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A numerical characterization of reduction ideals

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1 Introduction

Let $(A, \mathfrak{m})$ be an equidimensional complete local noetherian ring and let $J \subseteq I$ be two ideals in A. Recall that J is called a *reduction* of I if $JI^n = I^{n+1}$ for sufficiently large n. If $J \subseteq I$ are $\mathfrak{m}$-primary and J is a reduction of I then it is well known and easy to see that the Hilbert-Samuel multiplicities $e(J, A)$ and $e(I, A)$ are equal. By an important theorem of Rees [Ree] the converse also holds: *if $J \subseteq I$ are $\mathfrak{m}$-primary ideals with $e(J, A) = e(I, A)$ then J is a reduction of I.*

Now assume that $J \subseteq I$ are arbitrary ideals with the same radical $\sqrt{J} = \sqrt{I}$. If J is a reduction of I then we have always $e(J_\mathfrak{p}, A_\mathfrak{p}) = e(I_\mathfrak{p}, A_\mathfrak{p})$ for all minimal primes of I. However, the converse is not true, in general, as is seen by simple counterexamples. Under an additional assumption E. Böger [Boe] was able to prove a converse. To describe his result recall that the *analytic spread* $l(J)$ is the dimension of $G_J(A)/\mathfrak{m}G_J(A)$, where $G_J(A)$ denotes the associated graded ring. Then Böger's theorem is as follows: *let $J \subseteq I$ be arbitrary ideals of A having the same radical. If the analytic spread $l(J)$ is equal to the height of J and if $e(I_\mathfrak{p}, A_\mathfrak{p}) = e(J_\mathfrak{p}, A_\mathfrak{p})$ for all minimal primes of I then J is a reduction of I.*

There is an interesting generalization of Böger's theorem which is essentially due to Ulrich (see [FOV, 3.6]): *let $(A, \mathfrak{m})$ be as above and let $J \subseteq I$ be ideals. Then either $\mathrm{ht}(JI^{n-1} : I^n) \leq l(J)$ for all $n \geq 1$, or J is a reduction*

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of I. This shows in particular that J is a reduction of I if and only if $J_{\mathfrak{p}}$ is a reduction of $I_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p}$ with $\operatorname{ht}\mathfrak{p} = l(J_{\mathfrak{p}})$.

In this paper we will give a numerical characterization of reduction ideals which generalizes Böger's theorem to arbitrary ideals $J \subseteq I$. For this we use the j-multiplicity $j(I, A)$ introduced in [AMa]. It is equal to the Hilbert-Samuel multiplicity if the ideal I is $\mathfrak{m}$-primary. In the general case we can describe it in a geometric way roughly as follows: assume that $d := \dim A > 0$ and let $p : Y \to X$ be the blowing up of $X := \operatorname{Spec} A$ along the subscheme Z defined by the ideal I. Consider the union, say, E of all irreducible components of the exceptional set $p^{-1}(Z)$ that are set-theoretically contained in the special fibre $p^{-1}(\mathfrak{m})$. This is a projective scheme over $A/\mathfrak{m}^n$ for some n. The $(d-1)$-dimensional degree of E is by definition the multiplicity $j(I, A)$; see Sect. 2 for further details.

With these notations the main result of this paper is as follows.

Theorem. *Let $J \subseteq I$ be ideals in an equidimensional complete local noetherian ring A. Then the following are equivalent.*

1. J is a reduction of I;
2. $j(J_{\mathfrak{p}}, A_{\mathfrak{p}}) = j(I_{\mathfrak{p}}, A_{\mathfrak{p}})$ for all prime ideals $\mathfrak{p} \in \operatorname{Spec} A$;
3. $j(J_{\mathfrak{p}}, A_{\mathfrak{p}}) \leq j(I_{\mathfrak{p}}, A_{\mathfrak{p}})$ for all prime ideals $\mathfrak{p} \in \operatorname{Spec} A$.

Another somewhat technical generalization of Böger's theorem using Buchsbaum-Rim multiplicities was obtained by Kleiman and Thorup [KT]; their results were recently considerably simplified by Simis and Ulrich. However, this generalization does not give a complete numerical characterization of reduction ideals.

We add a few remarks about the contents of this paper. In Sect. 2 we introduce some basic notations and facts about reductions and the j-multiplicity. In Sect. 3 we will derive the main theorem which we will prove more generally in a module theoretic version, see 3.3.

2 Review of some known results

2.1 Let $(A, \mathfrak{m})$ be a local noetherian ring and $I \subseteq A$ an arbitrary ideal. Consider the associated graded ring

$$G := G(A) := G_I(A) := \bigoplus_{n\in\mathbb{N}} (I^n/I^{n+1})T^n.$$

If M is a finite A-module then the associated graded module

$$G(M) := G_I(M) := \bigoplus_{n\in\mathbb{N}} (I^n M/I^{n+1}M)T^n$$

is a module over $G(A)$ in a natural way, and its homogeneous components are finite (A/I)-modules. We recall the following useful result; see [FOV, 1.2.19] for a proof.

Proposition 2.2 *Let M, M' be finitely generated A-modules. Then the following hold.*

(a) *If $\operatorname{supp} M' \subseteq \operatorname{supp} M$ then the support of $G(M')$ (as module over G) is contained in the support of $G(M)$.*

(b) *Assume that $G(M)$ is equidimensional. Let $x \in I$ be an element with $\dim G(M)/x^*G(M) < d$, where $x^* = \bar{x}T$ denotes the initial form of x in G. Then the supports of the modules $G(M/xM)$ and $G(M)/x^*G(M)$ are equal.*

Generalizing the well known notion of a reduction ideal we will say that an ideal $J \subseteq I$ forms a *reduction* of (M, I) if

$$\dim G_I(M)/(J^*G_I(M) + \mathfrak{m}G_I(M)) = 0,$$

where J^* denotes the initial ideal of J, i.e. the ideal in G generated by the initial forms x^* of degree 1 elements in J. Note that then J is generated by at least $l_M(I) := \dim G_I(M)/\mathfrak{m}G_I(M)$ elements.

The ideal J will be called a *minimal reduction* for (M, I) if furthermore J is generated by $l_M(I)$ elements. If $A/\mathfrak{m}$ is infinite then this is equivalent to J being minimal among the reductions of (M, I). Thus J is a (minimal) reduction of (A, I) if and only if it is a (minimal) reduction of I in the usual sense. It is well known that such minimal reductions always exist if the residue field $A/\mathfrak{m}$ is infinite (cf. also 2.9 below).

Observe that by Nakayama's lemma J is a reduction of (M, I) if and only if

$$JI^nM = I^{n+1}M \quad \text{for } n \gg 0.$$

For later use we note the following simple facts, see e.g. [FMa].

Lemma 2.3 *Let M, N be finite A-modules. Then the following hold.*

(a) J is a reduction of I if and only if the Rees module $R_I(M) := \bigoplus_{n\geq 0} I^nM$ is finite over the Rees ring $R_J := \bigoplus_{n\geq 0} J^n$ of J.

(b) If $\operatorname{supp} N \subseteq \operatorname{supp} M$ and J is a reduction of (M, I) then J is also a reduction of (N, I).

Proof. For the convenience of the reader we repeat the simple argument. (a) follows from the graded version of Nakayama's lemma. In order to prove (b), note that by 2.2 (a) the support of $G_I(N)$ is contained in the support of $G_I(M)$. Hence

$$\dim G_I(N)/J^*G_I(N) + \mathfrak{m}G_I(N) \leq \dim G_I(M)/J^*G_I(M) + \mathfrak{m}G_I(M)$$

and so J is as well a reduction of (N, I). $\square$

In order to be able to formulate the main results in an efficient way, we need a generalization of the notions of height and analytic spread of an ideal to the case of modules (see [FMa]). We call the number

$$\operatorname{ht}_M I := \min\{\dim M_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{supp} M \cap V(I)\}$$

the *M-height* of I. Moreover, the number

$$l_M(I) := \dim G_I(M)/\mathfrak{m}G_I(M)$$

is called *the analytic spread of* (M, I).

In the case $M = A$ we also write in brief $l(I)$ and $\operatorname{ht} I$ instead of $l_A(I)$, $\operatorname{ht}_A I$, respectively. We need the following elementary properties of these numbers.

Proposition 2.4 *1. If N, M are A-modules with $\operatorname{supp} N \subseteq \operatorname{supp} M$ then $l_N(I) \le l_M(I)$ and $\operatorname{ht}_N I \ge \operatorname{ht}_M I$.*

2. $\operatorname{ht}_M I \le l_M(I) \le \min\{\dim M, \mu(I)\}$, where $\mu(I)$ denotes the minimal number of generators for the ideal I.

3. If J is a reduction of (M, I) then $l_M(I) = l_M(J)$ and $\operatorname{ht}_M I = \operatorname{ht}_M J$.

Proof. The first part of (1) follows easily from 2.2 (a). Moreover, the inequality $\operatorname{ht}_N I \le \operatorname{ht}_M I$ is immediate from the definition.

The remaining assertions (2), (3) are well known in case that $M = A$ (see e.g. [FOV, 3.6.4]). In view of (1) this proves the proposition. □

2.5 Let $(A, \mathfrak{m})$ be a local Noetherian ring and $I \subseteq A$ an ideal. Then one can assign to every finite A-module a generalized multiplicity $j(I, M)$ which was introduced in the case $M = A$ by [AMa] and in the general case in [FOV, Sect. 6.1]. Let us recall the definition of these multiplicities. Let G be as before the associated graded ring of A and $G(M)$ the associated graded module of M. Let $\Gamma G(M)_j := H^0_{\mathfrak{m}}(G(M)_j)$ denote the submodule of elements supported on $\mathfrak{m}$. Their direct sum

$$\Gamma G(M) = \bigoplus_{j \ge 0} \Gamma G(M)_j$$

is a graded G-submodule of $G(M)$ which has homogeneous components of finite length. Moreover, this module is annihilated by a sufficiently high power $\mathfrak{m}^k$ of $\mathfrak{m}$ and so may be considered as a module over the graded ring $\bar{G} := G \otimes_A A/\mathfrak{m}^k$. Hence its multiplicity $e(\Gamma G(M)) := e(\bar{G}_+, \Gamma G(M))$ is well defined, where $\bar{G}_+$ is the ideal in $\bar{G}$ of elements of positive degree. For a number $d \ge \dim M$ we set

$$j_d(I, M) := \begin{cases} e(\Gamma G(M)) & \text{if } d = \dim \Gamma G(M) \\ 0 & \text{if } d > \dim \Gamma G(M)\,. \end{cases}$$

Moreover, we set $j(I,M) := j_{\dim M}(I,M)$. We note the following simple facts which follow almost immediately from the definition.

1. $j(I,M) \neq 0$ if and only if $l_M(I) = \dim M$ (see e.g. [FOV, 6.1.6 (1)]).
2. If $\dim M > 0$ and $I^n M = 0$ for some n then $j(I,M) = 0$.

This number shares many properties of usual multiplicities. For instance, we have the following result; see [FOV, 6.1.7].

Lemma 2.6 *(Additivity) Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of A-modules and $d \geq \dim M$. Then*

$$j_d(I,M) = j_d(I,M') + j_d(I,M'').$$

What is also important in the following is the behaviour of j under taking hyperplane sections; see [FOV, 6.1.10].

Proposition 2.7 *Assume that $x \in I$ is an element satisfying the following conditions.*

1. $\dim G(M)/x^* G(M) < d := \dim M$
2. $\dim \tilde{G}(M)/(x^*\tilde{G}(M) + \mathfrak{m}\tilde{G}(M)) < d-1$, *where $\tilde{G}(M)$ denotes the quotient $G(M)/\Gamma G(M)$.*

Then $j_d(I,M) + j_{d-1}(I, \operatorname{Ann}_M x) = j_{d-1}(I, M/xM)$.

Clearly, if $\operatorname{ht}_M I > 0$ then the above conditions (1), (2) are satisfied for a sufficiently general element of I. The following standard generic element construction will provide such sufficiently generic elements.

2.8 Let $J \subseteq I$ be an ideal such that $J_{\mathfrak{p}}$ is a reduction of $(M_{\mathfrak{p}}, I_{\mathfrak{p}})$ for every prime $\mathfrak{p} \neq \mathfrak{m}$. Assume that $J = (x_1, \ldots, x_k)$ and let $U_1, \ldots, U_k$ be indeterminates. Consider

$$x := \sum_{i=1}^{k} U_i x_i$$

as an element of the localization, say A', of $A[U_1, \ldots, U_k]$ with respect to the ideal $\mathfrak{m}_A[U_1, \ldots, U_k]$. Set $M' := M \otimes_A A'$ and $I' := IA'$. Then $G(M') := G_{I'}(M') \cong G(M) \otimes_A A'$. This easily implies the following simple fact.

1. $j(I', M') = j(I,M)$.

The minimal (resp. associated) primes of M' are just the primes $\mathfrak{p}A'$ with $\mathfrak{p} \in \operatorname{Min} M$ (resp. $\mathfrak{p} \in \operatorname{Ass} M$). Therefore we have:

2. $\operatorname{Ann}_{M'} x$ is concentrated on $V(I')$.

The ring $G' := G_{I'}(A')$ is a localization of $G(A)[U_1, \ldots, U_k]$ and x^* corresponds to the generic linear combination $\sum U_i x_i^*$. Similarly $G(M')$ and the module $\tilde{G}(M')$ introduced in 2.7 (2) are localizations of $G(M)[U_1, \ldots, U_k]$, $\tilde{G}(M)[U_1, \ldots, U_k]$ respectively. Hence we have:

3. The associated primes of $G(M')$ are the primes $\mathfrak{p}G'$ with $\mathfrak{p} \in \operatorname{Ass} G(M)$, and the associated primes of $\widetilde{G}(M')$ are the primes $\mathfrak{p}G'$ with $\mathfrak{p} \in \operatorname{Ass} \widetilde{G}(M)$.

Lemma 2.9 *Assume that* $\operatorname{depth}_I M > 0$. *Then the following hold.*

1. x is not a zerodivisor on M'.

2. $\operatorname{Ann}_{G(M')} x^*$ is concentrated on $V(J^*G') \subseteq V(\mathfrak{m}G') \cup V(G'_+)$.

3. Assume moreover that J is a reduction of (M, I). Then we have $j_d(I', M') = j_{d-1}(I', M'/xM')$, where $d := \dim M$. In particular, $j_d(JA', M') = j_{d-1}(JA', M'/xM')$.

Proof. (1), (2) are an easy consequences of 2.8 (2), (3) respectively.

To prove the first part of (3) we note that the assumptions of 2.7 are satisfied as follows easily from (2) and 2.8 (3). Hence $j_d(I', M') = j_{d-1}(I', M'/xM')$. Applying this to the case $I = J$, the second part also follows. □

In the following, the base change $A \to A'$ will be suppressed, and we will speak simply about sufficiently generic elements.

In the next section we will also need the following observation.

Proposition 2.10 *Assume that J is a reduction of (M, I). Then we have $j(J, M) = j(I, M)$.*

Proof. We proceed by induction on $d := \dim M$. For $\dim M = 0$ the assertion is obvious. So assume that $d > 0$. We note first that J is a reduction of (M, I) if and only if it is a reduction of $(I^n M, I)$. Moreover, by 2.5 (2) $j_d(I, M/I^n M) = j_d(J, M/I^n M) = 0$ and so, applying the additivity of j to the exact sequence

$$0 \to I^n M \to M \to M/I^n M \to 0,$$

we have that $j(I, I^n M) = j(I, M)$. The same argument yields that $j(J, I^n M) = j(J, M)$. Therefore, replacing M by $I^n M$, $n \gg 0$, we may assume that $\operatorname{depth}_I M > 0$. Take a sufficiently general element $x \in J$ (see 2.8 and 2.9). By 2.9 (3), $j(I, M) = j(I, M/xM)$ and $j(J, M) = j(J, M/xM)$. Applying the induction hypothesis the result follows. □

3 The main result

The aim of this section is to prove the theorem stated in the introduction. In the following let $(A, \mathfrak{m})$ be a local noetherian ring and let M be a finite A-module. The key step is the following lemma.

Lemma 3.1 *Let $J \subseteq I \subseteq \mathfrak{m}$ be ideals, M a formally equidimensional A-module with $\operatorname{depth}_I M > 0$ and assume that the following condition is satisfied:*

$(*)$ *$J_\mathfrak{p}$ is a reduction of $(M_\mathfrak{p}, I_\mathfrak{p})$ for all $\mathfrak{p} \neq \mathfrak{m}$.*
Let $x \in J$ be sufficiently generic. Then

$$j(I, M/xM) \geq j(I, M).$$

Moreover, if equality holds then the the modules $G_I(M)/x^*G_I(M)$ and $G_I(M/xM)$ over $G = G_I(A)$ have the same support.

Proof. We may assume that A is complete. Consider the extended Rees ring

$$R := R_I(A) := \bigoplus_{n \in \mathbb{Z}} I^n T^n \subseteq A[T, T^{-1}],$$

where as usual $I^n := A$ for $n \leq 0$. Similarly let

$$N := R_I(M) \quad \text{and} \quad \bar{N} := R_I(\bar{M}) \quad \text{with } \bar{M} = M/xM$$

denote the Rees modules associated to M and $\bar{M}$, respectively. Letting

$$N' := \ker(N \to \bar{N})$$

we consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N'(1) & \longrightarrow & N(1) & \longrightarrow & \bar{N}(1) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle T^{-1}} & & \downarrow{\scriptstyle T^{-1}} & & \downarrow{\scriptstyle T^{-1}} & & \\ 0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & \bar{N} & \longrightarrow & 0\,. \end{array}$$

This gives an exact sequence of cokernels

(1) $$0 \to G' := N'/T^{-1}N' \to G(M) \to G(\bar{M}) \to 0\,,$$

where $G(M) = G_I(M)$, $G(\bar{M}) = G_I(\bar{M})$ are the associated graded modules. Denote the cokernel of the natural injection $N \overset{xT}{\hookrightarrow} N'$ by L. Using the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \xrightarrow{xT} & N'(1) & \longrightarrow & L(1) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle T^{-1}} & & \downarrow{\scriptstyle T^{-1}} & & \downarrow{\scriptstyle T^{-1}} & & \\ 0 & \longrightarrow & N(-1) & \xrightarrow{xT} & N' & \longrightarrow & L & \longrightarrow & 0 \end{array}$$

the snake-lemma yields an exact sequence

(2) $$0 \to U \to G(M)(-1) \to G' \to V \to 0,$$

where U, V are the kernel and cokernel of $T^{-1} : L(1) \to L$, respectively, i.e. the sequence

$$(3) \qquad 0 \to U \to L(1) \xrightarrow{T^{-1}} L \to V \to 0$$

is exact. It follows from (1) and (2) that

$$(4) \qquad \begin{aligned} U &= \ker(G(M) \overset{xT}{\to} G(M)) \\ V &= \ker(G(M)/xTG(M) \to G(\bar{M})). \end{aligned}$$

Since x is generic and $(*)$ is satisfied the kernel of the map

$$G(M)_{n-1} \overset{xT}{\to} G(M)_n, \quad n \gg 0,$$

has support in $\mathfrak{m}$, see 2.9 (2). Hence U_n has finite length for $n \gg 0$.

By the lemma of Artin-Rees the module $L = \bigoplus_\nu (xM \cap I^\nu M)/I^\nu M$ is annihilated by T^{-k} for $k \gg 0$. Hence U, V and L have the same support. It also follows that V_n and L_n have finite length for $n \gg 0$. Applying the functor $\Gamma := H^0_{\mathfrak{m}}$ to the sequence (2) in degree n gives that

$$(2)' \qquad 0 \to U_n \to \Gamma G(M)_{n-1} \to \Gamma(G')_n \to V_n \to 0.$$

is exact for $n \gg 0$; note that $H^0_{\mathfrak{m}}(\Lambda) = \Lambda$ and $H^i_{\mathfrak{m}}(\Lambda) = 0$ for $i \geq 1$ and every A-module Λ of finite length. Applying Γ to the sequence (1) in degree n gives an exact sequence for all n

$$(1)' \qquad 0 \to \Gamma(G')_n \to \Gamma G(M)_n \to \Gamma G(\bar{M})_n.$$

Thus the corresponding Hilbert functions satisfy the inequality

$$(5) \qquad H_{\Gamma G(\bar{M})}(n) \geq H_{\Gamma G(M)}(n) - H_{\Gamma G'}(n)$$

for all n. By (2)' we have for $n \gg 0$ that

$$H_{\Gamma G'}(n) = H_{\Gamma G(M)}(n-1) + H_V(n) - H_U(n).$$

Inserting this into (5) gives

$$(6) \qquad H_{\Gamma G(\bar{M})}(n) \geq \Delta H_{\Gamma G(M)}(n) - H_V(n) + H_U(n),$$

where Δ is the difference operator, i.e. $\Delta H(n) = H(n) - H(n-1)$ for a function $H : \mathbb{Z} \to \mathbb{R}$. The sequence (3) yields

$$H_U(n) - H_V(n) = H_L(n+1) - H_L(n) = \Delta H_L(n+1).$$

Combining this with (6) we finally get

$$(7) \qquad H_{\Gamma G(\bar{M})}(n) \geq \Delta H_{\Gamma G(M)}(n) + \Delta H_L(n+1) \geq \Delta H_{\Gamma G(M)}(n)$$

for $n \gg 0$. Comparing the coefficients of highest degree the first part of the lemma follows.

Now assume that $j(I, M/xM) = j(I, M)$. By (7) $\Delta H_L(n+1)$, $n \gg 0$, is necessarily a polynomial of degree at most $d-3$, where $d := \dim M$. Therefore

$$\dim \Gamma(L) = \dim \Gamma(U) = \dim \Gamma(V) \leq d-2,$$

i.e. the kernel of $xT : G(M) \to G(M)$ has support in $V(G_+) \cup \Sigma$, where $\dim \Sigma \leq d-2$. In particular, $\dim \ker(xT : G(M) \to G(M)) < \dim G(M)$. Hence 2.2(b) implies that the modules $G(M/xM)$ and $G(M)/x^*G(M)$ have the same support, as required. □

Lemma 3.2 *Let $J \subseteq I \subseteq \mathfrak{m}$ be ideals, M a finite formally equidimensional A-module and assume that condition $(*)$ of 3.1 is satisfied. Then*

$$j(J, M) \geq j(I, M).$$

Proof. We may assume that A is complete. We proceed by induction on $d := \dim M$. For $\dim M = 0$ the assertion is obvious. So assume that $d > 0$. With the same argument as in the proof of 2.10 we may replace M by $I^n M$, $n \gg 0$, and are thus reduced to the case that $\operatorname{depth}_I M > 0$. Then a sufficiently general element $x \in J$ is not a zero divisor for M. Applying 3.1 and the induction hypothesis we get the following chain of inequalities:

$$j(J, M) = j(J, M/xM) \geq j(I, M/xM) \geq j(I, M).$$

This gives the result. □

We can now prove a module theoretic version of the theorem stated in the introduction.

Theorem 3.3 *Let $(A, \mathfrak{m})$ be a local noetherian ring and M a formally equidimensional finite A-module. Let $J \subseteq I \subseteq \mathfrak{m}$ be ideals. Then the following is equivalent.*

1. *J is a reduction of (M, I).*
2. *$j(J_\mathfrak{p}, M_\mathfrak{p}) = j(I_\mathfrak{p}, M_\mathfrak{p})$ for all prime ideals $\mathfrak{p} \in \operatorname{Spec} A$.*
3. *$j(J_\mathfrak{p}, M\mathfrak{p}) \leq j(I_\mathfrak{p}, M_\mathfrak{p})$ for all prime ideals $\mathfrak{p} \in \operatorname{Spec} A$.*

Proof. The implication $(1) \Rightarrow (2)$ follows from 2.10, and $(2) \Rightarrow (3)$ is trivial. To show $(3) \Rightarrow (1)$ we may assume that A is complete. We proceed by induction on $\dim M$. If $\dim M = 0$ then the assertion is obvious. So assume in the following that $\dim M > 0$. As in the proof of 2.10 we can reduce to the case that $\operatorname{depth}_I M = \operatorname{depth}_J M > 0$. Take now a sufficiently generic element x in J (see 2.8 and 2.9). By induction hypothesis condition $(*)$ of 3.1 is satisfied and so

$$(1) \qquad j(I, M/xM) \geq j(I, M) \geq j(J, M) = j(J, M/xM),$$

where the last equality follows from 2.9 (3). Condition $(*)$ of 3.1 also implies that $J_{\mathfrak{p}}$ is a minimal reduction for $((M/xM)_{\mathfrak{p}}, I_{\mathfrak{p}})$ for all primes $\mathfrak{p} \neq \mathfrak{m}$. Using the induction hypothesis, J is a reduction of $(M/xM, I)$. By definition this means that

$$G_I(M/xM)/J^*G_I(M/xM) + \mathfrak{m}G_I(M/xM)$$

has dimension zero. Moreover by 2.10 all inequalities in (1) are equalities. Hence by 3.1 the modules $G_I(M/xM)$ and $G_I(M)/x^*G_I(M)$ have the same support. It follows that $G_I(M)/J^*G_I(M) + \mathfrak{m}G_I(M)$ also has dimension zero, as required. □

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