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A spectral multiplier theorem for a sublaplacian on $\mathrm{SU}(2)^\star$

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Abstract. We prove a Hörmander-type spectral multiplier theorem for a sublaplacian on $\mathrm{SU}(2)$, with critical index determined by the Euclidean dimension of the group. This result is the analogue for $\mathrm{SU}(2)$ of the result for the Heisenberg group obtained by D. Müller and E.M. Stein and by W. Hebisch.

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1. Introduction

Suppose that X is a measure space, equipped with a measure μ, and that L is a self-adjoint positive definite operator on $L^2(X)$. Then L has a spectral resolution:

$$L = \int_0^\infty \lambda \,\mathrm{d}E_L(\lambda),$$

where the $E_L(\lambda)$ are spectral projectors. For any bounded Borel function $F\colon [0,\infty) \to \mathbf{C}$, we define the operator $F(L)$ by the formula

$$F(L) = \int_0^\infty F(\lambda)\,\mathrm{d}E_L(\lambda).$$

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By the spectral theorem, $F(L)$ is well defined and bounded on $L^2(X)$. Spectral multiplier theorems give sufficient conditions on F under which the operator $F(L)$ extends to a bounded operator on $L^p(X)$ for some range of p. Once and for all, fix a nonzero cut-off function η in the Schwartz space $\mathcal{S}(\mathbf{R})$ supported in $\mathbf{R}^+$. Our theorem, like many, will be phrased in terms of the "local Sobolev norm"

$$\sup_{t\in\mathbf{R}^+} \left\|\eta\, F_{(t)}\right\|_{H_s},$$

where H_s is the Sobolev space of order s, and $F_{(t)}$ is given by

$$F_{(t)}(\lambda) = F(t\lambda) \qquad \forall\lambda \in [0,\infty).$$

The main goal of this article is to prove a spectral multiplier theorem for a sublaplacian on $\mathrm{SU}(2)$, the group of 2×2 complex unitary matrices of determinant 1. Its Lie algebra $\mathfrak{su}(2)$ consists of the 2×2 complex skew-adjoint matrices of trace 0. Define X, Y, and Z in $\mathfrak{su}(2)$ by

$$X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad \text{and} \qquad Z = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{1.1}$$

These form a basis of $\mathfrak{su}(2)$. We identify X, Y and Z with the corresponding left-invariant vector fields on $\mathrm{SU}(2)$, and define L by the formula

$$L = -(X^2 + Y^2). \tag{1.2}$$

Then L is a positive definite self-adjoint left-invariant second-order subelliptic differential operator on $L^2(\mathrm{SU}(2))$. The main result of this paper is the following spectral multiplier theorem.

Theorem 1.1. *Suppose that $s > 3/2$ and that $F\colon [0,\infty) \to \mathbf{C}$ is a bounded Borel function such that*

$$\sup_{t\in[1,\infty)} \left\|\eta\, F_{(t)}\right\|_{H_s} < \infty.$$

Then $F(L)$ is of weak type $(1,1)$ and bounded on $L^p(\mathrm{SU}(2))$ when $1<p<\infty$.

The subject of spectral multiplier theorems for differential operators is very broad, and it is impossible to give a complete bibliography here. We therefore only mention work directly related to our results. We start with the standard Laplace operator Δ_d on $\mathbf{R}^d$. Assume that $s > d/2$ and that $F\colon [0,\infty) \to \mathbf{C}$ satisfies the condition that

$$\sup_{t\in\mathbf{R}^+} \left\|\eta\, F_{(t)}\right\|_{H_s} < \infty. \tag{1.3}$$

Then L. Hörmander's multiplier theorem [19], specialised to the radial multipliers, shows that the operator $F(\Delta_d)$ is of weak type $(1,1)$ and bounded on $L^p(\mathbf{R}^d)$ for p in $(1,\infty)$. The order of differentiability is optimal, in the sense that, if $s < d/2$, then we can find a function F such that (1.3) holds but $F(\Delta_d)$ is not of weak type $(1,1)$ (see [6]). Since this condition holds, we say that $d/2$ is the *critical index*.

Now suppose that L is a homogeneous sublaplacian on a stratified nilpotent Lie group of homogeneous dimension d. A. Hulanicki and E.M. Stein [22] (see also [16, Theorem 6.25]) proved that if (1.3) holds for some s in $(3d/2+2,\infty)$, then $F(L)$ is of weak type $(1,1)$ and bounded on L^p when $1 < p < \infty$. L. De Michele and G. Mauceri [12] improved this result and proved that the same conclusions hold if $s > d/2+1$. Next, M. Christ [6], and independently Mauceri and S. Meda [23], proved that differentiability of order greater than $d/2$ is sufficient; see also [30]. Then X.T. Duong [13] proved that for some nilpotent groups of step 2, the order of differentiability required in the multiplier theorem is less then $d/2$. Finally, Müller and Stein [25] proved that the Hörmander multiplier theorem holds for some generalised Heisenberg groups when $s > n/2$, where n is the Euclidean dimension of the group. Independently Hebisch [18] proved the same result for all generalised Heisenberg groups. Müller and Stein [25] also proved that $n/2$ is the critical index.

At about the same time, spectral multiplier theorems on Lie groups of polynomial growth were investigated by G. Alexopoulos [3]. In his result, the required order of differentiability is connected with the volume growth of the ball $B(e,r)$ with centre e and radius r. More precisely, assume that $\mu\big(B(x,r)\big) \sim r^d$ when $r \le 1$ and $\mu\big(B(x,r)\big) \sim r^D$ when $r \ge 1$. Denote by $\Lambda_s(\mathbf{R})$ the space of Lipschitz (Hölder) continuous functions of order s. If $s > \max(d,D)/2$ and $F\colon [0,\infty) \to \mathbf{C}$ is bounded and satisfies the condition that

$$\sup_{t\in\mathbf{R}^+} \big\|\eta\, F_{(t)}\big\|_{\Lambda_s} < \infty,$$

then $F(L)$ is of weak type $(1,1)$ and is bounded on L^p when $1 < p < \infty$. Alexopoulos' multiplier theorem, applied to the operator defined by (1.2), yields a result which is weaker than Theorem 1.1; Alexopoulos' method requires Λ_s for $s > 2$ instead of H_s for $s > 3/2$. In Sect. 3, we give an alternative proof of Alexopoulos' multiplier theorem. In fact we obtain a more general version, valid not only in the Lie group setting, but also for abstract operators with the finite speed propagation property.

As we see, the critical index in multiplier theorems is often determined by the volume growth rate of the ball, or the dimension of the corresponding semigroup (which at least in principle are the same—see [37]). For elliptic operators the dimension of the corresponding semigroup coincides with the Euclidean dimension of the underlying space. However, for subelliptic operators, this semigroup dimension is strictly greater than the Euclidean dimension. Theorem 1.1 provides another example of a subelliptic operator for which the critical index in the spectral multiplier theorem is determined by the Euclidean dimension of the underlying space, not by the dimension of the corresponding semigroup. So we may view Theorem 1.1 as an extension of the multiplier theorems of [18] and [25]. Note that the groups investigated by [18] and [25] are all nilpotent of step 2, while $\mathrm{SU}(2)$ is simple. However, in this context it is interesting to note the connection between the Heisenberg group and $\mathrm{SU}(2)$ (see [26, 27]).

Multiplier theorems on compact Lie groups, in particular $\mathrm{SU}(2)$, were investigated by N.J. Weiss [39], R.R. Coifman and G. Weiss [10], J.-L. Clerc [8, 9], A. Bonami and Clerc [4], and others. However only the result of [10] is applicable to subelliptic operators, and the multiplier theorem of [10] is weaker then Theorem 1.1.

The proof of Theorem 1.1 has three main ingredients. First, using a Calderón–Zygmund type argument, we show in Theorem 3.3 that, in order to prove a weak-type $(1,1)$ estimate for the operator $F(\sqrt{L})$, it suffices to show that

$$\sup_{r\in\mathbf{R}^+}\sup_{y\in X}\int_{B(y,r)^c}\left|K_{F(1-\Phi_{(r)})(\sqrt{L})}(x,y)\right|\,\mathrm{d}\mu(x)\le C,$$

where K_T is the kernel of the operator T, and $\Phi_{(r)}$ is a damping factor. Next, in Lemma 3.4, we show how to estimate integrals outside a ball. In Theorem 3.5, we show how one very simple Plancherel type estimate may be combined with Theorem 3.3 and Lemma 3.4 to prove Alexopoulos' multiplier theorem. As noted, this is a weaker result than Theorem 1.1. To prove our main theorem, we need one more ingredient, namely a sharper weighted Plancherel estimate, established in Sect. 4. In Sect. 5, we observe that our Theorem 1.1 implies the result of Müller and Stein and of Hebisch for the Heisenberg group $\mathbb{H}_1$ by a contraction argument.

2. Preliminaries

The purpose of this section is to introduce some notation, describe the hypotheses under which we work, and prove a few lemmas which will be useful in our investigation of multiplier theorems.

2.1. Some notation

Assume that (X,ρ) is a metric space, equipped with a regular Borel measure μ. The Lebesgue spaces $L^p(X)$ are constructed relative to this measure. Let $B(y,r)$ denote the ball $\{x\in X\colon \rho(x,y)\le r\}$; $B(y,r)^c$ will denote its complement in X.

Suppose that T is a bounded operator from $L^p(X)$ to $L^q(X)$. We write $\|T\|_{L^p \to L^q}$ for the usual operator norm of T. If T is of weak type $(1,1)$, i.e., if

$$\mu\left(\{x \in X : |Tf(x)| > \lambda\}\right) \leq C\, \frac{\|f\|_{L^1}}{\lambda} \qquad \forall \lambda \in \mathbf{R}^+ \qquad \forall f \in L^1(X),$$

then we write $\|T\|_{L^1 \to L^{1,\infty}}$ for the least possible value of C in the above inequality; this is often called the "operator norm", though in fact it is not a norm.

If there is a locally integrable function $K_T \colon X \times X \to \mathbf{C}$ such that

$$\langle Tf_1, f_2 \rangle = \int_X Tf_1\, \overline{f_2}\, \mathrm{d}\mu = \int_X K_T(x,y)\, f_1(y)\, \overline{f_2(x)}\, \mathrm{d}\mu(y)\, \mathrm{d}\mu(x)$$

for all f_1 and f_2 in $C_c(X)$, then we say that T is a *kernel operator* with kernel K_T. It is well known that if T is bounded from $L^1(X)$ to $L^q(X)$, where $q > 1$, then T is a kernel operator, and

$$\|T\|_{L^1 \to L^q} = \sup_{y \in X} \|K_T(\cdot, y)\|_{L^q}\,;$$

vice versa, if T is a kernel operator and the right hand side of the above inequality is finite, then T is bounded from $L^1(X)$ to $L^q(X)$, even if $q = 1$.

Given an operator T from $L^p(X)$ to $L^q(X)$, we write

$$\operatorname{supp} K_T \subseteq \{(x,y) \in X \times X : \rho(x,y) \leq r\}$$

if $\langle Tf_1, f_2 \rangle = 0$ whenever f_n is in $C(X)$ and $\operatorname{supp} f_n \subseteq B(x_n, r_n)$ when $n = 1, 2$, and $r_1 + r_2 + r < \rho(x_1, x_2)$. This definition makes sense even if T is not a kernel operator, in the sense of the previous definition.

Observe that, if F is in $L^\infty(\mathbf{R})$, then the adjoint of the operator $F(\sqrt{L})$ is $\bar{F}(\sqrt{L})$. This implies that, in order to prove that $F(\sqrt{L})$ is of weak type $(1,1)$ and bounded on $L^r(X)$ when $1 < r < \infty$, for all F is some class of bounded functions which is closed under conjugation, it suffices to prove that $F(\sqrt{L})$ is of weak type $(1,1)$. For $F(\sqrt{L})$ is bounded on $L^2(X)$ by the spectral theorem, and the boundedness of $F(\sqrt{L})$ on $L^r(X)$ for r in $(1,2)$ follows by interpolation and for r in $(2,\infty)$ by duality.

2.2. Hypotheses on the ambient space X

We make two assumptions about the measured metric space (X, μ, ρ).

Assumption 2.1. We suppose throughout that the "doubling condition" holds, i.e., there exists a constant C such that

$$\mu(B(x, 2r)) \leq C\, \mu(B(x,r)) \qquad \forall x \in X \quad \forall r \in \mathbf{R}^+.$$

For d and D in $[0,\infty)$, we define $V_{d,D}\colon \mathbf{R}^+ \to \mathbf{R}^+$ by the formula

$$V_{d,D}(t) = \begin{cases} t^d & \text{when } t \le 1 \\ t^D & \text{when } t \ge 1. \end{cases}$$

We will also use $V_{D,d}$, with the roles of d and D reversed, in light of the well-known principle that local and global behaviour in the spatial variables correspond to global and local behaviour respectively in the spectral variables. Note that $V_{d,D}(r) = V_{D,d}(r^{-1})^{-1}$.

Assumption 2.2. We always suppose that there is a (d,D) regular weight on X, by which we mean a nonnegative measurable function $w\colon X \times X \to \mathbf{R}^+$, possibly 1, such that

$$\int_{B(y,r)} w^{-1}(x,y)\,\mathrm{d}\mu(x) \le C\,V_{d,D}(r) \qquad \forall r \in \mathbf{R}^+ \quad \forall y \in X. \tag{2.1}$$

By Hölder's inequality, this implies that

$$\begin{aligned} &\int_{B(y,r)} |k(x,y)|\ \mathrm{d}\mu(x) \\ &\quad\le \left(\int_{B(y,r)} w^{-1}(x,y)\,\mathrm{d}\mu(x)\right)^{1/2} \\ &\qquad\times \left(\int_{B(y,r)} |k(x,y)|^2\, w(x,y)\,\mathrm{d}\mu(x)\right)^{1/2} \\ &\quad\le C\left(V_{d,D}(r)\int_{B(y,r)} |k(x,y)|^2\, w(x,y)\,\mathrm{d}\mu(x)\right)^{1/2}. \end{aligned} \tag{2.2}$$

In particular, if $D = 0$, then taking limits as r tends to ∞ shows that

$$\int_X |k(x,y)|\ \mathrm{d}\mu(x) \le C\left(\int_X |k(x,y)|^2\, w(x,y)\,\mathrm{d}\mu(x)\right)^{1/2}$$

for all y in X, so that

$$\|T\|_{L^1\to L^1} \le C\sup_{y\in X}\left(\int_X |K_T(x,y)|^2\, w(x,y)\,\mathrm{d}\mu(x)\right)^{1/2}. \tag{2.3}$$

2.3. Hypotheses on the operator L

Let L be a self-adjoint positive definite operator on $L^2(X)$. We make two assumptions throughout this paper about L.

Assumption 2.3. We suppose that L has the finite propagation speed property:

$$\operatorname{supp} K_{\cos(t\sqrt{L})} \subseteq \{(x,y) \in X \times X : \rho(x,y) \leq t\},$$

Assumption 2.4. We suppose that there is a constant C and a positive integer k such that L satisfies the Sobolev-type estimate

$$\|f\|_{L^\infty} \leq C\,\mu\big(B(x,r)\big)^{-1} \left\|\big(1 + r^2 L\big)^k f\right\|_{L^1}$$

for all f on X with support in $B(x,r)$, for all x in X and r in $\mathbf{R}^+$.

We now give a well known and useful consequence of Assumption 2.3, which goes back to [5].

Lemma 2.1. *Assume that $\hat{F}$ is the Fourier transform of a bounded even Borel function F and that* $\operatorname{supp} \hat{F} \subseteq [-r, r]$. *Then*

$$\operatorname{supp} K_{F(\sqrt{L})} \subseteq \{(x,y) \in X \times X : \rho(x,y) \leq r\}.$$

Proof. If F is an even function, then by the Fourier inversion formula,

$$F(\sqrt{L}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{F}(t) \cos(t\sqrt{L})\,\mathrm{d}t.$$

But $\operatorname{supp} \hat{F} \subseteq [-r, r]$, so Lemma 2.1 follows from Assumption 2.3. □

2.4. Even functions

The result of Lemma 2.1 is key to our work. In order to be able to use it, we must deal with even functions on $\mathbf{R}$ rather than functions on $[0, \infty)$. Of course, since the spectrum of L is contained in $[0, \infty)$, the operator $F(\sqrt{L})$ depends only on the restriction of F to this set.

We denote by $\mathcal{B}(\mathbf{R})$ the space of bounded even complex-valued Borel functions on $\mathbf{R}$, and by $\mathcal{B}_R(\mathbf{R})$ the subspace of $\mathcal{B}(\mathbf{R})$ of functions which vanish outside $[-R, R]$.

2.5. Plancherel type hypotheses

Given a function $F \colon \mathbf{R} \to \mathbf{C}$ and R in $\mathbf{R}^+$, we denote by $F_{(R)} \colon \mathbf{R} \to \mathbf{C}$ the function $x \mapsto F(Rx)$.

Assumption 2.5. Throughout this paper, we will suppose that

$$\begin{aligned} &\sup_{y \in X} \left(\int_X \left| K_{F(\sqrt{L})}(x,y) \right|^2 w(x,y)\,\mathrm{d}\mu(x) \right)^{1/2} \\ &\qquad \leq C\, V_{D,d}(R)^{1/2} \left\| F_{(R)} \right\|_{L^\infty}, \end{aligned} \tag{2.4}$$

for all R in $\mathbf{R}^+$ and all F in $\mathcal{B}_R(\mathbf{R})$, where w is a (d, D) regular weight.

Sometimes we will replace $\left\| F_{(R)} \right\|_{L^\infty}$ with $\left\| F_{(R)} \right\|_{L^p}$, where p is in $[1, \infty)$; this is a stronger assumption.

When $D = 0$, Assumption 2.5 is equivalent (up to a change in constants) to the apparently weaker assumption that

$$\sup_{y \in X} \left(\int_X \left| K_{F(\sqrt{L})}(x,y) \right|^2 w(x,y) \, \mathrm{d}\mu(x) \right)^{1/2} \leq C \, N^{d/2} \left\| F_{(N)} \right\|_{L^\infty} ,$$

for all N in $\mathbf{Z}^+$ and all F in $\mathcal{B}_N(\mathbf{R})$. We sometimes suppose that this inequality holds when $\left\| F_{(N)} \right\|_{L^\infty}$ is replaced by the mixed norm, $\left\| F_{(N)} \right\|_{N,p}$, given by

$$\begin{aligned} \|G\|_{N,p} &= \left(\frac{1}{N} \sum_{i=1}^{N} \Big(\sup_{|\lambda| \in [\frac{i-1}{N}, \frac{i}{N}]} |G(\lambda)| \Big)^p \right)^{1/p} \\ &= \left(\frac{1}{N} \sum_{i=1}^{N} \sup_{|\lambda| \in [\frac{i-1}{N}, \frac{i}{N}]} |G(\lambda)|^p \right)^{1/p} \end{aligned} \tag{2.5}$$

where p is in $[1, \infty)$ and N is a positive integer; in this definition, to obtain a norm, we must require that $\operatorname{supp} G \subseteq [-1, 1]$.

2.6. Examples

First, suppose that $w = 1$, that the uniform ball size condition

$$C \, V_{d,D}(r) \leq \mu\big(B(x,r)\big) \leq C' \, V_{d,D}(r) \qquad \forall r \in \mathbf{R}^+ \quad \forall x \in X$$

holds, and that L satisfies the heat kernel estimate

$$\|\exp(-tL)\|_{L^1 \to L^2} \leq C \, V_{d,D}(t)^{-1/4} \qquad \forall t \in \mathbf{R}^+. \tag{2.6}$$

Then for F in $\mathcal{B}_R(\mathbf{R})$, we see that

$$\begin{aligned} \left\| F(\sqrt{L}) \right\|_{L^1 \to L^2} &\leq \left\| F(\sqrt{L}) e^{R^{-2} L} \right\|_{L^2 \to L^2} \|\exp(-R^{-2} L)\|_{L^1 \to L^2} \\ &\leq C \, V_{D,d}(R)^{1/2} \, \|F\|_{L^\infty} , \end{aligned}$$

and Assumption 2.5 holds. Next, from the formulae

$$\begin{aligned} \|f\|_{L^\infty} &\leq \left\| (1 + r^2 L)^{-m} \right\|_{L^1 \to L^\infty} \left\| (1 + r^2 L)^m f \right\|_{L^1} , \\ (I + r^2 L)^{-m} &= \frac{1}{\Gamma(m)} \int_0^\infty e^{-t} \, t^{m-1} \exp(-t r^2 L) \, \mathrm{d}t \end{aligned}$$

and

$$\|\exp(-tL)\|_{L^1 \to L^\infty} \le C\, V_{d,D}(t)^{-1/2} \qquad \forall t \in \mathbf{R}^+,$$

(which is a consequence of (2.6)), Assumption 2.4 follows. It is perhaps worth pointing out that, when (2.6) holds, then Assumption 2.3 is equivalent to having Gaussian bounds for the heat kernel—see [31] for more details. Examples where these hypotheses hold include Lie groups of polynomial growth.

Second, when the space X is a Lie group, and L is a left-invariant differential operator, then the operator $F(\sqrt{L})$ is given by convolution with a kernel, $\widetilde{K}_{F(\sqrt{L})}$ say, i.e.,

$$F(\sqrt{L})f(g) = f * \widetilde{K}_{F(\sqrt{L})}(g) = \int_X f(h)\, \widetilde{K}_{F(\sqrt{L})}(h^{-1}g)\, \mathrm{d}h,$$

and

$$\left\| F(\sqrt{L}) \right\|_{L^1 \to L^2} = \left\| \widetilde{K}_{F(\sqrt{L})} \right\|_{L^2}.$$

The Plancherel formula for the commutative subalgebra of $L^1(X)$ generated by L gives rise to a formula of the form

$$\left\| \widetilde{K}_{F(\sqrt{L})} \right\|_{L^2} = \left(\int_0^\infty |F(\lambda)|^2 \,\mathrm{d}\pi(\lambda) \right)^{1/2}$$

for some Plancherel measure π (see, e.g., [6]). For a homogeneous sublaplacian on a homogeneous group of homogeneous dimension Q, it is immediate that $\mathrm{d}\pi(\lambda)$ is a multiple of $\lambda^{Q-1}\,\mathrm{d}\lambda$. Hence, in this case,

$$\left\| F(\sqrt{L}) \right\|_{L^1 \to L^2} \le C\, V_{Q,Q}(R)^{1/2} \left\| F_{(R)} \right\|_{L^2},$$

for all F in $\mathcal{B}_R(\mathbf{R})$. On $\mathbf{T}^n$, the description of the Plancherel measure involves number theory, and for a general subelliptic operator on a compact Lie group, one cannot be very specific about the Plancherel measure. However, the case where L is the Laplacian is covered in the next example.

Third, for a general positive definite elliptic pseudo-differential operator on a compact manifold, Assumption 2.4 holds by general elliptic regularity theory. Further, one has the Avakumovič–Agmon–Hörmander theorem.

Theorem 2.2. *Let L be a positive definite elliptic pseudo-differential operator of order m on a compact manifold X of dimension d. Then*

$$\left\| \chi_{[r-1,r)}(L^{1/m}) \right\|_{L^1 \to L^2} \le C\, r^{(d-1)/2} \qquad \forall r \in \mathbf{R}^+. \tag{2.7}$$

Theorem 2.2 was proved by Hörmander in [21]; see also [1,2,20]. This theorem has a useful corollary.

Corollary 2.3. *Let L be a positive definite elliptic pseudo-differential operator of order 2 on a compact manifold X of dimension d. Then*

$$\left\| F(\sqrt{L}) \right\|_{L^1 \to L^2} \le C\, N^{d/2} \left\| F_{(N)} \right\|_{N,2} \qquad \forall N \in \mathbf{Z}^+ \quad \forall F \in \mathcal{B}(\mathbf{R})_N. \tag{2.8}$$

Proof. By the spectral theorem,

$$\begin{aligned}\left\|F(\sqrt{L})\right\|_{L^1\to L^2} &\le \left(\sum_{i=1}^{N}\left\|\chi_{[i-1,i]}\,F(\sqrt{L})\right\|^2_{L^1\to L^2}\right)^{1/2}\\ &\le CN^{d/2}\left\|F_{(N)}\right\|_{N,2},\end{aligned}$$

as required. □

The importance of the estimate (2.7) for multiplier theorems was noted by C.D. Sogge [32], who used it to establish the convergence of Riesz means up to the critical exponent $(d-1)/2$, see also [7]. The following theorem appears to be due to A. Seeger and Sogge [29]; see also Hebisch [17].

Theorem 2.4. *Suppose that L is the Laplace–Beltrami operator on a compact Riemannian manifold X of dimension d. Assume that $s > d/2$ and that $F\colon [0,\infty) \to \mathbf{C}$ is a bounded function such that*

$$\sup_{t\in[1,\infty)}\left\|\eta\,F_{(t)}\right\|_{H_s} < \infty.$$

Then $F(\sqrt{L})$ is of weak type $(1,1)$ and bounded on $L^p(X)$ when $1 < p < \infty$.

This result is a consequence of Theorem 3.6 below and Corollary 2.3.

Theorem 2.4, applied to the Laplace operator on a compact Lie group, gives a stronger result then Alexopoulos' multiplier theorem. However, we do not know whether the Avakumovič–Agmon–Hörmander condition holds for subelliptic operators. Hence Alexopoulos' result gives the best known result for a sublaplacian on a compact Lie group other than $\mathrm{SU}(2)$.

2.7. The projection $E_L(0)$

In spectral multiplier theory, it is often necessary to consider the possibility that the projection $E_L(0)$ is nontrivial, and this paper is no exception.

Lemma 2.5. *The projection $E_L(0)$ is zero if $D > 0$, and is bounded on all the spaces $L^p(X)$ for p in $[1,\infty]$ if $D = 0$.*

Proof. Observe that, for all small positive ϵ, we have

$$\begin{aligned}&\sup_{y\in X}\left(\int_X \left|K_{E_L(0)}(x,y)\right|^2 w(x,y)\,\mathrm{d}\mu(x)\right)^{1/2}\\ &\qquad = \sup_{y\in X}\left(\int_X \left|K_{\chi_{\{0\}}(\sqrt{L})}(x,y)\right|^2 w(x,y)\,\mathrm{d}\mu(x)\right)^{1/2} \qquad (2.9)\end{aligned}$$

$$\leq C\, V_{D,d}(\epsilon),$$

from Assumption 2.5. It follows that if $D > 0$ then $E_L(0) = 0$.

If $D = 0$, then the left hand side of inequality (2.9) is bounded. Combining this fact with (2.3), we see that

$$\|E_L(0)\|_{L^1 \to L^1} \leq \sup_{y \in X} \int_X \left|K_{E_L(0)}(x,y)\right| \,\mathrm{d}\mu(x) \leq C.$$

By duality and interpolation, $E_L(0)$ is bounded on all the spaces $L^p(X)$ for p in $[1,\infty]$. □

2.8. Besov spaces

We will phrase our results in terms of Besov spaces. For the reader's convenience, we recall the definitions here.

Fix $[0,1]$-valued functions ϕ_0 and ϕ in $\mathcal{S}(\mathbf{R})$ supported in $(-4,4)$ and $(1,4)$ respectively, such that $\phi_0(\lambda) + \sum_{k \in \mathbf{Z}^+} \phi_k(\lambda) = 1$ in $\mathbf{R}$, where $\phi_k(\lambda) = \phi(2^{-k}|\lambda|)$ for k in $\mathbf{Z}^+$. Then $\phi_0 = 1$ on $[-2,2]$ and $\operatorname{supp}\phi_j \subseteq [2^j, 2^{j+2}] \cup [-2^{j+2}, -2^j]$ for j in $\mathbf{N}$. We define the operators T_{ϕ_j} on $\mathcal{S}'(\mathbf{R})$ by the formula

$$(T_{\phi_j} F)\widehat{\ } = \phi_j \widehat{F}, \tag{2.10}$$

for j in $\mathbf{N}$.

For s in $\mathbf{R}^+$ and p and q in $[1,\infty]$, the Besov space $B_s^{p,q}(\mathbf{R})$ is defined to be the set of all locally integrable functions F on $\mathbf{R}$ such that $\|F\|_{B_s^{p,q}} < \infty$, where

$$\|F\|_{B_s^{p,q}} = \Big(\sum_{j \in \mathbf{N}} 2^{jsq} \left\| T_{\phi_j} F \right\|_{L^p}^q \Big)^{1/q}$$

if $1 \leq q < \infty$, with the usual modification if $q = \infty$. Clearly $B_s^{p,q}(\mathbf{R}) \subseteq B_{\bar{s}}^{p,\bar{q}}(\mathbf{R})$ if $s > \bar{s}$ or if $s = \bar{s}$ and $q < \bar{q}$. It is known that the Besov space $B_s^{p,q}(\mathbf{R})$ is "close to" the potential space $W_s^p(\mathbf{R})$ of functions f in $L^p(\mathbf{R})$ such that $\Delta^{s/2} f$ also lies in $L^p(\mathbf{R})$, with the norm $\|f\|_{W_s^p}$ given by $\|f\|_{L^p} + \left\|\Delta^{s/2} f\right\|_{L^p}$. In particular, $B_s^{2,2}(\mathbf{R}) = H_s(\mathbf{R})$ and $\Lambda_s(\mathbf{R}) \subseteq B_s^{\infty,\infty}(\mathbf{R})$. See, e.g., [33, Chap. V] or [36, Chap. I and II] for more details.

Locally, Besov spaces are invariant under composition with diffeomorphisms. This means that it is equivalent to show that $F(\sqrt{L})$ is bounded from $L^u(X)$ to $L^v(X)$ for all F such that $\eta F_{(R)}$ is in $B_s^{p,q}(\mathbf{R})$ for all R in $\mathbf{R}^+$, and to show that $F(L)$ is bounded from $L^u(X)$ to $L^v(X)$ for all F in the same class.

3. General multiplier theorems

We fix an even function $\varPhi$ in $\mathcal{S}(\mathbf{R})$ such that $\varPhi(0) = 1$, whose Fourier transform $\hat{\varPhi}$ is supported in $[-1, 1]$; we let $\varPhi_{(r)}$ denote the dilated function $\varPhi(r\cdot)$ and $\varPhi^{(l)}$ denote the l^{th} derivative of $\varPhi$. For later purposes, note that for any fixed odd positive integer k, we may assume that $\varPhi^{(l)}(0) = 0$ when $1 \leq l \leq k$. It then follows that there is a constant C such that

$$\max\left\{\left|(\varPhi_{(r)} - 1)^{(l)}(x)\right| : 1/4 \leq x \leq 1, 0 \leq l \leq k\right\} \leq C\,\frac{r^{k+1}}{1 + r^{k+1}} \qquad \forall r \in \mathbf{R}^+. \tag{3.1}$$

Indeed, because $\varPhi$ is in $\mathcal{S}(\mathbf{R})$, it follows, for all x in $\mathbf{R}$ and r in $\mathbf{R}^+$, that if $l = 0$, then

$$\left|\varPhi_{(r)}(x) - 1\right| \leq 1 + \|\varPhi\|_{L^\infty},$$

while if $l > 0$, then

$$\left|(\varPhi_{(r)} - 1)^{(l)}(x)\right| = r^l\left|\varPhi^{(l)}(rx)\right| \leq C\,r^l(1 + |rx|)^{-l}.$$

Further, $\varPhi$ extends to an entire function in $\mathbf{C}$ of exponential type 1, and we may write

$$\varPhi(x) = 1 + \frac{c_{k+1}}{(k+1)!}x^{k+1} + \dots \qquad \forall x \in \mathbf{C},$$

where the coefficients c_m are uniformly bounded; it is easy to use this fact to show that

$$\left|(\varPhi_{(r)} - 1)^{(l)}(x)\right| \leq C\,r^{k+1}$$

when $|x| \leq 1$ and $0 < r \leq 1$.

The following lemma is crucial to our paper.

Lemma 3.1. *With $\varPhi$ in $\mathcal{S}(\mathbf{R})$ chosen as above, the kernel $K_{\varPhi_{(r)}(\sqrt{L})}$ of the self-adjoint operator $\varPhi_{(r)}(\sqrt{L})$ satisfies*

$$\operatorname{supp} K_{\varPhi_{(r)}(\sqrt{L})} \subseteq \{(x,y) \in X \times X : \rho(x,y) \leq r\}.$$

Further, if $\operatorname{supp} b \subseteq B(x,r)$, then for all q in $[1,\infty]$,

$$\left\|\varPhi_{(r)}(\sqrt{L})b\right\|_{L^q} \leq C\,\mu\big(B(x,2r)\big)^{-1/q'}\,\|b\|_{L^1} \qquad \forall r \in \mathbf{R}^+.$$

Proof. The first part of the lemma follows from Lemma 2.1.

Now we show that

$$\left\|\varPhi_{(r)}(\sqrt{L})\right\|_{L^1 \to L^1} \leq C \qquad \forall r \in \mathbf{R}^+. \tag{3.2}$$

Take f in $L^1(X)$. Then

$$
\begin{aligned}
&\left\|\Phi_{(r)}(\sqrt{L})f\right\|_{L^1} \\
&\quad = \int_X \left|\int_X K_{\Phi_{(r)}(\sqrt{L})}(x,y) f(y)\,\mathrm{d}\mu(y)\right| \,\mathrm{d}\mu(x) \\
&\quad \le \int_X \int_X \left|K_{\Phi_{(r)}(\sqrt{L})}(x,y) f(y)\right| \,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) \\
&\quad \le \sup_{y\in X} \int_X \left|K_{\Phi_{(r)}(\sqrt{L})}(x,y)\right| \,\mathrm{d}\mu(x)\, \|f\|_{L^1} \\
&\quad \le \sup_{y\in X} \left(V_{d,D}(r) \int_X \left|K_{\Phi_{(r)}(\sqrt{L})}(x,y)\right|^2 w(x,y)\,\mathrm{d}\mu(x)\right)^{1/2} \|f\|_{L^1} ,
\end{aligned}
$$

by Fubini's theorem and (2.2). We conclude that

$$
\begin{aligned}
&\left\|\Phi_{(r)}(\sqrt{L})\right\|_{L^1\to L^1} \\
&\quad \le \sup_{y\in X} \left(V_{d,D}(r) \int_X \left|K_{\Phi_{(r)}(\sqrt{L})}(x,y)\right|^2 w(x,y)\,\mathrm{d}\mu(x)\right)^{1/2} . \qquad (3.3)
\end{aligned}
$$

An integration by parts shows that

$$
\begin{aligned}
\Phi_{(r)}(\sqrt{L}) &= \int_0^\infty \Phi(r\sqrt{\lambda})\,\mathrm{d}E_L(\lambda) \\
&= -\Phi_{(r)}(0)\,E_L(0) - \int_0^\infty \frac{r}{2\sqrt{\lambda}}\,\Phi'(r\sqrt{\lambda}) \\
&\qquad \times \int_0^\infty \chi_{[0,\lambda]}(\lambda')\,\mathrm{d}E_L(\lambda')\,\mathrm{d}\lambda \\
&= -E_L(0) - \int_0^\infty \frac{r}{2\sqrt{\lambda}}\,\Phi'(r\sqrt{\lambda})\,\chi_{[0,\lambda]}(L)\,\mathrm{d}\lambda,
\end{aligned}
$$

so

$$
K_{\Phi_{(r)}(\sqrt{L})} = -K_{E_L(0)} - \int_0^\infty \frac{r}{2\sqrt{\lambda}}\,\Phi'(r\sqrt{\lambda})\,K_{\chi_{[0,\lambda]}(L)}\,\mathrm{d}\lambda. \qquad (3.4)
$$

Suppose that $D > 0$, so that $E_L(0) = 0$, by Lemma 2.5. We deduce from formulae (3.3) and (3.4), Minkowski's inequality, and the basic Plancherel assumption Assumption 2.5 that

$$
\begin{aligned}
&\left\|\Phi_{(r)}(\sqrt{L})\right\|_{L^1\to L^1} \\
&\quad \le \sup_{y\in X} \int_0^\infty \left|\frac{r}{2\sqrt{\lambda}}\,\Phi'(r\sqrt{\lambda})\right| \\
&\qquad \times \left(V_{d,D}(r) \int_X \left|K_{\chi_{[0,\sqrt{\lambda}]}(\sqrt{L})}(x,y)\right|^2 w(x,y)\,\mathrm{d}\mu(x)\right)^{1/2} \mathrm{d}\lambda
\end{aligned}
$$

$$
\begin{aligned}
&\leq \int_0^\infty \left| \frac{r}{2\sqrt{\lambda}} \, \Phi'(r\sqrt{\lambda}) \right| \left(V_{d,D}(r) V_{D,d}(\sqrt{\lambda}) \right)^{1/2} \mathrm{d}\lambda \\
&= \int_0^\infty \left| \Phi'(rs) \right| \left(V_{d,D}(r) \, V_{D,d}(s) \right)^{1/2} r \, \mathrm{d}s,
\end{aligned}
$$

by a change of variable. If $r \leq 1$, then this is at most

$$
\begin{aligned}
&\int_0^1 \left| \Phi'(rs) \right| r^{d/2} s^{D/2} r \, \mathrm{d}s + \int_1^\infty \left| \Phi'(rs) \right| r^{d/2} s^{d/2} r \, \mathrm{d}s \\
&\quad \leq \int_0^1 \left| \Phi'(rs) \right| (rs)^{\min(d/2,D/2)} r \, \mathrm{d}s \\
&\qquad + \int_1^\infty \left| \Phi'(rs) \right| (rs)^{d/2} r \, \mathrm{d}s \\
&\quad \leq \int_0^\infty \left| \Phi'(rs) \right| (rs)^{\min(d/2,D/2)} r \, \mathrm{d}s \\
&\qquad + \int_0^\infty \left| \Phi'(rs) \right| (rs)^{\max(d/2,D/2)} r \, \mathrm{d}s \\
&\quad = \int_0^\infty \left| \Phi'(t) \right| \left(t^{d/2} + t^{D/2} \right) \mathrm{d}t \\
&\quad < \infty,
\end{aligned}
$$

while if $r \geq 1$, then we can show similarly that the same bound holds. Thus (3.2) holds in this case.

On the other hand, if $D = 0$, then $\Phi_{(r)}(\sqrt{L})$ involves an extra term, namely, $E_L(0)$, which is bounded on $L^1(X)$ by Lemma 2.5, and combining this with the previous argument proves (3.2) in this case too.

To finish the proof of the lemma, take b in $L^1(X)$ supported in $B(x,r)$, and let Ψ be the function $x \mapsto (1+x^2)^k \Phi(x)$. Then $\Psi_{(r)}(\sqrt{L}) = (1 + r^2 L)^k \Phi_{(r)}(\sqrt{L})$. The argument to prove (3.2) also shows that

$$
\left\| (1 + r^2 L)^k \Phi_{(r)}(\sqrt{L}) \right\|_{L^1 \to L^1} = \left\| \Psi_{(r)}(\sqrt{L}) \right\|_{L^1 \to L^1} \leq C,
$$

so from Assumption 2.4, we deduce that

$$
\left\| \Phi_{(r)}(\sqrt{L}) b \right\|_{L^\infty} \leq C \mu\big(B(x, 2r)\big)^{-1} \left\| b \right\|_{L^1} .
$$

The general result follows from Hölder's inequality. □

We now recall the Calderón–Zygmund decomposition.

Theorem 3.2. *There exist constants C and k such that, for all f in $L^1(X)$ and λ in $\mathbf{R}^+$ such that $\lambda \, \mu(X) > \int_X |f| \, \mathrm{d}\mu$, there exists a sequence of balls $\{B(x_n, r_n) \colon n \in \mathbf{N}\}$ and a decomposition of f:*

$$f = g + b = g + \sum_{n\in\mathbf{N}} b_n$$

such that

(a) $\|g\|_{L^1} \le C\,\|f\|_{L^1}$
(b) $\|g\|_{L^\infty} \le C\,\lambda$
(c) $\operatorname{supp} b_n \subseteq B(x_n, r_n)$ *for all* n *in* $\mathbf{N}$
(d) $\int_X |b_n| \,\mathrm{d}\mu \le C\lambda\,\mu\big(B(x,r)\big)$ *for all* n *in* $\mathbf{N}$
(e) $\sum_{n\in\mathbf{N}} \mu\big(B(x_n, r_n)\big) \le C\,\lambda^{-1} \int_X |f| \,\mathrm{d}\mu$
(f) $\sum_{n\in\mathbf{N}} \chi_{B(x_n, 2r_n)} \le k.$

The proof is a variant of the standard arguments, for which see, e.g., [10, p. 66] or [34, p. 8], and we omit it. The parameter λ is called the *level* of the decomposition.

To prove that an operator is of weak type $(1,1)$, we usually use estimates for the gradient of the kernel. The following theorem replaces the gradient estimates in our setting (see [11, 14, 15, 17] for other variants of this).

Theorem 3.3. *Suppose that* F *is in* $\mathcal{B}(\mathbf{R})$, *that* $\|F\|_{L^\infty} \le A$, *and that*

$$\sup_{r\in\mathbf{R}^+}\sup_{y\in X}\int_{B(y,r)^c} \left| K_{F(1-\Phi_{(r)})(\sqrt{L})}(x,y)\right| \,\mathrm{d}\mu(x) \le A. \tag{3.5}$$

Then

$$\left\| F(\sqrt{L})\right\|_{L^1\to L^{1,\infty}} \le CA.$$

Proof. It is enough to prove that

$$\mu\left(\left\{x : \left|F(\sqrt{L})f(x)\right| \ge 3\lambda\right\}\right) \le CA\,\frac{\|f\|_{L^1}}{\lambda}$$

for all f in $L^1(X)$ and λ in $\mathbf{R}^+$ such that $\lambda\,\mu(X) > A\,\int_X |f| \,\mathrm{d}\mu$.

Fix such an f and λ, and let $\{B(x_n, r_n) : n \in \mathbf{N}\}$ and $f = g + \sum_{n\in\mathbf{N}} b_n$ be the corresponding Calderón–Zygmund sequence of balls and decomposition of f at level λ/A. We define the "nearly good" and "very bad" functions $\tilde{g}$ and $\tilde{b}$ by

$$\tilde{g} = \sum_{n\in\mathbf{N}} \Phi_{(r_n)}(\sqrt{L})b_n \qquad\text{and}\qquad \tilde{b} = \sum_{n\in\mathbf{N}} \big(b_n - \Phi_{(r_n)}(\sqrt{L})b_n\big).$$

Then $f = g + \tilde{g} + \tilde{b}$, so $\left\{x : \left|F(\sqrt{L})f(x)\right| \ge 3\lambda\right\}$ is a subset of

$$\begin{aligned}&\left\{x : \left|F(\sqrt{L})g(x)\right| \ge \lambda\right\} \cup \left\{x : \left|F(\sqrt{L})\tilde{g}(x)\right| \ge \lambda\right\}\\ &\qquad\qquad \cup \left\{x : \left|F(\sqrt{L})\tilde{b}(x)\right| \ge \lambda\right\}.\end{aligned} \tag{3.6}$$

To estimate the measure of the first set, recall that $F(\sqrt{L})$ is bounded on L^2, by spectral theory. Thus, by the Chebyshev inequality,

$$\mu\left(\left\{x : \left|F(\sqrt{L})g(x)\right| \geq \lambda\right\}\right) \leq \frac{\left\|F(\sqrt{L})g\right\|_{L^2}^2}{\lambda^2} \\ \leq \frac{\|F\|_{L^\infty}^2 \|g\|_{L^2}^2}{\lambda^2} \\ \leq CA \, \frac{\|f\|_{L^1}}{\lambda},$$

since $\|g\|_{L^2}^2 \leq C\,\lambda\,\|f\|_{L^1}/A$.

To deal with the set involving $\tilde{g}$ similarly, it will suffice to show that

$$\Big\|\sum_{n\in\mathbf{N}} \Phi_{(r_n)}(\sqrt{L})b_n\Big\|_{L^2}^2 \leq C\,\lambda\,\frac{\|f\|_{L^1}}{A}.$$

Now by Lemma 3.1, $\operatorname{supp}\Phi_{(r_n)}(\sqrt{L})b_n \subseteq B(x_n, 2r_n)$, and by the Calderón–Zygmund decomposition, no point of X belongs to more than k balls $B(x_n, 2r_n)$. Thus, by Lemma 3.1,

$$\Big\|\sum_{n\in\mathbf{N}} \Phi_{(r_n)}(\sqrt{L})b_n\Big\|_{L^2}^2 \leq k\sum_{n\in\mathbf{N}} \Big\|\Phi_{(r_n)}(\sqrt{L})b_n\Big\|_{L^2}^2 \\ \leq C\sum_{n\in\mathbf{N}} \frac{\|b_n\|_{L^1}^2}{\mu\big(B(x_n, r_n)\big)} \\ \leq C'\,\frac{\lambda}{A}\sum_{n\in\mathbf{N}} \|b_n\|_{L^1} \\ \leq C''\lambda\,\frac{\|f\|_{L^1}}{A},$$

as required.

It remains to deal with the third term in (3.6). Now

$$\mu\bigg(\bigg\{x : \Big|\sum_{n\in\mathbf{N}} F(1-\Phi_{(r_n)})(\sqrt{L})b_n(x)\Big| \geq \lambda\bigg\}\bigg) \\ \leq \sum_{n\in\mathbf{N}} \mu\big(B(x_n, 2r_n)\big) \\ \quad + \mu\bigg(\bigg\{x : \Big|\sum_{n\in\mathbf{N}} F(1-\Phi_{(r_n)})(\sqrt{L})b_n(x)\Big| \geq \lambda\bigg\} \setminus \bigcup_{n\in\mathbf{N}} B(x_n, 2r_n)\bigg).$$

However, by the properties of the Calderón–Zygmund decomposition and hypothesis (3.5),

$$\sum_{n\in\mathbf{N}} \mu\big(B(x_n, 2r_n)\big) \leq CA\,\frac{\|f\|_{L^1}}{\lambda},$$

and

$$\mu\Big(\Big\{x : \Big|\sum_{n\in\mathbf{N}} F(1-\Phi_{(r_n)})(\sqrt{L})b_n(x)\Big| \geq \lambda\Big\} \setminus \bigcup_{n\in\mathbf{N}} B(x_n, 2r_n)\Big)$$

$$\leq \frac{1}{\lambda}\int_{X\setminus\bigcup_{n\in\mathbf{N}} B(x_n,2r_n)} \Big|\sum_{n\in\mathbf{N}} F(1-\Phi_{(r_n)})(\sqrt{L})b_n(x)\Big|\, \mathrm{d}\mu(x)$$

$$\leq \frac{1}{\lambda}\sum_{n\in\mathbf{N}} \int_{X\setminus B(x_n,2r_n)} \Big|F(1-\Phi_{(r_n)})(\sqrt{L})b_n(x)\Big|\, \mathrm{d}\mu(x)$$

$$\leq \frac{1}{\lambda}\sum_{n\in\mathbf{N}} \|b_n\|_{L^1} \sup_{y\in X} \int_{B(y,r_n)^c} \Big|K_{F(1-\Phi_{(r_n)})(\sqrt{L})}(x,y)\Big|\, \mathrm{d}\mu(x)$$

$$\leq CA\,\frac{\|f\|_{L^1}}{\lambda},$$

as required. □

The next step is to estimate the expression (3.5). A key reduction in the difficulty of the problem can be effected using the finite propagation speed hypothesis (Assumption 2.3) and Fourier analysis. To formulate this, we recall the definition of the Besov space $B^{p,q}_s(\mathbf{R})$ from Sect. 2.8 and the mixed norm $\|\cdot\|_{N,p}$ from (2.5).

Lemma 3.4. *Suppose that* $w\colon X\times X \to \mathbf{R}^+$ *is nonnegative, and that* L *satisfies Assumption 2.3 (the finite propagation speed property).*
(a) If

$$\sup_{y\in X}\Big(\int_X \Big|K_{F(\sqrt{L})}(x,y)\Big|^2 w(x,y)\,\mathrm{d}\mu(x)\Big)^{1/2} \leq C\, V_{D,d}(R)^{1/2}\,\big\|F_{(R)}\big\|_{L^p}, \tag{3.7}$$

for all R *in* $\mathbf{R}^+$ *and all* F *in* $\mathcal{B}_R(\mathbf{R})$*, then for all* s *in* $\mathbf{R}^+$ *there exists a constant* C_s *such that*

$$\sup_{y\in X}\Big(\int_{B(y,r)^c} \Big|K_{F(\sqrt{L})}(x,y)\Big|^2 w(x,y)\,\mathrm{d}\mu(x)\Big)^{1/2} \leq C_s\frac{V_{D,d}(R)^{1/2}}{(1+rR)^s}\,\big\|F_{(R)}\big\|_{B^{p,\infty}_s} \tag{3.8}$$

for all r *and* R *in* $\mathbf{R}^+$ *and all* F *in* $\mathcal{B}_R(\mathbf{R})$*.*
(b) If

$$\sup_{y\in X}\Big(\int_X \Big|K_{F(\sqrt{L})}(x,\cdot\,)\Big|^2 w(x,y)\,\mathrm{d}\mu(x)\Big)^{1/2}$$

$$\leq C\,N^{d/2}\left\|F_{(N)}\right\|_{N,p} \tag{3.9}$$

for all N *in* $\mathbf{Z}^+$ *and all* F *in* $\mathcal{B}_N(\mathbf{R})$*, then for all even functions* ξ *in* $\mathcal{S}(\mathbf{R})$ *supported in* $[-1,1]$ *and all* s *in* $\mathbf{R}^+$ *there exists a constant* $C_{s,\xi}$ *such that*

$$\begin{aligned}\sup_{y\in X}&\left(\int_{B(y,r)^c}\left|K_{\xi*F(\sqrt{L})}(x,y)\right|^2 w(x,y)\,\mathrm{d}\mu(x)\right)^{1/2}\\ &\leq C_{s,\xi}\,\frac{N^{d/2}}{(1+rN)^s}\left\|F_{(N)}\right\|_{B_s^{p,\infty}}\end{aligned} \tag{3.10}$$

for all r *in* $\mathbf{R}^+$*, all* N *in* $\mathbf{Z}^+$*, and all* F *in* $\mathcal{B}_N(\mathbf{R})$*.*

Remark. Observe that (2.6) implies (3.7) where $w=1$ and $p=\infty$. Other Plancherel type inequalities imply other forms of (3.7) or (3.9).

Observe also that hypothesis (3.9) is a slightly weaker version of hypothesis (3.7). The price we pay for the weaker hypothesis is a weaker conclusion, in as much as F is replaced by $\xi * F$; this effectively damps the kernel of the corresponding operator far away from the diagonal.

Proof. To prove (3.8), we fix r and R, such that $rR>1$, for otherwise the result is trivial. Recall that ϕ_0 and ϕ_j in $\mathcal{S}(\mathbf{R})$ are $[0,1]$-valued even functions supported in $(-4,4)$ and $[2^j,2^{j+2}]\cup[-2^{j+2},-2^j]$ respectively. Further, $\phi_0(\lambda)+\sum_{k\in\mathbf{Z}^+}\phi_k(\lambda)=1$ in $\mathbf{R}$, and $\phi_0=1$ on $[-2,2]$. We define ψ to be $\phi_0(\,\cdot\,/4r)$ and ψ_0 to be $\phi_0(\,\cdot\,/4rR)$. Define T_{ϕ_j} by (2.10) and T_ψ and T_{ψ_0} analogously, e.g., $(T_\psi F)\widehat{\ } = \psi\widehat{F}$.

Take F in $\mathcal{B}_R(\mathbf{R})$. First, $\operatorname{supp}\psi\subseteq[-r,r]$, so from Lemma 2.1,

$$\operatorname{supp}K_{T_\psi F(\sqrt{L})}\subseteq\{(x,y)\in X\times X:\rho(x,y)\leq r\}\,,$$

hence

$$K_{F(\sqrt{L})}(x,y)=K_{[F-T_\psi F](\sqrt{L})}(x,y)$$

for all x,y such that $\rho(x,y)>r$, and so

$$\begin{aligned}&\left(\int_{B(y,r)^c}\left|K_{F(\sqrt{L})}(x,y)\right|^2 w(x,y)\,\mathrm{d}\mu(x)\right)^{1/2}\\ &\qquad\leq\left(\int_X\left|K_{[F-T_\psi F](\sqrt{L})}(x,y)\right|^2 w(x,y)\,\mathrm{d}\mu(x)\right)^{1/2}.\end{aligned} \tag{3.11}$$

Now

$$F-T_\psi F=\sum_{j\in\mathbf{N}}[\phi_j]_{(R^{-1})}[F-T_\psi F] \tag{3.12}$$

$$= [\phi_0]_{(R^{-1})}[F - T_\psi F] - \sum_{j\in\mathbf{Z}^+} [\phi_j]_{(R^{-1})} T_\psi F,$$

since if $j \geq 1$, then $\operatorname{supp}[\phi_j]_{(R^{-1})} \subseteq [-2^{j+2}R, -2^jR] \cup [2^jR, 2^{j+2}R]$, and $\operatorname{supp}(F) \subseteq [-R, R]$, so that $[\phi_j]_{(R^{-1})}F = 0$. It follows that

$$\begin{aligned}
&\left(\int_{B(y,r)^c} \left|K_{F(\sqrt{L})}(x,y)\right|^2 w(x,y)\,\mathrm{d}\mu(x)\right)^{1/2} \\
&\leq \left(\int_X \left|K_{[\phi_0]_{(R^{-1})}[F-T_\psi F](\sqrt{L})}(x,y)\right|^2 w(x,y)\,\mathrm{d}\mu(x)\right)^{1/2} \\
&\qquad + \sum_{j\in\mathbf{Z}^+} \left(\int_X \left|K_{[\phi_j]_{(R^{-1})}T_\psi F(\sqrt{L})}(x,y)\right|^2 w(x,y)\,\mathrm{d}\mu(x)\right)^{1/2}.
\end{aligned} \tag{3.13}$$

To deal with the first term, recall that $\operatorname{supp}(\phi_0) \subseteq [-4,4]$, so that $\operatorname{supp}[\phi_0]_{(R^{-1})} \subseteq [-4R, 4R]$, and by hypothesis (3.7),

$$\begin{aligned}
&\left(\int_X \left|K_{[\phi_0]_{(R^{-1})}[F-T_\psi F](\sqrt{L})}(x,y)\right|^2 w(x,y)\,\mathrm{d}\mu(x)\right)^{1/2} \\
&\qquad \leq C\,V_{D,d}(4R)^{1/2} \left\|\phi_0[F - T_\psi F]_{(R)}\right\|_{L^p} \\
&\qquad \leq C\,V_{D,d}(4R)^{1/2} \left\|F_{(R)} - T_{\psi_0}F_{(R)}\right\|_{L^p} \\
&\qquad = C\,V_{D,d}(4R)^{1/2} \left\|\sum_{n\in\mathbf{N}} T_{\phi_n}[I - T_{\psi_0}]F_{(R)}\right\|_{L^p}.
\end{aligned}$$

Now $\phi_n[1 - \psi_0] = \phi_n[1 - \phi_0(\,\cdot\,/4rR)]$, and this is zero unless $2^n \geq 2rR$. Consequently, $T_{\phi_n}[I - T_{\psi_0}]F_{(R)} = 0$ unless $n \geq N_0$, where $N_0 = \log_2(2rR)$, and

$$\begin{aligned}
&\left(\int_X \left|K_{[\phi_0]_{(R^{-1})}[F-T_\psi F](\sqrt{L})}(x,y)\right|^2 w(x,y)\,\mathrm{d}\mu(x)\right)^{1/2} \\
&\qquad \leq C\,V_{D,d}(4R)^{1/2} \sum_{n\geq N_0} \left\|T_{\phi_n}[I - T_{\psi_0}]F_{(R)}\right\|_{L^p} \\
&\qquad \leq C'\,V_{D,d}(R)^{1/2} \sum_{n\geq N_0} \left\|T_{\phi_n}F_{(R)}\right\|_{L^p} \\
&\qquad \leq C'\,V_{D,d}(R)^{1/2}\, 2^{-N_0 s} \sum_{n\geq N_0} 2^{ns} \left\|T_{\phi_n}F_{(R)}\right\|_{L^p} \\
&\qquad \leq C''\,V_{D,d}(R)^{1/2}\,(rR)^{-s} \left\|F_{(R)}\right\|_{B_s^{p,\infty}}.
\end{aligned} \tag{3.14}$$

Now we treat the summed term in formula (3.13). Since

$$\operatorname{supp}([\phi_j]_{(R^{-1})}T_\psi F) \subseteq [-2^{j+2}R, 2^{j+2}R],$$

Hypothesis (3.7) implies that

$$
\begin{aligned}
&\left(\int_X \left|K_{[\phi_j]_{(R^{-1})}T_\psi F(\sqrt{L})}(x,y)\right|^2 w(x,y)\,\mathrm{d}\mu(x)\right)^{1/2} \\
&\qquad \le C\,V_{D,d}(2^{j+2}R)^{1/2}\left\|[[\phi_j]_{(R^{-1})}T_\psi F]_{(2^{j+2}R)}\right\|_{L^p} \\
&\qquad \le C\,V_{D,d}(2^{j+2}R)^{1/2}\left\|[\phi_j T_{\psi_0}F_{(R)}]_{(2^{j+2})}\right\|_{L^p} \\
&\qquad \le C\,V_{D,d}(R)^{1/2}\,2^{\max(d,D)(j+2)/2}\,2^{-(j+2)/p}\left\|\phi_j T_{\psi_0}F_{(R)}\right\|_{L^p}.
\end{aligned}
\tag{3.15}
$$

Choose l such that $l > \max(d,D,2s)/2+1$. Then, since ϕ_0 is in $\mathcal{S}(\mathbf{R})$, there exists a constant C_l such that $\widehat{\phi_0}(s) \le C_l(1+|s|)^{-l}$ for all s in $\mathbf{R}$, and so

$$
\widehat{\psi_0}(s) \le \frac{4rR\,C_l}{(1+4rR|s|)^l} \qquad \forall s\in\mathbf{R}.
$$

Thus, if $t\ge 2$, then

$$
\begin{aligned}
\left|T_{\psi_0}F_{(R)}(t)\right| &\le \int_{\mathbf{R}} \left|F_{(R)}(t-s)\right| \frac{4rR\,C_l}{(1+4rR|s|)^l}\,\mathrm{d}s \\
&\le 2C_l\left\|F_{(R)}\right\|_{L^1}\frac{4rR}{(1+4rR|t-1|)^l} \\
&\le 4C_l\left\|F_{(R)}\right\|_{L^p}\frac{1}{(1+4rR|t-1|)^{l-1}}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\left\|\phi_j T_{\psi_0}F_{(R)}\right\|_{L^p} &\le 4C_l\left\|F_{(R)}\right\|_{L^p}\left(\int_{2^j}^{2^{j+2}}\frac{1}{(1+4rR|t-1|)^{p(l-1)}}\,\mathrm{d}t\right)^{1/p} \\
&\le 4C_l\left\|F_{(R)}\right\|_{L^p}3^{1/p}\,2^{j/p}\frac{1}{(1+4rR|2^j-1|)^{l-1}}.
\end{aligned}
\tag{3.16}
$$

Combining estimates (3.15) and (3.16) and the fact that $l > \max(d,D,2s)/2$ $+\,1$, we conclude that

$$
\begin{aligned}
&\sum_{j\in\mathbf{Z}^+}\left(\int_X \left|K_{[\phi_j T_{\psi_0}F_{(R)}]_{(R^{-1})}(\sqrt{L})}(x,y)\right|^2 w(x,y)\,\mathrm{d}\mu(x)\right)^{1/2} \\
&\quad \le C\,V_{D,d}(R)^{1/2}\sum_{j\in\mathbf{Z}^+}2^{\max(d,D)(j+2)/2}\,2^{-(j+2)/p}\left\|\phi_j T_{\psi_0}F_{(R)}\right\|_{L^p} \\
&\quad \le 4C\,C_l\,(3/4)^{1/p}\,V_{D,d}(R)^{1/2}\left\|F_{(R)}\right\|_{L^p} \\
&\qquad \times \sum_{j\in\mathbf{Z}^+}\frac{2^{\max(d,D)(j+2)/2}}{(1+4rR|2^j-1|)^{l-1}}
\end{aligned}
$$

$$\le C' \left\| F_{(R)} \right\|_{L^p} V_{D,d}(R)^{1/2} \, (rR)^{-s}. \tag{3.17}$$

Combining estimates (3.13), (3.14) and (3.17) proves (3.8).

To prove (3.10), we note that if $\operatorname{supp} F \subseteq [-N, N]$ then $\operatorname{supp}(\xi * F) \subseteq [-N-1, N+1]$. Further,

$$|\xi * F(\lambda)| \le \|\xi\|_{L^{p'}} \left(\int_{\lambda-1}^{\lambda+1} \left| F(\lambda') \right|^p \, \mathrm{d}\lambda' \right)^{1/p}, \tag{3.18}$$

so

$$\begin{aligned}
&\left\| (\xi * F)_{(N+1)} \right\|_{N+1,p} \\
&\quad = \left(\frac{1}{N+1} \sum_{i=1}^{N+1} \sup_{\lambda \in [\frac{i-1}{N+1}, \frac{i}{N+1}]} |\xi * F((N+1)\lambda)|^p \right)^{1/p} \\
&\quad \le \frac{\|\xi\|_{L^{p'}}}{(N+1)^{1/p}} \left(\sum_{i=1}^{N+1} \int_{i-2}^{i+1} \left| F(\lambda') \right|^p \, \mathrm{d}\lambda' \right)^{1/p} \\
&\quad \le \frac{\|\xi\|_{L^{p'}} \, (3N)^{1/p}}{(N+1)^{1/p}} \left\| F_{(N)} \right\|_{L^p} \\
&\quad \le C \left\| F_{(N)} \right\|_{L^p}. \qquad (3.19)
\end{aligned}$$

Then hypothesis (3.9) implies that

$$\left(\int_X \left| K_{\xi * F(\sqrt{L})}(x, \cdot) \right|^2 w(x,y) \, \mathrm{d}\mu(x) \right)^{1/2} \le C \, N^{d/2} \left\| F_{(N)} \right\|_{L^p} \tag{3.20}$$

for all positive integers N. In order to prove (3.10), we may assume that $rN > 1$, since otherwise the matter is again trivial. We now repeat the proof of (3.8), with F replaced by $\xi * F$ on the left hand side, and R replaced by N.

The argument leading to (3.11) shows that

$$\begin{aligned}
&\left(\int_{B(y,r)^c} \left| K_{\xi * F(\sqrt{L})}(x,y) \right|^2 w(x,y) \, \mathrm{d}\mu(x) \right)^{1/2} \\
&\qquad \le \left(\int_X \left| K_{\xi * [F - T_\psi F](\sqrt{L})}(x,y) \right|^2 w(x,y) \, \mathrm{d}\mu(x) \right)^{1/2}.
\end{aligned}$$

The analogue of formula (3.12) is

$$\begin{aligned}
\xi * [F - T_\psi F] &= \xi * ([\phi_0]_{(N^{-1})} [F - T_\psi F]) \\
&\qquad - \sum_{j \in \mathbf{Z}^+} \xi * ([\phi_j]_{(N^{-1})} [T_\psi F]),
\end{aligned}$$

and so

$$\Bigl(\int_{B(y,r)^c} \Bigl|K_{\xi * F(\sqrt{L})}(x,y)\Bigr|^2 w(x,y)\,\mathrm{d}\mu(x)\Bigr)^{1/2}$$
$$\le \Bigl(\int_X \Bigl|K_{\xi*([\phi_0]_{(N^{-1})}[F-T_\psi F])(\sqrt{L})}(x,y)\Bigr|^2 w(x,y)\,\mathrm{d}\mu(x)\Bigr)^{1/2}$$
$$+ \sum_{j\in\mathbf{Z}^+} \Bigl(\int_X \Bigl|K_{\xi*([\phi_j]_{(N^{-1})}T_\psi F)(\sqrt{L})}(x,y)\Bigr|^2 w(x,y)\,\mathrm{d}\mu(x)\Bigr)^{1/2}. \tag{3.21}$$

To deal with the first term, observe that $\mathrm{supp}([\phi_0]_{(N^{-1})}[F - T_\psi F]) \subseteq [-4N, 4N]$, so that, by estimate (3.20),

$$\Bigl(\int_X \Bigl|K_{\xi*([\phi_0]_{(N^{-1})}[F-T_\psi F])(\sqrt{L})}(x,y)\Bigr|^2 w(x,y)\,\mathrm{d}\mu(x)\Bigr)^{1/2}$$
$$\le C\,N^{d/2}\,\bigl\|\phi_0[F-T_\psi F]_{(N)}\bigr\|_{L^p}$$
$$\le C'\,N^{d/2}\,(rN)^{-s}\,\bigl\|F_{(N)}\bigr\|_{B_s^{p,\infty}}\,,$$

by the argument leading to (3.14).

Now we treat the summed term in formula (3.21). Since

$$\mathrm{supp}([\phi_j]_{(N^{-1})}T_\psi F) \subset [-2^{j+2}N, 2^{j+2}N],$$

the estimate (3.20) implies that

$$\Bigl(\int_X \Bigl|K_{\xi*([\phi_j]_{(N^{-1})}T_\psi F)(\sqrt{L})}(x,y)\Bigr|^2 w(x,y)\,\mathrm{d}\mu(x)\Bigr)^{1/2}$$
$$\le C\,(2^{j+2}N)^{d/2}\,\bigl\|[\phi_j T_{\psi_0} F_{(N)}]_{(2^{j+2})}\bigr\|_{L^p}\,.$$

These terms may be estimated and summed as before, and (3.10) follows. This ends the proof of Lemma 3.4. □

Now we show how to use Theorem 3.3 and Lemma 3.4 to prove a general multiplier theorem. Assumptions 2.1, 2.2, 2.3 and 2.4 are all needed.

Theorem 3.5. *Suppose that* $s > \max(d, D)/2$, *and that*

$$\sup_{y\in X}\Bigl(\int_X \Bigl|K_{F(\sqrt{L})}(x,y)\Bigr|^2 w(x,y)\,\mathrm{d}\mu(x)\Bigr)^{1/2}$$
$$\le C\,V_{D,d}(R)^{1/2}\,\bigl\|F_{(R)}\bigr\|_{L^p}\,, \tag{3.22}$$

for all R *in* $\mathbf{R}^+$ *and all* F *in* $\mathcal{B}_R(\mathbf{R})$. *Then for all bounded Borel functions* F *such that* $\sup_{t\in\mathbf{R}^+}\bigl\|\eta\,F_{(t)}\bigr\|_{B_s^{p,\infty}} < \infty$, *the operator* $F(\sqrt{L})$ *is of weak type* $(1,1)$ *and is bounded on* $L^r(X)$ *for all* r *in* $(1,\infty)$; *further,*

$$\Bigl\|F(\sqrt{L})\Bigr\|_{L^1\to L^{1,\infty}} \le C\Bigl(\sup_{t\in\mathbf{R}^+}\bigl\|\eta\,F_{(t)}\bigr\|_{B_s^{p,\infty}} + \|F\|_{L^\infty}\Bigr). \tag{3.23}$$

Remark. If we take p equal to ∞ and w equal to 1, then we obtain Alexopoulos' multiplier theorem. Indeed, Assumption 2.5 is (3.22) with p equal to ∞. Recall that the Lipschitz space Λ_s is included in the Besov space $B_s^{\infty,\infty}$, so that our result implies the result formulated in [3].

Proof. By the remark at the end of Sect. 2.1, it suffices to prove the weak type $(1,1)$ estimate (3.23). In light of Theorem 3.3, it suffices to prove that

$$\sup_{y\in X}\int_{B(y,r)^c}\left|K_{F(1-\Phi_{(r)})(\sqrt{L})}(x,y)\right|\,\mathrm{d}\mu(x)\leq C\sup_{t\in\mathbf{R}^+}\left\|\eta\,F_{(t)}\right\|_{B_s^{p,q}}.$$

By Lemma 3.4,

$$\begin{aligned}&\left(\int_{B(y,r)^c}\left|K_{F(\sqrt{L})}(x,y)\right|^2 w(x,y)\,\mathrm{d}\mu(x)\right)^{1/2}\\ &\leq C_s\,\frac{V_{D,d}(R)^{1/2}}{(1+rR)^s}\,\left\|F_{(R)}\right\|_{B_s^{p,q}}\end{aligned}\tag{3.24}$$

for all r and R in $\mathbf{R}^+$, all y in X, and all F in $\mathcal{B}_R(\mathbf{R})$. Our first step is to show that

$$\begin{aligned}&\int_{B(y,r)^c}\left|K_{F(\sqrt{L})}(x,y)\right|\,\mathrm{d}\mu(x)\\ &\leq C_s\,(1+rR)^{\max(d,D)/2-s}\,\left\|F_{(R)}\right\|_{B_s^{p,q}}\end{aligned}\tag{3.25}$$

for all r and R in $\mathbf{R}^+$, all y in X, and all F in $\mathcal{B}_R(\mathbf{R})$. To prove this, we first suppose that $rR>1$. Fix y in X and write A_k for the annulus $\left\{x\in X : 2^k r<\rho(x,y)\leq 2^{k+1}r\right\}$. Then, by the Cauchy–Schwarz inequality and the definition of (d,D) regular weights (2.1),

$$\begin{aligned}&\int_{B(y,r)^c}\left|K_{F(\sqrt{L})}(x,y)\right|\,\mathrm{d}\mu(x)\\ &\leq\sum_{k\in\mathbf{N}}\int_{A_k}\left|K_{F(\sqrt{L})}(x,y)\right|\,\mathrm{d}\mu(x)\\ &\leq\sum_{k\in\mathbf{N}}\left(\int_{A_k}w^{-1}(x,y)\,\mathrm{d}\mu(x)\right)^{1/2}\\ &\quad\times\left(\int_{A_k}\left|K_{F(\sqrt{L})}(x,y)\right|^2 w(x,y)\,\mathrm{d}\mu(x)\right)^{1/2}\\ &\leq\sum_{k\in\mathbf{N}}\left(V_{d,D}(2^{k+1}r)\int_{B(y,2^k r)^c}\left|K_{F(\sqrt{L})}(x,y)\right|^2 w(x,y)\,\mathrm{d}\mu(x)\right)^{1/2}.\end{aligned}$$

Now

$$V_{d,D}(2^{k+1}r)=V_{d,D}(2^{k+1}rR/R)$$

$$\leq (2^{k+1}rR)^{\max(d,D)}\, V_{d,D}(R^{-1}) = (2^{k+1}rR)^{\max(d,D)}\, V_{D,d}(R)^{-1},$$

and so, from the last two inequalities and (3.24),

$$\begin{aligned}\int_{B(y,r)^c} &\left|K_{F(\sqrt{L})}(x,y)\right| \,\mathrm{d}\mu(x) \\ &\leq C_s \sum_{k\in\mathbf{N}} \frac{(2^{k+1}rR)^{\max(d,D)/2}}{(1+2^k rR)^s} \left\|F_{(R)}\right\|_{B^{p,q}_s} \\ &\leq C'_s\,(1+rR)^{\max(d,D)/2-s} \left\|F_{(R)}\right\|_{B^{p,q}_s},\end{aligned}$$

proving (3.25) in this case. When $rR \leq 1$, we define the annuli A_k using R^{-1} instead of r, and write A for the set $\left\{x \in X : r < \rho(x,y) \leq R^{-1}\right\}$. Then

$$\begin{aligned}&\int_{B(y,r)^c} \left|K_{F(\sqrt{L})}(x,y)\right| \,\mathrm{d}\mu(x) \\ &\leq \int_A \left|K_{F(\sqrt{L})}(x,y)\right| \,\mathrm{d}\mu(x) + \sum_{k\in\mathbf{N}} \int_{A_k} \left|K_{F(\sqrt{L})}(x,y)\right| \,\mathrm{d}\mu(x);\end{aligned} \tag{3.26}$$

the additional integral is treated in the same way as the integrals over the annuli A_k and the general case of (3.25) follows.

Choose an even function ω in $\mathcal{S}(\mathbf{R})$ supported in $[1/4, 1] \cup [-1/4, -1]$ such that

$$\sum_{n\in\mathbf{Z}} \omega(2^n\lambda) = 1 \qquad \forall \lambda \in \mathbf{R}^+,$$

and let ω_n denote the function $\omega(2^{-n}\cdot)$. Then

$$F(1-\Phi_{(r)})(\sqrt{L}) = \sum_{n\in\mathbf{Z}} \omega_n F(1-\Phi_{(r)})(\sqrt{L}).$$

From (3.25),

$$\begin{aligned}\sup_{y\in X} \int_{B(y,r)^c} &\left|K_{F(1-\Phi_{(r)})(\sqrt{L})}(x,y)\right| \,\mathrm{d}\mu(x) \\ &\leq \sum_{n\in\mathbf{Z}} \sup_{y\in X} \int_{B(y,r)^c} \left|K_{\omega_n F(1-\Phi_{(r)})(\sqrt{L}))}(x,y)\right| \,\mathrm{d}\mu(x) \\ &\leq C_s \sum_{n\in\mathbf{Z}} (1+2^n r)^{\max(d,D)/2-s} \left\|\omega_n F(1-\Phi_{(r)})_{(2^n)}\right\|_{B^{p,q}_s}.\end{aligned}$$

Now for any Besov space $B^{p,q}_s(\mathbf{R})$, if k is an integer greater than s, then

$$\left\|[\omega_n F(1-\Phi_{(r)})]_{(2^n)}\right\|_{B^{p,q}_s}$$

$$\le C \left\| [\omega_n F]_{(2^n)} \right\|_{B_s^{p,q}} \left\| [1 - \Phi_{(r)}]_{(2^n)} \right\|_{\Lambda_k([1/4,1])}$$

(see [36, Corollary 4.2.2]), and from inequality (3.1),

$$\left\| [1 - \Phi_{(r)}]_{(2^n)} \right\|_{\Lambda_k([1/4,1])} \le C \, \frac{(2^n r)^{k+1}}{1 + (2^n r)^{k+1}}.$$

It follows that

$$\begin{aligned} &\sup_{y \in X} \int_{B(y,r)^c} \left| K_{F(1-\Phi_{(r)})(\sqrt{L})}(x,y) \right| \, \mathrm{d}\mu(x) \\ &\qquad \le C \sum_{n \in \mathbf{Z}} \frac{(2^n r)^{k+1}}{1 + (2^n r)^{k+1}} \, (1 + 2^n r)^{\max(d,D)/2 - s} \left\| [\omega_n F]_{(2^n)} \right\|_{B_s^{p,q}} \\ &\qquad \le C \sup_{n \in \mathbf{Z}} \left\| [\omega_n F]_{(2^n)} \right\|_{B_s^{p,q}} , \end{aligned}$$

as required to prove the theorem. □

Our next general theorem relates to the case where $D = 0$. Again, Assumptions 2.1, 2.2, 2.3 and 2.4 are all needed.

Theorem 3.6. *Suppose that* $D = 0$, *that* $s > \max(d/2, 1/p)$, *and that* $q \le \min(p, 2)$. *Suppose also that*

$$\left(\int_X \left| K_{F(\sqrt{L})}(x, \cdot\,) \right|^2 w(x,y) \, \mathrm{d}\mu(x) \right)^{1/2} \le C \, N^{d/2} \left\| F_{(N)} \right\|_{N,p} \tag{3.27}$$

for all positive integers N *and all* F *in* $\mathcal{B}_N(\mathbf{R})$. *Then for all bounded Borel functions* F *such that* $\sup_{t \in \mathbf{R}^+} \left\| \eta \, F_{(t)} \right\|_{B_s^{p,q}} < \infty$, *the operator* $F(\sqrt{L})$ *is of weak type* $(1,1)$ *and is bounded on* $L^r(X)$ *for all* r *in* $(1,\infty)$; *further,*

$$\left\| F(\sqrt{L}) \right\|_{L^1 \to L^{1,\infty}} \le C \left(\sup_{t \in \mathbf{R}^+} \left\| \eta \, F_{(t)} \right\|_{B_s^{p,\infty}} + \|F\|_{L^\infty} \right). \tag{3.28}$$

Remark. In light of (3.27) and (2.3), $F(\sqrt{L})$ is bounded on $L^1(X)$ if F is bounded and $\operatorname{supp} F$ is compact. Thus we can replace the supremum in (3.28) by $\sup_{t > T} \left\| \eta \, F_{(t)} \right\|_{B_s^{p,\infty}}$ for any finite T.

Proof. Without loss of generality, we may assume that $p < \infty$, since otherwise the result is a consequence of the previous theorem. As in the previous theorem, it suffices to prove the weak type $(1,1)$ estimate (3.28). Choose ξ in $\mathcal{S}(\mathbf{R})$ which is even and has support in $[-1,1]$, and such that $\widehat{\xi}(0) = 1$ and $\widehat{\xi}^{(l)}(0) = 0$ if $1 \le l \le k-1$ for some even positive integer k greater than s. By Lemma 3.4,

$$\left(\int_{B(y,r)^c} \left| K_{\xi * F(\sqrt{L})}(x,y) \right|^2 w(x,y) \, \mathrm{d}\mu(x) \right)^{1/2}$$

$$\leq C_s \frac{V_{D,d}(R)^{1/2}}{(1+rR)^s} \left\| F_{(R)} \right\|_{B_s^{p,q}}$$

for all r and R in $\mathbf{R}^+$ and all F in $\mathcal{B}_R(\mathbf{R})$.

By repeating the proof of the previous theorem, we may easily show that

$$\sup_{y\in X} \int_{B(y,r)^c} \left| K_{\xi * F(\sqrt{L})}(x,y) \right| \, \mathrm{d}\mu(x) \leq C_s (1+rR)^{d/2-s} \left\| F_{(R)} \right\|_{B_s^{p,q}}$$

for all r and R in $\mathbf{R}^+$ and all F in $\mathcal{B}_R(\mathbf{R})$, and hence deduce that

$$\left\| \xi * F(\sqrt{L}) \right\|_{L^1 \to L^{1,\infty}} \leq C \sup_{t\in\mathbf{R}^+} \left\| \eta \, F_{(t)} \right\|_{B_s^{p,q}} .$$

To complete the proof, we will show that

$$\left\| F(\sqrt{L}) - \xi * F(\sqrt{L}) \right\|_{L^1 \to L^1} \leq C \sup_{t\in\mathbf{R}^+} \left\| \eta \, F_{(t)} \right\|_{B_s^{p,q}} .$$

Suppose that N is in $\mathbf{Z}^+$, that $\operatorname{supp} G \subseteq [-N, N]$, and that $\left\| G_{(N)} \right\|_{B_s^{p,q}} < \infty$. We claim that $\operatorname{supp}[G - \xi * G] \subseteq [-N-1, N+1]$, and that

$$\left\| [G - \xi * G]_{(N+1)} \right\|_{N+1,p} \leq C \, (N+1)^{-s} \left\| G_{(N+1)} \right\|_{B_s^{p,q}} . \tag{3.29}$$

Assuming this claim for the moment, then the theorem follows. Indeed, write H_n for $\phi_n F - \xi * (\phi_n F)$; then $F - \xi * F = \sum_{n\in\mathbf{N}} H_n$. Since $\operatorname{supp} H_n \subseteq [-2^{n+2}-1, 2^{n+2}+1]$, it follows from (2.3) and (3.27) that

$$\begin{aligned}
\left\| H_n(\sqrt{L}) \right\|_{L^1 \to L^1} &\leq C \sup_{y\in X} \left(\int_X \left| K_{H_n(\sqrt{L})}(x,y) \right|^2 w(x,y) \, \mathrm{d}\mu(x) \right)^{1/2} \\
&\leq C' (2^{n+2}+1)^{d/2} \left\| [H_n]_{(2^{n+2}+1)} \right\|_{2^{n+2}+1,p} ,
\end{aligned}$$

and from our claim it then follows that

$$\begin{aligned}
&\left\| F(\sqrt{L}) - \xi * F(\sqrt{L}) \right\|_{L^1 \to L^1} \\
&\quad \leq \sum_{n\in\mathbf{N}} \left\| H_n(\sqrt{L}) \right\|_{L^1 \to L^1} \\
&\quad \leq \sum_{n\in\mathbf{N}} C' (2^{n+2}+1)^{d/2} \left\| [\phi_n F - \xi * (\phi_n F)]_{(2^{n+2}+1)} \right\|_{2^{n+2}+1,p} \\
&\quad \leq \sum_{n\in\mathbf{N}} C' (2^{n+2}+1)^{d/2} \, (2^{n+2}+1)^{-s} \left\| [\phi_n F]_{(2^{n+2}+1)} \right\|_{B_s^{p,q}} \\
&\quad \leq C \sup_{t\in\mathbf{R}^+} \left\| \eta \, F_{(t)} \right\|_{B_s^{p,q}} ,
\end{aligned}$$

as required.

To prove our claim (3.29), we write ζ for the function on $\mathbf{R}$ defined by the condition that

$$\widehat{\zeta} = \big(1 - \widehat{\xi}\,\big)\, |\cdot|^{-s}\,.$$

Observe first that

$$\Big(\sum_{i\in\mathbf{Z}} \sup_{t\in[i-1,i]} |\zeta * H|^p\Big)^{1/p} \le C\,\|H\|_{L^p} \qquad \forall H \in L^p(\mathbf{R}). \tag{3.30}$$

Indeed, Fourier analysis shows that $|\zeta(t)| \le C_1 |t|^{s-1}$ when $|t| \le 1$ and $|\zeta(t)| \le C_2 |t|^{s-k-1}$ when $|t| \ge 1$. Therefore we may write ζ as $\sum_{j\in\mathbf{Z}} \zeta_j(\cdot - j)$, where $\operatorname{supp}\zeta_j \subseteq [-1,1]$ and $\sum_{j\in\mathbf{Z}} \|\zeta_j\|_{L^{p'}} < \infty$ (this is where we require that $s > 1/p$). The argument of (3.18) and (3.19) then shows that (3.30) holds.

The proof of our claim (3.29) is now straightforward. Indeed,

$$\begin{aligned}
&\big\|[G - \xi * G]_{(N+1)}\big\|_{N+1,p} \\
&\quad = \Big(\frac{1}{N+1} \sum_{i=1}^{N+1} \sup_{t\in[\frac{i-1}{N+1},\frac{i}{N+1}]} |[G - \xi * G]((N+1)t)|^p\Big)^{1/p} \\
&\quad \le (N+1)^{-1/p} \Big(\sum_{i=-\infty}^{\infty} \sup_{t\in[i-1,i]} |\zeta * IG(t)|^p\Big)^{1/p},
\end{aligned}$$

where ζ is as above and $(IG)\widehat{\ } = |\cdot|^s \widehat{G}$. Therefore, by (3.30),

$$\begin{aligned}
\big\|[G - \xi * G]_{(N+1)}\big\|_{N+1,p} &\le C\,(N+1)^{-1/p}\,\|IG\|_{L^p} \\
&\le C\,(N+1)^{-s}\,\big\|I[G_{(N+1)}]\big\|_{L^p} \\
&\le C\,(N+1)^{-s}\,\big\|G_{(N+1)}\big\|_{B_s^{p,q}}\,,
\end{aligned}$$

since $\|IG\|_{L^p} \le C\,\|G\|_{B_s^{p,q}}$ when $q \le \min(p,2)$ (see, e.g., [33, p. 155]). This proves our claim and hence the theorem. □

4. Spectral multipliers on $\mathrm{SU}(2)$

The Euler angles are the usual coordinates on $\mathrm{SU}(2)$. However, to study the operator L defined by (1.2) it is much more convenient to use another coordinate system, which we now describe.

Let B be the ball in $\mathbf{R}^2$ of radius $\pi/2$ and centre 0. For $(x,y,z) \in B \times [-\pi,\pi]$, we write

$$\Psi(x,y,z) = \exp(xX + yY)\exp(zZ),$$

where X, Y, Z are defined by (1.1). Next, we write $x = r\cos\theta$ and $y = r\sin\theta$, and define Φ by the formula

$$\Phi(r,\theta,z) = \Psi(x,y,z).$$

Now we compute the operator L in the coordinates given by Φ. First we note that

$$\exp(xX + yY) = \exp \begin{pmatrix} 0 & e^{i\theta} r \\ -e^{-i\theta} r & 0 \end{pmatrix} = \begin{pmatrix} \cos r & e^{i\theta} \sin r \\ -e^{-i\theta} \sin r & \cos r \end{pmatrix}$$

and

$$\exp(xX + yY) \exp(zZ) = \begin{pmatrix} e^{iz} \cos r & e^{i(\theta - z)} \sin r \\ -e^{-i(\theta - z)} \sin r & e^{-iz} \cos r \end{pmatrix}.$$

Now

$$\begin{aligned} &\exp(xX + yY) \exp(tX) \\ &\quad = \begin{pmatrix} \cos r \cos t - e^{i\theta} \sin r \sin t & \cos r \sin t + e^{i\theta} \sin r \cos t \\ - \cos r \sin t - e^{-i\theta} \sin r \cos t & \cos r \cos t - e^{-i\theta} \sin r \sin t \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} &\exp(xX + yY) \exp(tY) \\ &\quad = \begin{pmatrix} \cos r \cos t + ie^{i\theta} \sin r \sin t & i \cos r \sin t + e^{i\theta} \sin r \cos t \\ i \cos r \sin t - e^{i\theta} \sin r \cos t & \cos r \cos t - ie^{i\theta} \sin r \sin t \end{pmatrix}. \end{aligned}$$

Further,

$$\begin{aligned} &\exp(xX + yY) \exp(zZ) \exp(tX) \\ &\quad = \exp(xX + yY) \exp(tAd_{zZ} X) \exp(zZ). \end{aligned}$$

It follows that

$$\begin{aligned} X &= \cos(-\theta + 2z)\, \partial_r + \sin(-\theta + 2z)(\tan r(\partial_z + \partial_\theta) + \cot r\, \partial_\theta), \\ Y &= - \sin(2z - \theta)\partial_r + \cos(2z - \theta)(\tan r(\partial_z + \partial_\theta) + \cot r\, \partial_\theta). \end{aligned}$$

Thus (see [27]), $X^2 + Y^2$ is equal to

(4.1)
$$\partial_r^2 + (\cot r - \tan r)\, \partial_r + \cot^2 r\, \partial_\theta^2 + 2\partial_\theta(\partial_z + \partial_\theta) + \tan^2 r\, (\partial_z + \partial_\theta)^2.$$

Note that L commutes with ∂_θ. This implies that the convolution kernel $\widetilde{K}_{F(\sqrt{L})}$ associated to a function of the sublaplacian is independent of θ. Further, we also note that $Z = \partial_z$ and that Haar measure $\mathrm{d}g$ is given by the formula

$$\mathrm{d}g = \sin(2r(g))\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}z. \tag{4.2}$$

For future purposes, observe that, for a smooth function ϕ on $[0, \pi/2]$,

$$
\begin{aligned}
&\int_0^{\pi/2} \phi(r)\,\big(\phi''(r) + (\cot r - \tan r)\,\phi'(r)\big)\,\sin(2r)\,\mathrm{d}r \\
&\quad = \int_0^{\pi/2} \big(\phi(r)\,\sin(2r)\big)\,\phi''(r)\,\mathrm{d}r \\
&\qquad +2\int_0^{\pi/2} \phi(r)\,\phi'(r)\,\big(\cos^2 r - \sin^2 r\big)\,\mathrm{d}r \\
&\quad = -\int_0^{\pi/2} \big(\phi(r)\,\sin(2r)\big)'\,\phi'(r)\,\mathrm{d}r \\
&\qquad +2\int_0^{\pi/2} \phi(r)\,\phi'(r)\cos(2r)\,\mathrm{d}r \\
&\quad = -\int_0^{\pi/2} \sin(2r)\,\phi'(r)^2\,\mathrm{d}r \\
&\quad \le 0. \qquad (4.3)
\end{aligned}
$$

We recall briefly the representation theory of $\mathrm{SU}(2)$; see, e.g., [35] or [38] for more details. The action of $\mathrm{SU}(2)$ on $\mathbf{C}^2$ induces an action π_l on the space $\mathcal{H}_l$ of homogeneous polynomials of degree l in two complex variables. The obvious basis for this space is composed of the polynomials $z_1^j z_2^{l-j}$, where $j = 0, 1, \dots, l$. The operator $\mathrm{d}\pi(Z)$ is represented by a diagonal matrix in this basis, with entries $-il, i(2-l), \dots, im, \dots, il$; the integer m is known as a *weight*. The operator $-\mathrm{d}\pi(X^2+Y^2+Z^2)$ acts as the scalar $l(l+2)$ on $\mathcal{H}_l$, whence $\mathrm{d}\pi(L)$ acts by multiplying vectors of weight m by $l(l+2)-m^2$. In particular, this implies that

$$
\int_{\mathrm{SU}(2)} \left|\widetilde{K}_{F(\sqrt{L})}(h)\right|^2 \mathrm{d}h = \sum_{(l,m)\in\Lambda} (l+1)\left|F(\sqrt{l(l+2)-m^2})\right|^2, \tag{4.4}
$$

where Λ is the set of all (l,m) in $\mathbf{N}\times\mathbf{Z}$ such that $|m|\le l$ and $l-m$ is even, and (as before) $\widetilde{K}_{F(\sqrt{L})}$ denotes the convolution kernel of the operator $F(\sqrt{L})$.

Let μ^n be the measure on $\mathrm{SU}(2)$ given by

$$
\mu^n(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\exp(zZ))\,e^{-inz}\,\mathrm{d}z \qquad \forall f\in C(\mathrm{SU}(2)).
$$

Then $\pi_l(\mu^n)$ is the projection onto the vectors of weight n in $\mathcal{H}_l$. This implies that the operator $\pi_l(\widetilde{K}_{F(\sqrt{L})} * \mu^n)$ annihilates all the weight vectors of weight different from n, and multiplies vectors of weight n by $F(\sqrt{l(l+2)-n^2})$. The next lemma is the new ingredient needed for the proof of Theorem 1.1.

Lemma 4.1. *Suppose that N is in $\mathbf{Z}^+$, that F is in $\mathcal{B}_N(\mathbf{R})$, and that α is in $(0,1)$. Then*

$$\int_{\mathrm{SU}(2)} \left|\widetilde{K}_{F(\sqrt{L})}(g)\right|^2 |r(g)|^\alpha \,\mathrm{d}g \le CN^{4-\alpha} \left\|F_{(N)}\right\|_{N,p}^2 .$$

Proof. We write $\widetilde{K}^n_{F(\sqrt{L})}$ for $\widetilde{K}_{F(\sqrt{L})} * \mu^n$. By Fourier series,

$$\int_{\mathrm{SU}(2)} \left|\widetilde{K}_{F(\sqrt{L})}(h)\right|^2 r(h)^\alpha \,\mathrm{d}h = \sum_{n\in\mathbf{Z}} \int_{\mathrm{SU}(2)} \left|\widetilde{K}^n_{F(\sqrt{L})}(h)\right|^2 r(h)^\alpha \,\mathrm{d}h.$$

Fix N in $\mathbf{Z}^+$. We write $\mathcal{S}$ for the integer interval $(-N/2, N/2) \cap \mathbf{Z}$, and $\mathcal{T}$ for its complement in $\mathbf{Z}$, i.e., $\mathbf{Z} \setminus (-N/2, N/2)$. For n in $\mathcal{S}$, we use the simple estimate that

$$\int_{\mathrm{SU}(2)} \left|\widetilde{K}^n_{F(\sqrt{L})}(h)\right|^2 r(h)^\alpha \,\mathrm{d}h \le \int_{\mathrm{SU}(2)} \left|\widetilde{K}^n_{F(\sqrt{L})}(h)\right|^2 \,\mathrm{d}h. \tag{4.5}$$

For n in $\mathcal{T}$, we use a more subtle estimate. From (4.1), (4.2), and (4.3), it follows that, for a smooth function f on $\mathrm{SU}(2)$, if $\partial_\theta f = 0$ and $P_n f = f$, then

$$\langle Lf, f\rangle \ge n^2 \left\|f \tan r\right\|_{L^2}^2$$

and so

$$\langle L^\alpha f, f\rangle \ge n^{2\alpha} \left\|f \tan^\alpha r\right\|_{L^2}^2$$

when α is in $[0,1]$. Indeed, for any quadratic forms A and B, if $A \ge B \ge 0$ then $A^\alpha \ge B^\alpha$ for all α in $[0,1]$. Hence

$$\begin{aligned}\int_{\mathrm{SU}(2)} \left|\widetilde{K}^n_{F(\sqrt{L})}(h)\right|^2 r(h)^\alpha \,\mathrm{d}h &\le \int_{\mathrm{SU}(2)} \left|\widetilde{K}^n_{F(\sqrt{L})}(h)\right|^2 \tan^\alpha r(h) \,\mathrm{d}h \\ &\le \frac{1}{n^\alpha} \langle L^{\alpha/2} \widetilde{K}^n_{F(\sqrt{L})}, \widetilde{K}^n_{F(\sqrt{L})}\rangle \\ &= \frac{1}{n^\alpha} \left\|L^{\alpha/4} \widetilde{K}^n_{F(\sqrt{L})}\right\|_{L^2}^2 \\ &= \frac{1}{n^\alpha} \left\|\widetilde{K}^n_{G(\sqrt{L})}\right\|_{L^2}^2 ,\end{aligned} \tag{4.6}$$

where $G(\lambda) = \lambda^{\alpha/2} F(\lambda)$.

Define the regions H_k, S and T by the formulae

$$\begin{aligned} H_k &= \Big\{(x,y) \in \mathbf{R}^2 \colon ((k-1)^2 + y^2 + 1)^{1/2} - 1 < x \\ &\qquad \le (k^2 + y^2 + 1)^{1/2} - 1\Big\}, \\ S &= \Big\{(x,y) \in \mathbf{R}^2 \colon |y| < \lceil N/2 \rceil,\ \ |y| \le x \le (N^2 + y^2 + 1)^{1/2} - 1\Big\},\end{aligned}$$

$$T = \left\{ (x,y) \in \mathbf{R}^2 \colon |y| \geq \lceil N/2 \rceil, \ \ |y| \leq x \leq (N^2 + y^2 + 1)^{1/2} - 1 \right\}.$$

The integer lattice points in S and T will be denoted by Σ and T respectively, and T^+ will denote the subset of T in the first quadrant. The "bottom right hand corner" of T^+, which is also the "top right hand corner" of S, is the point (u,v), where $v = \lceil N/2 \rceil$ and $u = (N^2 + \lceil N/2 \rceil^2 + 1)^{1/2} - 1$.

By virtue of (4.5) and (4.4),

$$\begin{aligned}
&\sum_{n \in \mathcal{S}} \int_{\mathrm{SU}(2)} \left| \widetilde{K}^n_{F(\sqrt{L})}(h) \right|^2 r(h)^\alpha \,\mathrm{d}h \\
&\quad \leq \sum_{(l,n) \in \Sigma} (l+1) \left| F\Big(\sqrt{l(l+2) - n^2}\Big) \right|^2 \\
&\quad = \sum_{k=0}^{N} \sum_{(l,n) \in \Sigma \cap H_k} (l+1) \left| F\Big(\sqrt{l(l+2) - n^2}\Big) \right|^2 \\
&\quad \leq \sum_{k=0}^{N} \sum_{(l,n) \in \Sigma \cap H_k} (l+1) \sup_{t \in [k-1,k]} |F(t)|^2 \\
&\quad \leq N \, \|F\|_{N,2}^2 \max_{0 \leq k \leq N} \sum_{(l,n) \in \Sigma \cap H_k} (l+1). \qquad (4.7)
\end{aligned}$$

To estimate $\sum_{(l,n) \in \Sigma \cap H_k} (l+1)$, observe that the line $y = n$ meets H_k in a segment of length

$$\left(\sqrt{k^2 + n^2 + 1} - 1 \right) - \left(\sqrt{(k-1)^2 + n^2 + 1} - 1 \right) < 2.$$

Thus at any fixed height, there are at most two points of Λ inside H_k. Further, if $(x,y) \in S$, then

$$x + 1 < u + 1 < 2N,$$

and so

$$\sum_{(l,n) \in \Sigma \cap H_k} (l+1) \leq \sum_{n = -\lceil N/2 \rceil + 1}^{\lceil N/2 \rceil - 1} 4N \leq 4N^2;$$

combining this with (4.7) shows that

$$\sum_{n \in \mathcal{S}} \int_{\mathrm{SU}(2)} \left| \widetilde{K}^n_{F(\sqrt{L})}(h) \right|^2 r(h)^\alpha \,\mathrm{d}h \leq 4N^3 \, \|F\|_{N,2}^2 \,.$$

Similarly, by virtue of (4.6) and (4.4),

$$\sum_{n \in \mathcal{T}} \int_{\mathrm{SU}(2)} \left| \widetilde{K}^n_{F(\sqrt{L})}(h) \right|^2 r(h)^\alpha \,\mathrm{d}h$$

$$\le \sum_{(l,n)\in \mathrm{T}} (l+1) \frac{\big(l(l+2)-n^2\big)^{\alpha/2}}{|n|^\alpha} \left| F\Big(\sqrt{l(l+2)-n^2}\Big)\right|^2$$
$$\le 4 \sum_{(l,n)\in \mathrm{T}} (l+1)^{1-\alpha} \big(l(l+2)-n^2\big)^{\alpha/2} \left| F\Big(\sqrt{l(l+2)-n^2}\Big)\right|^2,$$

since, if $(x,y) \in T$, then $3|y| \ge x$ and $|y| \ge 1$, so that $4|y| \ge x+1$. Thus, by the argument to prove (4.7),

$$\begin{aligned}
&\sum_{n\in\mathcal{T}} \int_{\mathrm{SU}(2)} \left| \widetilde{K}^n_{F(\sqrt{L})}(h)\right|^2 r(h)^\alpha \,\mathrm{d}h \\
&\qquad \le 4 \sum_{k=1}^{N} \sum_{(l,n)\in \mathrm{T}\cap H_k} (l+1)^{1-\alpha} \big(l(l+2)-n^2\big)^{\alpha/2} \\
&\qquad\qquad \left| F\Big(\sqrt{l(l+2)-n^2}\Big)\right|^2 \\
&\qquad \le 4 \sum_{k=1}^{N} k^\alpha \sum_{(l,n)\in \mathrm{T}\cap H_k} (l+1)^{1-\alpha} \left| F\Big(\sqrt{l(l+2)-n^2}\Big)\right|^2 \\
&\qquad \le 4 N^{\alpha+1} \left\| F\right\|^2_{N,2} \max_{1\le k\le N} \sum_{(l,n)\in \mathrm{T}\cap H_k} (l+1)^{1-\alpha}. \qquad (4.8)
\end{aligned}$$

To prove the lemma, it therefore remains to show that, if $1 \le k \le N$, then

$$\sum_{(l,n)\in \mathrm{T}\cap H_k} (l+1)^{1-\alpha} \le C\, N^{3-2\alpha}. \qquad (4.9)$$

Observe that, if $h \ge 0$, then the line $y = x - 2h$ meets H_k in the line segment $L_{h,k}$, where

$$(k-1)^2 < (x+1)^2 - (x-2h)^2 - 1 \le k^2.$$

This inequality implies that

$$\frac{(k-1)^2 + 4h^2}{4h+2} < x \le \frac{k^2+4h^2}{4h+2},$$

so the number of points in $\Lambda \cap L_{h,k}$ is at most $(2k-1)/(4h+2)$. It also implies that

$$(2h+1)(2x-2h+1) = (x+1)^2 - (x-2h)^2 \le k^2+1,$$

whence, for (x,y) in $L_{h,k}$,

$$2x+1 \le \frac{k^2+1}{2h+1} + 2h.$$

Using these facts, inequality (4.8), and symmetry, we conclude that

$$
\begin{aligned}
\sum_{(l,n)\in \mathrm{T}\cap H_k} (l+1)^{1-\alpha} &= 2 \sum_{(l,n)\in \mathrm{T}^+\cap H_k} (l+1)^{1-\alpha} \\
&\le 2 \sum_{h=0}^{\lfloor (u-v)/2 \rfloor} \frac{2k-1}{4h+2} \left(\frac{k^2+1}{2h+1} + 2h \right)^{1-\alpha} \\
&\le 2 \sum_{h=0}^{\lfloor (u-v)/2 \rfloor} \left(\frac{k^{3-2\alpha}}{(2h+1)^{2-\alpha}} + \frac{k}{(2h+1)^{\alpha}} \right) \\
&\le 2 \sum_{h=0}^{\lfloor (u-v)/2 \rfloor} \left(\frac{N^{3-2\alpha}}{(2h+1)^{2-\alpha}} + \frac{N}{(2h+1)^{\alpha}} \right) \\
&\le C_\alpha \, N^{3-2\alpha},
\end{aligned}
$$

as $\alpha \in [0,1)$. This ends the proof of (4.9) and Lemma 4.1. □

Proof of Theorem 1.1. Take ρ to be the left-invariant control distance associated with the sublaplacian L on $\mathrm{SU}(2)$; in cylindrical coordinates, this is equivalent to the left-invariant metric ρ' defined by the condition

$$\rho'(h,e) = (r(h)^4 + z(h)^2)^{1/4} \qquad \forall h \in \mathrm{SU}(2).$$

Assumption 2.1 (the doubling condition) holds for the control metric, as for all sublaplacians on groups of polynomial growth.

Fix α in $[0,1)$, and define the weight w by the condition $w(x,y) = \tilde{w}(y^{-1}x)$, where, in cylindrical coordinates,

$$\tilde{w}(h) = r(h)^\alpha.$$

It is easy to check Assumption 2.2 for this weight. Assumption 2.3 holds for the operator L and metric ρ; see [24,31]. Finally, Assumption 2.4 follows from the standard estimates for the heat kernel associated to L on $\mathrm{SU}(2)$; see [28,37]. Together with Lemma 4.1, we thus have all the conditions necessary to apply Theorem 3.6, and Theorem 1.1 is proved.

5. Remarks and comments on the Heisenberg group

Let $\mathbb{H}_1$ be the Heisenberg group and $L_{\mathbb{H}_1}$ be the homogeneous sublaplacian on $\mathbb{H}_1$, see, e.g., [25]. It is shown in [27] (see the proposition on p. 587 and the theorem on p. 574) that if the operator L is defined by (1.2) then

$$\left\| F(\sqrt{L}_{\mathbb{H}_1}) \right\|_{L^p(\mathbb{H}_1)\to L^p(\mathbb{H}_1)} \le C \limsup_{t\to 0} \|F(tL)\|_{L^p(\mathrm{SU}(2))\to L^p(\mathrm{SU}(2))}$$

for any p in $[1,\infty)$. Thus from Theorem 1.1, we get the following corollary.

Corollary 5.1. *Suppose that $s > 3/2$ and that $F\colon \mathbf{R} \to \mathbf{C}$ is a continuous function such that*

$$\sup_{t\in\mathbf{R}^+} \left\|\eta\, F_{(t)}\right\|_{H_s} < \infty.$$

Then $F(\sqrt{L_{\mathbb{H}_1}})$ is bounded on $L^p(\mathbb{H}_1)$ when $1 < p < \infty$.

This gives an alternative proof of the spectral multiplier theorem for Heisenberg group of Hebisch and of Müller and Stein. In [25], it is shown that Corollary 5.1 is sharp, in the sense that it is false for any $s < 3/2$. It follows that Theorem 1.1 is sharp as well. Finally we note that the proof of [25] may be extended to show the following result.

Theorem 5.2. *Suppose that G is a direct product of the form $G_1 \times \ldots \times G_k$, where each factor G_j is a Heisenberg group $\mathbb{H}_{n_j}$, a Euclidean group $\mathbf{R}^{n_j}$, or $\mathrm{SU}(2)$, and that L is a sum $L_1 + \ldots + L_k$ of sublaplacians L_j on G_j. If $s > (1/2)\dim G$ and F is bounded and $\sup_{t\in\mathbf{R}^+} \left\|\eta\, F_{(t)}\right\|_{H_s} < \infty$, then $F(\sqrt{L})$ is of weak type $(1,1)$ and is bounded on $L^p(G)$ when $1 < p < \infty$.*

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