-\nu}(\mathbb{R}) \times H^1_{\nu}(\mathbb{R}) \to \mathbb{C}$

$$\Psi(\varepsilon)[u,v] = \int_{\mathbb{R}} \left\{ a\left(\frac{x}{\varepsilon}\right) \frac{du}{dx}(x) \overline{\frac{dv}{dx}(x)} + a_0\left(\frac{x}{\varepsilon}\right) u(x)\overline{v(x)} \right\} dx. \tag{2.7}$$

Proposition 2.2 *There exist positive constants ν_0, C and η such that for all $\nu \in (0,\nu_0)$ and all $\varepsilon > 0$,*

1. $|\Psi(\varepsilon)[u,v]| \le C\|u\|_{1,-\nu}\|v\|_{1,\nu}$,
2. $\inf_{\|u\|_{1,-\nu}=1} \sup_{\|v\|_{1,\nu}=1} |\Psi(\varepsilon)[u,v]| \ge \eta > 0$,
3. $\sup_{u\in H^1_{-\nu}(\mathbb{R})} |\Psi(\varepsilon)[u,v]| > 0$ *for all* $0 \ne v \in H^1_{\nu}(\mathbb{R})$,
4. *for all $f \in \left(H^1_\nu(\mathbb{R})\right)^*$, there exists a unique weak solution u^ε of (2.1), i.e.*

$$u^\varepsilon \in H^1_{-\nu}(\mathbb{R}) : \quad \Psi(\varepsilon)[u^\varepsilon, v] = \langle f, v\rangle_{(H^1_\nu)^*\times H^1_\nu}, \quad \forall\, v \in H^1_\nu(\mathbb{R}). \tag{2.8}$$

Moreover, u^ε admits the integral representation (2.3) and the following a-priori estimate holds

$$\|u^\varepsilon\|_{H^1_{-\nu}} \le (1/\eta)\|f\|_{(H^1_\nu)^*}.$$

A proof of these statements is given in [10].

Next, we define

$$\psi(y,\varepsilon,t) := \phi(y,\varepsilon,t)\mathrm{e}^{\mathrm{i}t\varepsilon y}. \tag{2.9}$$

With the above notations, for every $t \in \mathbb{R}$ the kernel $\psi(\cdot/\varepsilon,\varepsilon,t) \in H^1_{-\nu}(\mathbb{R})$ is the unique weak solution of the problem

$$\Psi(\varepsilon)\left[\psi\left(\frac{\cdot}{\varepsilon},\varepsilon,t\right), v\right] = \langle \mathrm{e}^{\mathrm{i}t(\cdot)}, v\rangle_{(H^1_\nu)^*\times H^1_\nu}, \quad \forall\, v \in H^1_\nu(\mathbb{R}). \tag{2.10}$$

In the remainder of this paper, we will show how the kernels $\phi(y,\varepsilon,t)$, $\psi(y,\varepsilon,t)$ can be used to design FE-approximations of (1.1), (1.2) which encode the microstructure and coefficient regularity. A crucial role in establishing exponential convergence will be played by the kernels' analyticity.

2.2 Analyticity of the kernels

It has already been shown in [11] that the kernel $\phi(\cdot,\varepsilon,t)$ can be continued analytically with respect to (ε,t) in a neighbourhood $\hat{G}\subset\mathbb{C}^2$ of $\mathbb{R}^2$, with values in $H^1_{\mathrm{per}}(0,2\pi)$, $H^1_{-\nu}(\mathbb{R})$. We show here that for every fixed $\varepsilon>0$, $\phi(\cdot,\varepsilon,t)$ and $\psi(\cdot,\varepsilon,t)$ can be continued analytically with respect to t in a strip neighbourhood of $\mathbb{R}$, and the width of the strip is independent of ε.

For $d>0$ let us use the notation

$$\mathcal{D}_d := \{t\in\mathbb{C} \text{ such that } |\mathrm{Im}\, t|<d\}. \tag{2.11}$$

Then the following theorem holds

Theorem 2.3 *For every $\nu\in(0,\nu_0)$, the mappings*

$$\mathcal{D}_{\nu/2}\ni t\to\psi\left(\frac{\cdot}{\varepsilon},\varepsilon,t\right)\in H^1_{-\nu}(\mathbb{R}), \tag{2.12}$$

$$\mathcal{D}_{\nu/2}\ni t\to\phi\left(\frac{\cdot}{\varepsilon},\varepsilon,t\right)\in H^1_{-2\nu}(\mathbb{R}), \tag{2.13}$$

are holomorphic in $\mathcal{D}_{\nu/2}$ with values in the Banach spaces $H^1_{-\nu}(\mathbb{R})$, $H^1_{-2\nu}(\mathbb{R})$ respectively. Moreover, for all $k\geq 0$, $\varepsilon>0$ and $t\in\mathcal{D}_{\nu/2}$ holds

$$\left\|\frac{d^k}{dt^k}\psi\left(\frac{\cdot}{\varepsilon},\varepsilon,t\right)\right\|_{1,-\nu}\leq\frac{\sqrt{(2k)!}}{\gamma\nu^k\sqrt{\nu/2}}, \tag{2.14}$$

$$\left\|\frac{d^k}{dt^k}\phi\left(\frac{\cdot}{\varepsilon},\varepsilon,t\right)\right\|_{1,-2\nu}\leq C(1+|t|)\frac{k!}{(\nu/2)^k}. \tag{2.15}$$

For the Proof of Theorem 2.3 we refer to Appendix A.

3 Finite dimensional approximation in the nonperiodic setting

Our objective is to construct generalized FE spaces $\mathcal{V}^{\mathrm{N}}_\varepsilon$ which incorporate the microstructure of the solution u^ε. To this end, we approximate the representation formula (2.3) by expressions of the form $\sum_{k=1}^{\mathrm{N}} c_k(f)\phi_k(x/\varepsilon)$ where $c_k(\cdot)$ may depend on f and on ε in a complicated way. The first approach is based on asymptotic expansion of $\phi(y,\varepsilon,t)$ in ε and is, as we show, nothing but classical homogenization. The second, spectral approach exploits the analyticity of $\phi(y,\varepsilon,t)$ in ε and t. It allows to obtain exponential convergence rates, independent of ε.

3.1 Asymptotic approximation of (2.3) (expansion in ε)

3.1.1 Derivation of the expansion. Since the kernel $\phi(\cdot,\varepsilon,t)$ is analytic with respect to ε with values in $H^1_{\mathrm{per}}(-\pi,\pi)$, it can be expanded in powers of ε, for every fixed $t\in\mathbb{R}$, i.e.,

$$\phi(\cdot,\varepsilon,t)=\sum_{k=0}^{\infty}\varepsilon^k\phi_k(\cdot,t). \tag{3.1}$$

Note that the convergence radius depends on t, and the coefficients $\phi_k(\cdot,t)$ are in $H^1_{\mathrm{per}}(-\pi,\pi)$ and depend holomorphically on t. Setting

$$\begin{aligned}\mathit{\Phi}(\varepsilon,t)[\phi,v]:=\int_{-\pi}^{\pi}a(y)\frac{d}{dy}\left(\phi(y)\mathrm{e}^{\mathrm{i}\varepsilon yt}\right)\overline{\frac{d}{dy}\left(v(y)\mathrm{e}^{\mathrm{i}\varepsilon yt}\right)}\\ +\varepsilon^2a_0(y)\phi(y)\overline{v(y)}\,dy,\end{aligned} \tag{3.2}$$

we may write

$$\mathit{\Phi}(\varepsilon,t)=\mathit{\Phi}_0+\varepsilon\mathit{\Phi}_1(t)+\varepsilon^2\mathit{\Phi}_2(t), \tag{3.3}$$

in which

$$\begin{aligned}\mathit{\Phi}_0[\phi,v]&:=\int_{-\pi}^{\pi}a(y)\frac{d\phi}{dy}(y)\overline{\frac{dv}{dy}}(y)\,dy,\\ \mathit{\Phi}_1(t)[\phi,v]&:=\mathrm{i}\int_{-\pi}^{\pi}a(y)\left(t\phi(y)\overline{\frac{dv}{dy}}(y)-t\frac{d\phi}{dy}(y)\overline{v(y)}\right)\,dy,\\ \mathit{\Phi}_2(t)[\phi,v]&:=\int_{-\pi}^{\pi}\left(|t|^2a(y)+a_0(y)\right)\phi(y)\overline{v(y)}\,dy,\end{aligned}$$

for all $\phi,v\in H^1_{\mathrm{per}}(-\pi,\pi),t\in\mathbb{R}$. We note that

$$\mathit{\Phi}_k(t)[\phi,v]=\sum_{k'=0}^{k}(\mathrm{i}t)^{k'}\mathit{\Phi}_{k;k'}[\phi,v], \tag{3.4}$$

with $\mathit{\Phi}_{k;k'}[\cdot,\cdot]$ independent of ε and t. Denote by

$$W^1_{\mathrm{per}}=\left\{\phi\in H^1_{\mathrm{per}}(-\pi,\pi)\;:\;\int_{-\pi}^{\pi}\phi(y)\,dy=0\right\}.$$

Since $\phi(\cdot,\varepsilon,t)$ is the weak solution of the variational problem

$$\phi(\cdot,\varepsilon,t)\in H^1_{\mathrm{per}}(-\pi,\pi):\quad \Phi(\varepsilon,t)[\phi(\cdot,\varepsilon,t),v]=\varepsilon^2\int_{-\pi}^{\pi}\overline{v(y)}\,dy, \tag{3.5}$$

after substituting the expansion (3.1) into (3.3) and equating like powers of ε, the following expressions for $\phi_k(\cdot,t)$ can be derived (for the proof we refer to [11] for example)

$$\phi_k(\cdot,t)=\begin{cases} g_0(t), & \text{if } k=0\\ \sum_{j=0}^{k-1} g_j(t)\chi_{k-j}(\cdot,t)+g_k(t), & \text{if } k\geq 1,\end{cases} \tag{3.6}$$

where for each $k\geq 1$, $\chi_k(\cdot,t)\in W^1_{\mathrm{per}}$ is the solution of

$$\Phi_0[\chi_k(\cdot,t),v]=\begin{cases} -\Phi_1(t)[1,v], & \text{if } k=1\\ -\Phi_1(t)[\chi_1(\cdot,t),v]-\Phi_2(t)[1,v], & \text{if } k=2\\ -\Phi_1(t)[\chi_{k-1}(\cdot,t),v]-\Phi_2(t)[\chi_{k-2}(\cdot,t),v], & \text{if } k\geq 3,\end{cases} \tag{3.7}$$

and the $g_k(t)\in\mathbb{C}$ are defined recursively by

$$g_k(t)=\begin{cases} \dfrac{2\pi}{\Phi_1(t)[\chi_1,1]+\Phi_2(t)[1,1]}, & \text{if } k=0\\ -\dfrac{g_0(t)}{2\pi}\sum_{j=0}^{k-1} g_j(t)\left(\Phi_1(t)[\chi_{k+1-j}(\cdot,t),1]\right. & \\ \left.+\Phi_2(t)[\chi_{k-j}(\cdot,t),1]\right), & \text{if } k\geq 1.\end{cases} \tag{3.8}$$

Let now $\chi_{1;1}(\cdot)\in W^1_{\mathrm{per}}$ be the unique weak solution of

$$\Phi_0[\chi_{1;1}+y,v]=0,\quad \forall\, v\in W^1_{\mathrm{per}}. \tag{3.9}$$

Then, $\chi_{1;1}(\cdot)$ is a real valued function and it can be deduced directly from the definition of $\chi_1(\cdot,t)$ that

$$\chi_1(\cdot,t)=\mathrm{i}t\chi_{1;1}(\cdot). \tag{3.10}$$

Substituting (3.10) in the definition of g_0 we get

$$g_0(t) = \frac{1}{A|t|^2 + A_0}, \tag{3.11}$$

where

$$A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} a_0(y)\, dy,$$

$$A = \frac{1}{2\pi} \Phi_0(\chi_{1;1} + y, y) = \frac{1}{2\pi} \Phi_0(\chi_{1;1} + y, \chi_{1;1} + y). \tag{3.12}$$

Replacing in the integral representation (2.3) of $u^\varepsilon(x)$ the kernel $\phi(\cdot, \varepsilon, t)$ by its asymptotic expansion (3.1), we get the formal expansion for u^ε in powers of ε

$$u^\varepsilon(x) = \sum_{k \geq 0} \varepsilon^k u^\varepsilon_{(k)}(x).$$

The leading term $u^\varepsilon_{(0)}(x) = u_{(0)}(x)$ is by (3.6) and (3.11) independent of ε and the unique weak solution of the homogenized differential equation, with constant coefficients A and A_0 defined by the averaging formulas (3.12)

$$-\frac{d}{dx}\left(A \frac{du_{(0)}}{dx} \right) + A_0 u_{(0)}(x) = f(x).$$

If f satisfies the usual assumptions, the coefficients $u^\varepsilon_{(k)}(x)$, $k \geq 1$, may be represented as Bochner integrals with kernel $\phi_k(\cdot/\varepsilon, t)$

$$u^\varepsilon_{(k)}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}}^{(B)} \hat{f}(t) \mathrm{e}^{\mathrm{i}xt} \phi_k\left(\frac{x}{\varepsilon}, t\right) dt. \tag{3.13}$$

Solving for $\phi_1(\cdot, t)$ now yields

$$\phi_1(\cdot, t) = g_0(t)\chi_1(\cdot, t) + g_1(t) = \mathrm{i}t g_0(t)\chi_{1;1}(\cdot) + g_1(t).$$

Therefore, by (3.13)

$$u^\varepsilon_{(1)}(x) = \frac{du^\varepsilon_{(0)}}{dx} \chi_{1;1}\left(\frac{x}{\varepsilon}\right) + \tilde{u}^\varepsilon_{(1)}(x),$$

where

$$\tilde{u}^\varepsilon_{(1)}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(t) g_1(t) \mathrm{e}^{\mathrm{i}tx} dt.$$

Similarly,

$$\begin{aligned}\phi_2(\cdot,t) &= g_0(t)\chi_2(\cdot,t) + g_1(t)\chi_1(\cdot,t) + g_2(t)\\ &= g_0(t)\left((\mathrm{i}t)^2\chi_{2;2}(\cdot) + \chi_{2;0}(\cdot)\right) + \mathrm{i}t g_1(t)\chi_{1;1}(\cdot) + g_2(t),\end{aligned}$$

where

$$\varPhi_0\left[\chi_{2;2},v\right] = -\int_{-\pi}^{\pi} a(y)\chi_{1;1}(y)\overline{\frac{dv}{dy}(y)}dy + \tag{3.14}$$
$$\int_{-\pi}^{\pi} a(y)\frac{d}{dy}\left(\chi_{1;1}+y\right)\overline{v(y)}dy,$$

$$\varPhi_0\left[\chi_{2;0},v\right] = -\int_{-\pi}^{\pi} a_0(y)\overline{v(y)}dy, \quad \forall\, v\in W^1_{\mathrm{per}}. \tag{3.15}$$

Hence,

$$\begin{aligned}u^\varepsilon_{(2)}(x) &= \frac{d^2u_{(0)}}{dx^2}(x)\chi_{2;2}\left(\frac{x}{\varepsilon}\right)\\ &\quad + u_{(0)}(x)\chi_{2;0}\left(\frac{x}{\varepsilon}\right) + \frac{d\tilde{u}^\varepsilon_{(1)}}{dx}(x)\chi_{1;1}\left(\frac{x}{\varepsilon}\right) + \tilde{u}^\varepsilon_{(2)}(x),\end{aligned}$$

where

$$\tilde{u}^\varepsilon_{(2)}(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \hat{f}(t)g_2(t)\mathrm{e}^{\mathrm{i}tx}dt.$$

By (3.4) and an induction argument it can be directly derived from (3.7) that

$$\chi_k(\cdot,t) = \sum_{l=0}^{\left[\frac{k}{2}\right]} \chi_{k;k-2l}(\cdot)(\mathrm{i}t)^{k-2l}, \tag{3.16}$$

where $\chi_{k;j}(\cdot)$ are real valued functions which are independent of ε, t.

Writing $\varPhi_1(t)[\cdot,\cdot] = \mathrm{i}t\varPhi_{1;1}[\cdot,\cdot]$ and $\varPhi_2(t)[\cdot,\cdot] = (\mathrm{i}t)^2\varPhi_{2;2}[\cdot,\cdot]+\varPhi_{2;0}[\cdot,\cdot]$ we can easily find a recursive system of equations for $\chi_{k;k-2l}(\cdot)$

$$\begin{aligned}\sum_{l=0}^{\left[\frac{k}{2}\right]}(\mathrm{i}t)^{k-2l}\varPhi_0[\chi_{k;k-2l},v] \;&=\; -\sum_{j=0}^{\left[\frac{k-1}{2}\right]}(\mathrm{i}t)^{k-2j}\varPhi_{1;1}[\chi_{k-1;k-1-2j},v]\\ &\quad -\sum_{m=0}^{\left[\frac{k-2}{2}\right]}\Big\{(\mathrm{i}t)^{k-2m}\varPhi_{2;2}[\chi_{k-2;k-2-2m},v]\\ &\quad +(\mathrm{i}t)^{k-2-2m}\varPhi_{2;0}[\chi_{k-2;k-2-2m},v]\Big\}.\end{aligned}$$

Equating like equal powers of it in both sides we get a recursive set of variational problems for $\chi_{k;k-2l}(\cdot)$, for $k \geq 1$ ($\chi_{-1;-1} \equiv 0$, $\chi_{0;0} \equiv 1$)

$$\Phi_0[\chi_{k;k-2l}, v] = \begin{cases} -\Phi_{1;1}[\chi_{k-1;k-1}, v] \\ -\Phi_{2;2}[\chi_{k-2;k-2}, v], & \text{if } l = 0, \\ -\Phi_{1;1}[\chi_{k-1;k-1-2l}, v] \\ -\Phi_{2;2}[\chi_{k-2;k-2-2l}, v] \\ -\Phi_{2;0}[\chi_{k-2;k-2l}, v], & \text{if } 1 \leq l \leq \left[\frac{k}{2}\right] - 1, \\ -\Phi_{1;1}[\chi_{k-1;k-1-2l}, v] \\ -\Phi_{2;0}[\chi_{k-2;k-2l}, v], & \text{if } l = \left[\frac{k}{2}\right],\ k \text{ odd}, \\ -\Phi_{2;0}[\chi_{k-2;k-2l}, v], & \text{if } l = \left[\frac{k}{2}\right],\ k \text{ even}. \end{cases}$$
(3.17)

Moreover, it can be also seen that

$$g_k(t) = (g_0(t))^{k+1}\, p_{3k}^{(k)}(\mathrm{i}t),$$

where $p_j^{(k)}(\cdot)$ denotes a polynomial with real coefficients of degree j.

Proposition 3.1 *For $k \geq 0$ and any $t \in \mathbb{R}$,*

$$\phi_k(\cdot, t) \in \text{Span } \{\chi_{k-j;\ k-j-2l}(\cdot)\}_{0 \leq j \leq k, 0 \leq 2l \leq k-j}\,,$$

with $\chi_{i;j}(\cdot)$ defined recursively by (3.17) and the convention that $\chi_{0;0} \equiv 1$.

Proof. By substituting (3.16) in (3.6) we can write $\phi_k(\cdot, t)$ in the following form

$$\phi_k(\cdot, t) = \sum_{j=0}^{k-1} g_j(t) \sum_{l=0}^{\left[\frac{k-j}{2}\right]} \chi_{k-j;k-j-2l}(\cdot)(\mathrm{i}t)^{k-j-2l}$$
$$+ g_k(t) \in \text{span} \{\chi_{k-j;\ k-j-2l}(\cdot)\}_{0 \leq j \leq k, 0 \leq 2l \leq k-j}\,. \qquad \square$$

3.1.2 Justification. Taking the Taylor expansion of $\phi(\cdot, \varepsilon, t)$ with integral representation for the remainder, we can write

$$\phi(\cdot, \varepsilon, t) = \sum_{k=0}^{L} \varepsilon^k \phi_k(\cdot, t) + \frac{\varepsilon^{L+1}}{L!} \int_{(0,1)}^{(B)} (1-s)^L \frac{d^{L+1}\phi}{d\varepsilon^{L+1}}(\cdot, s\varepsilon, t)\, ds$$
$$= \sum_{k=0}^{L} \varepsilon^k \phi_k(\cdot, t) + \frac{\varepsilon^{L+1}}{(L+1)!} \frac{d^{L+1}\phi}{d\varepsilon^{L+1}}(\cdot, \theta(\varepsilon), t),$$

for some intermediate point $0 < \theta(\varepsilon) < \varepsilon$. Therefore,

$$u^\varepsilon(x) = u^{\varepsilon,L}(x) + \frac{\varepsilon^{L+1}}{\sqrt{2\pi}(L+1)!} \int\limits_{\mathbb{R}}^{(B)} \hat{f}(t) \mathrm{e}^{\mathrm{i}xt} \frac{d^{L+1}\phi}{d\varepsilon^{L+1}} \left(\frac{x}{\varepsilon}, \theta(\varepsilon), t\right) dt,$$

where $u^{\varepsilon,L}(x) = \sum_{k=0}^{L} \varepsilon^k u^\varepsilon_{(k)}(x)$. Assume now that $k \geq 2$ and take the k-th derivative with respect to ε in the variational definition (3.5) of $\phi(\cdot, \varepsilon, t)$. It follows that $\frac{d^k\phi}{d\varepsilon^k}(\cdot, \varepsilon, t) \in H^1_{\mathrm{per}}(-\pi, \pi)$ is the weak solution of

$$\begin{aligned}
\Phi(\varepsilon, t) \left[\frac{1}{k!} \frac{d^k\phi}{d\varepsilon^k}(\cdot, \varepsilon, t), v\right] = & -\Phi_1(t) \left[\frac{1}{(k-1)!} \frac{d^{k-1}\phi}{d\varepsilon^{k-1}}(\cdot, \varepsilon, t), v\right] \\
& -2\,\varepsilon \Phi_2(t) \left[\frac{1}{(k-1)!} \frac{d^{k-1}\phi}{d\varepsilon^{k-1}}(\cdot, \varepsilon, t), v\right] \\
& -\Phi_2(t) \left[\frac{1}{(k-2)!} \frac{d^{k-2}\phi}{d\varepsilon^{k-2}}(\cdot, \varepsilon, t), v\right] \\
& +2\,\delta_2^k \int\limits_{-\pi}^{\pi} \overline{v(y)}\, dy,
\end{aligned}$$

with δ_2^k denoting the Kronecker symbol. By an induction argument it can be shown that

$$\left\| \frac{1}{k!} \frac{d^k\phi}{d\varepsilon^k}(\cdot, \varepsilon, t) \right\|_{H^1(-\pi,\pi)} \leq C\eta^k (1 + |t|)^{3k+1} \tag{3.18}$$

uniformly with $t \in \mathbb{R}$, where the constants $C > 0, \eta > 1$ are independent of t and k. Assume now that $s > 0$ and $f \in H^s(\mathbb{R})$. Then,

$$\begin{aligned}
\|u^\varepsilon - u^{\varepsilon,L}\|_{1,-\nu} \leq & \ \varepsilon^L \|f\|_{H^s(\mathbb{R})} \\
& \times \left(\int\limits_{\mathbb{R}} \left\| \frac{1}{(L+1)!} \frac{d^{L+1}\phi}{d\varepsilon^{L+1}} \left(\frac{\cdot}{\varepsilon}, \theta(\varepsilon), t\right) \right\|_{1,-\nu}^2 (1 + |t|^2)^{-s}\, dt \right)^{1/2}.
\end{aligned}$$

From the estimate (3.18) it follows that

$$\|u^\varepsilon - u^{\varepsilon,L}\|_{1,-\nu} \leq M\varepsilon^L \eta^{L+1} \|f\|_{H^s(\mathbb{R})} = C_{\mathrm{L}} \varepsilon^L \|f\|_{H^s(\mathbb{R})}, \tag{3.19}$$

for f sufficiently smooth. In conclusion, for sufficiently smooth data f and any $\varepsilon > 0$, the solution $u^\varepsilon(x)$ can be approximated to any order L in ε from the subspace

$$\mathrm{Span} \left\{ \chi_{k;l} \left(\frac{x}{\varepsilon}\right) \right\}$$

where $\chi_{k;l}(y)$ are the functions arising in the classical homogenization approach (see, e.g., [12]).
One might therefore consider choosing $\mathrm{Span}\{\chi_{k;l}(y)\}$ as local FE approximation spaces. This has indeed been tried (see, e.g., [3]) and gives reasonable results in special cases. However, there are severe disadvantages of this approach: i) the number of $\chi_{k;l}(y)$ necessary to achieve an error of order ε^L grows like L^2 (and worse in higher dimensions), ii) in practice, $\varepsilon > 0$ is given and not at our disposal; therefore, there is no guarantee that at fixed $\varepsilon > 0$ the inclusion of further terms in the asymptotic expansion will decrease the error, iii) the constant in the error estimate (3.19) in general increases quickly with L.

3.2 Spectral approximation of (2.3)

The error estimate (3.19) is in analogy to h-type FEM based on Taylor series expansion of the exact solution with $\varepsilon > 0$ assuming the role of h (there, we can reduce h, but here we cannot choose $\varepsilon > 0$, however). Taylor series will, in general, not give error estimates which are optimal in terms of the polynomial degree of the approximation. We will therefore derive in the present section a different system of microscale shape functions and establish spectral approximation results for them.

The main idea is to approximate the Fourier-Bochner integral (2.3) by a finite sum by truncating a (generalized) Poisson summation formula. To this end, let $h > 0$ and k be an integer, and define $S(k,h)$ by

$$S(k,h)(x) = \frac{\sin[\pi(x-kh)/h]}{\pi(x-kh)/h}. \tag{3.20}$$

We shall refer to $S(k,h)$ as the k'th $Sinc$ function, with step size h, evaluated at x.

Lemma 3.2

$$S(k,h)(x) = \frac{1}{2\pi} \int_{-\pi/h}^{\pi/h} h e^{\mathrm{i}kht - \mathrm{i}xt} dt, \tag{3.21}$$

and

$$\int_{\mathbb{R}} S(k,h)(x) S(l,h)(x) dx = \frac{h^2}{2\pi} \int_{-\pi/h}^{\pi/h} e^{\mathrm{i}(k-l)ht} dt = h\delta_{k-l}. \tag{3.22}$$

Proof. See, eg., [13], Theorem 1.10.1.

Definition 3.3 For a Banach space X, we denote by

$$\boldsymbol{H}^p(\mathcal{D}_d;X) = \{\, g : \mathcal{D}_d \to X \,|\, g \text{ is analytic in } \mathcal{D}_d \text{ and } \boldsymbol{N}_p(g,\mathcal{D}_d;X) < \infty \,\}, \tag{3.23}$$

where

$$\boldsymbol{N}_p(g,\mathcal{D}_d;X) = \begin{cases} \lim\limits_{\delta\to 0^+} \left(\displaystyle\int_{\partial\mathcal{D}_d(\delta)} \|g(z)\|_X^p |dz| \right)^{1/p} &, \quad \text{if } 1 \le p < \infty \\ \lim\limits_{\delta\to 0^+} \sup\limits_{z\in\mathcal{D}_d(\delta)} \|g(z)\|_X &, \quad \text{if } p = \infty, \end{cases}$$

and for $0 < \delta < 1$, $\mathcal{D}_d(\delta)$ is defined by

$$\mathcal{D}_d(\delta) = \{z \in \mathbb{C} \,:\, |\mathrm{Re}\,(z)| < 1/\delta,\ |\mathrm{Im}\,(z)| < d(1-\delta)\}.$$

Definition 3.4 We say, a function f fulfills the 'usual assumptions', if $f \in L^2(\mathbb{R})$, and its Fourier transformation $\hat{f}(\cdot)$ can be extended to a holomorphic function in the strip $\mathcal{D}_d$, with $d = d(\nu) = \nu/2$ and $\hat{f}$ satisfies the following growth condition :

$$|\hat{f}(z)| \le C(f)\mathrm{e}^{-\alpha|z|}, \quad \forall\, z \in \mathcal{D}_d, \tag{3.24}$$

for some positive constants $C(f), \alpha > 0$.

Then the following theorem holds :

Theorem 3.5 *Under the 'usual assumptions' on f the mapping*

$$\mathcal{D}_d \ni t \to g(t,\cdot) = g_\varepsilon(t,\cdot) := \frac{1}{\sqrt{2\pi}}\hat{f}(t)\phi\left(\frac{\cdot}{\varepsilon},\varepsilon,t\right) \in H^1_{-2\nu}(\mathbb{R}) \tag{3.25}$$

is in $\boldsymbol{H}^p(\mathcal{D}_d; H^1_{-2\nu}(\mathbb{R}))$, for all $1 \le p \le \infty$.
Moreover, there exists $C(\gamma,\nu) > 0$ such that $g_\varepsilon(t,\cdot)$ satisfies the growth condition:

$$\|g_\varepsilon(t,\cdot)\|_{1,-2\nu} \le C(\gamma,\nu)C(f)\left(1+\frac{1}{\alpha}\right)\mathrm{e}^{-\frac{\alpha}{2}|t|}, \quad \forall\, t \in \mathcal{D}_d, \tag{3.26}$$

where α and $C(f)$ are as in (3.24).

Proof. Strictly speaking, g in (3.25) depends on ε. However, all estimates which follow will be robust with respect to ε and we therefore do not write the dependence on ε explicitly.
From the usual assumptions on f and from (2.15) it follows easily that there exists a positive constant $C = C(\gamma,\nu) > 0$ such that

$$\|g(t,\cdot)\|_{1,-2\nu} \le C(\gamma,\nu)C(f)(1+|t|)\mathrm{e}^{-\alpha|t|}, \qquad \forall\, t \in \mathcal{D}_d. \tag{3.27}$$

It follows therefore that

$$\|g(t,\cdot)\|_{1,-2\nu} \leq C(\gamma,\nu)C(f)\left(1+\frac{1}{\alpha}\right)\mathrm{e}^{-\frac{\alpha}{2}|t|}, \qquad \forall\, t \in \mathcal{D}_d. \tag{3.28}$$

Then, for $1 \leq p < \infty$, we have that

$$\begin{aligned} \boldsymbol{N}_p(g,\mathcal{D}_d;H^1_{1,-2\nu}) &= \left(\int\limits_{\partial\mathcal{D}_d} \|g(z,\cdot)\|^p_{1,-2\nu}|dz|\right)^{1/p} \\ &\leq C(\gamma,\nu)C(f)\left(1+\frac{1}{\alpha}\right)\left(\int\limits_{\partial\mathcal{D}_d} \mathrm{e}^{-\frac{p\alpha}{2}|z|}|dz|\right)^{1/p} \\ &\leq C(\gamma,\nu)C(f)\left(1+\frac{1}{\alpha}\right)\left(\frac{8}{\alpha p}\right)^{1/p}. \end{aligned}$$

The case $p=\infty$ is treated analogously. □

Let $L \geq 1$ and assume in what follows that $\pi/h \geq 2\,L$, i.e., $h \leq \pi/(2\,L)$. Define, for $z \in \mathcal{D}_d$,

$$\begin{aligned} C(g,h)(z,x) &:= \sum_{k=-\infty}^{\infty} g(kh,x)S(k,h)(z), \\ C_{\mathrm{N}}(g,h)(z,x) &:= \sum_{k=-N}^{\mathrm{N}} g(kh,x)S(k,h)(z) \end{aligned}$$

in $H^1_{-2\nu}(\mathbb{R})$, and set

$$\begin{aligned} E(f,h)(z,x) &:= g(z,x) - C(g,h)(z,x), \\ E_{\mathrm{N}}(f,h)(z,x) &:= g(z,x) - C_{\mathrm{N}}(g,h)(z,x) \end{aligned} \tag{3.29}$$

in $H^1_{-2\nu}(\mathbb{R})$. Define $\delta(f,h)(\cdot)$, $\delta_{\mathrm{N}}(f,h)(\cdot) \in H^1_{-2\nu}(-L,L) \cap H^0_{-2\nu}(\mathbb{R})$ formally as

$$\delta(f,h)(x) = \lim_{\delta\to 0^+} \int\limits_{\mathbb{R}}^{(B)} \mathrm{e}^{-\delta|t|}\mathrm{e}^{\mathrm{i}xt}E(f,h)(t,x)\,dt, \tag{3.30}$$

$$\delta_{\mathrm{N}}(f,h)(x) = \lim_{\delta\to 0^+} \int\limits_{\mathbb{R}}^{(B)} \mathrm{e}^{-\delta|t|}\mathrm{e}^{\mathrm{i}xt}E_{\mathrm{N}}(f,h)(t,x)\,dt. \tag{3.31}$$

It will be shown that the above definitions make sense, and that the limits in (3.30) and (3.31) are well defined as Bochner integrals of $H^1_{-2\nu}(-L,L)$, respectively $H^0_{-2\nu}(\mathbb{R})$-valued functions. Notice that the weighted Sobolev spaces $H^1_{-2\nu}(-L,L)$ are continuously embedded in $H^1(-L,L)$, and

$$\|F(\cdot)\|_{H^1(-L,L)} \le \mathrm{e}^{\nu L}\|F(\cdot)\|_{H^1_{-2\nu}(-L,L)}, \quad \forall\, F(\cdot) \in H^1_{-2\nu}(-L,L). \tag{3.32}$$

From the properties of the $Sinc$ functions $S(k,h)(\cdot)$ in Lemma 3.2 it will be seen that

$$\begin{aligned}\delta_{\mathrm{N}}(f,h)(x) &= \lim_{\delta\to 0} \overset{(B)}{\int_{\mathbb{R}}} \mathrm{e}^{-\delta|t|}\mathrm{e}^{\mathrm{i}xt}\left\{g(t,x) - \sum_{k=-N}^{\mathrm{N}} g(kh,x)S(k,h)(t)\right\} dt \\ &= \begin{cases} \overset{(B)}{\displaystyle\int_{\mathbb{R}}} \mathrm{e}^{\mathrm{i}xt}g(t,x)\,dt - h\displaystyle\sum_{k=-N}^{\mathrm{N}} g(kh,x)\mathrm{e}^{\mathrm{i}khx} \quad , & \text{if } |x| < \dfrac{\pi}{h}, \\ \displaystyle\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}xt}g(t,x)\,dt \quad , & \text{if } |x| > \dfrac{\pi}{h}, \end{cases}\end{aligned} \tag{3.33}$$

in $H^0_{-2\nu}(\mathbb{R}) \cap H^1_{-2\nu}(-L,L)$. To this end, define the following trapezoidal approximation of (2.3)

$$\begin{aligned}u^{\varepsilon}_{N,h}(x) &= \mathbf{1}_{[-\frac{\pi}{h},\frac{\pi}{h}]}(x)\frac{1}{\sqrt{2\pi}}h\sum_{k=-N}^{\mathrm{N}} \phi\left(\frac{x}{\varepsilon},\varepsilon,kh\right)\hat{f}(kh)\mathrm{e}^{\mathrm{i}khx} \\ &= \mathbf{1}_{[-\frac{\pi}{h},\frac{\pi}{h}]}(x)\frac{1}{\sqrt{2\pi}}h\sum_{k=-N}^{\mathrm{N}} \psi\left(\frac{x}{\varepsilon},\varepsilon,kh\right)\hat{f}(kh).\end{aligned} \tag{3.34}$$

Remark 3.6 Since

$$\hat{f}(-\xi) = \overline{\hat{f}(\xi)}, \quad \psi\left(\frac{\cdot}{\varepsilon},\varepsilon,-\xi\right) = \overline{\psi\left(\frac{\cdot}{\varepsilon},\varepsilon,\xi\right)},$$

it follows that

$$\begin{aligned}u^{\varepsilon}_{N,h}(x) &= \mathbf{1}_{[-\frac{\pi}{h},\frac{\pi}{h}]}(x)\frac{1}{\sqrt{2\pi}}h\psi\left(\frac{x}{\varepsilon},\varepsilon,0\right)\hat{f}(0) + 2\mathbf{1}_{[-\frac{\pi}{h},\frac{\pi}{h}]}(x)\frac{1}{\sqrt{2\pi}}h \\ &\quad\times\left\{\sum_{k=1}^{\mathrm{N}} \mathrm{Re}\,\psi\left(\frac{x}{\varepsilon},\varepsilon,kh\right)\mathrm{Re}\,\hat{f}(kh) - \mathrm{Im}\,\psi\left(\frac{x}{\varepsilon},\varepsilon,kh\right)\mathrm{Im}\,\hat{f}(kh)\right\},\end{aligned}$$

and the solution of (2.3) can formally be written as $u^{\varepsilon}(\cdot) = u^{\varepsilon}_{N,h}(\cdot) + \delta_{\mathrm{N}}(f,h)(\cdot)$.

3.3 Exponential convergence

We will now show that (3.34) approximates u^ε in (2.3) at an exponential rate, independent of ε. We start with the following result.

Lemma 3.7 *Assume that f satisfies the 'usual assumptions', g is as in (3.25) and $z \in \mathcal{D}_d$ is arbitrary. Then holds the representation*

$$\begin{aligned} E(f,h)(z,x) &= g(z,x) - C(g,h)(z,x) \\ &= \frac{\sin(\pi z/h)}{2\pi i} \int_{\mathbb{R}}^{(B)} \left\{ \frac{g(t - \mathrm{i}d^-, x)}{(t - z - \mathrm{i}d)\sin[\pi(t - \mathrm{i}d)/h]} \right. \\ &\qquad \left. - \frac{g(t + id^-, x)}{(t - z + id)\sin[\pi(t + id)/h]} \right\} dt, \end{aligned}$$

where this equality has to be understood as equality between two elements of the Banach space $H^1_{-2\nu}(\mathbb{R})$ and the integral as a Bochner integral of $H^1_{-2\nu}(\mathbb{R})$-valued functions.

Proof. Let $0 < \delta < d$, let n denote a positive integer, let $\mathcal{D}(n,\delta)$ denote the region

$$\mathcal{D}(n,\delta) = \left\{ z \in \mathbb{C} \mid |\mathrm{Re}\, z| < \left(n + \frac{1}{2}\right) h,\ |\mathrm{Im}\, z| < \delta \right\} \tag{3.35}$$

and consider, for $z = a + ib \in \mathcal{D}_d$ fixed, $\zeta = \xi + i\eta$, the following Bochner-integral in $H^1_{-2\nu}(\mathbb{R})$

$$E(n,\delta,f)(z,x) = \frac{\sin(\pi z/h)}{2\pi i} \int_{\partial\mathcal{D}(n,\delta)}^{(B)} \frac{g(\zeta,x)}{(\zeta - z)\sin(\pi\zeta/h)}\, d\zeta. \tag{3.36}$$

Then, for n sufficiently large and δ sufficiently close to d, z is in $\mathcal{D}(n,\delta)$ and $|z - \zeta| \geq \min\left\{ \left(n + \frac{1}{2}\right) h - |a|, \delta - |b| \right\} > 0$.

Along the vertical segments of the boundary $\partial\mathcal{D}(n,\delta)$

$$\zeta = \pm\left(n + \frac{1}{2}\right) h + iy$$

and therefore $|\sin(\pi\zeta/h)| = \cosh(\pi y/h) \geq 1$. Then, the $H^1_{-2\nu}$-norm of the integral (3.36) along these segments is bounded by

$$\frac{|\sin(\pi z/h)|}{2\pi} \int_{-\delta}^{\delta} \left\{ \frac{\left\| g\left(\left(n + \frac{1}{2}\right) h + iy, \cdot\right) \right\|_{1,-2\nu}}{\left|(n + \frac{1}{2})h - a\right|} \right.$$

$$
\begin{aligned}
&+ \frac{\left\| g\left(\left(-n-\frac{1}{2}\right)h + iy, \cdot\right)\right\|_{1,-2\nu}}{\left|(-n-\frac{1}{2})h - a\right|} \Bigg\} \, dy \\
&\le \frac{|\sin(\pi z/h)|}{2\pi} (2\delta)^{1/q} \boldsymbol{N}_p(g, \mathcal{D}_d; H^1_{-2\nu}(\mathbb{R})) \\
&\times \left\{ \frac{1}{\left|\left(n+\frac{1}{2}\right)h - a\right|} + \frac{1}{\left|\left(n+\frac{1}{2}\right)h + a\right|} \right\},
\end{aligned}
$$

which tends to zero as $n \to \infty$ (here $1/p + 1/q = 1$). Now, since the following relations hold

$$
\begin{aligned}
\sinh(\pi\delta/h) &\le \left[\cosh^2(\pi\delta/h) - \cos^2(\pi t/h)\right]^{1/2} \\
&= |\sin[\pi(t \pm i\delta)/h]| \le \cosh(\pi\delta/h),
\end{aligned}
$$

and along the horizontal segments of $\partial\mathcal{D}_d$

$$
|z - \zeta| = [(a-\xi)^2 + (|b| - \delta)^2]^{1/2},
$$

the $H^1_{-2\nu}$-norm of the integral along these segments has the bound

$$
\begin{aligned}
&\frac{|\sin(\pi z/h)|}{2\pi \sinh(\pi\delta/h)} \boldsymbol{N}_p(g, \mathcal{D}_d; H^1_{-2\nu}(\mathbb{R})) \\
&\times \left\{ \int_{\mathbb{R}} \frac{1}{[(a-\xi)^2 + (|b|-\delta)^2]^{q/2}} d\xi \right\}^{1/q}. \qquad (3.37)
\end{aligned}
$$

This implies that $E(n, \delta, f)(z, \cdot) \in H^1_{-2\nu}(\mathbb{R})$ admits the representation

$$
\begin{aligned}
E(n, \delta, f)(z, \cdot) &= g(z, \cdot) - \sin(\pi z/h) \sum_{k=-n}^{n} \frac{(-1)^k g(kh, \cdot)}{\pi(z - kh)/h} \\
&= g(z, \cdot) - \sum_{k=-n}^{n} \frac{\sin\left[\pi \dfrac{(z-kh)}{h}\right]}{\pi \dfrac{z-kh}{h}} g(kh, \cdot) \\
&= g(z, \cdot) - \sum_{k=-n}^{n} S(k, h)(z) g(kh, \cdot).
\end{aligned}
$$

Also, the limits $n \to \infty$ and $\delta \to d$ exist in $H^1_{-2\nu}(\mathbb{R})$ in both sides and the lemma follows. □

Remark 3.8 We do not actually need the strong ‘usual assumptions’ on f to deduce the above integral representation for $E(f, h)(z, \cdot)$ for z in the strip $\mathcal{D}_d$. These assumptions on f just imply that the integrand g defined in (3.25) is in $\boldsymbol{H}^p(\mathcal{D}_d; H^1_{-2\nu}(\mathbb{R}))$, for every $1 \le p \le \infty$, as shown in Theorem 3.5. For the proof of Lemma 3.7 it is sufficient to know that $g \in \boldsymbol{H}^p(\mathcal{D}_d; H^1_{-2\nu}(\mathbb{R}))$, for some $1 \le p \le \infty$, and such a property on g holds under more general hypothesis on f than the 'usual assumptions', such as $f \in H^s_{comp}(\mathbb{R})$ for some $s > 1$. In this case $g \in \boldsymbol{H}^\infty(\mathcal{D}_d; H^1_{-2\nu}(\mathbb{R}))$ and again the representation in Lemma 3.7 is valid.

Theorem 3.9 *Let f satisfy the 'usual assumptions' in Definition 3.4 with some $\alpha, d > 0$ and let $L > 0$ be arbitrary. Define*

$$h = \left(\frac{\pi d}{\alpha N}\right)^{1/2}, \tag{3.38}$$

and assume $N \ge (4dL^2)/(\alpha\pi)$, i.e., such that $\pi/h \ge 2\,L$.

Then, with $E(f,h)(t,\cdot)$ as in Lemma 3.7 we have the following representation

$$\begin{aligned}\delta(f,h)(\cdot) &:= \lim_{\delta\to 0^+} \int_{\mathbb{R}}^{(B)} \mathrm{e}^{\mathrm{i}t(\cdot)}\mathrm{e}^{-\delta|t|}E(f,h)(t,\cdot)\,dt \\ &= \int_{\mathbb{R}}^{(B)} f_1(t,\cdot)g(t-\mathrm{i}d^-,\cdot)\,dt + \int_{\mathbb{R}}^{(B)} f_2(t,\cdot)g(t+id^-,\cdot)\,dt,\end{aligned}$$

in $H^0_{-2\nu}(\mathbb{R}) \cap H^1_{-2\nu}(-L,L)$. Here, the kernels f_1 and f_2 are defined by

$$f_1(t,x) = \begin{cases} \mathrm{e}^{xd+ixt} & , \quad \text{if } x < -\dfrac{\pi}{h} \\ \dfrac{i}{2}\dfrac{\mathrm{e}^{\left(x-\frac{\pi}{h}\right)(d+it)}}{\sin[\pi(t-\mathrm{i}d)/h]} & , \quad \text{if } -\dfrac{\pi}{h} < x < \dfrac{\pi}{h} \\ 0 & , \quad \text{if } x > \dfrac{\pi}{h}, \end{cases} \tag{3.39}$$

and

$$f_2(t,x) = \begin{cases} 0 & , \quad \text{if } x < -\dfrac{\pi}{h} \\ -\dfrac{i}{2}\dfrac{\mathrm{e}^{-\left(x+\frac{\pi}{h}\right)(d-\mathrm{i}t)}}{\sin[\pi(t+id)/h]} & , \quad \text{if } -\dfrac{\pi}{h} < x < \dfrac{\pi}{h} \\ \mathrm{e}^{-xd+ixt} & , \quad \text{if } x > \dfrac{\pi}{h}. \end{cases} \tag{3.40}$$

Moreover, there exists a constant $C = C(\gamma,\nu)C(f)\,(1+1/\alpha)^2\,(1/\alpha) > 0$, *which depends on* f, α, d, γ, *but is independent of* N *and* L, *such that*

$$\|\delta(f,h)(\cdot)\|_{0,-2\nu} + \|\delta(f,h)(\cdot)\|_{H^1_{-2\nu}(-L,L)} \le C\mathrm{e}^{-(\pi d\alpha N)^{1/2}}.$$

For the proof of this theorem we refer to Appendix B.

Our main result on the trapezoidal approximation $u^{\varepsilon}_{N,h}(x)$ of the Fourier-integral (2.3) is :

Theorem 3.10 *Under the assumptions in Theorem 3.9, the error* $\delta_{\mathrm{N}}(f,h)(\cdot) = u^{\varepsilon}(\cdot) - u^{\varepsilon}_{N,h}(\cdot)$, *with* $u^{\varepsilon}_{N,h}(\cdot)$ *as in (3.34), decays exponentially with respect to* N *and uniformly with respect to* ε *in the* $\|\cdot\|_{0,-2\nu}$, $\|\cdot\|_{H^1_{-2\nu}(-L,L)}$*-norms:*

$$\begin{aligned}&\|\delta_{\mathrm{N}}(f,h)(\cdot)\|_{0,-2\nu} + \|\delta_{\mathrm{N}}(f,h)(\cdot)\|_{H^1_{-2\nu}(-L,L)}\\ &\le C(\gamma,\nu)C(f)\left(1+\frac{1}{\alpha}\right)^2\frac{1}{\alpha}\mathrm{e}^{-(\pi\alpha d N)^{1/2}}.\end{aligned}\tag{3.41}$$

The constants $C(\gamma,\nu), C(f)$ *are independent of* ε, N, L.

Proof. From the definitions (3.30) and (3.31) of $\delta(f,h)(\cdot)$ and $\delta_{\mathrm{N}}(f,h)(\cdot)$ and the properties of the $Sinc$ functions $S(k,h)(\cdot)$ in Lemma 3.2 it follows that

$$\begin{aligned}\delta_{\mathrm{N}}(f,h)(\cdot) &= \delta(f,h)(\cdot) + \sum_{|k|>N}\lim_{\delta\to 0^+}\overset{(B)}{\int_{\mathbb{R}}}\mathrm{e}^{-\delta|t|}\mathrm{e}^{\mathrm{i}t(\cdot)}g(kh,\cdot)S(k,h)(t)\,dt\\ &= \delta(f,h)(\cdot) + \frac{1}{\sqrt{2\pi}}\mathbf{1}_{[-\frac{\pi}{h},\frac{\pi}{h}]}(\cdot)\sum_{|k|>N}h\hat{f}(kh)\phi\left(\frac{\cdot}{\varepsilon},\varepsilon,kh\right)\mathrm{e}^{\mathrm{i}kh(\cdot)},\end{aligned}\tag{3.42}$$

in $H^0_{-2\nu}(\mathbb{R})$, respectively

$$\delta_{\mathrm{N}}(f,h)(\cdot) = \delta(f,h)(\cdot) + \frac{1}{\sqrt{2\pi}}\sum_{|k|>N}h\hat{f}(kh)\phi\left(\frac{\cdot}{\varepsilon},\varepsilon,kh\right)\mathrm{e}^{\mathrm{i}kh(\cdot)},\tag{3.43}$$

in $H^1_{-2\nu}(-L,L)$. It follows therefore that

$$\begin{aligned}\|\delta_{\mathrm{N}}(f,h)(\cdot)\|_{0,-2\nu} &\le \|\delta(f,h)(\cdot)\|_{0,-2\nu}\\ &\quad + \frac{1}{\sqrt{2\pi}}\sum_{|k|>N}h|\hat{f}(kh)|\left\|\phi\left(\frac{\cdot}{\varepsilon},\varepsilon,kh\right)\mathrm{e}^{\mathrm{i}(\cdot)kh}\right\|_{0,-2\nu}\\ &\le \|\delta(f,h)(\cdot)\|_{0,-2\nu} + C\sum_{|k|>N}h|\hat{f}(kh)|,\end{aligned}$$

$$\begin{aligned}\|\delta_{\mathrm{N}}(f,h)(\cdot)\|_{H^1_{-2\nu}(-L,L)} &\leq \|\delta(f,h)(\cdot)\|_{H^1_{-2\nu}(-L,L)} \\ &\quad + \frac{1}{\sqrt{2\pi}}\sum_{|k|>N} h|\hat{f}(kh)| \left\|\phi\left(\frac{\cdot}{\varepsilon},\varepsilon,kh\right)\mathrm{e}^{\mathrm{i}(\cdot)kh}\right\|_{1,-2\nu} \\ &\leq \|\delta(f,h)(\cdot)\|_{H^1_{-2\nu}(-L,L)} + C\sum_{|k|>N} h|\hat{f}(kh)|,\end{aligned}$$

since it has been shown in Lemma A.2 that $\|\phi(\cdot/\varepsilon,\varepsilon,t)\mathrm{e}^{\mathrm{i}(\cdot)t}\|_{1,-2\nu} \leq C(\gamma,\nu)$, for all $t \in \mathcal{D}_d \subset \mathbb{C}$, therefore in particular for all $t \in \mathbb{R}$. Since $|\hat{f}(kh)| \leq C(f)\mathrm{e}^{-\alpha|k|h}$,

$$h\sum_{|k|>N} |\hat{f}(kh)| \leq 2\,C(f)h\,\mathrm{e}^{-\alpha Nh}\frac{\mathrm{e}^{-\alpha h}}{1-\mathrm{e}^{-\alpha h}} \leq 2\,C(f)\frac{1}{\alpha}\mathrm{e}^{-\alpha Nh}, \tag{3.44}$$

which implies with our choice of h that the sum in (3.42) satisfies the estimate (3.41). It is therefore enough to show that $\|\delta(f,h)(\cdot)\|_{0,-2\nu} + \|\delta(f,h)(\cdot)\|_{H^1_{-2\nu}(-L,L)}$ satisfy (3.41), and this is just the statement of Theorem 3.9. □

As a corollary, the following approximation result holds

Corollary 3.11 *Let us assume that f satisfies the usual assumptions and that*

$$h = \left(\frac{\pi d}{\alpha N}\right)^{1/2}, \quad N \geq \frac{4dL^2}{\alpha\pi}. \tag{3.45}$$

Let

$$\mathcal{W}^{\mathrm{N}}_\varepsilon := \mathrm{Span}\left\{\mathrm{Re}\,\psi\left(\frac{\cdot}{\varepsilon},\varepsilon,kh\right), \mathrm{Im}\,\psi\left(\frac{\cdot}{\varepsilon},\varepsilon,kh\right)\ :\ 0 \leq k \leq N\right\}. \tag{3.46}$$

Then

$$\begin{aligned}&\inf_{v\in\mathcal{W}^{\mathrm{N}}_\varepsilon} \|u^\varepsilon - v\|_{H^1_{-2\nu}(-L,L),H^0_{-2\nu}(\mathbb{R})} \\ &\leq C(\gamma,\nu)C(f)\left(1+\frac{1}{\alpha}\right)^2\frac{1}{\alpha}\mathrm{e}^{-(\pi\alpha dN)^{1/2}},\end{aligned} \tag{3.47}$$

where $C(f)$ and $\alpha = \alpha(f)$ are those from Definition 3.4.

3.4 Spectral convergence

In this section we assume that f in (2.1) is in $H^s_{\mathrm{comp}}(\mathbb{R})$. We will show that for any $\varepsilon > 0$ the solution u^ε can be approximated by

$$u^\varepsilon_{\mathrm{N}} \in \mathrm{Span}\,\{\mathrm{Re}\,\psi(\cdot/\varepsilon,\varepsilon,kh), \mathrm{Im}\,\psi(\cdot/\varepsilon,\varepsilon,kh)\ :\ |k| \leq N\}$$

with respect to $\|\cdot\|_{1,-\nu}$ at an algebraic convergence rate independent of ε.

Proposition 3.12 *Assume that f in (2.1) is $H^s_{\text{comp}}(\mathbb{R})$ with $s > 1$ and let* $\operatorname{supp} f \subset (-M, M)$*, with $M > 0$. Let $d := \min\{1/M, \nu/2\}$ and $N \geq 1$. Then, for any $\varepsilon > 0$ and all N*

$$\inf_{v \in \mathcal{W}^{\mathrm{N}}_\varepsilon} \|u^\varepsilon - v\|_{1,-\nu} \leq C_{\nu,s} M^{1/2} N^{-(s-1)/2} \|f\|_{H^s(\mathbb{R})}, \tag{3.48}$$

where $C_{\nu,s} > 0$ is independent of ε, N and M.

Proof. By a density argument it can be assumed that $f \in C^\infty_0(-M, M)$. It is known then that the Fourier transform of f can be continued analytically in $\mathbb{C}$ and $\hat{f}$ is uniformly bounded in a strip of width $1/M$. Therefore, the integrand in the Bochner integral representation (2.3) of u^ε is analytic in a strip with values in the Banach space $H^1_{-\nu}(\mathbb{R})$

$$g(t,\cdot) = \hat{f}(t)\psi\left(\frac{\cdot}{\varepsilon}, \varepsilon, t\right) \in \mathcal{A}\left(\mathcal{D}_{\min\{1/M,\nu/2\}}; H^1_{-\nu}(\mathbb{R})\right).$$

Define $d = \min\{1/M, \nu/2\}$, and let $h = \sqrt{d/N}$. Let us split the solution u^ε again as

$$u^\varepsilon(\cdot) = \sum_{|k| \leq N} \frac{1}{\sqrt{2\pi}} h\, \hat{f}(kh)\psi\left(\frac{\cdot}{\varepsilon}, \varepsilon, kh\right)$$

$$+ \sum_{|k| \geq N+1} \frac{1}{\sqrt{2\pi}} h\, \hat{f}(kh)\psi\left(\frac{\cdot}{\varepsilon}, \varepsilon, kh\right) + \int_{\mathbb{R}}^{(B)} g(t,\cdot)\, dt - h \sum_{k \in \mathbb{Z}} g(kh,\cdot). \tag{3.49}$$

We define $u^\varepsilon_{\mathrm{N}}$ as the first sum in the right hand side of (3.49). The regularity of f implies that

$$\begin{aligned} \left|t^\alpha \hat{f}(t)\right| &= \frac{1}{\sqrt{2\pi}} \left| \int_{supp\, f} \mathrm{e}^{-\mathrm{i}ty} \frac{d^\alpha f}{dy^\alpha}(y) \right| \\ &\leq C M^{1/2} \|f\|_{H^s(\mathbb{R})}, \quad \forall\, t \in \mathbb{R}, \quad \forall\, \alpha \leq s. \end{aligned}$$

It has been shown in Theorem 2.3 that $\psi(\cdot/\varepsilon, \varepsilon, t)$ is analytic in $\mathcal{D}_{\nu/2}$ with values in $H^1_{-\nu}(\mathbb{R})$ and uniformly bounded. Moreover, the norm $\|\psi(\cdot/\varepsilon, \varepsilon, t)\|_{L^\infty(\mathcal{D}_{\nu/2}; H^1_{-\nu}(\mathbb{R}))}$ is bounded uniformly with respect to ε. Hence, the second sum in (3.49) satisfies the estimate in (3.48)

$$\begin{aligned} &\left\| \sum_{|k| \geq N+1} \frac{1}{\sqrt{2\pi}} h\, \hat{f}(kh)\psi\left(\frac{\cdot}{\varepsilon}, \varepsilon, kh\right) \right\|_{1,-\nu} \\ &\leq C_\nu M^{1/2} \|f\|_{H^s(\mathbb{R})} h \sum_{|k| \geq N+1} (kh)^{-s} \\ &\leq C_{\nu,s} M^{1/2} \|f\|_{H^s(\mathbb{R})} (dN)^{-(s-1)/2}. \end{aligned}$$

It remains to find a similar bound for the remainder in (3.49). Define by

$$\delta(g) = \int_{\mathbb{R}}^{(B)} g(t,\cdot)\,dt - \sum_{k\in\mathbb{Z}} h\,g(kh,\cdot).$$

We can write $\delta(g)$ as

$$\delta(g) = \int_{\mathbb{R}}^{(B)} \left\{ g(t,\cdot) - \sum_{k\in\mathbb{Z}} g(kh,\cdot) S(k,h)(t) \right\} dt. \tag{3.50}$$

Since $\sup_{t\in\mathcal{D}_d} \|g(t,\cdot)\|_{1,-\nu} \le C_\nu \sup_{t\in\mathcal{D}_d} |\hat{f}(t)| \le C_\nu M^{1/2} \|f\|_{H^s(\mathbb{R})}$, by Definition 3.3 $g(t,\cdot) \in \boldsymbol{H}^\infty\left(\mathcal{D}_d; H^1_{-\nu}(\mathbb{R})\right)$. Therefore, as pointed out in Remark 3.8, the integrand in (3.50) can be written as

$$\begin{aligned} g(t,\cdot) &- \sum_{k\in\mathbb{Z}} g(kh,\cdot) S(k,h)(t) \\ &= \frac{\sin(\pi t/h)}{2\pi i} \int_{\mathbb{R}}^{(B)} \frac{g(\tau - id^-,\cdot)}{(\tau - t - id)\sin[\pi(\tau - id)/h]} \\ &\quad - \frac{g(\tau + id^-,\cdot)}{(\tau - t + id)\sin[\pi(\tau + id)/h]}\,d\tau. \end{aligned} \tag{3.51}$$

Substituting (3.51) in (3.50), changing the order of integration and integrating with respect to t first, we get that

$$\begin{aligned} \delta(g) = \int_{\mathbb{R}}^{(B)} \Bigg\{ &\frac{i}{2} \frac{\mathrm{e}^{-\pi/h(d+i\tau)}}{\sin[\pi(\tau - id)/h]} g(\tau - id^-,\cdot) \\ &- \frac{i}{2} \frac{\mathrm{e}^{-\pi/h(d-i\tau)}}{\sin[\pi(\tau + id)/h]} g(\tau + id^-,\cdot) \Bigg\}\,d\tau. \end{aligned} \tag{3.52}$$

Taking the $\|\cdot\|_{1,-\nu}$ norm of $\delta(g)$ in (3.52) we can estimate it as follows

$$\|\delta(g)\|_{1,-\nu} \le C_{\nu,s} \mathrm{e}^{-\pi\sqrt{dN}} M^{1/2} \|f\|_{H^s(\mathbb{R})},$$

and conclude the proof. $\square$

4 Generalized p-FEM in homogenization

4.1 Convergence results

We return to the problem (1.1) on the bounded domain $\Omega := (-1, 1)$: let $f(\cdot) \in L^2(\Omega)$ and $\varepsilon > 0$ fixed. Denote by $u^\varepsilon(\cdot) \in H_0^1(\Omega)$ the weak solution of the following boundary value problem

$$-\frac{d}{dx}\left(a\left(\frac{x}{\varepsilon}\right)\frac{du^\varepsilon}{dx}(x)\right) + a_0\left(\frac{x}{\varepsilon}\right)u^\varepsilon(x) = f(x) \quad \text{in} \quad \Omega,$$

$$u^\varepsilon(-1) = u^\varepsilon(1) = 0. \tag{4.1}$$

FE-convergence results for (4.1) can be deduced from the unbounded domain case. We start with a spectral convergence result.

Theorem 4.1 *Let $f \in H^s(\Omega)$ for some $s > 1$ and consider $\tilde{\mathcal{W}}_\varepsilon^\mu := \mathcal{W}_\varepsilon^\mu \cap H_0^1(\Omega)$, with $\mathcal{W}_\varepsilon^\mu$ as in (3.46). Then, there exists a constant $C > 0$ depending only on Ω and s, such that*

$$\inf_{v \in \tilde{\mathcal{W}}_\varepsilon^\mu} \|u^\varepsilon - v\|_{1,\Omega} \le C\mu^{-(s-1)/2}\|f\|_{s,\Omega}. \tag{4.2}$$

Proof. The proof is based on Proposition 3.12 and on a well known extension result for Sobolev functions. There exists a continuous extension operator $\Sigma : H^s(\Omega) \to H^s(\mathbb{R})$, such that $supp\, \Sigma g \subset \tilde{\Omega}, \forall\, g \in H^s(\Omega)$, with $\overline{\Omega} \subset \tilde{\Omega}$ and $\tilde{\Omega}$ compactly embedded in $\mathbb{R}$. Let us denote by $\overline{f}$ the extension Σf of f. Then, by the continuity of Σ, $\|\overline{f}\|_{s,\mathbb{R}} \le C\|f\|_{s,\Omega}$, with $C > 0$ a constant depending only on s, Ω and $\tilde{\Omega}$, but independent on f. Let $\overline{u}^\varepsilon \in H^1_{-\nu}$ be the solution of (2.1) corresponding to $\overline{f}$. Then, its restriction $\overline{u}^\varepsilon|_\Omega$ solves the differential equation in (4.1), but does not fulfill the boundary conditions. They can be enforced by solving two extra problems (2.1) with right hand sides $f_1, f_2 \in C_0^\infty(\mathbb{R})$, such that

$$(supp\, f_1 \cup supp\, f_2) \cap \overline{\Omega} = \emptyset. \tag{4.3}$$

Let $\overline{u}_1^\varepsilon, \overline{u}_2^\varepsilon$ be the corresponding solutions (on $\mathbb{R}$) of (2.1) with respect to f_1, f_2. Then, because of (4.3), their restrictions $\overline{u}_1^\varepsilon|_\Omega, \overline{u}_2^\varepsilon|_\Omega$ solve the differential equation in (4.1) with homogeneous right hand side. Denoting by

$$u^\varepsilon := \overline{u}^\varepsilon|_\Omega + c_1\overline{u}_1^\varepsilon|_\Omega + c_2\overline{u}_2^\varepsilon|_\Omega,$$

then there exist unique constants $c_1, c_2 \in \mathbb{R}$, such that u^ε satisfies the homogeneous boundary conditions in (4.1). Moreover, it can be seen that $|c_1| + |c_2| \le C_{s,\nu}\|f\|_{s,\Omega}$, with the constant $C_{s,\nu} > 0$ depending only on s, ν.

By Proposition 3.12, $\overline{u}^{\varepsilon}, \overline{u}_1^{\varepsilon}, \overline{u}_2^{\varepsilon}$ can be approximated in $H^1_{-\nu}(\mathbb{R})$ at an algebraic rate of convergence $\mu^{-(s-1)/2}$ by elements of the FE space $\mathcal{W}_\varepsilon^\mu$, and therefore their restrictions to Ω too. □

No exponential convergence can be proved in this way, since for analytic $f \in [-1,1]$, Σf is not an analytic function on $\mathbb{R}$ anymore; however, the following result shows that *if subspaces are designed corresponding to solutions of (4.1) with polynomial right hand side, exponential convergence is achieved*. To this end, associated with the kernel $\psi(\cdot,\varepsilon,t)$ as in (2.9), we introduce the FE-spaces $\tilde{\mathcal{V}}_\varepsilon^\mu \subset H_0^1(-1,1)$:

$$\mathcal{V}_\varepsilon^\mu := \text{span}\left\{ \text{Re}\, \frac{d^l\psi}{dt^l}\left(\frac{\cdot}{\varepsilon},\varepsilon,0\right), 0 \le l \le \mu,\ l = 2k, \right.$$
$$\left. \text{Im}\, \frac{d^l\psi}{dt^l}\left(\frac{\cdot}{\varepsilon},\varepsilon,0\right), 0 \le l \le \mu,\ l = 2k+1 \right\},$$

$$\tilde{\mathcal{V}}_\varepsilon^\mu := (\mathcal{V}_\varepsilon^\mu + \text{span}\,\{v_1^\varepsilon, v_2^\varepsilon\}) \bigcap H_0^1(-1,1), \tag{4.4}$$

where $v_1^\varepsilon(\cdot), v_2^\varepsilon(\cdot)$ are the solutions of (4.1) with homogeneous data $f = 0$ and the following inhomogeneous boundary conditions:

$$v_1^\varepsilon(-1) = 1, \quad v_1^\varepsilon(1) = 0, \quad \text{resp.} \quad v_2^\varepsilon(-1) = 0, \quad v_2^\varepsilon(1) = 1.$$

Theorem 4.2 *Let f be analytic in $[-1,1]$ and let u^ε be the weak solution of (4.1). There exist constants $C, b > 0$, depending only on f, such that for $\mu \in \mathbb{N}$ sufficiently large*

$$\inf_{v \in \tilde{\mathcal{V}}_\varepsilon^\mu} \|u^\varepsilon - v\|_{H_0^1(-1,1)} \le C\mathrm{e}^{-b\mu}. \tag{4.5}$$

With other words, the error with respect to the FE-space $\tilde{\mathcal{V}}_\varepsilon^\mu$ decays exponentially with respect to μ, uniformly in ε.

Remark 4.3 We observe that $\mathcal{V}_\varepsilon^\mu$ is spanned by products of the “micro” shape functions $\frac{d^l}{dt^l}\phi\left(\frac{x}{\varepsilon},\varepsilon,t\right)\Big|_{t=0}$ times polynomials of degree at most μ. In particular, we see that increasing the number of “micro” shape functions must be accompanied by some increase in the macroscopic polynomial degree p to achieve (4.5). We will address this computationally below.

Before giving the proof of Theorem 4.2 we need the following preparatory lemma.

Lemma 4.4 *Let $L_k(\cdot)$, $k \in \mathbb{N}$ denote the k-th Legendre polynomial, and consider $f \in \mathcal{A}([-1,1])$ and its Legendre series $f(x) = \sum_{k=0}^\infty a_k L_k(x)$. Then,*

$$\sum_{k=0}^\infty 2\frac{|a_k|^2}{2k+1} = \|f\|^2_{L^2(-1,1)} \tag{4.6}$$

and there exist $\tilde{C}(f), b > 0$ *such that*

$$\|f - f^{(p)}\|_{L^2(-1,1)} \le \tilde{C}(f)\mathrm{e}^{-bp}, \tag{4.7}$$

where $f^{(p)}$ *is the truncated Legendre series*

$$f^{(p)} := \sum_{k=0}^{p} a_k L_k. \tag{4.8}$$

The constant $b > 0$ *depends on the domain of analyticity of* $f(\cdot)$.

For a proof of this result see e.g. [6].

Proof of Theorem 4.2. Denoting by $\phi^{\varepsilon}_{(k)}(\cdot)$ the weak solution in $H^1_0(-1,1)$ of (4.1) which corresponds to $f = L_k$, we get that

$$u^{\varepsilon}_{(\mu)}(\cdot) := \sum_{k=0}^{\mu} a_k \phi^{\varepsilon}_{(k)}(\cdot) \tag{4.9}$$

solves (4.1) with the right hand side $f^{(\mu)}$. By Lemma 4.4 the error with respect to the exact solution u^ε satisfies the following bound

$$\|u^\varepsilon - u^\varepsilon_{(\mu)}\|_{H^1_0(-1,1)} \le C(\gamma)\|f - f^{(\mu)}\|_{L^2(-1,1)} \le C(\gamma, f)\mathrm{e}^{-b\mu}. \tag{4.10}$$

It is therefore enough to show that $u^\varepsilon_{(\mu)}(\cdot) \in \tilde{\mathcal{V}}^{\mu}_{\varepsilon}$. To this end, recall that $d^l/dt^l \psi(\cdot/\varepsilon, \varepsilon, 0)$ are solutions of (2.1) corresponding to $f = (\mathrm{i}x)^l$. Since $\psi(\cdot/\varepsilon, \varepsilon, -t) = \overline{\psi(\cdot/\varepsilon, \varepsilon, t)}$,

$$\begin{aligned} \frac{d^l\psi}{dt^l}\left(\frac{\cdot}{\varepsilon}, \varepsilon, 0\right) &= \mathrm{Re}\,\frac{d^l\psi}{dt^l}\left(\frac{\cdot}{\varepsilon}, \varepsilon, 0\right), \text{ if } l = 2k, k \in \mathbb{N}, \\ \frac{d^l\psi}{dt^l}\left(\frac{\cdot}{\varepsilon}, \varepsilon, 0\right) &= \mathrm{iIm}\,\frac{d^l\psi}{dt^l}\left(\frac{\cdot}{\varepsilon}, \varepsilon, 0\right), \text{ if } l = 2k+1, k \in \mathbb{N}. \end{aligned}$$

Therefore, $(-\mathrm{i})^l d^l/dt^l \psi(\cdot/\varepsilon, \varepsilon, 0)$ solves (2.1) with $f = x^l$ and takes in all cases real values. □

4.2 Selection of the micro shape functions

We have seen so far that collocation of the kernel $\psi(x/\varepsilon, \varepsilon, t)$ at various sets of collocation points $\mathcal{N} = \{t_j\}_j$ gives systems of shape functions with very favorable approximation properties for elliptic problems with microstructure. In the present section, we present a FEM for the solution of the unit cell problem and a methodology to derive a well conditioned set of shape functions from the collocated kernels $\psi(x/\varepsilon, \varepsilon, t_j)$, $t_j \in \mathcal{N}$. This will be based on the SVD of the matrix of coefficient vectors of the Finite Element approximations to the $\phi(y,\varepsilon,t_j)$, $t_j \in \mathcal{N}$.

Let $\mathcal{N} = \{t_j \,:\, j = 1,\ldots,\hat{\mu}\}$ be any set of collocation points in $\mathbb{C}$. Given a partition $\mathcal{T}$ of the unit cell $Q = (-\pi,\pi)$ into intervals K, for an arbitrary $t_j \in \mathcal{N}$, compute the FE approximations

$$\tilde{\phi}(y,\varepsilon,t_j) \in S^{k,1}_{\mathrm{per}}(Q,\mathcal{T}) \,:\, \Phi(\varepsilon,t_j)[\tilde{\phi},v] = \varepsilon^2 \int_Q \overline{v(y)}\,dy, \qquad \forall\, v \in S^{k,1}_{\mathrm{per}}(Q,\mathcal{T}), \tag{4.11}$$

where $\Phi(\varepsilon,t)[\cdot,\cdot]$ is as in (3.2) and the FE space $S^{k,1}_{\mathrm{per}}(Q,\mathcal{T})$ is defined by

$$S^{k,1}_{\mathrm{per}}(Q,\mathcal{T}) := \left\{ u \in H^1_{\mathrm{per}}(Q) \,:\, u\Big|_K \in \mathcal{P}_k(K),\ \forall\, K \in \mathcal{T} \right\}, \tag{4.12}$$

and $\mathcal{P}_k(K)$ is a space of polynomials of degree at most k on K. Since the sesquilinear form $\Phi(\varepsilon,t)[\cdot,\cdot]$ is coercive (in the sense that a Gårding inequality holds and the unit cell solution operator is injective), there exists a unique solution $\tilde{\phi}(y,\varepsilon,t_j) \in S^{k,1}_{\mathrm{per}}(Q,\mathcal{T})$ of (4.11).

Several questions arise in practice:

1. How to design the mesh $\mathcal{T}$ in $S^{k,1}_{\mathrm{per}}(Q,\mathcal{T})$ for the computation of the unit-cell problem?

2. How to choose the collocation points t_j?

3. Are the functions $\tilde{\phi}(y,\varepsilon,t_j)$ suitable as basis functions for FE calculations?

4. How does $\mathrm{span}\,\{\tilde{\phi}(y,\varepsilon,t_j) \,:\, t_j \in \mathcal{N}\}$ depend on $\mathcal{N}$?

We have found the following answers:

1. If the coefficient functions $a(y), a_0(y)$ in (4.1) are piecewise analytic functions of y, so are the $\tilde{\phi}(y,\varepsilon,t_j)$. Therefore, $\mathcal{T}$ is selected such that the elements coincide with pieces of analyticity of $a(\cdot), a_0(\cdot)$.

2. In agreement with Theorem 3.9, we choose $t_j(\mu) = jh$ where $h = 1/\sqrt{\mu}$ with $j = 0,1,...,\mu-1$. Notice that the values of d and α in (3.38) are generally not available. Therefore, the choice of t_j is to some extent heuristic (see, however, item 4. below).

3. By Theorem 2.3, $\phi(y,\varepsilon,t)$ is analytic in t at $t = 0$. As μ increases, the collocation points t_j will cluster near $t = 0$ (as, e.g., in $\mathcal{W}^{\mu}_{\varepsilon}$ in (3.46)) resulting in almost linear dependence of the shape functions $\tilde{\phi}(y,\varepsilon,t_j)$; these functions are hence not well-suited as basis for a generalized p-FEM. Some orthogonalization is needed to obtain a well-conditioned basis. In addition, the points $t_j(\mu)$ depend on μ meaning that the shape functions $\tilde{\phi}(y,\varepsilon,t_j(\mu))$ are not hierarchical.

We propose therefore an *oversampling*, i.e. to select $\hat{\mu} > \mu$ sufficiently large and

$$\mathcal{N} = \{t_j(\hat{\mu}) \,:\, j = 1, \dots, \hat{\mu}\},$$

and to perform an *orthogonalization* as follows:

Algorithm 4.5 Let $\underline{N}(y)$ be a basis of $S^{k,1}_{\mathrm{per}}(Q, \mathcal{T})$. Then $\tilde{\phi}(y, \varepsilon, t_j) = \Phi_j(\varepsilon)^\top \underline{N}(y), \quad j = 1, \dots, \hat{\mu}$. Compute the SVD

$$[\Phi_1(\varepsilon), \dots, \Phi_{\hat{\mu}}(\varepsilon)] = \underline{U}\,\mathrm{diag}(\sigma_1, \dots, \sigma_{\hat{\mu}})\underline{V}^\top$$

with $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \dots \geq \sigma_{\hat{\mu}} \geq 0$ and set

$$\hat{\mathcal{V}}^\mu_\varepsilon := \mathrm{span}\left\{\phi_j\left(\frac{x}{\varepsilon}, \varepsilon\right) := \underline{U}_j^\top \underline{N}\left(\frac{x}{\varepsilon}\right),\ j = 1, \dots, \mu\right\}, \tag{4.13}$$

with $\underline{U}_j$ being the j-th column of $\underline{U}$.

Ignoring roundoff, this orthogonalization changes only the basis, but not the span of the shape functions if $\mu = \hat{\mu}$. If $\mu < \hat{\mu}$, however, the definition (4.13) will change the span. Nevertheless, if $\sigma_j < eps$ for $\mu < j \leq \hat{\mu}$ with eps of the order of machine precision, this change will be negligible.

4. If $|kh| \leq \rho_0 < 1$, with ρ_0 being the radius of convergence of the power series of $\psi(\cdot/\varepsilon, \varepsilon, t)$ at $t = 0$, then the elements $\mathrm{Re}\,\psi(\cdot/\varepsilon, \varepsilon, kh)$, $\mathrm{Im}\,\psi(\cdot/\varepsilon, \varepsilon, kh)$ of the FE space $\mathcal{W}^{\mathrm{N}}_\varepsilon$ in (3.46) can be, up to an exponentially decaying remainder $\mathrm{e}^{-b\mu}$, approximated by elements in the FE space $\mathcal{V}^\mu_\varepsilon$ introduced in (4.4), with μ equal to the number of k such that $|k|h \leq \rho_0$. Since the kernel $\psi(\cdot/\varepsilon, \varepsilon, t)$ is analytic in t, for any set of collocation points $\{t_j\}$ which are close to the origin, span $\{\psi(\cdot/\varepsilon, \varepsilon, t_j)\}$ is practically independent of the choice of the collocation points. Therefore the precise choice of t_j will not matter much, as long as with increasing μ they cover the interval $[-\sqrt{\mu}, \sqrt{\mu}]$ and are spaced as $1/\sqrt{\mu}$ by Theorem 3.10.

We present in Fig. 2 the shape functions $\{\phi_j(y, \varepsilon)\}_{j=1}^{\mu}$ obtained with Algorithm 4.5 for the case when $a_0 \equiv 1$, $a(\cdot)$ is as in (4.14), $\varepsilon = 0.001$. Based on Theorem 3.10 the set of collocation points is $\mathcal{N} = \{t_j(\hat{\mu}) = j/\sqrt{\hat{\mu}} \,:\, j = 0, \dots, \hat{\mu},\ \hat{\mu} = 64\}$. In this case the number of j such that the corresponding singular values $\sigma_j > eps = 10^{-10}$ is $\mu = 5$. Hence the orthogonalization has, as a byproduct, also reduced the number of micro shape functions substantially. We clearly see the low regularity of these shape functions at the jumps of $a(\cdot)$ at $y = \pm\pi/2$. Note also that, unlike the kernels $\phi(y, \varepsilon, t_j)$, the $\phi_j(y, \varepsilon)$ are piecewise polynomials.

$$a(y) = \begin{cases} 10 & \text{if } |y| \leq \dfrac{\pi}{2}, \\ 1 & \text{else}, \end{cases} \qquad a_0(y) = \begin{cases} 1 & \text{if } |y| \leq \dfrac{\pi}{2}, \\ 50 & \text{else}\,. \end{cases} \tag{4.14}$$

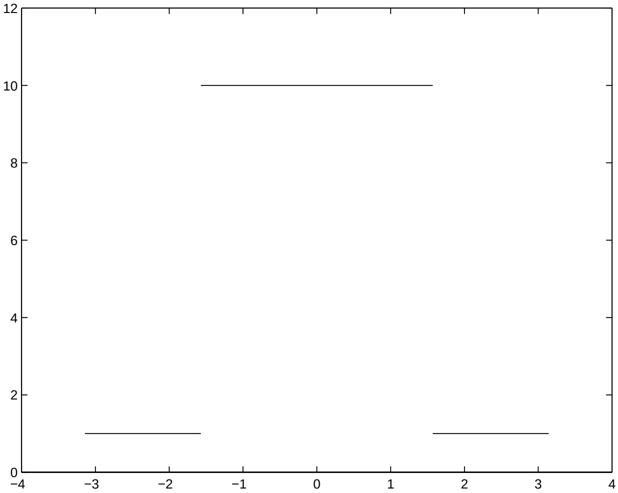

Fig. 1. The coefficient $a(\cdot)$

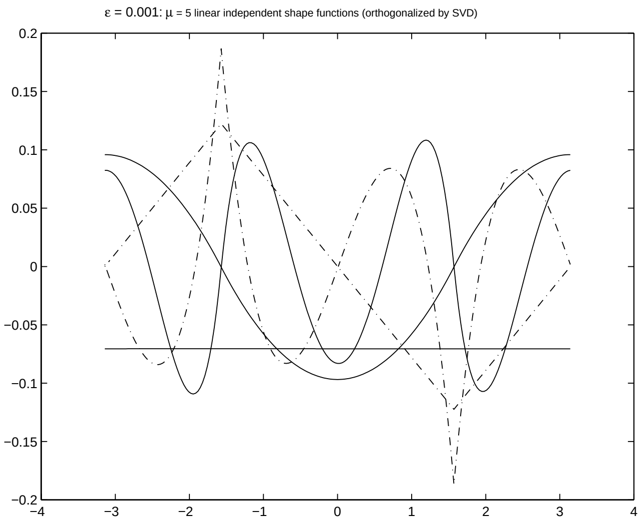

Fig. 2. $\phi_j(\cdot, \varepsilon),\ j = 1, \ldots, 5$

Remark 4.6 We see in Fig. 2 that $\phi_1(y, \varepsilon) \equiv\ const$; this is due to $a_0 \equiv 1$, in fact if $a_0 \not\equiv 1$, then the solution of (4.11) for $t = 0$ is not the constant function equal to 1. To illustrate this, we choose $a(\cdot)$ and $a_0(\cdot)$ as in (4.14). Our numerical results indicate that in this case we have $\phi_1(y, \varepsilon) = const + O(\varepsilon)\phi_2(y, \varepsilon) + h.o.t$, see Fig. 3.

Remark 4.7 In numerical experiments we found that Algorithm 4.5 is very robust with respect to the choice of collocation points. After the SVD the first shape functions associated with the largest singular values are practi-

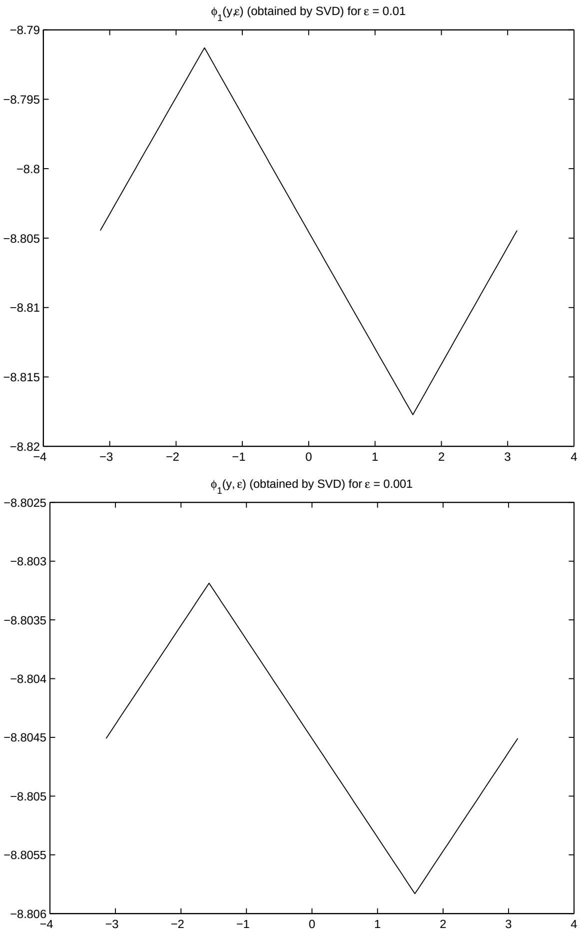

Fig. 3. $10^2\phi_1(y,\varepsilon)$ in the case when the absolute term $a_0(\cdot)$ is not a constant, but piecewise constant

cally independent of the number and of the choice of t_j. The shape functions $\phi_j(\cdot,\varepsilon)$ resulting from Algorithm 4.5 are therefore, at least numerically, *hierarchical*, and enable *hierarchic modeling* of problems with microstructure.

4.3 Generalized p-FEM

We consider now the problem (1.3) with absolute terms $a_0 \equiv 1$, $a_0 \equiv 0$, respectively. Since $\tilde{\mathcal{V}}_\varepsilon^\mu$ in (4.4) is not available (because the computation of the boundary correctors $v_1^\varepsilon, v_2^\varepsilon$ is as expensive as that of the solution itself), we construct a space $S_0^{\underline{p},\underline{\mu}}(\Omega,\mathcal{T})\subset H_0^1(\Omega)$ with analogous properties:

$$S_0^{\underline{p},\underline{\mu}}(\Omega,\mathcal{T})=\Bigg\{u\in H_0^1(\Omega)\,:\,u\Big|_K=\sum_{j=1}^{p_K+1}\sum_{\mu=1}^{\mu_K+1}u_{j,\mu}^{[K]}\nu_j^{[K]}(x)\phi_\mu\left(\frac{x}{\varepsilon},\varepsilon\right),$$

$$u_{j,\mu}^{[K]}\in\mathbb{R},\ \forall\,K\in\mathcal{T},\,j=1,\dots,p_K+1,\,\mu=1,\dots,\mu_K+1\Bigg\}, \tag{4.15}$$

where $\nu_j^{[K]}(x)=N_j\left(\left(F^{[K]}\right)^{-1}(x)\right)$, $\forall\,K\in\mathcal{T}$. By $F^{[K]}\,:\,(-1,1)\to K$ we denote the linear mapping with respect to the element $K=(\underline{x}_K,\overline{x}_K)$

$$x=F^{[K]}(\xi)=\frac{1}{2}(1-\xi)\underline{x}_K+\frac{1}{2}(1+\xi)\overline{x}_K,\quad \forall\,\xi\in(-1,1),$$

and $\{N_j(\xi)\}$ is the standard hierarchical polynomial basis

$$N_1(\xi)\;=\;(1-\xi)/2,\quad N_2(\xi)=(1+\xi)/2,$$

$$N_j(\xi)\;=\;\sqrt{\frac{2j-1}{2}}\int_{-1}^{\xi}L_{j-2}(t)\,dt,\quad \forall\,j\geq 3. \tag{4.16}$$

By the vector $\underline{p}=\{p_K\}_{K\in\mathcal{T}}$ we denote the 'macro' polynomial degree of the FE method, and $\underline{\mu}=\{\mu_K\}_{K\in\mathcal{T}}$ stands for the 'micro' degree of the spectral approximation.

The FE solution $u_{FE}^{\varepsilon}(x)$ is defined as usual:

$$u_{FE}^{\varepsilon}(\cdot)\in S_0^{\underline{p},\underline{\mu}}(\Omega,\mathcal{T})\;\;:\;\;\int_{-1}^{1}a\left(\frac{x}{\varepsilon}\right)\frac{du_{FE}^{\varepsilon}}{dx}(x)\frac{dv}{dx}(x)\,dx$$

$$=\int_{-1}^{1}f(x)v(x)\,dx,\quad \forall\,v\in S_0^{\underline{p},\underline{\mu}}(\Omega,\mathcal{T}). \tag{4.17}$$

We see from (4.15) that each element contains products of standard polynomial shape functions (4.16) and the first μ_K+1 micro shape functions. We used in all our computations the orthonormalized micro shape functions $\phi_j(y,\varepsilon)$ in (4.13) from the unit cell problem with absolute term $a_0\equiv 1$. The mesh $\mathcal{T}=\mathcal{T}_{\rm b}\bigcup\mathcal{T}_0$ is selected to have the following properties:

– if $K\in\mathcal{T}_{\rm b}$ (which means that K is a boundary element and the length of K is $O(\varepsilon)$), then we choose the standard p - FEM elements, since the microscale is resolved by $\mathcal{T}_{\rm b}$, i.e., $\mu_K=0$ and $p_K=p$; these elements are needed to accommodate the homogeneous boundary conditions and could be omitted for the Neumann problem.

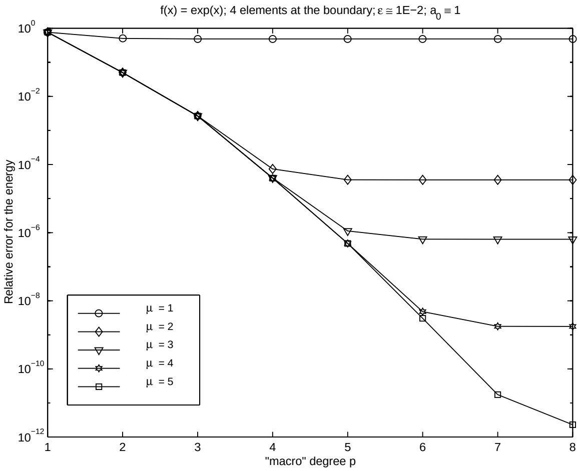

Fig. 4. Exponential rate of convergence for the FE energy. $f(x) = \exp(x)$

– if $K \in \mathcal{T}_0$, then we take $\mu_K = \mu$ and $p_K = p$, which corresponds to the PUM using $\mathcal{V}_\varepsilon^\mu$ as local approximation spaces. With this choice, the FE functions $u \in S_0^{\underline{p},\underline{\mu}}(\Omega, \mathcal{T})$ will provide excellent approximation properties on the interior elements $K \in \mathcal{T}_0$ for the elements of $\mathcal{V}_\varepsilon^\mu$; it turns out that the boundary correctors $v_1^\varepsilon, v_2^\varepsilon$ are also very well approximated on these elements by $S_0^{\underline{p},\underline{\mu}}(\Omega, \mathcal{T})$.

Remark 4.8 Equivalently, we may choose $\mathcal{T}_{\mathrm{b}} = \emptyset$ and modify the shape functions $\phi_j(y, \varepsilon)$ in the elements $K \in \mathcal{T}_0$ abutting at the boundary, see [8] for details.

Remark 4.9 Computation of the stiffness matrix can be done with a fixed number of operations (independent of ε) exploiting the periodicity of the coefficients $a(\cdot), a_0(\cdot)$ and that of the special shape functions $\phi_j(y, \varepsilon)$. We must compute only once integrals of $\phi_j(y, \varepsilon)$ and its derivatives times monomials on the unit cell. This is the reason to use $\phi_j(y, \varepsilon)$ times monomials instead of $\psi(y, \varepsilon, t_j)$. Full details can be found in [8].

4.4 Numerical results

We implemented the generalized p-FEM described in the previous section for (4.1) with $a(\cdot)$ as in (4.14) and absolute terms $a_0 \equiv 1$, $a_0 \equiv 0$, respectively. Two different right hand sides were chosen, namely

$$f_1(x) = 1, \quad f_2(x) = \mathrm{e}^x. \tag{4.18}$$

The exact solution $u^\varepsilon(x)$ corresponding to $a_0 \equiv 0$ and $f(x) = f_1(x)$ is piecewise cubic, for $f = f_2$ the solution $u^\varepsilon(x)$ is piecewise analytic but

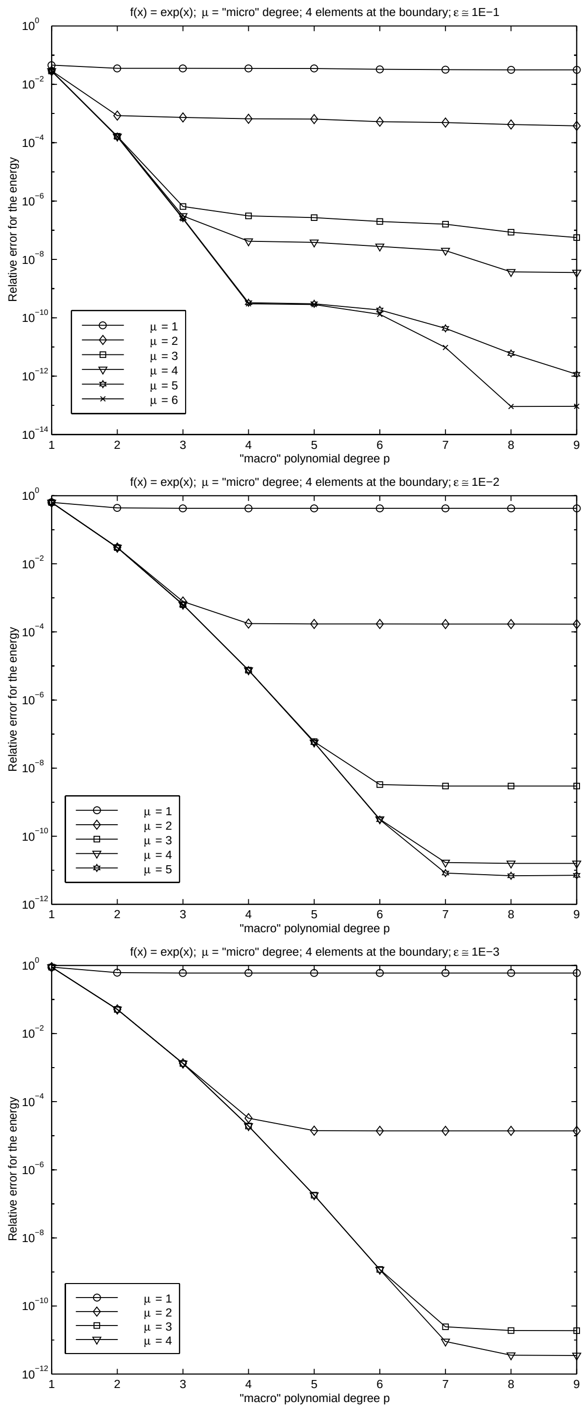

Fig. 5. Convergence rate for fixed micro degree μ and increasing macro polynomial degree p. $f(x) = f_2(x)$

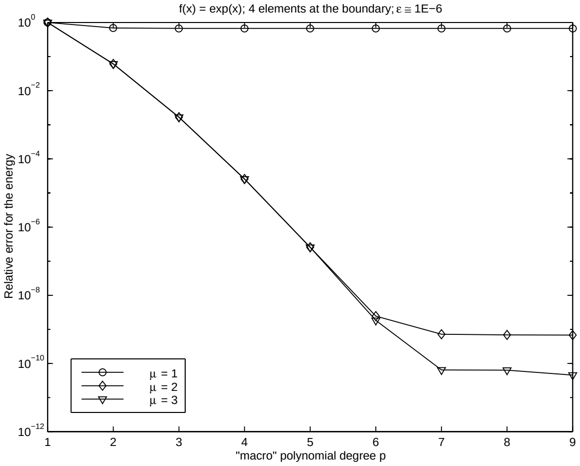

Fig. 5. (continued)

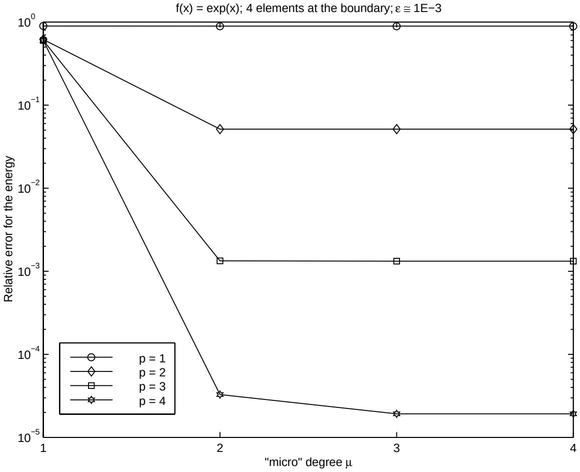

Fig. 6. Convergence rate for fixed macro polynomial degree p and increasing micro degree μ. $f(x) = f_2(x)$

non-polynomial on the microscale. The goal of the numerical experiments is to show $a)$ that exponential convergence can be achieved (with subspaces (4.15)), $b)$ that this convergence is indeed independent of ε, $c)$ that the particular choice of the subspace span $\left\{\phi_\mu\left(\frac{x}{\varepsilon}, \varepsilon\right), \mu = 1, \ldots, \mu_K + 1\right\}$ needs to take into account only the principal part of the operator (4.1) and $d)$ to investigate combination of p_K and μ_K necessary to obtain exponential convergence. Note that our mathematical theory does not allow to draw conclusions on $c)$ and $d)$.

In all experiments p is increased on a fixed mesh $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{T}_\mathrm{b}$ with $\mathcal{T}_\mathrm{b}$ covering 4 periods of length $2\pi\varepsilon$ at each boundary point for various values

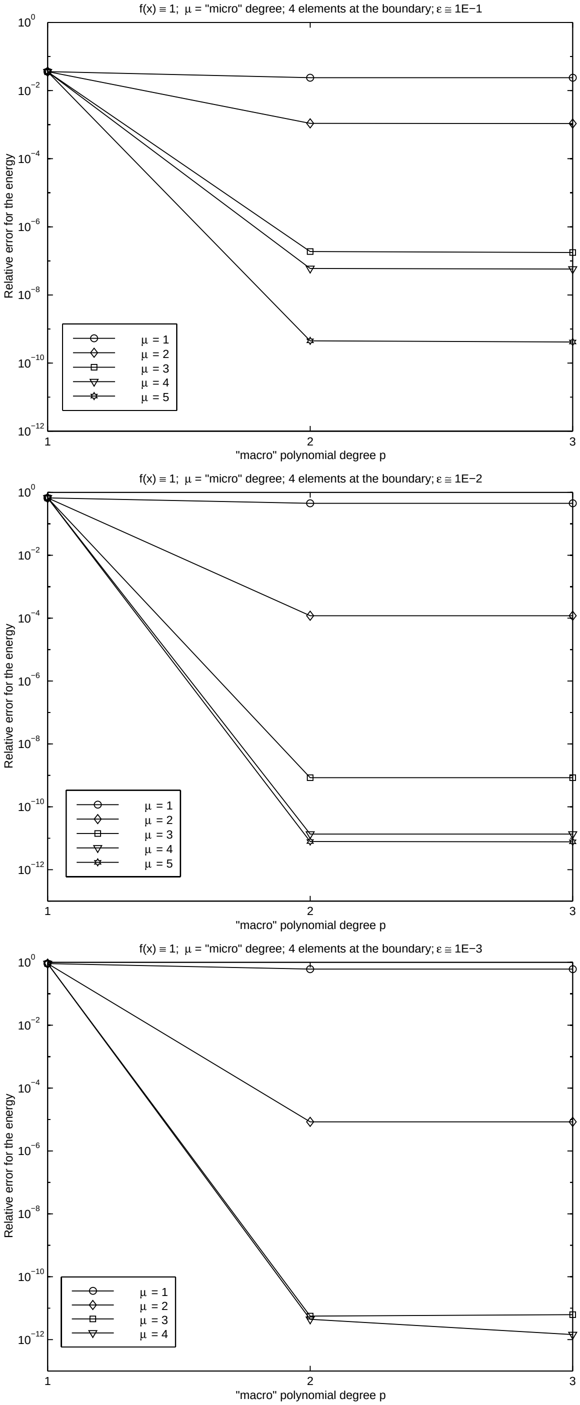

Fig. 7. Convergence rate for fixed micro degree μ and increasing macro polynomial degree p. $f(x) = f_1(x)$

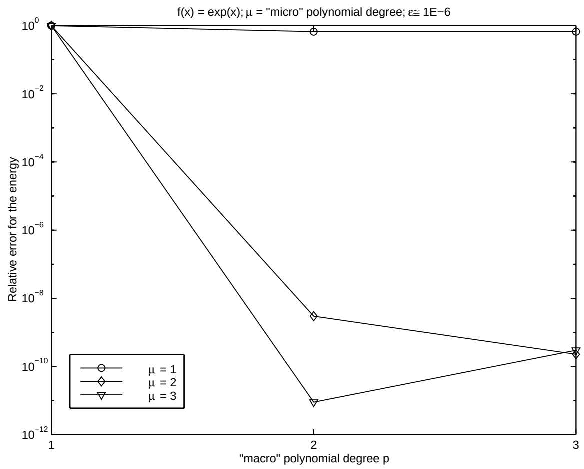

Fig. 7. (continued)

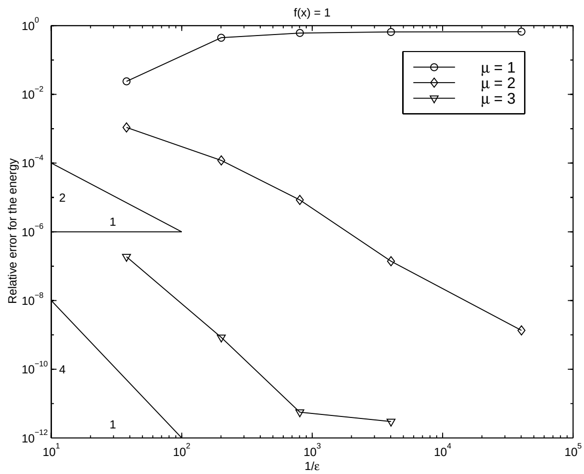

Fig. 8. Relative error for the energy versus $1/\varepsilon$ for increasing micro degree μ

of μ. Figure 4 shows the convergence of the generalized p-FEM for $a_0 \equiv 1$, $f(x) = f_2(x)$ and $\varepsilon \cong 10^{-2}$. The curves corresponding to $\mu = 1$ show the error when only macroscopic shape functions, i.e. global polynomials, are used (recall that $a_0 \equiv 1$ and that $\phi_1(y, \varepsilon) \equiv const$, see Remark 4.6).

We see that for fixed $\mu > 1$ and increasing p, first exponential convergence is apparent, however a saturation occurs at a p-level which depends on the micro degree μ. Exponential convergence requires therefore the joint increase of the micro degree μ with the macro degree p.

So far, our theory concerned the case when $a_0 > 0$. In practice, however, also the case $a_0 = 0$ is of interest, for example in diffusion problems. For $a_0 = 0$, our mathematical results require several technical modifications.

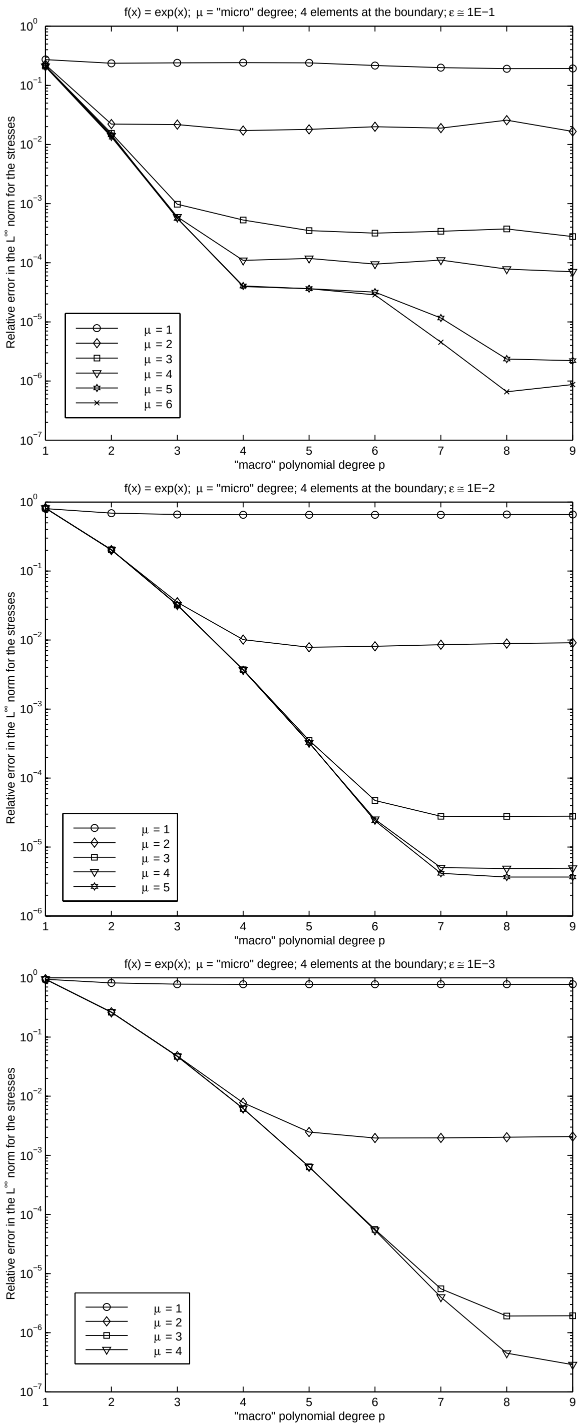

Fig. 9. Exponential rate of convergence in the L^∞ norm for the stresses. $f(x) = f_2(x)$

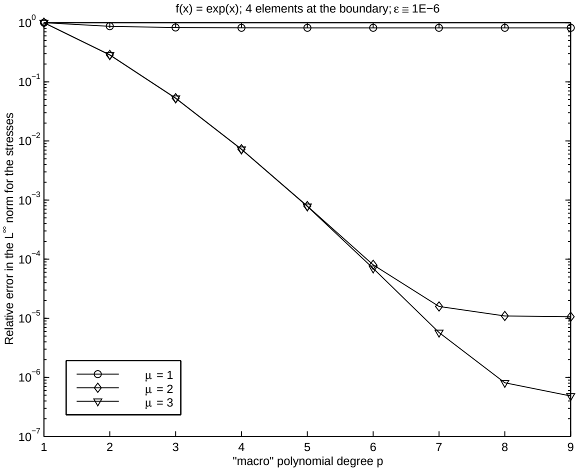

Fig. 9. (continued)

Since a change in a_0 does not affect the principal part of the differential operator which strongly influences the shape functions, we investigate next the performance of shape functions corresponding to $a_0 = 1$ for the problem (4.1) without absolute term a_0.

In Fig. 5 we show analogous results for $a_0 \equiv 0$, $f(x) = f_2(x)$ (with respect to the same mesh) and different microscales ε, varying from $\cong 10^{-6}$ up to $\cong 10^{-1}$. We note that for $\mu = 1$ and for $\varepsilon \cong 10^{-1}$ a very slow convergence is apparent - here the scales are resolved, but the low solution regularity stalls the spectral convergence. As before, one can see from the results in Fig. 6 that keeping p fixed and increasing μ does not lead to exponential convergence, in agreement with Remark 4.3. Rather, Figs. 4, 5, 6 show again that μ must be increased together with p to obtain exponential convergence that is *robust*, i.e. independent of ε.

Comparing the error plots in Fig. 5 for several ε and the same fixed value of μ we see that the saturation level appears to be proportional to some power of ε. This is more clearly visible in Fig. 8 and indicates that our finite elements with the choice $\mu > 1$ can represent the correctors in classical homogenization theory and are consistent with the homogenized problem at $\varepsilon = 0$ of higher order in ε.

In Fig. 7 we show analogous results for $f(x) = f_1(x)$, $a_0 \equiv 0$. Since the exact solution $u^\varepsilon(x)$ is piecewise cubic, for small ε no change occurs when μ is increased beyond $\mu = 4$, despite our shape functions being obtained for $a_0 \equiv 1$ rather than for $a_0 \equiv 0$. We conclude that the micro shapefunctions of the problem (4.1) with $a_0 = 1$ perform equally well if used for the operator without absolute term.

Finally, in Fig. 9 we show the pointwise error

$$\left\| \frac{d}{dx} \left(u^\varepsilon - u^\varepsilon_{FE} \right) \right\|_{L^\infty(-1,1)} \tag{4.19}$$

for $f(x) = f_2(x)$ and various ε. We see that the above conclusions apply also to these errors with respect to the (stronger) $W^{1,\infty}$-norm.

A Proof of Theorem 2.3

Lemma A.1 *The mapping*

$$\mathcal{D}_{\nu/2} := \{t \in \mathbb{C} \text{ such that } |\mathrm{Im}\, t| < \nu/2\} \ni t \to G(t) := \mathrm{e}^{\mathrm{i}tx} \in \left(H^1_\nu(\mathbb{R})\right)^* \tag{A.1}$$

is holomorphic in $\mathcal{D}_{\nu/2}$ with values in the Banach space $\left(H^1_\nu(\mathbb{R})\right)^*$. Moreover, $G_k(t) := (\mathrm{i}x)^k \mathrm{e}^{\mathrm{i}tx} \in \left(H^1_\nu(\mathbb{R})\right)^*$ is the $k-$th derivative with respect to t of the $\left(H^1_\nu(\mathbb{R})\right)^*$-valued mapping $G(t)$ and its norm has the following bound

$$\forall\, t \in \mathcal{D}_{\nu/2} : \quad \|G_k(t)\|_{(H^1_\nu(\mathbb{R}))^*} \le \frac{\sqrt{(2k)!}}{\nu^k \sqrt{\nu/2}}, \quad k = 0, 1, 2, \dots . \tag{A.2}$$

Proof. It is sufficient to show that $\forall\, v \in H^1_\nu(\mathbb{R})$ the application

$$\mathcal{D}_{\nu/2} \ni t \to \langle G(t), v \rangle_{(H^1_\nu(\mathbb{R}))^* \times H^1_\nu(\mathbb{R})} \in \mathbb{C} \tag{A.3}$$

is $\mathbb{C}$-differentiable. Let $t_0 \in \mathcal{D}_{\nu/2}$ arbitrary, fixed, and $t \in \mathcal{D}_{\nu/2}$ such that $|t - t_0| \le \nu/4$. Then,

$$\begin{aligned} &\left| \frac{1}{t - t_0} \langle G(t) - G(t_0), v \rangle_{(H^1_\nu(\mathbb{R}))^* \times H^1_\nu(\mathbb{R})} - \langle G_1(t_0), v \rangle_{(H^1_\nu(\mathbb{R}))^* \times H^1_\nu(\mathbb{R})} \right| \\ &\qquad \le \|v\|_{1,\nu} \left[\int_{\mathbb{R}} \left| \frac{\mathrm{e}^{\mathrm{i}xt} - \mathrm{e}^{\mathrm{i}xt_0}}{t - t_0} - \mathrm{i}x\mathrm{e}^{\mathrm{i}xt_0} \right|^2 \mathrm{e}^{-2\nu|x|} dx \right]^{1/2} \\ &\qquad \le C(\nu)|t - t_0| \|v\|_{1,\nu}. \end{aligned}$$

This implies that the limit

$$\lim_{t \to t_0} \frac{1}{t - t_0} \langle G(t) - G(t_0), v \rangle_{(H^1_\nu(\mathbb{R}))^* \times H^1_\nu(\mathbb{R})} \tag{A.4}$$

exists and is equal to $\langle G_1(t_0), v \rangle_{(H^1_\nu(\mathbb{R}))^* \times H^1_\nu(\mathbb{R})}$.

Let us take $v \in H^1_\nu$ and estimate $|\langle G_k(t), v\rangle_{(H^1_\nu(\mathbb{R}))^* \times H^1_\nu(\mathbb{R})}|$:

$$
\begin{aligned}
|\langle G_k(t), v\rangle_{(H^1_\nu(\mathbb{R}))^* \times H^1_\nu(\mathbb{R})}| &= \left| \int_{\mathbb{R}} (\mathrm{i}x)^k \mathrm{e}^{\mathrm{i}tx} v(x)\, dx \right| \\
&\leq \|v\|_{1,\nu} \left(\int_{\mathbb{R}} |x|^{2k} \mathrm{e}^{2|\mathrm{Im}(t)\, x|} \mathrm{e}^{-2\nu|x|} \right)^{1/2} \\
&\leq \|v\|_{1,\nu} \frac{\sqrt{(2k)!}}{\nu^k \sqrt{\nu/2}}. \qquad \square
\end{aligned}
$$

Now, for $t \in \mathcal{D}_{\nu/2}$, let $\psi^\varepsilon_k(t)$ be the weak solution in $H^1_{-\nu}(\mathbb{R})$ of the following problem

$$
\Psi(\varepsilon)[\psi^\varepsilon_k(t), v] = \langle G_k(t), v\rangle_{(H^1_\nu(\mathbb{R}))^* \times H^1_\nu(\mathbb{R})}, \quad \forall\, v \in H^1_\nu(\mathbb{R}). \tag{A.5}
$$

Lemma A.2 *The mapping*

$$
\mathcal{D}_{\nu/2} \ni t \to \psi\left(\frac{\cdot}{\varepsilon}, \varepsilon, t\right) \in H^1_{-\nu}(\mathbb{R}) \tag{A.6}
$$

is holomorphic in $\mathcal{D}_{\nu/2}$ with values in the Banach space $H^1_{-\nu}(\mathbb{R})$. Moreover,

$$
\psi^\varepsilon_k(t) = \frac{d^k}{dt^k}\psi\left(\frac{\cdot}{\varepsilon}, \varepsilon, t\right), \tag{A.7}
$$

and its norm

$$
\|\psi^\varepsilon_k(t)\|_{1,-\nu} \leq \frac{\sqrt{(2k)!}}{\gamma\nu^k\sqrt{\nu/2}}, \tag{A.8}
$$

uniformly with respect to $t \in \mathcal{D}_{\nu/2}$.

Proof. The proof is similar to that of Lemma A.1 and is based on the fact that

$$
\Psi(\varepsilon)\left[\psi\left(\frac{\cdot}{\varepsilon}, \varepsilon, t\right), v\right] = \langle G(t), v\rangle_{(H^1_\nu(\mathbb{R}))^* \times H^1_\nu(\mathbb{R})} \tag{A.9}
$$

and on the properties of the sesquilinear form $\Psi(\varepsilon)[\cdot,\cdot]$ stated in Proposition 2.2. In order to prove the analyticity of the $H^1_{-\nu}(\mathbb{R})$-valued mapping $t \to \psi(\cdot/\varepsilon, \varepsilon, t)$ in the strip $\mathcal{D}_{\nu/2}$, it is enough to show that for every $v \in H^1_\nu(\mathbb{R})$ the $\mathbb{C}$-valued function

$$
\mathcal{D}_{\nu/2} \ni t \to \Psi(\varepsilon)\left[\psi\left(\frac{\cdot}{\varepsilon}, \varepsilon, t\right), v\right] \in \mathbb{C} \tag{A.10}
$$

is holomorphic. From the definition of ψ_1^ε it follows that

$$\begin{aligned}
&\lim_{t\to t_0}\frac{1}{t-t_0}\Psi(\varepsilon)\left[\psi\left(\frac{\cdot}{\varepsilon},\varepsilon,t\right)-\psi\left(\frac{\cdot}{\varepsilon},\varepsilon,t_0\right),v\right]\\
&=\lim_{t\to t_0}\frac{1}{t-t_0}\langle G(t)-G(t_0),v\rangle_{(H^1_\nu)^*\times H^1_\nu}\\
&=\langle G_1(t_0),v\rangle_{(H^1_\nu)^*\times H^1_\nu}=\Psi(\varepsilon)[\psi_1^\varepsilon(t_0),v].
\end{aligned}$$

In addition, from (A.2)

$$\left\|\frac{d^k}{dt^k}\psi\left(\frac{\cdot}{\varepsilon},\varepsilon,t\right)\right\|_{1,-\nu}\le\frac{1}{\gamma}\|G_k(t)\|_{(H^1_\nu)^*}\le\frac{\sqrt{(2k)!}}{\gamma\nu^k\sqrt{\nu/2}},\tag{A.11}$$

uniformly with respect to $t\in\mathcal{D}_{\nu/2}$. □

Theorem A.3 *For a given $\nu>0$ there exists a positive $d=d(\nu)$ such that the mapping*

$$\mathcal{D}_d\ni t\to\phi\left(\frac{\cdot}{\varepsilon},\varepsilon,t\right)\in H^1_{-2\nu}(\mathbb{R})\tag{A.12}$$

is a holomorphic function of $t\in\mathcal{D}_d$ with values in the Banach space $H^1_{-2\nu}(\mathbb{R})$. Moreover,

$$\left\|\frac{d^k}{dt^k}\phi\left(\frac{\cdot}{\varepsilon},\varepsilon,t\right)\right\|_{1,-2\nu}\le C(1+|t|)\frac{k!}{(\nu/2)^k},\quad\forall t\in\mathcal{D}_d,\tag{A.13}$$

where the constant $C>0$ depends on ν, γ, but does not depend on $t\in\mathcal{D}_d$.

Proof. Let $d=d(\nu)=\nu/2$ and $t_0\in\mathcal{D}_d$ arbitrary, fixed. Then, since we can write $\phi(\cdot/\varepsilon,\varepsilon,t)=\mathrm{e}^{-\mathrm{i}\,t(\cdot)}\psi(\cdot/\varepsilon,\varepsilon,t)$ it follows that in the Banach space $H^1_{-2\nu}(\mathbb{R})\supset H^1_{-\nu}(\mathbb{R})$

$$\begin{aligned}
&\frac{d^k}{dt^k}\phi\left(\frac{x}{\varepsilon},\varepsilon,t\right)\Big|_{t=t_0}\\
&=\sum_{l=0}^{k}(-\mathrm{i}x)^l\binom{k}{l}\mathrm{e}^{-\mathrm{i}t_0x}\psi^{(k-l)}\left(\frac{x}{\varepsilon},\varepsilon,t_0\right).
\end{aligned}\tag{A.14}$$

Taking now the $\|\cdot\|_{1,-2\nu}$-norm in both sides we get that

$$\begin{aligned}
&\left\|\frac{d^k}{dt^k}\phi\left(\frac{x}{\varepsilon},\varepsilon,t\right)\Big|_{t=t_0}\right\|_{1,-2\nu}\\
&\le\left\|\sum_{l=0}^{k}(-\mathrm{i}x)^l\binom{k}{l}\mathrm{e}^{-\mathrm{i}t_0x}\psi^{(k-l)}\left(\frac{x}{\varepsilon},\varepsilon,t_0\right)\right\|_{0,-2\nu}
\end{aligned}$$

$$+\left\| \sum_{l=0}^{k} l\,(-\mathrm{i}x)^{l-1}(-\mathrm{i})\binom{k}{l}\mathrm{e}^{-\mathrm{i}t_0 x}\psi^{(k-l)}\left(\frac{x}{\varepsilon},\varepsilon,t_0\right)\right\|_{0,-2\nu}$$
$$+\left\| \sum_{l=0}^{k}(-\mathrm{i}x)^{l}\binom{k}{l}\mathrm{e}^{-\mathrm{i}t_0 x}\frac{d}{dx}\left(\psi^{(k-l)}\left(\frac{x}{\varepsilon},\varepsilon,t_0\right)\right)\right\|_{0,-2\nu}$$
$$+\left\| \sum_{l=0}^{k}(-\mathrm{i}x)^{l}\binom{k}{l}(-\mathrm{i}t_0)\mathrm{e}^{-\mathrm{i}t_0 x}\psi^{(k-l)}\left(\frac{x}{\varepsilon},\varepsilon,t_0\right)\right\|_{0,-2\nu}.$$

Let us estimate only the first term of the right hand side, the others can be treated analogously.

$$\left\| \sum_{l=0}^{k}(-\mathrm{i}x)^{l}\binom{k}{l}\mathrm{e}^{-\mathrm{i}t_0 x}\psi^{(k-l)}\left(\frac{x}{\varepsilon},\varepsilon,t_0\right)\right\|_{0,-2\nu}$$
$$\le \sum_{l=0}^{k}\binom{k}{l}\left(\int_{\mathbb{R}}|x|^{2l}\left|\psi^{(k-l)}\left(\frac{x}{\varepsilon},\varepsilon,t_0\right)\right|^2\mathrm{e}^{2|\mathrm{Im}(t_0)x|}\mathrm{e}^{-4\nu|x|}dx\right)^{1/2}$$
$$\le C\sum_{l=0}^{k}\binom{k}{l}\left(\frac{l}{\nu/2e}\right)^{l}\frac{(k-l)!}{(\nu/2)^{k-l}}\le C\frac{k!}{(\nu/2)^k}.$$

Here we used that $x^p\mathrm{e}^{-\nu x}\le (p/\nu\mathrm{e})^p$, $\forall\, x>0$, $p\in\mathbb{N}$ and the estimations for the $\|\cdot\|_{1,-\nu}$-norm of the derivatives with respect to t of $\psi(\cdot/\varepsilon,\varepsilon,t)$ from Lemma A.2. Summing up all the estimates it follows that

$$\left\|\frac{d^k}{dt^k}\phi\left(\frac{x}{\varepsilon},\varepsilon,t\right)\Big|_{t=t_0}\right\|_{1,-2\nu}\le C(1+|t_0|)\frac{k!}{(\nu/2)^k}. \tag{A.15}$$

This implies therefore that the series

$$\sum_{k=0}^{\infty}\frac{(t-t_0)^k}{k!}\frac{d^k}{dt^k}\phi\left(\frac{x}{\varepsilon},\varepsilon,t\right)\Big|_{t=t_0} \tag{A.16}$$

is absolutely convergent in the Banach space $H^1_{-2\nu}(\mathbb{R})$ for $|t-t_0|<\nu/2$ such that $t\in\mathcal{D}_d$. □

B Proof of Theorem 3.9

In this appendix we will present the proof of Theorem 3.9. Our aim is to approximate the Fourier-Bochner integral

$$u^\varepsilon(\cdot)=\frac{1}{\sqrt{2\pi}}\overset{(B)}{\int_{\mathbb{R}}}\hat{f}(t)\mathrm{e}^{\mathrm{i}t(\cdot)}\phi\left(\frac{\cdot}{\varepsilon},\varepsilon,t\right)dt=\overset{(B)}{\int_{\mathbb{R}}}\mathrm{e}^{\mathrm{i}t(\cdot)}g(t,\cdot)\,dt, \tag{B.1}$$

where

$$g(t,\cdot) = \frac{1}{\sqrt{2\pi}} \hat{f}(t)\phi\left(\frac{\cdot}{\varepsilon}, \varepsilon, t\right) \in \mathcal{A}(\mathcal{D}_d; H^1_{-2\nu}(\mathbb{R})).$$

The integrals in (B.1) have to be understood as Bochner-integrals of $H^1_{-2\nu}(\mathbb{R})$-valued functions. Recall that by Theorem 3.5

$$\|g(t,\cdot)\|_{1,-2\nu} \leq C(\gamma,\nu)C(f)\left(1+\frac{1}{\alpha}\right)\mathrm{e}^{-\frac{\alpha}{2}|t|} \quad \forall\, t \in \mathcal{D}_d.$$

Define the approximations

$$u^{\varepsilon}_{N,h}(\cdot) := \mathbf{1}_{[-\frac{\pi}{h},\frac{\pi}{h}]}(\cdot)\frac{1}{\sqrt{2\pi}} h \sum_{k=-N}^{\mathrm{N}} \hat{f}(kh)\psi\left(\frac{\cdot}{\varepsilon}, \varepsilon, kh\right) \tag{B.2}$$

and its error

$$\delta_{\mathrm{N}}(f,h)(\cdot) := u^{\varepsilon}(\cdot) - u^{\varepsilon}_{N,h}(\cdot). \tag{B.3}$$

Proposition B.1 *Assume that* $f \in L^{\infty}(\mathbb{R}^2) \cap L^1(\mathbb{R}; L^{\infty}(\mathbb{R}))$ *and* $g \in H^0_{-2\nu}(\mathbb{R})$. *Then*

$$\overset{(B)}{\int_{\mathbb{R}}} g(\cdot) f(t,\cdot)\, dt = g(\cdot)\int_{\mathbb{R}} f(t,\cdot)\, dt. \tag{B.4}$$

Proof. Let us verify that the expressions in (B.4) have sense. The Bochner integral is well defined, since $\|g(\cdot)f(t,\cdot)\|_{0,-2\nu} \leq \|g(\cdot)\|_{0,-2\nu}\|f(t,\cdot)\|_{L^{\infty}}$ and $\|f(t,\cdot)\|_{L^{\infty}}$ is, as a function of t, in $L^1(\mathbb{R})$. Then, the right hand side of (B.4) is an element of $H^0_{-2\nu}(\mathbb{R})$ since

$$\int_{\mathbb{R}} f(t,\cdot)dt \in L^{\infty}(\mathbb{R}).$$

We consider two cases:
Case 1: if $g \in C^{\infty}_0(\mathbb{R})$, the assertion is obvious.
Case 2: $g \in H^0_{-2\nu}(\mathbb{R}) = \overline{C^{\infty}_0(\mathbb{R})}^{\|\cdot\|_{0,-2\nu}}$, then take $(g_n)_n \subset C^{\infty}_0(\mathbb{R})$, such that $g_n \to g$ in $H^0_{-2\nu}(\mathbb{R})$, as $n \to \infty$. Then,

$$\left\| \overset{(B)}{\int_{\mathbb{R}}} (g_n - g)(\cdot) f(t,\cdot)\, dt \right\|_{0,-2\nu} \leq \int_{\mathbb{R}} \|g_n(\cdot) - g(\cdot)\|_{0,-2\nu}\|f(t,\cdot)\|_{L^{\infty}} dt.$$

Now, the integrand $\|g_n(\cdot) - g(\cdot)\|_{0,-2\nu}\|f(t,\cdot)\|_{L^\infty} \to 0$, as $n \to \infty$, for almost every $t \in \mathbb{R}$, and is bounded by an L^1 application $C\|f(t,\cdot)\|_{L^\infty}$, uniformly with respect to n. It follows therefore that

$$\overset{(B)}{\int_{\mathbb{R}}} g_n(\cdot)f(t,\cdot)\,dt \to \overset{(B)}{\int_{\mathbb{R}}} g(\cdot)f(t,\cdot)\,dt \quad \text{in } H^0_{-2\nu},$$

as $n \to \infty$. Since

$$g_n(\cdot)\int_{\mathbb{R}} f(t,\cdot)\,dt \to g(\cdot)\int_{\mathbb{R}} f(t,\cdot)\,dt \quad \text{in } H^0_{-2\nu}(\mathbb{R})$$

as $n \to \infty$ the proposition follows. □

Proposition B.2 *Assume that*

$$f \in L^\infty(\mathbb{R}; W^{1,\infty}(-L,L)) \cap L^1(\mathbb{R}; W^{1,\infty}(-L,L))$$

and $g \in H^1_{-2\nu}(\mathbb{R})$. *Then*

$$\overset{(B)}{\int_{\mathbb{R}}} g(\cdot)f(t,\cdot)\,dt = g(\cdot)\int_{\mathbb{R}} f(t,\cdot)\,dt \tag{B.5}$$

in $H^1_{-2\nu}(-L,L)$.

Proof. First of all, let us convince ourselves that the expressions in (B.5) have sense. The Bochner integral is well defined, since $\|g(\cdot)f(t,\cdot)\|_{H^1_{-2\nu}(-L,L)} \leq \|g(\cdot)\|_{1,-2\nu}\|f(t,\cdot)\|_{W^{1,\infty}(-L,L)}$, which is in $L^1(\mathbb{R})$ as a function of t. Then, the right hand side of (B.4) is an element of $H^1_{-2\nu}(-L,L)$, since

$$\int_{\mathbb{R}} f(t,\cdot)dt \in W^{1,\infty}(-L,L).$$

As before, we use a density argument :
Case 1: if $g \in C_0^\infty(\mathbb{R})$, the assertion is obvious.
Case 2: $g \in H^1_{-2\nu}(\mathbb{R}) = \overline{C_0^\infty(\mathbb{R})}^{\|\cdot\|_{1,-2\nu}}$, then take $(g_n)_n \subset C_0^\infty(\mathbb{R})$, such that $g_n \to g$ in $H^1_{-2\nu}(\mathbb{R})$, as $n \to \infty$. Then

$$\Big\| \overset{(B)}{\int_{\mathbb{R}}} (g_n - g)(\cdot)f(t,\cdot)\,dt \Big\|_{H^1_{-2\nu}(-L,L)} \leq \int_{\mathbb{R}} \|g_n(\cdot) - g(\cdot)\|_{1,-2\nu}\|f(t,\cdot)\|_{W^{1,\infty}(-L,L)}dt.$$

Now, the integrand $\|g_n(\cdot) - g(\cdot)\|_{1,-2\nu}\|f(t,\cdot)\|_{W^{1,\infty}(-L,L)} \to 0$, as $n \to \infty$, for almost every $t \in \mathbb{R}$, and is bounded by an L^1 with respect to t application $C\|f(t,\cdot)\|_{W^{1,\infty}(-L,L)}$, for all n. It follows therefore that

$$\int_{\mathbb{R}}^{(B)} g_n(\cdot) f(t,\cdot)\, dt \to \int_{\mathbb{R}}^{(B)} g(\cdot) f(t,\cdot)\, dt \quad \text{in } H^1_{-2\nu}(-L,L),$$

as $n \to \infty$, and the proposition follows since

$$g_n(\cdot) \int_{\mathbb{R}} f(t,\cdot)\, dt \to g(\cdot) \int_{\mathbb{R}} f(t,\cdot)\, dt \quad \text{in } H^1_{-2\nu}(-L,L),$$

as $n \to \infty$. $\square$

Recall now that

$$\begin{aligned}\delta_{\mathrm{N}}(f,h)(\cdot) &= u^\varepsilon(\cdot) - u^\varepsilon_{N,h}(\cdot) \\ &= \int_{\mathbb{R}}^{(B)} \mathrm{e}^{\mathrm{i}t(\cdot)} g(t,\cdot)\, dt - \sum_{k=-N}^{\mathrm{N}} h\mathbf{1}_{[-\frac{\pi}{h},\frac{\pi}{h}]}(\cdot) g(kh,\cdot)\mathrm{e}^{\mathrm{i}kh(\cdot)}.\end{aligned}$$

Proposition B.3 *Let us assume that $L \geq 1$ is given, and $h = (\pi d/\alpha N)^{1/2}$ satisfies*

$$\frac{\pi}{h} \geq 2L,$$

i.e., $N \geq 4dL^2/\alpha\pi$. Then,

$$\lim_{\delta\to 0^+} g(kh,\cdot) \int_{\mathbb{R}} \mathrm{e}^{-\delta|t|} \mathrm{e}^{\mathrm{i}t(\cdot)} S(k,h)(t)\, dt = h\mathbf{1}_{[-\frac{\pi}{h},\frac{\pi}{h}]}(\cdot) g(kh,\cdot)\mathrm{e}^{\mathrm{i}kh(\cdot)}$$

(B.6)

in $H^0_{-2\nu}(\mathbb{R}) \cap H^1_{-2\nu}(-L,L)$.

Proof. First, let us notice that since $g(kh,\cdot) \in H^1_{-2\nu}(\mathbb{R})$ and

(B.7) $$F_\delta(\cdot) := \int_{\mathbb{R}} \mathrm{e}^{-\delta|t|} \mathrm{e}^{\mathrm{i}t(\cdot)} S(k,h)(t)\, dt \in W^{1,\infty}(\mathbb{R}),$$

(B.8) $$F_0(\cdot) = F(\cdot) := \mathbf{1}_{[-\frac{\pi}{h},\frac{\pi}{h}]}(\cdot) h\mathrm{e}^{\mathrm{i}kh(\cdot)} \in L^\infty(\mathbb{R}) \cap W^{1,\infty}(-L,L),$$

the terms in (B.6) are well defined as elements of $H^0_{-2\nu}(\mathbb{R}) \cap H^1_{-2\nu}(-L, L)$. Then,

$$\begin{aligned}
F_\delta(x) &= \int_{\mathbb{R}} \mathrm{e}^{-\delta|t|} \mathrm{e}^{\mathrm{i}tx} \frac{1}{2\pi} \left(\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} h \mathrm{e}^{\mathrm{i}kh\tau - \mathrm{i}\tau t} d\tau \right) dt \\
&= \frac{h}{2\pi} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \mathrm{e}^{\mathrm{i}kh\tau} \left(\int_{\mathbb{R}} \mathrm{e}^{-\delta|t|} \mathrm{e}^{\mathrm{i}tx - \mathrm{i}\tau t} \, dt \right) d\tau \\
&= \frac{h}{\pi} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \mathrm{e}^{\mathrm{i}kh\tau} \frac{\delta}{\delta^2 + (x-\tau)^2} \, d\tau = \frac{h}{\pi} \int_{-\frac{\pi}{h}-x}^{\frac{\pi}{h}-x} \mathrm{e}^{\mathrm{i}kh(s+x)} \frac{\delta}{\delta^2 + s^2} \, ds \\
&= \frac{h}{\pi} \mathrm{e}^{\mathrm{i}khx} \int_{-\frac{\pi}{h}-x}^{\frac{\pi}{h}-x} \mathrm{e}^{\mathrm{i}khs} \frac{\delta}{\delta^2 + s^2} \, ds = \frac{h}{\pi} \mathrm{e}^{\mathrm{i}khx} \int_{\left(-\frac{\pi}{h}-x\right)/\delta}^{\left(\frac{\pi}{h}-x\right)/\delta} \mathrm{e}^{\mathrm{i}kh\delta\tau} \frac{1}{1+\tau^2} \, d\tau.
\end{aligned}$$

It the following it will be shown that uniformly with respect to x and δ

(B.9) $$|F_\delta(x)| \le h, \quad \forall\, x \in \mathbb{R},$$

(B.10) $$\left| \frac{d}{dx} F_\delta(x) \right| \le \frac{h}{\pi L} + kh^2, \quad \forall\, x \in (-L, L),$$

and

(B.11) $$F_\delta(x) \to \mathbf{1}_{\left[-\frac{\pi}{h}, \frac{\pi}{h}\right]}(x) h \mathrm{e}^{\mathrm{i}khx}, \text{ as } \delta \to 0^+, \text{ for a.e. } x \in \mathbb{R},$$

(B.12) $$\frac{d}{dx} F_\delta(x) \to \frac{d}{dx} F(x) = \mathrm{i}kh^2 \mathrm{e}^{\mathrm{i}khx}, \text{ for a.e. } x \in (-L, L).$$

The assertions (B.9) and (B.11) follow immediately from the representation of $F_\delta(\cdot)$:

$$F_\delta(x) = \frac{\pi}{h} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \mathrm{e}^{\mathrm{i}kh\tau} \frac{\delta}{\delta^2 + (x-\tau)^2} \, d\tau$$

(B.13) $$= \frac{h}{\pi} \mathrm{e}^{\mathrm{i}khx} \int_{\left(-\frac{\pi}{h}-x\right)/\delta}^{\left(\frac{\pi}{h}-x\right)/\delta} \mathrm{e}^{\mathrm{i}kh\delta\tau} \frac{1}{1+\tau^2} \, d\tau.$$

From (B.13) it can be deduced that

$$\begin{aligned}\frac{d}{dx}F_\delta(x) &= \frac{h}{\pi}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \mathrm{e}^{\mathrm{i}kh\tau}\frac{-2(x-\tau)\delta}{[\delta^2+(x-\tau)^2]^2}\,d\\ &= \frac{h}{\pi}\mathrm{e}^{\mathrm{i}khx}\int_{-\frac{\pi}{h}-x}^{\frac{\pi}{h}-x} \mathrm{e}^{\mathrm{i}khs}\frac{2\delta s}{(\delta^2+s^2)^2}\,ds\\ &= \frac{h}{\pi}\mathrm{e}^{\mathrm{i}khx}\int_{\left(-\frac{\pi}{h}-x\right)/\delta}^{\left(\frac{\pi}{h}-x\right)/\delta} \mathrm{e}^{\mathrm{i}kh\delta\tau}\frac{2\tau}{\delta(1+\tau^2)^2}\,d\tau\\ &= -\frac{h}{\pi}\mathrm{e}^{\mathrm{i}khx}\frac{1}{\delta}\mathrm{e}^{\mathrm{i}kh\tau\delta}\frac{1}{1+\tau^2}\Bigg|_{\tau=\left(-\frac{\pi}{h}-x\right)/\delta}^{\tau=\left(\frac{\pi}{h}-x\right)/\delta}\\ &\quad+\frac{h}{\pi}\mathrm{e}^{\mathrm{i}khx}\int_{\left(-\frac{\pi}{h}-x\right)/\delta}^{\left(\frac{\pi}{h}-x\right)/\delta} \mathrm{e}^{\mathrm{i}kh\delta\tau}ikh\frac{1}{1+\tau^2}\,d\tau.\end{aligned}$$

It follows therefore that for $x\in(-L,L)\subset[-\pi/h,\pi/h]$,

$$\begin{aligned}\frac{d}{dx}F_\delta(x) &= -\frac{h}{\pi}\mathrm{e}^{\mathrm{i}khx}\mathrm{e}^{\mathrm{i}kh(\pi/h-x)}\frac{1/\delta}{1+[(\pi/h-x)/\delta]^2}\\ &\quad+\frac{h}{\pi}\mathrm{e}^{\mathrm{i}khx}\mathrm{e}^{\mathrm{i}kh(-\pi/h-x)}\frac{1/\delta}{1+[(-\pi/h-x)/\delta]^2}\\ &\quad+\mathrm{i}\frac{kh^2}{\pi}\mathrm{e}^{\mathrm{i}khx}\int_{\left(-\frac{\pi}{h}-x\right)/\delta}^{\left(\frac{\pi}{h}-x\right)/\delta} \mathrm{e}^{\mathrm{i}kh\delta\tau}\frac{1}{1+\tau^2}\,d\tau\\ &= -\frac{h}{\pi}\mathrm{e}^{\mathrm{i}k\pi}\frac{1/\delta}{1+[(\pi/h-x)/\delta]^2}+\frac{h}{\pi}\mathrm{e}^{-\mathrm{i}k\pi}\frac{1/\delta}{1+[(-\pi/h-x)/\delta]^2}\\ &\quad+\mathrm{i}\frac{kh^2}{\pi}\mathrm{e}^{\mathrm{i}khx}\int_{\left(-\frac{\pi}{h}-x\right)/\delta}^{\left(\frac{\pi}{h}-x\right)/\delta} \mathrm{e}^{\mathrm{i}kh\delta\tau}\frac{1}{1+\tau^2}\,d\tau.\end{aligned} \tag{B.14}$$

Since $|\,\pi/h\pm x\,|\geq L$,

$$\frac{1/\delta}{1+[(\pi/h\pm x)/\delta]^2}\leq\frac{1/\delta}{1+(L/\delta)^2}\to 0,$$

as $\delta \to 0^+$. This will imply that the first two terms in (B.14) converge to 0, as $\delta \to 0^+$, uniformly with respect to $x \in (-L, L)$. Furthermore, it can be easily seen that the last term in (B.14) converges to $\mathrm{i}kh^2\mathrm{e}^{\mathrm{i}khx}$, as $\delta \to 0^+$, and is uniformly with respect to δ and $x \in (-L, L)$ bounded by kh^2. Moreover, since

$$\frac{1/\delta}{1+(L/\delta)^2} \le \frac{1}{2\,L}, \quad \forall\, \delta > 0,$$

it follows that

$$\left|\frac{d}{dx}F_\delta(x)\right| \le \frac{h}{\pi L} + kh^2, \quad \forall\, \delta > 0, \quad \forall\, x \in (-L, L).$$

Then,

$$\lim_{\delta\to 0^+} \|g(kh,\cdot)(F_\delta(\cdot) - F(\cdot))\|^2_{0,-2\nu} = 0, \tag{B.15}$$

since

$$\begin{aligned}\lim_{\delta\to 0^+} &\|g(kh,\cdot)(F_\delta(\cdot) - F(\cdot))\|^2_{0,-2\nu} \\ &= \lim_{\delta\to 0^+} \int_{\mathbb{R}} |g(kh,x)|^2 \mathrm{e}^{-4\nu|x|} |F_\delta(x) - F(x)|^2\, dx,\end{aligned}$$

which is 0 because of Lebesgue Theorem on dominated convergence. Indeed, the integrand is in $L^1(\mathbb{R})$, converges to 0 for almost every $x \in \mathbb{R}$, and is bounded by an integrable function $|2\,h\,g(kh,x)\mathrm{e}^{-2\nu|x|}|^2$.
With similar arguments it can be shown that

$$\lim_{\delta\to 0^+} \|g(kh,\cdot)(F_\delta(\cdot) - F(\cdot))\|^2_{H^1_{-2\nu}(-L,L)} = 0, \tag{B.16}$$

since

$$\begin{aligned}\|g(kh,\cdot)&(F_\delta(\cdot) - F(\cdot))\|^2_{H^1_{-2\nu}(-L,L)} \\ &\le \int_{(-L,L)} |g(kh,x)|^2 \mathrm{e}^{-4\nu|x|} \left[\left|\frac{d}{dx}F_\delta(x) - F(x)\right|^2 \right. \\ &\qquad \left. + |F_\delta(x) - F(x)|^2\right]\, dx \\ &+ \int_{(-L,L)} \left|\frac{d}{dx}g(kh,x)\right|^2 \mathrm{e}^{-4\nu|x|} |F_\delta(x) - F(x)|^2\, dx.\end{aligned} \tag{B.17}$$

The integrands in (B.17) are in $L^1(-L, L)$, converge to 0 a.e. in $(-L, L)$, as $\delta \to 0^+$ and are uniformly (with respect to $\delta > 0$) bounded by an

$L^1(-L,L)$ function. The Lebesgue Theorem on dominated convergence implies therefore that

$$\lim_{\delta\to 0^+} \|g(kh,\cdot)(F_\delta(\cdot)-F(\cdot))\|_{H^1_{-2\nu}(-L,L)} = 0. \qquad \square \tag{B.18}$$

Now, it follows that

$$\begin{aligned}\delta_{\mathrm{N}}(f,h)(\cdot) = &\overset{(B)}{\int_{\mathbb{R}}} \mathrm{e}^{\mathrm{i}t(\cdot)} g(t,\cdot)\,dt \\ &- \sum_{k=-N}^{\mathrm{N}} \lim_{\delta\to 0^+} g(kh,\cdot) \int_{\mathbb{R}} \mathrm{e}^{-\delta|t|}\mathrm{e}^{\mathrm{i}t(\cdot)} S(k,h)(t)\,dt\end{aligned} \tag{B.19}$$

in $H^0_{-2\nu}(\mathbb{R})\cap H^1_{-2\nu}(-L,L)$. By Propositions B.1, B.2 the errors $\delta_{\mathrm{N}}(f,h)(\cdot)$ can be now interpreted as the following Bochner-integrals in $H^0_{-2\nu}(\mathbb{R})$, respectively $H^1_{-2\nu}(-L,L)$:

$$\begin{aligned}\delta_{\mathrm{N}}(f,h)(\cdot) &= \lim_{\delta\to 0^+} \overset{(B)}{\int_{\mathbb{R}}} \mathrm{e}^{\mathrm{i}t(\cdot)}\mathrm{e}^{-\delta|t|} \left(g(t,\cdot) - \sum_{k=-N}^{\mathrm{N}} g(kh,\cdot)S(k,h)(t)\right) dt \\ &= \lim_{\delta\to 0^+} \overset{(B)}{\int_{\mathbb{R}}} \mathrm{e}^{\mathrm{i}t(\cdot)}\mathrm{e}^{-\delta|t|} \left(g(t,\cdot) - \sum_{k=-\infty}^{\infty} g(kh,\cdot)S(k,h)(t)\right) dt \\ &\quad + \sum_{|k|\geq N+1} \lim_{\delta\to 0^+} \overset{(B)}{\int_{\mathbb{R}}} \mathrm{e}^{\mathrm{i}t(\cdot)}\mathrm{e}^{-\delta|t|} g(kh,\cdot)S(k,h)(t)\,dt \\ &= \lim_{\delta\to 0^+} \overset{(B)}{\int_{\mathbb{R}}} \mathrm{e}^{\mathrm{i}t(\cdot)}\mathrm{e}^{-\delta|t|} E(f,h)(t,\cdot)\,dt \\ &\quad + \sum_{|k|\geq N+1} \mathbf{1}_{[-\frac{\pi}{h},\frac{\pi}{h}]}(\cdot) h\mathrm{e}^{\mathrm{i}kh(\cdot)} g(kh,\cdot),\end{aligned}$$

in $H^0_{-2\nu}(\mathbb{R})$, respectively

$$\begin{aligned}\delta_{\mathrm{N}}(f,h)(\cdot) &= \lim_{\delta\to 0^+} \overset{(B)}{\int_{\mathbb{R}}} \mathrm{e}^{\mathrm{i}t(\cdot)}\mathrm{e}^{-\delta|t|} \left(g(t,\cdot) - \sum_{k=-N}^{\mathrm{N}} g(kh,\cdot)S(k,h)(t)\right) dt \\ &= \lim_{\delta\to 0^+} \overset{(B)}{\int_{\mathbb{R}}} \mathrm{e}^{\mathrm{i}t(\cdot)}\mathrm{e}^{-\delta|t|} E(f,h)(t,\cdot)\,dt + \sum_{|k|\geq N+1} h\mathrm{e}^{\mathrm{i}kh(\cdot)} g(kh,\cdot),\end{aligned}$$

in $H^1_{-2\nu}(-L,L)$. Now, it has already been shown that

$$\Big\| \sum_{|k|\geq N+1} \mathbf{1}_{[-\frac{\pi}{h},\frac{\pi}{h}]}(\cdot) h \mathrm{e}^{\mathrm{i}kh(\cdot)} g(kh,\cdot) \Big\|_{0,-2\nu} \leq C(\gamma,\nu)\frac{C(f)}{\alpha}\mathrm{e}^{-\sqrt{\pi d\alpha N}}, \tag{B.20}$$

$$\Big\| \sum_{|k|\geq N+1} h \mathrm{e}^{\mathrm{i}kh(\cdot)} g(kh,\cdot) \Big\|_{H^1_{-2\nu}(-L,L)} \leq C(\gamma,\nu)\frac{C(f)}{\alpha}\mathrm{e}^{-\sqrt{\pi d\alpha N}}, \tag{B.21}$$

if $h = (\pi d/\alpha N)^{1/2}$.
It remains to find similar estimates for the $\|\cdot\|_{0,-2\nu}$, $\|\cdot\|_{H^1_{-2\nu}(-L,L)}$-norms of

$$\int_{\mathbb{R}}^{(B)} \mathrm{e}^{\mathrm{i}t(\cdot)}\mathrm{e}^{-\delta|t|} E(f,h)(t,\cdot)\,dt, \tag{B.22}$$

which are uniform with respect to δ, as $\delta \to 0^+$.

By Theorem 3.7, $E(f,h)(t,\cdot)$ has the following representation as Bochner-integral of $H^1_{-2\nu}(\mathbb{R})$-valued functions

$$E(f,h)(t,\cdot) = \frac{\sin(\pi t/h)}{2\pi \mathrm{i}} \int_{\mathbb{R}}^{(B)} \left\{ \frac{g(\tau - \mathrm{i}d^-,\cdot)}{(\tau - t - \mathrm{i}d)\sin[\pi(\tau - \mathrm{i}d)/h]} - \frac{g(\tau + \mathrm{i}d^-,\cdot)}{(\tau - t + \mathrm{i}d)\sin[\pi(\tau + \mathrm{i}d)/h]} \right\} d\tau,$$

and

$$\|E(f,h)(t,\cdot)\|_{1,-2\nu} \leq C, \tag{B.23}$$

uniformly with respect to $t \in \mathbb{R}$. It follows therefore that

$$\int_{\mathbb{R}}^{(B)} \mathrm{e}^{\mathrm{i}t(\cdot)}\mathrm{e}^{-\delta|t|} E(f,h)(t,\cdot)\,dt = \int_{(\tau,t)\in\mathbb{R}^2}^{(B)} \mathrm{e}^{\mathrm{i}t(\cdot)}\mathrm{e}^{-\delta|t|} \frac{\sin(\pi t/h)}{2\pi \mathrm{i}} \times \left\{ \frac{g(\tau - \mathrm{i}d^-,\cdot)}{(\tau - t - \mathrm{i}d)\sin[\pi(\tau - \mathrm{i}d)/h]} - \frac{g(\tau + \mathrm{i}d^-,\cdot)}{(\tau - t + \mathrm{i}d)\sin[\pi(\tau + \mathrm{i}d)/h]} \right\} d\tau\,dt.$$

Here, the integrals will be alternatively considered as Bochner integrals of $H^0_{-2\nu}(\mathbb{R})$, respectively $H^1_{-2\nu}(-L,L)$-valued functions. Now, since the

$H^0_{-2\nu}(\mathbb{R})$, respectively $H^1_{-2\nu}(-L,L)$-norms of the integrands are in $L^1(\mathbb{R}^2)$, we can change the order of the integration and we get

$$\begin{aligned} &\int_{\mathbb{R}}^{(B)} \mathrm{e}^{\mathrm{i}t(\cdot)}\, \mathrm{e}^{-\delta|t|}\, E(f,h)(t,\cdot)\, dt \\ &\qquad = \frac{1}{2\pi i} \int_{\mathbb{R}}^{(B)} \Big\{ \int_{\mathbb{R}}^{(B)} \sin(\pi t/h)\, \frac{\mathrm{e}^{\mathrm{i}t(\cdot)}\mathrm{e}^{-\delta|t|}}{\tau - t - \mathrm{i}d}\, \frac{g(\tau - \mathrm{i}d^-,\cdot)}{\sin[\pi(\tau - \mathrm{i}d)/h]} \\ &\qquad - \sin(\pi t/h)\, \frac{\mathrm{e}^{\mathrm{i}t(\cdot)}\mathrm{e}^{-\delta|t|}}{\tau - t + \mathrm{i}d}\, \frac{g(\tau + \mathrm{i}d^-,\cdot)}{\sin[\pi(\tau + \mathrm{i}d)/h]}\, dt \Big\}\, d\tau. \end{aligned}$$

We shall restrict ourselves to the first term, the second can be treated in an analogous fashion. To this end, using Propositions B.1, B.2 we get that

$$\int_{\mathbb{R}}^{(B)} \sin(\pi t/h)\, \frac{\mathrm{e}^{\mathrm{i}t(\cdot)}\mathrm{e}^{-\delta|t|}}{\tau - t - \mathrm{i}d}\, \frac{g(\tau - \mathrm{i}d^-,\cdot)}{\sin[\pi(\tau - \mathrm{i}d)/h]}\, dt = \frac{g(\tau - \mathrm{i}d^-,\cdot)}{\sin[\pi(\tau - \mathrm{i}d)/h]} F_\delta(\tau,\cdot) \tag{B.24}$$

in $H^0_{-2\nu}(\mathbb{R}) \cap H^1_{-2\nu}(-L,L)$, where

$$F_\delta(\tau,\cdot) := \int_{\mathbb{R}} \sin(\pi t/h)\, \frac{\mathrm{e}^{\mathrm{i}t(\cdot)}\mathrm{e}^{-\delta|t|}}{\tau - t - \mathrm{i}d}\, dt. \tag{B.25}$$

We get therefore that

$$\begin{aligned} &\int_{\mathbb{R}}^{(B)} \Big\{ \int_{\mathbb{R}}^{(B)} \sin(\pi t/h)\, \frac{\mathrm{e}^{\mathrm{i}t(\cdot)}\mathrm{e}^{-\delta|t|}}{\tau - t - \mathrm{i}d}\, \frac{g(\tau - \mathrm{i}d^-,\cdot)}{\sin[\pi(\tau - \mathrm{i}d)/h]}\, dt \Big\}\, d\tau \\ &\qquad = \int_{\mathbb{R}}^{(B)} \frac{g(\tau - \mathrm{i}d^-,\cdot)}{\sin[\pi(\tau - \mathrm{i}d)/h]} F_\delta(\tau,\cdot)\, d\tau. \end{aligned} \tag{B.26}$$

Assume now that the following hold: for all $\tau \in \mathbb{R}$

$$F_\delta(\tau,x) \to F_0(\tau,x) = F(\tau,x), \text{ as } \delta \to 0^+, \text{ a.e. } x \in \mathbb{R}, \tag{B.27}$$

$$\frac{d}{dx} F_\delta(\tau,x) \to \frac{d}{dx} F(\tau,x), \text{ as } \delta \to 0^+, \text{ a.e. } x \in (-L,L), \tag{B.28}$$

where

$$F(\tau,x)=\begin{cases} 2\pi i\sin[\pi(\tau-\mathrm{i}d)/h]\mathrm{e}^{\mathrm{i}x(\tau-\mathrm{i}d)} & , \quad \text{if } x<-\dfrac{\pi}{h} \\ -\pi\mathrm{e}^{-\mathrm{i}(\pi/h-x)(\tau-\mathrm{i}d)} & , \quad \text{if } -\dfrac{\pi}{h}<x<\dfrac{\pi}{h} \\ 0 & , \quad \text{if } x>\dfrac{\pi}{h}. \end{cases}$$

Moreover, assume that

$$|F_\delta(\tau,x)|\le 2\pi, \quad \forall\, x\in\mathbb{R} \tag{B.29}$$

$$\left|\frac{d}{dx}F_\delta(\tau,x)\right|\le C(1+|\tau|)+\frac{1}{L}, \quad \forall\, x\in(-L,L), \tag{B.30}$$

where $C>0$ depends on d,h, but does not depend on $x\in(-L,L),\delta$ or τ. Assuming that (B.27), (B.28), (B.29), (B.30) hold for $F_\delta(\tau,\cdot)$, we claim that

$$\lim_{\delta\to 0^+}\int_{\mathbb{R}}^{(B)}\frac{g(\tau-\mathrm{i}d^-,\cdot)}{\sin[\pi(\tau-\mathrm{i}d)/h]}F_\delta(\tau,\cdot)\,d\tau=\int_{\mathbb{R}}^{(B)}\frac{g(\tau-\mathrm{i}d^-,\cdot)}{\sin[\pi(\tau-\mathrm{i}d)/h]}F(\tau,\cdot)\,d\tau, \tag{B.31}$$

in $H^0_{-2\nu}(\mathbb{R})\cap H^1_{-2\nu}(-L,L)$. In order to prove (B.31), under the assumptions (B.27), (B.28), (B.29), (B.30), let us estimate first

$$\left\|g(\tau-\mathrm{i}d^-,\cdot)\frac{F_\delta(\tau,\cdot)-F(\tau,\cdot)}{\sin[\pi(\tau-\mathrm{i}d)/h]}\right\|^2_{0,-2\nu}$$

$$\le\int_{\mathbb{R}}\mathrm{e}^{-4\nu|x|}|g(\tau-\mathrm{i}d^-,x)|^2|F_\delta(\tau,x)-F(\tau,x)|^2\,dx, \tag{B.32}$$

$$\left\|g(\tau-\mathrm{i}d^-,\cdot)\frac{F_\delta(\tau,\cdot)-F(\tau,\cdot)}{\sin[\pi(\tau-\mathrm{i}d)/h]}\right\|^2_{H^1_{-2\nu}(-L,L)}$$

$$\le\int_{(-L,L)}\mathrm{e}^{-4\nu|x|}|g(\tau-\mathrm{i}d^-,x)|^2$$

$$\times\left[|F_\delta(\tau,x)-F(\tau,x)|^2+\left|\frac{d}{dx}F_\delta(\tau,x)-\frac{d}{dx}F(\tau,x)\right|^2\right]\,dx$$

$$+\int_{(-L,L)}\mathrm{e}^{-4\nu|x|}\left|\frac{d}{dx}g(\tau-\mathrm{i}d^-,x)\right|^2|F_\delta(\tau,x)-F(\tau,x)|^2\,dx. \tag{B.33}$$

Now, since the integrands converge to 0, as $\delta \to 0^+$, for a.e. $x \in \mathbb{R}$, respectively for a.e. $x \in (-L, L)$, and are bounded by integrable functions, we conclude by Lebesgue Theorem on dominated convergence that

$$\left\| g(\tau - \mathrm{i}d^-, \cdot) \frac{F_\delta(\tau, \cdot) - F(\tau, \cdot)}{\sin[\pi(\tau - \mathrm{i}d)/h]} \right\|_{0,-2\nu} \to 0, \tag{B.34}$$

respectively

$$\left\| g(\tau - \mathrm{i}d^-, \cdot) \frac{F_\delta(\tau, \cdot) - F(\tau, \cdot)}{\sin[\pi(\tau - \mathrm{i}d)/h]} \right\|_{H^1_{-2\nu}(-L,L)} \to 0, \tag{B.35}$$

as $\delta \to 0^+$, for a.e. $\tau \in \mathbb{R}$. Moreover,

$$\begin{aligned} &\left\| g(\tau - \mathrm{i}d^-, \cdot) \frac{F_\delta(\tau, \cdot) - F(\tau, \cdot)}{\sin[\pi(\tau - \mathrm{i}d)/h]} \right\|_{0,-2\nu} \\ &\qquad \le C \| g(\tau - \mathrm{i}d^-, \cdot) \|_{0,-2\nu} \| F_\delta(\tau, \cdot) - F(\tau, \cdot) \|_{L^\infty(\mathbb{R})} \\ &\qquad \le C(\gamma, \nu) C(f) \left(1 + \frac{1}{\alpha} \right) \mathrm{e}^{-\frac{\alpha}{2}|\tau|}, \end{aligned}$$

respectively

$$\begin{aligned} &\left\| g(\tau - \mathrm{i}d^-, \cdot) \frac{F_\delta(\tau, \cdot) - F(\tau, \cdot)}{\sin[\pi(\tau - \mathrm{i}d)/h]} \right\|_{H^1_{-2\nu}(-L,L)} \\ &\qquad \le C \| g(\tau - \mathrm{i}d^-, \cdot) \|_{1,-2\nu} \| F_\delta(\tau, \cdot) - F(\tau, \cdot) \|_{W^{1,\infty}(-L,L)} \\ &\qquad \le C(\gamma, \nu) C(f) \left(1 + \frac{1}{\alpha} \right)^2 \mathrm{e}^{-\frac{\alpha}{4}|\tau|}. \end{aligned}$$

The Lebesgue Theorem on dominated convergence implies therefore that

$$\int_{\mathbb{R}} \left\| \frac{g(\tau - \mathrm{i}d^-, \cdot)}{\sin[\pi(\tau - \mathrm{i}d)/h]} (F_\delta(\tau, \cdot) - F(\tau, \cdot)) \right\|_{0,-2\nu} \to 0, \tag{B.36}$$

$$\int_{\mathbb{R}} \left\| \frac{g(\tau - \mathrm{i}d^-, \cdot)}{\sin[\pi(\tau - \mathrm{i}d)/h]} (F_\delta(\tau, \cdot) - F(\tau, \cdot)) \right\|_{H^1_{-2\nu}(-L,L)} \to 0, \tag{B.37}$$

as $\delta \to 0^+$, which proves (B.31).

It remains to show (B.27), (B.28), (B.29) and (B.30) for $F_\delta(\tau, \cdot)$. As a Fourier transformation of a L^1 function, $F_\delta(\tau, \cdot)$ is continuous with respect to x, for all $\tau \in \mathbb{R}$ and all $\delta > 0$. One can see that

$$F_\delta(\tau, \cdot) = \mathcal{F}\left(\frac{g_\delta(t)}{\tau - t - \mathrm{i}d} \right)(\cdot), \tag{B.38}$$

where by $\mathcal{F}$ we denote the Fourier transformation and

$$g_\delta(t) = \mathrm{e}^{-\delta|t|}\sin(\pi t/h).$$

Then, $F_\delta(\tau,\cdot)$ satisfies the following first order differential equation (in x):

$$\frac{d}{dx}F_\delta(\tau,x) - \mathrm{i}(\tau-\mathrm{i}d)F_\delta(\tau,x) = -\mathrm{i}\mathcal{F}(g_\delta)(x)$$
$$= -\left[\frac{\delta}{\delta^2+\left(x+\frac{\pi}{h}\right)^2} - \frac{\delta}{\delta^2+\left(x-\frac{\pi}{h}\right)^2}\right]. \tag{B.39}$$

Therefore, for every $a\in\mathbb{R}$, $F_\delta(\tau,x)$ admits the following representation

$$F_\delta(\tau,x) = F_\delta(\tau,a)\mathrm{e}^{-\mathrm{i}(\tau-\mathrm{i}d)x}$$
$$-\int_a^x \mathrm{e}^{-\mathrm{i}(\tau-\mathrm{i}d)(s-x)}\left[\frac{\delta}{\delta^2+\left(s+\frac{\pi}{h}\right)^2} - \frac{\delta}{\delta^2+\left(s-\frac{\pi}{h}\right)^2}\right] ds. \tag{B.40}$$

Lemma B.4 *Let $\tau\in\mathbb{R}$ and $\delta>0$ be arbitrary. Then,*

$$\lim_{|x|\to\infty} F_\delta(\tau,x) = 0. \tag{B.41}$$

Proof. It is enough to show that

$$\lim_{|y|\to\infty}\int_{\mathbb{R}} \mathrm{e}^{-\delta|t|}\frac{\mathrm{e}^{\mathrm{i}ty}}{\tau-t-\mathrm{i}d}\,dt = 0. \tag{B.42}$$

Let us first notice that

$$\int_{\mathbb{R}} \mathrm{e}^{-\delta|t|}\frac{\mathrm{e}^{\mathrm{i}ty}}{\tau-t-\mathrm{i}d}\,dt = -\,2\left\{\int_0^\infty \mathrm{e}^{-\delta t}\frac{(\tau-\mathrm{i}d)\cos(ty)}{t^2+(d+\mathrm{i}\tau)^2}\,dt\right.$$
$$\left. +\,\mathrm{i}\int_0^\infty \mathrm{e}^{-\delta t}\frac{t\sin(ty)}{t^2+(d+\mathrm{i}\tau)^2}\,dt\right\}. \tag{B.43}$$

We will show that the second integral converges to 0 as $|y|\to\infty$, because the first one can be treated in the same fashion.

$$\int_0^\infty \mathrm{e}^{-\delta t}\frac{t\sin(ty)}{t^2+(d+\mathrm{i}\tau)^2}\,dt = -\frac{1}{y}\cos(ty)\frac{t\mathrm{e}^{-\delta t}}{t^2+(d+\mathrm{i}\tau)^2}\Bigg|_{t=0}^{t=\infty}$$
$$+\frac{1}{y}\int_0^\infty \cos(ty)\mathrm{e}^{-\delta t}\left[-\,\delta\frac{t}{t^2+(d+\mathrm{i}\tau)^2} + \frac{t^2+(d+\mathrm{i}\tau)^2-2\,t^2}{[t^2+(d+\mathrm{i}\tau)^2]^2}\right] dt,$$

which converges to 0 as $|y| \to \infty$. $\square$

It follows therefore that

$$F_\delta(\tau,x) = \int\limits_x^{\infty} \mathrm{e}^{-\mathrm{i}(\tau-\mathrm{i}d)(s-x)} \left[\frac{\delta}{\delta^2 + \left(s+\frac{\pi}{h}\right)^2} - \frac{\delta}{\delta^2 + \left(s-\frac{\pi}{h}\right)^2} \right] ds, \tag{B.44}$$

and this implies that $|F_\delta(\tau,x)| \leq 2\pi$. Moreover, through changes of variables we can rewrite (B.44) in the following form

$$\begin{aligned} F_\delta(\tau,x) = &\int\limits_{\left(\frac{\pi}{h}+x\right)/\delta}^{\infty} \mathrm{e}^{-\mathrm{i}(\tau-\mathrm{i}d)(\delta s - x - \pi/h)} \frac{1}{1+s^2}\, ds \\ &- \int\limits_{\left(-\frac{\pi}{h}+x\right)/\delta}^{\infty} \mathrm{e}^{-\mathrm{i}(\tau-\mathrm{i}d)(\delta s - x + \pi/h)} \frac{1}{1+s^2}\, ds, \end{aligned} \tag{B.45}$$

and now (B.27) and (B.29) follow straightforwardly. In order to show (B.28) and (B.30) we make use of (B.39):

$$\frac{d}{dx} F_\delta(\tau,x) = \mathrm{i}(\tau - \mathrm{i}d) F_\delta(\tau,x) - \left[\frac{\delta}{\delta^2 + \left(x+\frac{\pi}{h}\right)^2} - \frac{\delta}{\delta^2 + \left(x-\frac{\pi}{h}\right)^2} \right]. \tag{B.46}$$

Since

$$\frac{\delta}{\delta^2 + \left(x \pm \frac{\pi}{h}\right)^2} \leq \frac{\delta}{\delta^2 + L^2} \leq \frac{1}{2\,L}, \quad \forall\, x \in (-L,L), \quad \forall\, \delta > 0,$$

(B.28) and (B.30) are now immediate. With these results, we get that

$$\begin{aligned} \lim_{\delta\to 0^+} \int\limits_{\mathbb{R}}^{(B)} \mathrm{e}^{\mathrm{i}t(\cdot)} \mathrm{e}^{-\delta|t|} E(f,h)(t,\cdot)\, dt = &\int\limits_{\mathbb{R}}^{(B)} f_1(t,\cdot) g(t - \mathrm{i}d^-,\cdot)\, dt \\ &+ \int\limits_{\mathbb{R}}^{(B)} f_2(t,\cdot) g(t + \mathrm{i}d^-,\cdot)\, dt, \end{aligned}$$

in $H^0_{-2\nu}(\mathbb{R}) \cap H^1_{-2\nu}(-L,L)$, where $f_1(t,\cdot)$ and $f_2(t,\cdot)$ are as in (3.39), (3.40). Notice that

$$\|f_1(t,\cdot)\|_{L^\infty(\mathbb{R})}, \quad \|f_2(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\sinh(\pi d/h)},$$

respectively

$$\|f_1(t,\cdot)\|_{W^{1,\infty}(-L,L)}, \;\; \|f_2(t,\cdot)\|_{W^{1,\infty}(-L,L)} \leq C(d)(1+|t|)\frac{1}{\sinh(\pi d/h)}.$$

It follows therefore that

$$\begin{aligned}
\|\delta(f,h)(\cdot)\|_{0,-2\nu} &\leq \int_{\mathbb{R}} \|f_1(t,\cdot)\|_{L^\infty} \, \|g(t-\mathrm{i}d^-,\cdot)\|_{0,-2\nu} \, dt \\
&+ \int_{\mathbb{R}} \|f_2(t,\cdot)\|_{L^\infty} \, \|g(t+\mathrm{i}d^-,\cdot)\|_{0,-2\nu} \, dt, \\
&\leq \frac{1}{\sinh(\pi d/h)} \int_{\mathbb{R}} \|g(t-\mathrm{i}d^-,\cdot)\|_{0,-2\nu} \\
&+ \|g(t+\mathrm{i}d^-,\cdot)\|_{0,-2\nu} \, dt,
\end{aligned}$$

respectively

$$\begin{aligned}
\|\delta(f,h)(\cdot)\|_{H^1_{-2\nu}(-L,L)} &\leq \int_{\mathbb{R}} \|f_1(t,\cdot)\|_{W^{1,\infty}(-L,L)} \, \|g(t-\mathrm{i}d^-,\cdot)\|_{1,-2\nu} \, dt \\
&+ \int_{\mathbb{R}} \|f_2(t,\cdot)\|_{W^{1,\infty}(-L,L)} \, \|g(t+\mathrm{i}d^-,\cdot)\|_{1,-2\nu} \, dt, \\
&\leq \frac{C(d)}{\sinh(\pi d/h)} \int_{\mathbb{R}} \big(\|g(t-\mathrm{i}d^-,\cdot)\|_{1,-2\nu} \\
&+ \; \|g(t+\mathrm{i}d^-,\cdot)\|_{1,-2\nu}\big) \, (1+|t|) \, dt.
\end{aligned}$$

This implies that

$$\begin{aligned}
\|\delta(f,h)(\cdot)\|_{0,-2\nu} &\leq C\mathrm{e}^{-(\pi d\alpha N)^{1/2}} \int_{\mathbb{R}} \|g(t-\mathrm{i}d^-,\cdot)\|_{0,-2\nu} \\
&+ \|g(t+\mathrm{i}d^-,\cdot)\|_{0,-2\nu} \, dt \\
&\leq C\mathrm{e}^{-(\pi d\alpha N)^{1/2}} \int_{\mathbb{R}} |\hat{f}(t-\mathrm{i}d^-)|(1+|t-\mathrm{i}d|) \\
&+ \hat{f}(t+\mathrm{i}d^-)|(1+|t+\mathrm{i}d|) \, dt \\
&\leq C(\gamma,\nu)C(f)\left(1+\frac{1}{\alpha}\right)\frac{1}{\alpha}\mathrm{e}^{-(\pi d\alpha N)^{1/2}},
\end{aligned}$$

respectively

$$\begin{aligned}\|\delta(f,h)(\cdot)\|_{H^1_{-2\nu}(-L,L)} &\le C\mathrm{e}^{-(\pi d\alpha N)^{1/2}} \int\limits_{\mathbb{R}} \big(\|g(t-\mathrm{i}d^-,\cdot)\|_{1,-2\nu} \\ &+ \|g(t+\mathrm{i}d^-,\cdot)\|_{1,-2\nu}\big)\,(1+|t|)\,dt \\ &\le C\mathrm{e}^{-(\pi d\alpha N)^{1/2}} \int\limits_{\mathbb{R}} \Big(|\hat{f}(t-\mathrm{i}d^-)|(1+|t-\mathrm{i}d|) \\ &+ |\hat{f}(t+\mathrm{i}d^-)|(1+|t+\mathrm{i}d|)\Big)\,(1+|t|)\,dt \\ &\le C(\gamma,\nu)C(f)\left(1+\frac{1}{\alpha}\right)^2 \frac{1}{\alpha}\mathrm{e}^{-(\pi d\alpha N)^{1/2}}.\end{aligned}$$

The proof of Theorem 3.9 is now complete. □

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