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Global bifurcation for quasilinear elliptic equations on $\mathbb{R}^N$

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Abstract. In this paper we discuss the global behaviour of some connected sets of solutions (λ, u) of a broad class of second order quasilinear elliptic equations

$$-\sum_{\alpha,\beta=1}^{N} a_{\alpha\beta}(x, u(x), \nabla u(x))\partial_\alpha\partial_\beta u(x) + b(x, u(x), \nabla u(x), \lambda) = 0 \quad (1)$$

for $x \in \mathbb{R}^N$ where λ is a real parameter and the function u is required to satisfy the condition

$$\lim_{|x|\to\infty} u(x) = 0. \quad (2)$$

The basic tool is the degree for proper Fredholm maps of index zero in the form due to Fitzpatrick, Pejsachowicz and Rabier. To use this degree the problem must be expressed in the form $F : J \times X \to Y$ where J is an interval, X and Y are Banach spaces and F is a C^1 map which is Fredholm and proper on closed bounded subsets. We use the usual spaces $X = W^{2,p}(\mathbb{R}^N)$ and $Y = L^p(\mathbb{R}^N)$. Then the main difficulty involves finding general conditions on $a_{\alpha\beta}$ and b which ensure the properness of F. Our approach to this is based on some recent work where, under the assumption that $a_{\alpha\beta}$ and b are asymptotically periodic in x as $|x| \to \infty$, we have obtained simple conditions which are necessary and sufficient for $F(\lambda, \cdot) : X \to Y$ to be Fredholm and proper on closed bounded subsets of X. In particular, the nonexistence of nonzero solutions in X of the asymptotic problem plays a crucial role in this issue. Our results establish the bifurcation of global

branches of solutions for the general problem. Various special cases are also discussed. Even for semilinear equations of the form

$$-\Delta u(x) + f(x, u(x)) = \lambda u(x),$$

our results cover situations outside the scope of other methods in the literature.

1 Introduction

In this paper we discuss the global behaviour of some connected sets of solutions (λ, u) of a second order quasilinear elliptic equation

$$-\sum_{\alpha,\beta=1}^{N} a_{\alpha\beta}(x, u(x), \nabla u(x))\partial_\alpha \partial_\beta u(x) + b(x, u(x), \nabla u(x), \lambda) = 0 \quad (3)$$

for $x \in \mathbb{R}^N$. Here λ is a real parameter and the function u is required to satisfy the condition

$$\lim_{|x|\to\infty} u(x) = 0.$$

In addition to the ellipticity of the matrix $[a_{\alpha\beta}]$ of coefficients, we suppose that $b(x, 0, \lambda) = 0$ for all $(x, \lambda) \in \mathbb{R}^{N+1}$. Thus $u \equiv 0$ is a solution of the problem for every $\lambda \in \mathbb{R}$ and our results deal with components of non-trivial solutions bifurcating from this line of trivial solutions.

The programme for establishing results of this kind was laid down in the fundamental work by Rabinowitz, [28] and [29]. It involves writing the differential equation, together with the relevant boundary conditions, as the set of zeros of the operator $F : \mathbb{R} \times X \to Y$, between function spaces X and Y and then using an appropriate topological degree to obtain global properties of connected components of non-trivial solutions. For very general elliptic equations on bounded domains, Sobolev or Hölder spaces can be chosen in such a way that the classical degree of Leray and Schauder can be used to obtain the desired results. However, even for the simplest semilinear equations of the form

$$-\Delta u(x) + f(x, u(x)) - \lambda u(x) = 0 \quad (4)$$

on $\mathbb{R}^N$, this framework fails since the equation cannot be expressed as a compact perturbation of the identity. This fact is intimately related to the presence of an essential spectrum for the linear operator $-\Delta + V$ where V is a bounded potential. There are various ways of circumventing this difficulty, including approximation by problems on bounded domains and the use of weighted Sobolev spaces, [4], [1], [12], [21], [36],[6] but we prefer to use an extension of the Leray-Schauder degree since it seems to yield the most general results under natural hypotheses. For ordinary differential equations on $[0, \infty)$ this approach was first adopted in [33], [34] and [35] using respectively the degree for $k-$set contractions and Galerkin maps; and it was subsequently developed in various ways. Since these early contributions there has been significant progress in constructing topological degree theories, [18], [32],[21] and [36], which can be applied to problems on unbounded domains. A particularly attractive and natural option is offered by the degree for proper Fredholm maps of index zero which has been built on the fundamental notion of parity, [10], [11] and [23], and it is this tool which we shall exploit to deal with (3).

The degree for proper Fredholm maps of index zero was used in [16] to deal with semilinear equations of the form (4) on $\mathbb{R}^N$ in the setting of standard Sobolev spaces. In that work it becomes clear that the main effort must be devoted to finding conditions which ensure the properness of the corresponding differential operator between appropriate function spaces. The maximum principle is used in [16] to establish properness for equations of the form (4) for λ lying in an interval $(-\infty, \beta)$ below the essential spectrum of the linearization at $u = 0$. The conclusions about global bifurcation which follow from this are also confined to the interval $(-\infty, \beta)$. More recently we have used a different approach which, for a broad class of quasilinear elliptic equations of the form (3), gives conditions which are both necessary and sufficient for the corresponding differential operator to be proper and Fredholm between the relevant function spaces at a given value of λ. It is this work which we shall now exploit to derive global bifurcation results for equations the form (3). Even for semilinear equations like (4), our results go beyond the framework in [16] since they are not confined to intervals below the essential spectrum of the linearization of (4) at $u = 0$. When compared with previous work on quasilinear equations on $\mathbb{R}^N$, we observe that we deal with the general form of the equation and we do not require any decay or integrability of the coefficients of the kind used in [6]. However, in confining our attention to strictly elliptic equations, we exclude some familiar examples of degenerate elliptic equations such as those involving the $p-$Laplacian.

In [25] we consider the differential operator

$$F(\lambda, u)(x) =$$

$$-\sum_{\alpha,\beta=1}^{N} a_{\alpha\beta}(x, u(x), \nabla u(x))\partial_\alpha\partial_\beta u(x) + b(x, u(x), \nabla u(x), \lambda) \qquad (5)$$

and we formulate conditions which are necessary and sufficient for $F(\lambda, \cdot) : X_p \to Y_p$ to be a C^1 proper Fredholm map of index zero (the relevant definitions are given in Sect. 3) between the spaces $X_p = W^{2,p}(\mathbb{R}^N)$ and $Y_p = L^p(\mathbb{R}^N)$ where $p \in (N, \infty)$. There are two reasons for choosing these spaces :

(i) all elements of X_p vanish as $|x| \to \infty$ and
(ii) we can ensure that $F(\lambda, u) \in Y_p$ for all $u \in X_p$ without imposing restrictions on the growth of the functions $a_{\alpha\beta}(x, \xi)$ and $b(x, \xi, \lambda)$ as $|\xi| \to \infty$.
Furthermore, our results in Sect. 4, giving explicit conditions for global bifurcation, do not depend upon the choice of p within the range (N, ∞).

Our criteria for properness involve an asymptotic limit operator $F^\infty(\lambda, \cdot)$ defined by

$$F^\infty(\lambda, u) = - \sum_{\alpha,\beta=1}^{N} a^\infty_{\alpha\beta}(x, u(x), \nabla u(x)) \partial_\alpha \partial_\beta u(x) + b^\infty(x, u(x), \nabla u(x), \lambda) \tag{6}$$

where it is supposed that there are functions $a^\infty_{\alpha\beta}$ and b^∞ which are N-periodic in x on $\mathbb{R}^N$ such that $b^\infty(x, 0, \lambda) \equiv 0$ and

$$\begin{aligned} &\lim_{|x|\to\infty} \left\{ a_{\alpha\beta}(x, \xi) - a^\infty_{\alpha\beta}(x, \xi) \right\} \\ &= \lim_{|x|\to\infty} \left\{ \partial_{\xi_i} b(x, \xi, \lambda) - \partial_{\xi_i} b^\infty(x, \xi, \lambda) \right\} = 0 \end{aligned}$$

for $1 \le \alpha, \beta \le N$ and $i = 0, 1, .., N$. Roughly speaking (see Corollary 6.2 of [25] for a complete statement), if such an operator $F(\lambda, \cdot) : X_p \to Y_p$ is C^1, then it is a proper (on closed bounded sets) Fredholm map of index zero provided that

(C1) there is an element $v \in X$ such that the bounded linear operator $D_u F(\lambda, v) : X_p \to Y_p$ is Fredholm of index zero, and

(C2) if $u \in X_p$ and $F^\infty(\lambda, u) = 0$, then $u = 0$.

This characterization of quasilinear elliptic operators on $\mathbb{R}^N$ which are proper on closed bounded sets and Fredholm of index zero is our starting point for obtaining global bifurcation results. In deriving such conclusions we should also formulate explicit conditions on the functions $a_{\alpha\beta}$ and b which imply that the above properties (C1) and (C2) hold. These conditions should prove useful in other contexts where the properness of the differential operator is relevant.

The first step in the programme we have just sketched is to ensure that the operator $F : \mathbb{R} \times X_p \to Y_p$ has enough smoothness for the subsequent discussion. For fixed λ, this is already dealt with in [25] so here we need only study the smoothness of the Nemytskii operator $B : \mathbb{R} \times X_p \to Y_p$ defined by

$$B(\lambda, u)(x) = b(x, u(x), \nabla u(x), \lambda).$$

This is undertaken in Sect. 2.

The general results about global bifurcation are formulated in Sect. 3. Let X and Y be real Banach spaces. An open interval J is called admissible for the map $F : \mathbb{R} \times X \to Y$ if the restriction of F to $J \times X$ is a C^1-proper Fredholm map of index zero in the sense of Definition 2. For a point $\lambda_0 \in J$ across which the parity is equal to -1, Theorem 2 contains global information which is available about the branch of solutions bifurcating from the point $(\lambda_0, 0)$ in $J \times X$. In the context of the differential operator (5) and the Sobolev spaces X_p and Y_p, the relationship between admissible intervals and the properties (C1) and (C2) introduced above is spelt out in Theorem 4. Having done this we can henceforth concentrate on the main problems which have to be resolved in this paper, namely giving explicit conditions which enable us to calculate the parity across λ_0 and to verify that (C1) and (C2) are satisfied.

In studying the parity and checking the condition (C1) we are essentially concerned with linear differential operators, or one-parameter families of them. Our paper [26] contains some results about the L^p-spectral theory of Schrödinger operators which we use in Sect. 3 to resolve these issues. Thus Sect. 3 ends with two results about global bifurcation for the equation (3) under the hypothesis that the condition (C2) is satisfied.

The methods available for checking the condition (C2) depend heavily on the form of the differential operator F^∞. In Sect. 4 we have exploited three different techniques which seem appropriate for meaningful situations. The first method hinges on the maximum principle and is applicable provided that $b^\infty(x, s, 0, \lambda)$ always has the same sign as s. The second approach is based on variational identities of the type found by Pohozaev, and further developed in [8], [24], [17] and [37]. To use this method we require the differential operator $F^\infty(\lambda, \cdot)$ to be autonomous and to have a variational structure. Finally, if $F^\infty(\lambda, \cdot)$ is a linear differential operator for each $\lambda \in J$, spectral theory can again be used to check the condition (C2). Each method leads to a bifurcation theorem and they are formulated respectively as Theorems 7, 8 and 9. Some special cases make it easier to understand the different types of equation covered by these results.

Section 5 is devoted to examples illustrating the general results. In Example 1 we compare the first and second methods in the context of a family of problems which includes the cases where the principal part of F can be either the Laplacian or the mean curvature operator. The Example 2 deals with the case where $F^\infty(\lambda, \cdot)$ is a periodic Schrödinger operator. If the interval J lies in a spectral gap of this operator the first two methods do not seem to be applicable.

Although our main concern has been to obtain global bifurcation results for quasilinear operators, our approach covers situations which appear to be completely untouched even for semilinear equations of the type (4). This can be illustrated in a very explicit way by considering the nonlinear Schrödinger equation,

$$-\Delta u(x) + V(x)u(x) + r(x)\left|u(x)\right|^{\tau} u(x) - \lambda u(x) = 0. \tag{7}$$

Referring to Case 1 of Example 1 in Sect. 5, we see that, when $\lim_{|x|\to\infty} V(x) = V(\infty) \in \mathbb{R}, \tau > 0$ and $\lim_{|x|\to\infty} r(x) \geq 0$, the result that we obtain concerning bifurcation for λ in the interval $(-\infty, V(\infty))$ coincides with what can be deduced from the work in [16] in this situation. The approach in [16] fails when $\lim_{|x|\to\infty} r(x) < 0$, whereas we can still deal with this case, for λ in the interval $(-\infty, V(\infty))$, provided that $\lim_{|x|\to\infty} V(x) = V(\infty) \in \mathbb{R}$ and the exponent τ is supercritical. In an other direction, Examples 2 and 3 seem to be the first results about global bifurcation in spectral gaps for such nonlinear Schrödinger equations when $\tau > 0$, $\lim_{|x|\to\infty} \{V(x) - P(x)\} = 0$ for some $N-$periodic function P and $\lim_{|x|\to\infty} r(x) = 0$.

Our results describe the global behaviour of some connected sets of solutions of equation (3) for values of λ lying in what we call an admissible interval for the operator (5). This restriction arises because the degree theory which underlies our whole approach is only available in such intervals. It is natural to ask whether these branches of solutions in fact extend beyond an admissible interval and what might be an appropriate tool for establishing this. However, in some situations, one can show that the problem has no solutions outside the admissible intervals. Example 3 in Sect. 5 demonstrates this in a particularly simple setting.

The variational identities which are used in Sect. 4.2 are established in our paper [27] where we also give conditions ensuring that solutions of a second order quasilinear equation on $\mathbb{R}^N$ decay exponentially as $|x| \to \infty$. In Sect. 6 we use the results in [27] to establish the exponential decay of solutions of the equation (3) under appropriate conditions.

Finally, we wish to point out that, to avoid overburdening the exposition, we have not used the most general form of the results in [25]. The hypothesis (A) in Sect. 3 requires the limit operator (6) to be $N-$periodic in x. This restriction could be relaxed by using the results of Sect. 7 in [25].

2 Notation and definitions

We use the standard notation for the Lebesgue and Sobolev spaces. The usual norm on $W^{k,p}(\mathbb{R}^N)$ is denoted by $\|\cdot\|_{k,p}$ with $W^{0,p}(\mathbb{R}^N) = L^p(\mathbb{R}^N)$ and $\|\cdot\|_{0,p} = |\cdot|_p$.

For $p \in (N, \infty)$, we set

$$X_p = W^{2,p}(\mathbb{R}^N) \text{ and } Y_p = L^p(\mathbb{R}^N). \tag{8}$$

Then $X_p \subset C^1(\mathbb{R}^N)$ and there exists a constant $C = C(N, p)$ such that

$$|u|_\infty + |\nabla u|_\infty \leq C \|u\|_{2,p} \text{ for all } u \in X_p. \tag{9}$$

Furthermore

$$\lim_{|x|\to\infty} u(x) = 0 \text{ and } \lim_{|x|\to\infty} \nabla u(x) = 0 \tag{10}$$

for all $u \in X_p$.
All of these results are proved in Chapter IX of [3].

This section deals with the smoothness of some one parameter families of Nemytskii operators from X_p into Y_p for $p \in (N, \infty)$. Let $f : \mathbb{R}^N \times \mathbb{R}^{N+1} \to \mathbb{R}$ and $g : \mathbb{R}^N \times \mathbb{R}^{N+2} \to \mathbb{R}$. For $u \in X_p$ and $\lambda \in \mathbb{R}$, we consider the maps

$$u \mapsto f(\cdot, u(\cdot), \nabla u(\cdot)) \text{ and } (\lambda, u) \mapsto g(\cdot, u(\cdot), \nabla u(\cdot), \lambda).$$

Using the notation

$$f : \mathbb{R}^N \times \mathbb{R}^{N+1} \to \mathbb{R} \text{ with } (x, \eta) = (x, \xi_0, ..\xi_N) \longmapsto f(x, \xi_0, ..\xi_N)$$

and

$$g : \mathbb{R}^N \times \mathbb{R}^{N+2} \to \mathbb{R} \text{ with } (x, \eta) = (x, \xi_0, ..\xi_N, \lambda) \longmapsto g(x, \xi_0, ..\xi_N, \lambda),$$

we see that the variables x and η play markedly different roles when deriving the smoothness properties of the maps $u \mapsto f(\cdot, u(\cdot), \nabla u(\cdot))$ and $(\lambda, u) \mapsto g(\cdot, u(\cdot), \nabla u(\cdot), \lambda)$ from those of the functions f and g. The terminology "bundle map" provides a convenient way of handling this distinction where x is the "base" variable and η is the "fiber" variable. Note that since we require smoothness with respect to u and λ it is natural to treat λ as a fiber variable.

Definition 1 *A function $f : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}$ is called an equicontinuous C^0- bundle map if f is continuous and the collection $\left\{ f(x, \cdot) : x \in \mathbb{R}^N \right\}$ is equicontinuous at ξ for every $\xi \in \mathbb{R}^M$. For a positive integer k, we say that $f = f(x, \eta)$ is an equicontinuous $C^k_\eta-$bundle map if the partial derivatives $D^\alpha_\eta f$ exist and are equicontinuous C^0-bundle maps for all multi-indices α with $|\alpha| \leq k$.*

Remark 2.1 If $V \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $g \in C^k(\mathbb{R}^M)$ then the function $f(x,\eta) = V(x)g(\eta)$ is an equicontinuous $C^k_\eta-$bundle map, as are finite sums of such functions.

Remark 2.2 Equicontinuous C^0-bundle maps are uniformly equicontinuous on compact subsets of $\mathbb{R}^M$ in the following sense. Let $f : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}$ be an equicontinuous C^0-bundle map. Given a compact subset K of $\mathbb{R}^M$ and $\varepsilon > 0$, there exists $\delta(K,\varepsilon) > 0$ such that $|f(x,\xi) - f(x,\eta)| < \varepsilon$ for all $x \in \mathbb{R}^N$ and $\xi, \eta \in K$ with $|\xi - \eta| < \delta(K,\varepsilon)$. See Lemma 2.1 of [25].

We can now formulate the essential smoothness properties of the family of quasilinear second order differential operators defined by (5) where the functions $a_{\alpha\beta} : \mathbb{R}^N \times \mathbb{R}^{N+1} \to \mathbb{R}$ and $b : \mathbb{R}^N \times \mathbb{R}^{N+2} \to \mathbb{R}$ are bundle maps having the following properties.

(B) For $\alpha, \beta = 1,..,N$, the function $a_{\alpha\beta} = a_{\beta\alpha} : \mathbb{R}^N \times \mathbb{R}^{N+1} \to \mathbb{R}$ is an equicontinuous $C^1_\xi-$bundle map with

$$a_{\alpha\beta}(\cdot,0) \text{ and } \partial_{\xi_i} a_{\alpha\beta}(\cdot,0) \in L^\infty(\mathbb{R}^N) \text{ for } i = 0,1,..,N. \tag{11}$$

The function $b : \mathbb{R}^N \times \mathbb{R}^{N+2} \to \mathbb{R}$ is continuous and its partial derivatives $\partial_{\xi_i} b, \partial_\lambda b, \partial_\lambda \partial_{\xi_i} b$ and $\partial_{\xi_i}\partial_\lambda b$ exist and are continuous on $\mathbb{R}^N \times \mathbb{R}^{N+2}$ for $i = 0,1,..,N$. For each $\lambda \in \mathbb{R}$, $b(\cdot,\lambda) : \mathbb{R}^N \times \mathbb{R}^{N+1} \to \mathbb{R}$ is an equicontinuous $C^1_\xi-$bundle map and $\partial_\lambda \partial_{\xi_i} b : \mathbb{R}^N \times \mathbb{R}^{N+2} \to \mathbb{R}$ is an equicontinuous C^0-bundle map for $i = 0,1,..,N$. Furthermore, for all $x \in \mathbb{R}^N$ and $\lambda \in \mathbb{R}$,

$$b(x,0,\lambda) = 0 \tag{12}$$

and

$$\partial_{\xi_i} b(\cdot,0,\lambda) \text{ and } \partial_{\xi_i}\partial_\lambda b(\cdot,0,\lambda) \in L^\infty(\mathbb{R}^N) \text{ for } i = 0,..,N. \tag{13}$$

Remark 2.3 The hypothesis (B) ensures that $\partial_\lambda \partial_{\xi_i} b \equiv \partial_{\xi_i}\partial_\lambda b$ for $i = 0,1,..,N$ and that

$$\partial_\lambda b(x,0,\lambda) = 0 \text{ for } x \in \mathbb{R}^N \text{ and } \lambda \in \mathbb{R}.$$

Furthermore, it is easy to deduce from (B) that, for each $\lambda \in \mathbb{R}$, $\partial_\lambda b(\cdot,\lambda) : \mathbb{R}^N \times \mathbb{R}^{N+1} \to \mathbb{R}$ is an equicontinuous $C^1_\xi-$bundle map.

Lemma 1 *Let b satisfy the conditions in (B) and let W be a bounded subset of $\mathbb{R} \times X_p$ where $p \in (N,\infty)$.*
(i) There exists a constant M such that, for $i = 0,1,..,N$,

$$|\partial_\lambda \partial_{\xi_i} b(x,u(x),\nabla u(x),\lambda)| \leq M \text{ for all } x \in \mathbb{R}^N \text{and } (\lambda,u) \in W.$$

(ii) Given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon,W) > 0$ such that, for $i = 0,1,..,N$,

$$|\partial_\lambda \partial_{\xi_i} b(x,u(x),\nabla u(x),\lambda) - \partial_\lambda \partial_{\xi_i} b(x,v(x),\nabla v(x),\mu)| < \varepsilon$$

whenever $(\lambda,u),(\mu,v) \in W$ and $|\lambda - \mu| + \|u - v\|_{2,p} < \delta$.

Proof. There is a compact subset K of $\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}$ such that $(u(x), \nabla u(x), \lambda) \in K$ for all $x \in \mathbb{R}^N$ and $(\lambda, u) \in W$. The conclusions now follow from the fact that $\partial_\lambda \partial_{\xi_i} b : \mathbb{R}^N \times \mathbb{R}^{N+2} \to \mathbb{R}$ is an equicontinuous C^0-bundle map for $i = 0, 1, .., N$, with $\partial_\lambda \partial_{\xi_i} b(\cdot, 0, \lambda) = \partial_{\xi_i} \partial_\lambda b(\cdot, 0, \lambda) \in L^\infty(\mathbb{R}^N)$ by (13). (See Lemma 2.1 of [25].) □

Theorem 1 *Fix $p \in (N, \infty)$ and consider the operator F defined by (5) under the hypothesis (B). Then $F \in C^1(\mathbb{R} \times X_p, Y_p)$ and the partial derivatives (in the sense of Fréchet) $D_u D_\lambda F$ and $D_\lambda D_u F$ exist and are continuous on $\mathbb{R} \times X_p$. In particular,*

$$[D_u F(\lambda, 0)v](x) =$$

$$-\sum_{\alpha,\beta=1}^{N} a_{\alpha\beta}(x, 0)\partial_\alpha \partial_\beta v(x) + \sum_{i=1}^{N} \partial_{\xi_i} b(x, 0, \lambda)\partial_i v(x) + \partial_{\xi_0} b(x, 0, \lambda)v(x) \tag{14}$$

and

$$[D_\lambda D_u F(\lambda, 0)v](x) = [D_u D_\lambda F(\lambda, 0)v](x) \tag{15}$$

$$= \sum_{i=1}^{N} \partial_\lambda \partial_{\xi_i} b(x, 0, \lambda)\partial_i v(x) + \partial_\lambda \partial_{\xi_0} b(x, 0, \lambda)v(x).$$

for all $v \in X_p$. Furthermore, the mapping $F(\cdot, u) : \mathbb{R} \to Y_p$ is equicontinuous with respect to u in bounded subsets of X_p.

Proof. For $(\lambda, u) \in \mathbb{R} \times X_p$, set

$$B(\lambda, u)(x) = b(x, u(x), \nabla u(x), \lambda)$$

and

$$C(\lambda, u)(x) = \partial_\lambda b(x, u(x), \nabla u(x), \lambda).$$

It follows from Theorem 2.3 of [25] that

$$B(\lambda, \cdot) \text{ and } C(\lambda, \cdot) \in C^1(X_p, Y_p)$$

for all $\lambda \in \mathbb{R}$, with

$$D_u B(\lambda, u)v = \partial_{\xi_0} b(\cdot, u, \nabla u, \lambda)v + \sum_{i=1}^{N} \partial_{\xi_i} b(\cdot, u, \nabla u, \lambda)\partial_i v$$

and

$$D_u C(\lambda, u)v = \partial_{\xi_0} \partial_\lambda b(\cdot, u, \nabla u, \lambda)v + \sum_{i=1}^{N} \partial_{\xi_i} \partial_\lambda b(\cdot, u, \nabla u, \lambda)\partial_i v.$$

Also, by Lemma 3.2 of [25]

$$F(\lambda,\cdot)\in C^1(X_p,Y_p) \text{ for all } \lambda\in\mathbb{R}$$

and (14) holds.

Next we show that

(a) $B(\cdot,u)$ and $C(\cdot,u):\mathbb{R}\to Y_p$ are equicontinuous with respect to u in bounded subsets of X_p
and
(b) $D_uB(\cdot,u)$ and $D_uC(\cdot,u):\mathbb{R}\to L(X_p,Y_p)$ are equicontinuous with respect to u in bounded subsets of X_p.

Notice that (a) and (b) together with the continuity of $B(\lambda,\cdot)$, $C(\lambda,\cdot)$, $D_uB(\lambda,\cdot)$ and $D_uC(\lambda,\cdot)$ at fixed λ show that the mappings B, C, D_uB and D_uC are continuous with respect to (λ,u). Also, (a) implies the equicontinuity of $F(\cdot,u)$ claimed in the theorem.

Let V be a bounded subset of X_p. With no loss of generality, we shall suppose that V is convex and that $0\in V$. Given any compact interval $I\subset\mathbb{R}$, the set $W=I\times V$ is a bounded subset of $\mathbb{R}\times X_p$. Consider (λ,u) and $(\lambda+\mu,u)\in W$. Then,

(a) $B(\lambda+\mu,u)(x)-B(\lambda,u)(x)$

$$\begin{aligned}
&=\int_0^1\frac{d}{dt}b(x,u(x),\nabla u(x),\lambda+t\mu)dt\\
&=\mu\int_0^1\partial_\lambda b(x,u(x),\nabla u(x),\lambda+t\mu)dt\\
&=\mu\int_0^1\int_0^1\frac{d}{ds}\partial_\lambda b(x,su(x),s\nabla u(x),\lambda+t\mu)dsdt\\
&=\mu[\int_0^1\int_0^1\partial_{\xi_0}\partial_\lambda b(x,su(x),s\nabla u(x),\lambda+t\mu)dsdt]u(x)\\
&+\mu\sum_{i=1}^N[\int_0^1\int_0^1\partial_{\xi_i}\partial_\lambda b(x,su(x),s\nabla u(x),\lambda+t\mu)dsdt]\partial_i u(x)
\end{aligned}$$

and so, by Lemma 1, there is a constant M such that

$$|B(\lambda+\mu,u)(x)-B(\lambda,u)(x)|\le|\mu|\,M\left\{|u(x)|+\sum_{i=1}^N|\partial_i u(x)|\right\}$$

for all $x\in\mathbb{R}^N$. Thus

$$\begin{aligned}
|B(\lambda+\mu,u)-B(\lambda,u)|_p&\le|\mu|\,M\left|\left\{|u|+\sum_{i=1}^N|\partial_i u|\right\}\right|_p\\
&\le|\mu|\,M\,\|u\|_{2,p}\le|\mu|\,ML
\end{aligned}$$

for some constant L and all $(\lambda, u), (\lambda+\mu, u) \in W$.
This proves the equicontinuity of the functions $B(\cdot, u)$ with respect to $u \in V$.

Similarly,

$$\begin{aligned}
&C(\lambda+\mu, u)(x) - C(\lambda, u)(x) \\
&= \partial_\lambda b(x, u(x), \nabla u(x), \lambda+\mu) - \partial_\lambda b(x, u(x), \nabla u(x), \lambda) \\
&= \int_0^1 \frac{d}{ds}[\partial_\lambda b(x, su(x), s\nabla u(x), \lambda+\mu) \\
&\quad -\partial_\lambda b(x, su(x), s\nabla u(x), \lambda)]ds \\
&= \left\{ \int_0^1 [\partial_{\xi_0}\partial_\lambda b(x, su(x), s\nabla u(x), \lambda+\mu) \right. \\
&\quad \left. -\partial_{\xi_0}\partial_\lambda b(x, su(x), s\nabla u(x), \lambda)]ds \right\} u(x) \\
&\quad + \sum_{i=1}^N \left\{ \int_0^1 [\partial_{\xi_i}\partial_\lambda b(x, su(x), s\nabla u(x), \lambda+\mu) \right. \\
&\quad \left. -\partial_{\xi_i}\partial_\lambda b(x, su(x), s\nabla u(x), \lambda)]ds \right\} \partial_i u(x)
\end{aligned}$$

Given $\varepsilon > 0$, Lemma 1(ii) shows that there exists $\delta = \delta(\varepsilon, W) > 0$ such that

$$|\partial_{\xi_i}\partial_\lambda b(x, su(x), s\nabla u(x), \lambda+t\mu) - \partial_{\xi_i}\partial_\lambda b(x, su(x), s\nabla u(x), \lambda)| < \varepsilon \tag{16}$$

for all $i = 0, 1, .., N, x \in \mathbb{R}^N$, all $s, t \in [0, 1]$ and all $(\lambda, u), (\lambda+\mu, u) \in W = I \times V$ with $|\mu| < \delta$.

Using (16) with $t = 1$, we see that

$$|C(\lambda+\mu, u)(x) - C(\lambda, u)(x)| \le \varepsilon \left\{ |u(x)| + \sum_{i=1}^N |\partial_i u(x)| \right\}$$

for all $x \in \mathbb{R}^N$ and the equicontinuity of $C(\cdot, u)$ follows from this estimate as for $B(\cdot, u)$ above.

(b) For $v \in X_p$,

$$
\begin{aligned}
&[D_uB(\lambda+\mu,u)v - D_uB(\lambda,u)v]\,(x)\\
&= [\partial_{\xi_0} b(x,u(x),\nabla u(x),\lambda+\mu) - \partial_{\xi_0} b(x,u(x),\nabla u(x),\lambda)]\,v(x)\\
&\quad + \sum_{i=1}^{N} [\partial_{\xi_i} b(x,u(x),\nabla u(x),\lambda+\mu) - \partial_{\xi_i} b(x,u(x),\nabla u(x),\lambda)]\,\partial_i v(x)\\
&= \left[\int_0^1 \frac{d}{dt}\partial_{\xi_0} b(x,u(x),\nabla u(x),\lambda+t\mu)dt\right] v(x)\\
&\quad + \sum_{i=1}^{N} \left[\int_0^1 \frac{d}{dt}\partial_{\xi_i} b(x,u(x),\nabla u(x),\lambda+t\mu)dt\right] \partial_i v(x)\\
&= \mu \left[\int_0^1 \partial_\lambda\partial_{\xi_0} b(x,u(x),\nabla u(x),\lambda+t\mu)dt\right] v(x)\\
&\quad + \mu \sum_{i=1}^{N} \left[\int_0^1 \partial_\lambda\partial_{\xi_i} b(x,u(x),\nabla u(x),\lambda+t\mu)dt\right] \partial_i v(x)
\end{aligned}
$$

so, by Lemma 1,

$$
|[D_uB(\lambda+\mu,u)v - D_uB(\lambda,u)v]\,(x)| \le |\mu|\, M \left\{ |v(x)| + \sum_{i=1}^{N} |\partial_i v(x)| \right\}
$$

for all $x \in \mathbb{R}^N$ where the constant M depends only on W. Thus

$$
|[D_uB(\lambda+\mu,u) - D_uB(\lambda,u)]\,v|_p \le |\mu|\, M\, \|v\|_{2,p}\,,
$$

showing that $\lambda \mapsto D_uB(\lambda,u)$ is equicontinuous with respect to $u \in V$.

Finally,

$$
\begin{aligned}
&[D_uC(\lambda+\mu,u)v - D_uC(\lambda,u)v]\,(x)\\
&= [\partial_{\xi_0}\partial_\lambda b(x,u(x),\nabla u(x),\lambda+\mu) - \partial_{\xi_0}\partial_\lambda b(x,u(x),\nabla u(x),\lambda)]v(x)\\
&\quad + \sum_{i=1}^{N} [\partial_{\xi_i}\partial_\lambda b(x,u(x),\nabla u(x),\lambda+\mu)\\
&\quad - \partial_{\xi_i}\partial_\lambda b(x,u(x),\nabla u(x),\lambda)]\partial_i v(x)
\end{aligned}
$$

and the equicontinuity of $\lambda \mapsto D_u C(\lambda, u)$ follows from (16) by similar arguments to those used above.

We now show that B is differentiable with respect to λ and that $D_\lambda B = C$. For this we fix $(\lambda, u) \in \mathbb{R} \times X_p$ and consider $B(\lambda+\mu, u) - B(\lambda, u) - \mu C(\lambda, u)$ where $(\lambda, u), (\lambda+\mu, u) \in W = I \times V$. Then

$$\begin{aligned}
&|B(\lambda+\mu, u) - B(\lambda, u) - \mu C(\lambda, u)|_p \\
&= \left| \int_0^1 \frac{d}{dt} b(\cdot, u, \nabla u, \lambda + t\mu) dt - \mu \partial_\lambda b(\cdot, u, \nabla u, \lambda) \right|_p \\
&= |\mu| \left| \int_0^1 [\partial_\lambda b(\cdot, u, \nabla u, \lambda + t\mu) - \partial_\lambda b(\cdot, u, \nabla u, \lambda)] dt \right|_p \\
&\leq |\mu| \left\{ \int_{\mathbb{R}^N} \int_0^1 |\partial_\lambda b(x, u(x), \nabla u(x), \lambda + t\mu) \right. \\
&\qquad \left. - \partial_\lambda b(x, u(x), \nabla u(x), \lambda)|^p \, dt dx \right\}^{1/p}
\end{aligned}$$

by Hölder's inequality.

But, since $\partial_\lambda b(x, 0, \lambda + t\mu) \equiv 0$,

$$\begin{aligned}
&\partial_\lambda b(x, u(x), \nabla u(x), \lambda + t\mu) - \partial_\lambda b(x, u(x), \nabla u(x), \lambda) \\
&= \int_0^1 \frac{d}{ds} [\partial_\lambda b(x, su(x), s\nabla u(x), \lambda + t\mu) \\
&\quad - \partial_\lambda b(x, su(x), s\nabla u(x), \lambda)] ds \\
&= \int_0^1 [\partial_{\xi_0} \partial_\lambda b(x, su(x), s\nabla u(x), \lambda + t\mu) \\
&\quad - \partial_{\xi_0} \partial_\lambda b(x, su(x), s\nabla u(x), \lambda)] u(x) ds \\
&\quad + \sum_{i=1}^N [\partial_{\xi_i} \partial_\lambda b(x, su(x), s\nabla u(x), \lambda + t\mu) \\
&\quad - \partial_{\xi_i} \partial_\lambda b(x, su(x), s\nabla u(x), \lambda)] \partial_i u(x) ds.
\end{aligned}$$

Given $\varepsilon > 0$, it now follows from (16) that there exists $\delta > 0$ such that

$$\begin{aligned}
&|\partial_\lambda b(x, u(x), \nabla u(x), \lambda + t\mu) - \partial_\lambda b(x, u(x), \nabla u(x), \lambda)| \\
&< \varepsilon \left\{ |u(x)| + \sum_{i=1}^N |\partial_i u(x)| \right\}
\end{aligned}$$

for all $x \in \mathbb{R}^N$, all $t \in [0,1]$ and all $(\lambda, u), (\lambda+\mu, u) \in W = I \times V$ with $|\mu| < \delta$. Hence

$$|B(\lambda+\mu,u) - B(\lambda,u) - \mu C(\lambda,u)|_p$$
$$\leq \varepsilon\,|\mu| \left\{ \int_{\mathbb{R}^N} \int_0^1 \left\{ |u(x)| + \sum_{i=1}^N |\partial_i u(x)| \right\}^p dt dx \right\}^{1/p} \leq \varepsilon\,|\mu|\,\|u\|_{2,p}$$

showing that $\partial_\lambda B(\lambda,u)$ exists and is equal to $C(\lambda,u)$.

From the properties of B and C that have already been established we deduce that

$F \in C^1(\mathbb{R}\times X_p, Y_p)$ and that $D_u D_\lambda F$ exists and is continuous with $D_u D_\lambda F = D_u C \in C(\mathbb{R}\times X_p, Y_p)$.

Finally we show that $D_u B$ is differentiable with respect to λ and that $D_\lambda D_u B = D_u C$ (where $D_\lambda D_u F = D_u D_\lambda F$). For $\lambda, \mu \in \mathbb{R}$ and $u, v \in X_p$,

$$[D_u B(\lambda+\mu,u)v - D_u B(\lambda,u)v - \mu D_u C(\lambda,u)v]\,(x)$$
$$= \mu \left[\int_0^1 \partial_\lambda \partial_{\xi_0} b(x,u(x),\nabla u(x),\lambda+t\mu)dt \right] v(x)$$
$$+ \sum_{i=1}^N \left[\int_0^1 \partial_\lambda \partial_{\xi_i} b(x,u(x),\nabla u(x),\lambda+t\mu)dt \right] \partial_i v(x)$$
$$- \partial_{\xi_0}\partial_\lambda b(x,u(x),\nabla u(x),\lambda)v(x)$$
$$- \sum_{i=1}^N \partial_{\xi_i}\partial_\lambda b(x,u(x),\nabla u(x),\lambda)\partial_i v(x)\}$$
$$= \mu\{ \int_0^1 [\partial_\lambda \partial_{\xi_0} b(x,u(x),\nabla u(x),\lambda+t\mu)$$
$$- \partial_\lambda \partial_{\xi_0} b(x,u(x),\nabla u(x),\lambda)]dtv(x)$$
$$+ \sum_{i=1}^N \int_0^1 [\partial_\lambda \partial_{\xi_i} b(x,u(x),\nabla u(x),\lambda+t\mu)$$
$$- \partial_\lambda \partial_{\xi_i} b(x,u(x),\nabla u(x),\lambda)]dt\partial_i v(x)\}.$$

Given $\varepsilon > 0$, it now follows from (16) that there exists $\delta > 0$ such that

$$\left| \frac{[D_u B(\lambda+\mu,u)v - D_u B(\lambda,u)v - \mu D_u C(\lambda,u)v]\,(x)}{\mu} \right|$$
$$\leq \varepsilon \left\{ |v(x)| + \sum_{i=1}^N |\partial_i v(x)| \right\}$$

for all $x \in \mathbb{R}^N$ and for all $(\lambda, u), (\lambda + \mu, u) \in W$ with $|\mu| < \delta$. Thus

$$\left| \frac{[D_u B(\lambda + \mu, u) - D_u B(\lambda, u) - \mu D_u C(\lambda, u)]\, v}{\mu} \right|_p$$
$$\leq \varepsilon \left\{ |v|_p + \sum_{i=1}^{N} |\partial_i v|_p \right\} \leq \varepsilon \, \|v\|_{2,p}$$

and so

$$\left\| \frac{[D_u B(\lambda + \mu, u) - D_u B(\lambda, u) - \mu D_u C(\lambda, u)]}{\mu} \right\|_{L(X_p.Y_p)} \leq \varepsilon$$

for all $(\lambda, u), (\lambda + \mu, u) \in W$ with $|\mu| < \delta$.
Hence $D_\lambda D_u B(\lambda, u) = D_u C(\lambda, u)$. □

3 Properness and global bifurcation

In this section we formulate some general results about the bifurcation of global branches of solutions for quasilinear equations on $\mathbb{R}^N$. We begin with the abstract setting and then we use it to treat the equation (3).

Definition 2 *Let X and Y be real Banach spaces and consider a function $F \in C^1(J \times X, Y)$ where J is an open interval. Let $P(\lambda, u) = \lambda$ be the projection of $\mathbb{R} \times X$ onto $\mathbb{R}$. We say that J is an admissible interval for F provided that*
(i) for all $(\lambda, u) \in J \times X$, the bounded linear operator $D_u F(\lambda, u) : X \to Y$ is a Fredholm operator of index zero
and
(ii) for any compact subset K of Y and any closed bounded subset W of $\mathbb{R} \times X$ such that

$$\inf J < \inf PW \leq \sup PW < \sup J,$$

$F^{-1}(K) \cap W$ is a compact subset of $\mathbb{R} \times X$.

The conditions (i) and (ii) specify the appropriate versions of Fredholmness and properness which underlie the topological degree defined in [23]. That degree is based on the notion of the parity, denoted by $\pi\,(A(\lambda) : \lambda \in [a, b])$, of a continuous path, $\lambda \mapsto A(\lambda)$, of bounded linear Fredholm operators of index zero from X into Y. For such a path, a parametrix is any continuous function $B : [a, b] \to GL(Y, X)$ such that $B(\lambda)A(\lambda) : X \to X$

is a compact perturbation of the identity for each $\lambda \in [a,b]$. If $A(a)$ and $A(b) \in GL(X,Y)$, the parity of the path A on $[a,b]$ is defined by

$$\pi\,(A(\lambda) : \lambda \in [a,b]) = d_{LS}(B(a)A(a))d_{LS}(B(b)A(b))$$

where d_{LS} denotes the Leray-Schauder degree. This definition is justified by showing that a parametrix always exists and that $d_{LS}(B(a)A(a))$ $d_{LS}(B(b)A(b))$ is independent of the choice of parametrix B. In some circumstances the parity can be expressed in a form which is easier to check directly. In formulating our bifurcation theorems for (3) we shall only use the following criterion. Let $L(X,Y)$ denote the Banach space of all bounded linear operators from X into Y and let the kernel and range of a linear operator T be denoted by $\ker T$ and rge T, respectively.

Proposition 1 *Let $A : [a,b] \to L(X,Y)$ be a continuous path of bounded linear operators having the following properties.*

(i) $A \in C^1\left([a,b], L(X,Y)\right)$.
(ii) $A(\lambda) : X \to Y$ *is a Fredholm operator of index zero for all* $\lambda \in [a,b]$.
(iii) There exists $\lambda_0 \in (a,b)$ *such that*

$$A'(\lambda_0)[\ker\ A(\lambda_0)] \oplus \textit{rge}\ A(\lambda_0) = Y$$

in the sense of a topological direct sum.
Then there exists $\varepsilon > 0$ *such that* $[\lambda_0 - \varepsilon, \lambda_0 + \varepsilon] \subset [a,b]$*,*

$$A(\lambda) \in GL(X,Y) \textit{ for all } \lambda \in [\lambda_0 - \varepsilon, \lambda_0) \cup (\lambda_0, \lambda_0 + \varepsilon]$$

and

$$\pi\,(A(\lambda) : \lambda \in [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]) = (-1)^k$$

where $k = \dim \ker A(\lambda_0)$.

Proof. See Proposition 2.1 of [9] and Theorem 6.18 of [10].

Note that for any continuous path $A : [a,b] \to L(X,Y)$ and any $\lambda_0 \in (a,b)$ such that $A(\lambda) \in GL(X,Y)$ for all $\lambda \in [a,b] \setminus \{\lambda_0\}$, the parity $\pi\,(A(\lambda) : \lambda \in [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon])$ is the same for all $\varepsilon > 0$ provided that $[\lambda_0 - \varepsilon, \lambda_0 + \varepsilon] \subset [a,b]$. This quantity is called the parity of the path A across λ_0. The preceding proposition provides one way of calculating the parity across λ_0.

We can now state the main result about global bifurcation which can be derived using the above notions.

Theorem 2 *Let X and Y be real Banach spaces and consider a function $F \in C^1(J \times X, Y)$ where J is an open interval which is admissible for F. Suppose that $\lambda_0 \in J$ and that there exists $\varepsilon > 0$ such that*

$[\lambda_0-\varepsilon, \lambda_0+\varepsilon] \subset J$, *$D_uF(\lambda, 0) \in GL(X, Y)$ for $\lambda \in [\lambda_0-\varepsilon, \lambda_0+\varepsilon] \setminus \{\lambda_0\}$ and $\pi\,(D_uF(\lambda, 0), [\lambda_0-\varepsilon, \lambda_0+\varepsilon]) = -1$. Let $Z = \{(\lambda, u) \in J \times X : u \neq 0$ and $F(\lambda, u) = 0\}$ and let C denote the connected component of $Z \cup \{(\lambda_0, 0)\}$ containing $(\lambda_0, 0)$. Then C has at least one of the following properties.*

(i) *C is unbounded.*
(ii) *The closure of C contains a point $(\lambda_1, 0)$ where $\lambda_1 \in J \setminus [\lambda_0-\varepsilon, \lambda_0+\varepsilon]$ and $D_uF(\lambda_1, 0) \notin GL(X, Y)$.*
(iii) *The closure of PC intersects the boundary of J.*

The above statements refer to Z and C with the metric inherited from $\mathbb{R} \times X$. The basic procedure for proving a result like this is to suppose that C has none of the properties stated in the conclusion and then to derive a contradiction using the properties of whatever degree is available. Using the degree for proper Fredholm maps, variants of this result appear as Theorem 6.1 of [23] and Theorem 7.2 of [11]. In those results $J = \mathbb{R}$ so the property (iii) in the conclusion can be dropped. The form stated above can be proved using the degree defined in [23]. Note that, if we assume that C does not have the property (iii), the open set Ω used in the proof of Theorem 6.1 of [23] can be chosen so that

$$\inf J < \inf PC \leq \sup PC < \sup J$$

and then the properties of the degree lead to a contradiction in the usual way. See Theorem 7.2 of [11], for example.

Clearly the interest of this result hinges on the extent to which the parity can be calculated and sharp explicit conditions found for admissible intervals. The relationship between the parity and the spectral/transversality properties of the linearization of F has been thoroughly investigated [10] and, in the context of quasilinear equations, Theorem 1 enables us to exploit these results in a straight forward way. For a broad class of quasilinear elliptic operators on $\mathbb{R}^N$ we have characterized in [25] the Fredholm and properness properties which determine admissible intervals.

For the rest of this section we fix $p \in (N, \infty)$ and consider the differential operator $F : \mathbb{R} \times X_p \to Y_p$ defined by (5) under the assumption (B). By Theorem 1 this already ensures that $F \in C^1(\mathbb{R} \times X_p, Y_p)$. Our next results deal with a situation where the parity of the path $\lambda \mapsto D_uF(\lambda, 0)$ across a value λ_0 can be determined in a relatively explicit way. They use the following assumption (L) which ensures that $D_uF(\lambda, 0)$ has a particularly simple form. More general behaviour at $u = 0$ can be handled provided that the asymptotic behaviour required for the discussion of properness is assumed and we shall return to this in due course.

(L) There is a (constant) positive definite matrix $[A_{\alpha\beta}]$ such that

$$a_{\alpha\beta}(x,0) = A_{\alpha\beta} = A_{\beta\alpha} \text{ for all } x \in \mathbb{R}^N$$

and

$$\partial_{\xi_\alpha} b(x,0,\lambda) = 0 \text{ for all } x \in \mathbb{R}^N \text{ and } \lambda \in \mathbb{R}$$

for all $\alpha, \beta = 1, .., N$,

Under the hypotheses (B) and (L), it follows from Theorem 1 that

$$\begin{aligned}&[D_u F(\lambda,0)v](x)\\ &= -\sum_{\alpha,\beta=1}^{N} A_{\alpha\beta}\partial_\alpha\partial_\beta v(x) + \partial_{\xi_0} b(x,0,\lambda)v(x)\end{aligned}$$

which can be reduced to the form

$$[D_u F(\lambda,0)v](x) = -\Delta v(x) + \partial_{\xi_0} b(x,0,\lambda)v(x) \tag{17}$$

by a linear change of the variable x. In this case the parity can be calculated from the spectral properties of the operator $-\Delta + \partial_{\xi_0} b(x,0,\lambda)$ using the results we obtained in [26]. Let us fix some notation and terminology which will be used henceforth.

We refer to [7] for the notions of spectrum, discrete spectrum and essential spectrum of an unbounded self-adjoint operator acting on a Hilbert space. The discrete spectrum consists of the isolated points in the spectrum which are eigenvalues of finite multiplicity. Those points in the spectrum which do not belong to the discrete spectrum form the essential spectrum.

In [26] we discussed the Fredholm properties of the operator $-\Delta + V$ in $L^p(\mathbb{R}^N)$ for a class of potentials admitting singularities. To deal with (17) it is sufficient to recall the following special case which appears as Theorem 1 in [26].

Theorem 3 *Let $V \in L^\infty(\mathbb{R}^N)$. Then $S_2 = -\Delta + V : W^{2,2}(\mathbb{R}^N) \subset L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is a self-adjoint operator whose spectrum and discrete spectrum are denoted by σ and σ_d respectively. For $p \in (1,\infty)$, consider also the operator $S_p : W^{2,p}(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ defined by*

$$S_p u = (-\Delta + V)u \text{ for } u \in W^{2,p}(\mathbb{R}^N).$$

For every $p \in (1,\infty)$, the following conclusions are valid.

(i) $S_p - \lambda I : W^{2,p}(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ is an isomorphism if $\lambda \notin \sigma$, whereas, if $\lambda \in \sigma_d$, then

(ii) $S_p - \lambda I : W^{2,p}(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ is a Fredholm operator of index zero,

(iii) *ker* $(S_p - \lambda I) =$ *ker* $(S_2 - \lambda I)$*, and*
(iv) $L^p(\mathbb{R}^N) =$ *ker* $(S_p - \lambda I) \oplus$ *rge* $(S_p - \lambda I)$ *where* $\oplus$ *denotes a topological direct sum.*

We now use this result to discuss the parity of the path $\lambda \mapsto D_uF(\lambda, 0)$.

Lemma 2 *Suppose that the conditions (B) and (L) are satisfied and consider* $\lambda_0 \in \mathbb{R}$ *such that* 0 *does not belong to the essential spectrum of the self-adjoint operator* $-\Delta + \partial_{\xi_0} b(x, 0, \lambda_0) : X_2 \subset Y_2 \to Y_2$*. Let* $p \in (N, \infty)$*.*

(i) Then $D_uF(\lambda_0, 0) : X_p \to Y_p$ *is a Fredholm operator of index zero.*
(ii) Furthermore, if

$$\textit{either } \partial_\lambda \partial_{\xi_0} b(\cdot, 0, \lambda_0) \geq 0 \textit{ but } \not\equiv 0 \textit{ on } \mathbb{R}^N,$$
$$\textit{or } \partial_\lambda \partial_{\xi_0} b(\cdot, 0, \lambda_0) \leq 0 \textit{ but } \not\equiv 0 \textit{ on } \mathbb{R}^N,$$

then there exists $\varepsilon > 0$ *such that* $D_uF(\lambda, 0) \in GL(X_p, Y_p)$ *for all* $\lambda \in [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon] \setminus \{\lambda_0\}$ *and*

$$\pi(D_uF(\lambda, 0), [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]) = (-1)^k$$

where $k = \dim \ker[-\Delta + \partial_{\xi_0} b(x, 0, \lambda_0)]$.

Remark 3.1 By Theorem 3(iii), the kernel of the linear operator $-\Delta + \partial_{\xi_0} b(x, 0, \lambda) : X_p \to Y_p$ does not depend on the choice of $p \in (1, \infty)$.

Proof. The first statement follows immediately from Theorem 3 with $V = \partial_{\xi_0} b(\cdot, 0, \lambda)$. To show that the parity across λ_0 is well-defined and equal to $(-1)^k$ it suffices, by Proposition 1 with $A(\lambda) = D_uF(\lambda, 0)$, to prove that

$$D_\lambda D_uF(\lambda_0, 0)[\ker D_uF(\lambda_0, 0)] \oplus \mathrm{rge} D_uF(\lambda_0, 0) = Y_p.$$

Under the additional assumptions of part (ii), suppose first that

$$u \in D_\lambda D_uF(\lambda_0, 0)[\ker D_uF(\lambda_0, 0)] \cap \mathrm{rge} D_uF(\lambda_0, 0).$$

Then there exist $v, w \in X_p$ such that

$$u = \partial_\lambda \partial_{\xi_0} b(x, 0, \lambda_0) v = -\Delta w + \partial_{\xi_0} b(x, 0, \lambda_0) w$$
$$\text{and } -\Delta v + \partial_{\xi_0} b(x, 0, \lambda_0) v = 0.$$

By Theorem 3(iii), $v \in X_q$ for all $q \in (1, \infty)$ and so

$$\begin{aligned} 0 &= \int_{\mathbb{R}^N} [-\Delta v + \partial_{\xi_0} b(x, 0, \lambda_0) v] w dx \\ &= \int_{\mathbb{R}^N} v[-\Delta w + \partial_{\xi_0} b(x, 0, \lambda_0) w] dx \\ &= \int_{\mathbb{R}^N} \partial_\lambda \partial_{\xi_0} b(x, 0, \lambda_0) v^2 dx \end{aligned}$$

from which it follows that $\partial_\lambda \partial_{\xi_0} b(x, 0, \lambda_0) v^2 = 0$ on $\mathbb{R}^N$. But, since $\partial_\lambda \partial_{\xi_0} b(\cdot, 0, \lambda_0)$ is either nonnegative or nonpositive, this implies that $\partial_\lambda \partial_{\xi_0} b(x, 0, \lambda_0) = 0$ or $v(x) = 0$ at each point. Thus, $\partial_\lambda \partial_{\xi_0} b(\cdot, 0, \lambda_0) v = 0$, that is, $u = 0$. This shows that $D_\lambda D_u F(\lambda_0, 0)[\ker D_u F(\lambda_0, 0)] \cap \text{rge}\, D_u F(\lambda_0, 0) = \{0\}$.

Since we already know from part (i) that $D_u F(\lambda_0, 0)$ is a Fredholm operator of index zero, it remains to show that $\dim D_\lambda D_u F(\lambda_0, 0)[\ker D_u F(\lambda_0, 0)] = k$. But $D_\lambda D_u F(\lambda_0, 0)[\ker D_u F(\lambda_0, 0)] = \{\partial_\lambda \partial_{\xi_0} b(x, 0, \lambda_0) v : v \in \ker D_u F(\lambda_0, 0)\}$ is a closed subspace of Y_p whose dimension cannot exceed that of $\ker D_u F(\lambda_0, 0)$.

If $v \in \ker D_u F(\lambda_0, 0)$ and $\partial_\lambda \partial_{\xi_0} b(\cdot, 0, \lambda_0) v = 0$, the hypothesis that $\partial_\lambda \partial_{\xi_0} b(\cdot, 0, \lambda_0)$ is not identically 0 shows that there is a non-empty open set $\Omega \subset \mathbb{R}^N$ such that $v \equiv 0$ on Ω and so, by (17) and the unique continuation principle (see Theorem C.9.1 of [31] for example), $v \equiv 0$ on $\mathbb{R}^N$. This shows that the multiplication by $\partial_\lambda \partial_{\xi_0} b(\cdot, 0, \lambda_0)$ is one-to-one on $\ker D_u F(\lambda_0, 0)$ and hence that $\dim D_\lambda D_u F(\lambda_0, 0)[\ker D_u F(\lambda_0, 0)] = k = \dim \ker D_u F(\lambda_0, 0)$ and the proof is complete. □

The following condition characterizes a frequently occurring situation in which the interval J becomes a gap in the essential spectrum of a Schrödinger operator.

(LL) The condition (L) is satisfied and there is a constant $c \neq 0$ such that

$$\partial_\lambda \partial_{\xi_0} b(x, 0, \lambda) = c \text{ for all } x \in \mathbb{R}^N \text{ and } \lambda \in \mathbb{R}.$$

When (B) and (LL) are satisfied we can and shall suppose (by redefining λ) that there exists a function $V \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ such that

$$\partial_{\xi_0} b(x, 0, \lambda) = V(x) - \lambda \text{ for all } x \in \mathbb{R}^N \text{ and } \lambda \in \mathbb{R}.$$

In this case

$$D_u F(\lambda, 0) v = (-\Delta + V) v - \lambda v \tag{18}$$

where $-\Delta + V : X_2 \subset Y_2 \to Y_2$ is a self-adjoint operator.

Corollary 1 *Let the conditions (B) and (LL) be satisfied and let* $p \in (N, \infty)$. *Consider an open interval* $J \subset \mathbb{R} \backslash \sigma_e$ *where* σ_e *denotes the essential spectrum of the self-adjoint operator* $-\Delta + V : X_2 \subset Y_2 \to Y_2$. *Then* $D_u F(\lambda, 0) : X_p \to Y_p$ *is a Fredholm operator of index zero for all* $\lambda \in J$. *If* $\lambda_0 \in J$ *is an eigenvalue of odd multiplicity of* $-\Delta + V : X_2 \subset Y_2 \to Y_2$ *there exists* $\varepsilon > 0$ *such that* $D_u F(\lambda, 0) \in GL(X_p, Y_p)$ *for all* $\lambda \in [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon] \backslash \{\lambda_0\}$ *and* $\pi(D_u F(\lambda, 0), [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]) = -1$.

Proof. This follows immediately from Lemma 2. □

We now turn to the more substantial problem of determining admissible intervals for an operator of the form (5). For this we introduce assumptions concerning its ellipticity and asymptotic behaviour as $|x| \to \infty$. The asymptotic behaviour plays a crucial role in ensuring the properness of F, but as we shall see it also implies that $D_u F(\lambda, 0) : X_p \to Y_p$ is a Fredholm operator of index zero without requiring the special structure assumed in condition (L).

(E) The operator F is strictly elliptic in the sense that there exists a continuous function, $\nu : \mathbb{R}^N \times \mathbb{R}^{N+1} \to (0, \infty)$, such that

$$\sum_{\alpha,\beta=1}^{N} a_{\alpha\beta}(x,\xi)\eta_\alpha\eta_\beta \geq \nu(x,\xi)\,|\eta|^2$$

for all $\eta \in \mathbb{R}^N$ and $(x,\xi) \in \mathbb{R}^N \times \mathbb{R}^{N+1}$.

(A) There exist equicontinuous C^0-bundle maps $a^\infty_{\alpha\beta} = a^\infty_{\beta\alpha} : \mathbb{R}^N \times \mathbb{R}^{N+1} \to \mathbb{R}$ for $\alpha, \beta = 1,..,N$ and an equicontinuous $C^1_\eta-$bundle map $b^\infty : \mathbb{R}^N \times \mathbb{R}^{N+2} \to \mathbb{R}$ such that $b^\infty(x, 0, \lambda) \equiv 0$ and

$$\lim_{|x|\to\infty} [a_{\alpha\beta}(x,\xi) - a^\infty_{\alpha\beta}(x,\xi)] = \lim_{|x|\to\infty} [\partial_{\xi_i} b(x,\xi,\lambda) - \partial_{\xi_i} b^\infty(x,\xi,\lambda)] \\ = 0$$

uniformly for (ξ, λ) in bounded subsets of $\mathbb{R}^{N+2}$, where $1 \leq \alpha, \beta \leq N$ and $i = 0, 1, .., N$. Furthermore, $a^\infty_{\alpha\beta}(\cdot,\xi)$ and $b^\infty(\cdot,\xi,\lambda) : \mathbb{R}^N \to \mathbb{R}$ are $N-$periodic on $\mathbb{R}^N$ in the sense that, for some $T = (T_1, .., T_N)$ with $T_i > 0$ for all $i = 1, .., N$,

$$a^\infty_{\alpha\beta}(x_1, ..., x_i + T_i, ..., x_N, \xi) = a^\infty_{\alpha\beta}(x_1, ..., x_N, \xi)$$

and

$$b^\infty(x_1, ..., x_i + T_i, ..., x_N, \xi, \lambda) = b^\infty(x_1, ..., x_N, \xi, \lambda)$$

for all $(x, \xi, \lambda) \in \mathbb{R}^N \times \mathbb{R}^{N+2}$ and $i = 1, .., N$.

Under the assumptions (B) and (A) we define a differential operator, F^∞, by (6).

Theorem 4 *Let the conditions (B), (E) and (A) be satisfied. Choose $p \in (N, \infty)$ and consider the operator $F : \mathbb{R} \times X_p \to Y_p$ defined by (5). An open interval J is admissible for F provided that for all $\lambda \in J$,*

(i) the linear differential operator $D_u F(\lambda, 0) : X_p \to Y_p$ is Fredholm of index zero and

(ii) $\{u \in X_p : F^\infty(\lambda, u) = 0\} = \{0\}$.

Remark 3.2 The assumptions (B) and (A) imply that $F^\infty(\lambda, u) \in Y_p$ for all $(\lambda, u) \in \mathbb{R} \times X_p$ and that $F^\infty(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$.

Proof. By Theorem 1 we already know that $F \in C^1(\mathbb{R} \times X_p, Y_p)$ so we need only prove that the condition (ii) in Definition 2 is satisfied. With this in mind, let K be a compact subset of Y_p and W a closed bounded subset of $\mathbb{R} \times X_p$ such that

$$\inf J < \inf PW \leq \sup PW < \sup J.$$

Consider an arbitrary sequence $\{(\lambda_n, u_n)\} \subset F^{-1}(K) \cap W$. We only have to prove that $\{(\lambda_n, u_n)\}$ contains a convergent subsequence. Since $\{F(\lambda_n, u_n)\} \subset K$ there exist $\lambda \in J, v \in K$ and a subsequence such that

$$\lambda_{n_i} \to \lambda \text{ and } v_{n_i} = F(\lambda_{n_i}, u_{n_i}) \to v \text{ strongly in } Y_p.$$

Now

$$F(\lambda, u_{n_i}) = F(\lambda, u_{n_i}) - F(\lambda_{n_i}, u_{n_i}) + F(\lambda_{n_i}, u_{n_i})$$

Since $\{u_{n_i}\}$ is a bounded subset of X_p, the equicontinuity of $F(\cdot, u)$ for u in bounded subsets of X_p (Theorem 1) shows that $|F(\lambda, u_{n_i}) - F(\lambda_{n_i}, u_{n_i})|_p \to 0$ and hence we have that $F(\lambda, u_{n_i}) \to v$ strongly in Y_p. But by Theorem 6.1 of [25] the restriction of $F(\lambda, \cdot) : X_p \to Y_p$ to closed bounded subsets of X_p is proper. Since $\{u_{n_i}\}$ is a bounded subset of X_p, this implies that $\{u_{n_i}\}$ has a subsequence converging to an element u in X_p. Hence $\{(\lambda_{n_i}, u_{n_i})\}$ has a subsequence converging to (λ, u) as required. □

Lemma 2 and Corollary 1 furnish explicit assumptions on the operator (5) ensuring that condition (i) of Theorem 4 is satisfied. Assumptions implying condition (ii) can be derived in various ways depending on the form of the equation $F^\infty(\lambda, u) = 0$. As explained in the Introduction, we demonstrate three different approaches to doing this. Before doing so we show how the condition (i) can be verified by using properties of the asymptotic limit even when the operator (5) does not have the property (L).

First of all we recall from Sect. 6 of [25] that, although the assumption (A) does not guarantee the differentiability of the operator $F^\infty : \mathbb{R} \times X_p \to Y_p$, it does imply that $F^\infty(\lambda, \cdot) : X_p \to Y_p$ is differentiable (in the sense of Fréchet) at 0 with

$$D_u F^\infty(\lambda, 0)v$$
$$= -\sum_{\alpha,\beta=1}^{N} a_{\alpha\beta}^\infty(\cdot, 0)\partial_\alpha\partial_\beta v + \sum_{\alpha=1}^{N} \partial_{\xi_\alpha} b^\infty(\cdot, 0, \lambda)\partial_\alpha v + \partial_{\xi_0} b^\infty(\cdot, 0, \lambda)v$$

for all $v \in X_p$ and $\lambda \in \mathbb{R}$.
We note that $D_uF^\infty(\lambda, 0)$ is a linear second order differential operator with continuous $N-$periodic coefficients. In [25], Lemma 6.6 and Remark 6.2 describe some situations where it is a Fredholm operator of index zero. The following assumption isolates a particularly agreeable situation.

(L$^\infty$) There is a (constant) positive definite matrix $\left[A^\infty_{\alpha\beta}\right]$ such that

$$a^\infty_{\alpha\beta}(x,0) = A^\infty_{\alpha\beta} = A^\infty_{\beta\alpha} \text{ for all } x \in \mathbb{R}^N$$

and

$$\partial_{\xi_\alpha} b^\infty(x,0,\lambda) = 0 \text{ for all } x \in \mathbb{R}^N \text{ and } \lambda \in \mathbb{R}$$

for $1 \le \alpha, \beta \le N$.

Remark 3.3 When this condition is satisfied we can assume that $A^\infty_{\alpha\beta} = \delta_{\alpha\beta}$ for $1 \le \alpha, \beta \le N$ (by making a linear change of variable) and hence

$$D_uF^\infty(\lambda,0)v = -\Delta v + \partial_{\xi_0} b^\infty(\cdot,0,\lambda)v$$

for $v \in X_p$ and $\lambda \in \mathbb{R}$.

Lemma 3 *Suppose that the conditions (B), (A) and (L$^\infty$) are satisfied and consider $\lambda_0 \in \mathbb{R}$ such that the self-adjoint operator, $S(\lambda_0) : X_2 \subset Y_2 \to Y_2$, defined by*

$$S(\lambda_0)v = -\Delta v + \partial_{\xi_0} b^\infty(\cdot,0,\lambda_0)v \text{ for } v \in X_2$$

is an isomorphism. Let $p \in (N, \infty)$.

(i) $D_uF(\lambda_0, 0) : X_p \to Y_p$ *is a Fredholm operator of index zero.*
(ii) Let $\{\varphi_i \in X_p : i = 1, .., k\}$ and $\{\psi_i \in Y_q : i = 1, .., k\}$ be bases for $\ker D_uF(\lambda_0,0)$ and $\ker [D_uF(\lambda_0,0)]^*$ respectively, with $k = \dim \ker D_uF(\lambda_0,0)$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\det\left[\int_{\mathbb{R}^N} \psi_i \left\{D_\lambda D_u F(\lambda_0,0)\varphi_j\right\} dx\right] \neq 0 \tag{19}$$

if and only if

$$D_\lambda D_u F(\lambda_0,0)\left[\ker D_uF(\lambda_0,0)\right] \oplus \textit{rge}\, D_uF(\lambda_0,0) = Y_p. \tag{20}$$

When (19) is satisfied there exists $\varepsilon > 0$ such that $D_uF(\lambda, 0) \in GL(X_p, Y_p)$ for all $\lambda \in [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon] \backslash \{\lambda_0\}$ and

$$\pi\left(D_uF(\lambda,0), [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]\right) = (-1)^k.$$

Remark 3.4 This result gives the same conclusions as Lemma 2 without requiring $D_uF(\lambda, 0)$ to be a formally symmetric differential operator. Note that when the conditions (B), (E), (A) and (L) are satisfied then so is (L^∞) with $A^\infty_{\alpha\beta} = A_{\alpha\beta}$. We also observe that, since $\partial_{\xi_0} b^\infty(\cdot, 0, \lambda_0)$ is an $N-$ periodic function, $S(\lambda_0)$ is an isomorphism if and only if 0 does not belong to its essential spectrum. (See Theorem 5.4 of Chapter 3 in [2].) Moreover when there exists a continuous $N-$ periodic function P such that $\partial_{\xi_0} b^\infty(x, 0, \lambda_0) \equiv P(x) - \lambda_0, S(\lambda_0) = -\Delta + P - \lambda_0$ and it is an isomorphism if and only if λ_0 does not belong to the spectrum of the $N-$ periodic Schrödinger operator $-\Delta + P$. Thus a result analogous to Corollary 1 can easily be formulated. In particular, when the condition (LL) is satisfied with $\partial_{\xi_0} b(x, 0, \lambda) \equiv V(x) - \lambda$, the condition ($\mathrm{L}^\infty$) is also satisfied and $\partial_{\xi_0} b^\infty(x, 0, \lambda_0) \equiv P(x) - \lambda$ where P is a continuous $N-$periodic function. Since $\lim_{|x|\to\infty} \{V(x) - P(x)\} = 0$, it follows that the essential spectrum of the Schrödinger operator $-\Delta + V : X_2 \subset Y_2 \to Y_2$ is equal to the whole spectrum of the $N-$periodic Schrödinger operator $-\Delta + P : X_2 \subset Y_2 \to Y_2$ (see [2]).

Proof. (i) It follows from Theorem 3(ii) and (iii), with $V = \partial_{\xi_0} b^\infty(\cdot, 0, \lambda_0)$, that $D_uF^\infty(\lambda_0, 0) \in GL(X_p, Y_p)$.
There is a constant $\tau > 0$ such that

$$\sum_{\alpha,\beta=1}^{N} A^\infty_{\alpha\beta}\xi_\alpha\xi_\beta \geq \tau\,|\xi|^2 \text{ for all } \xi \in \mathbb{R}^N$$

and so, by (A), there is a constant $z > 0$ such that

$$\sum_{\alpha,\beta=1}^{N} a_{\alpha\beta}(x, 0)\xi_\alpha\xi_\beta \geq \frac{\tau}{2}\,|\xi|^2 \text{ for all } \xi \in \mathbb{R}^N$$

provided that $|x| \geq z$. Using (E), we see that there is a constant $\tau_0 > 0$ such that

$$\sum_{\alpha,\beta=1}^{N} a_{\alpha\beta}(x, 0)\xi_\alpha\xi_\beta \geq \tau_0\,|\xi|^2 \text{ for all } x, \xi \in \mathbb{R}^N.$$

It now follows from Lemma 6.5 in [25] that $D_uF(\lambda_0, 0) : X_p \to Y_p$ is a Fredholm operator of index zero.

(ii) Since $D_uF(\lambda_0, 0) : X_p \to Y_p$ is a Fredholm operator of index zero, $\mathrm{rge}\, D_uF(\lambda_0, 0)$ is a closed subspace of Y_p and hence

$$\mathrm{rge}\, D_uF(\lambda_0, 0) = \{w \in Y_p : \psi(w) = 0 \text{ for all } \psi \in \ker\,[D_uF(\lambda_0, 0)]^*\}$$

where $[D_uF(\lambda_0,0)]^* : [Y_p]^* \to [X_p]^*$. Hence

$$\text{rge } D_uF(\lambda_0,0) = \left\{ w \in Y_p : \int_{\mathbb{R}^N} \psi_i w dx = 0 \text{ for } i = 1,..,k \right\}$$

when we make the usual identification of $[Y_p]^*$ with Y_q.
Suppose first that

$$D_\lambda D_uF(\lambda_0,0)\,[\ker D_uF(\lambda_0,0)] \oplus \text{rge } D_uF(\lambda_0,0) = Y_p.$$

This implies $\dim D_\lambda D_uF(\lambda_0,0)\,[\ker D_uF(\lambda_0,0)] = \text{codim rge } D_uF(\lambda_0,0) = k$ and so $D_\lambda D_uF(\lambda_0,0)$ must be one-to-one on $\ker D_uF(\lambda_0,0)$. Furthermore, if $w \in \ker D_uF(\lambda_0,0)$ and $\int_{\mathbb{R}^N} \psi_i\,[D_\lambda D_uF(\lambda_0,0)]\,w dx = 0$ for $i = 1,..,k$, then $D_\lambda D_uF(\lambda_0,0)w = 0$. Since $D_\lambda D_uF(\lambda_0,0)$ is one-to-one on $\ker D_uF(\lambda_0,0)$, this means that $w = 0$, and, expressing w in the form $\sum_{j=1}^k \alpha_j\varphi_j$ using the basis $\{\varphi_j \in X_p : j = 1,..,k\}$, we conclude that

$$\sum_{j=1}^k M_{ij}\alpha_j = 0 \text{ for } i = 1,..,k$$

implies that $\alpha = (\alpha_1,..,\alpha_k) = 0$ where

$$M_{ij} = \int_{\mathbb{R}^N} \psi_i\,[D_\lambda D_uF(\lambda_0,0)]\,\varphi_j dx.$$

Thus $\det M \neq 0$ where M denotes the $(k \times k)-$matrix with elements M_{ij}. Conversely, suppose that $\det M \neq 0$. Then $M\alpha = 0$ implies that $\alpha = 0$ and so, if $w \in \ker D_uF(\lambda_0,0)$ and $\int_{\mathbb{R}^N} \psi_i\,[D_\lambda D_uF(\lambda_0,0)]\,w dx = 0$ for $i = 1,..,k$, we can conclude that $w = 0$. This shows that

$$D_\lambda D_uF(\lambda_0,0)\,[\ker D_uF(\lambda_0,0)] \cap \text{rge } D_uF(\lambda_0,0) = \{0\}\,.$$

But it also means that, if $w \in \ker D_uF(\lambda_0,0)$ and $[D_\lambda D_uF(\lambda_0,0)]\,w = 0$, then $w = 0$. Thus $D_\lambda D_uF(\lambda_0,0)$ is one-to-one on $\ker D_uF(\lambda_0,0)$ and so $\dim D_\lambda D_uF(\lambda_0,0)\,[\ker D_uF(\lambda_0,0)] = \dim\ker D_uF(\lambda_0,0) = k =$ codim rge $D_uF(\lambda_0,0)$. This proves the equivalence of (19) and (20).

As in the proof of Lemma 2, the proof is completed by appealing to Proposition 1. □

Combining the above results we obtain the following rather general bifurcation theorem.

Theorem 5 *Let the conditions (B),(E),(A) and (L^∞) be satisfied. Choose $p \in (N,\infty)$ and consider the operator $F : \mathbb{R} \times X_p \to Y_p$ defined by (5). Suppose that J is an open interval having the following properties.*

(a) For all $\lambda \in J$, $\{u \in X_p : F^\infty(\lambda,u) = 0\} = \{0\}$.

(b) *For all* $\lambda \in J$, *the self-adjoint operator* $-\Delta + \partial_{\xi_0} b^\infty(\cdot, 0, \lambda) : X_2 \subset Y_2 \to Y_2$ *is an isomorphism.*
(c) *There is a point* $\lambda_0 \in J$ *such that* $\dim \ker D_u F(\lambda_0, 0)$ *is odd and the condition (19) is satisfied.*

Let C *denote the connected component of* $Z \cup \{(\lambda_0, 0)\}$ *containing* $(\lambda_0, 0)$ *where* $Z = \{(\lambda, u) \in J \times X_p : u \neq 0$ *and* $F(\lambda, u) = 0\}$ *and* $Z \cup \{(\lambda_0, 0)\}$ *has the metric inherited from* $\mathbb{R} \times X_p$. *Then* C *has at least one of the following properties.*

(i) C *is an unbounded subset of* $J \times X_p$.
(ii) *The closure of* C *in* $\overline{J} \times X_p$ *contains a point* $(\lambda_1, 0)$ *where* $\lambda_1 \neq \lambda_0$.
(iii) *The closure of* $\{\lambda : (\lambda, u) \in C$ *for some* $u \in X_p\}$ *intersects the boundary of* J.

Proof. By the hypothesis (b) and Lemma 3(i), $D_u F(\lambda, 0) : X_p \to Y_p$ is a Fredholm operator of index zero for all $\lambda \in J$. Using this and the hypothesis (a), it follows from Theorem 4 that the interval J is admissible for F. Finally, the assumption (c) and Lemma 3(ii) show that all of the hypotheses of Theorem 2 are satisfied by $F : \mathbb{R} \times X_p \to Y_p$ and the result follows. □

Theorem 6 *Let the conditions (B),(E),(A) and (LL) be satisfied with*

$$\partial_{\xi_0} b(x, 0, \lambda) = V(x) - \lambda.$$

Choose $p \in (N, \infty)$ *and consider the operator* $F : \mathbb{R} \times X_p \to Y_p$ *defined by (5). Suppose that* J *is an open interval having the following properties.*

(a) *For all* $\lambda \in J, \{u \in X_p : F^\infty(\lambda, u) = 0\} = \{0\}$.
(b) $J \subset \mathbb{R} \backslash \sigma_e$, *where* σ_e *denotes the essential spectrum of the self-adjoint operator* $-\Delta + V : X_2 \subset Y_2 \to Y_2$.
(c) $\lambda_0 \in J$ *is an eigenvalue of odd multiplicity of* $-\Delta + V : X_2 \subset Y_2 \to Y_2$.

Let C *denote the connected component of* $Z \cup \{(\lambda_0, 0)\}$ *containing* $(\lambda_0, 0)$. *The conclusion of Theorem 5 holds.*

Remark 3.5 It follows from (LL) and (A) that (L^∞) is also satisfied with

$$\partial_{\xi_0} b^\infty(x, 0, \lambda) = P(x) - \lambda$$

where P is a continuous $N-$periodic function such that

$$\lim_{|x| \to \infty} \{V(x) - P(x)\} = 0.$$

As was pointed out in the Remark 3.4, $J \subset \mathbb{R} \backslash \sigma_e \iff J \subset \mathbb{R} \backslash \Sigma$ where Σ denotes the spectrum of the self-adjoint operator $-\Delta + P : X_2 \subset Y_2 \to Y_2$.

Proof. Using Corollary 1 and Theorem 4 this follows from Theorem 2. □

4 Special cases

In this section we give more explicit assumptions on the functions $a_{\alpha\beta}$ and b which imply that the hypotheses of Theorem 5 are satisfied.

4.1 Using the maximum principle

The maximum principle can be used to establish the condition (ii) of Theorem 4 provided that the functions $a^\infty_{\alpha\beta}$ and b^∞ have the following properties.

(M$_\lambda$) There exists a continuous function $\nu : \mathbb{R}^{N+1} \to (0,\infty)$ such that

$$\sum_{\alpha,\beta=1}^{N} a^\infty_{\alpha\beta}(x,s,0)\eta_\alpha\eta_\beta \geq \nu(x,s)\,|\eta|^2 \text{ for all } \eta \in \mathbb{R}^N$$

and for all $(x,s) \in \mathbb{R}^{N+1}$ and

$$b^\infty(x,s,0,\lambda)s > 0 \text{ for all } (x,s) \in \mathbb{R}^{N+1} \text{ with } s \neq 0.$$

Remark 4.1 It follows from this that $\partial_{\xi_0} b^\infty(x,0,\lambda) \geq 0$ for all $x \in \mathbb{R}^N$.

Theorem 7 *Let the conditions (B),(E) and (A) be satisfied and let* $p \in (N,\infty)$. *Suppose that* J *is an open interval such that (M$_\lambda$) is satisfied for all* $\lambda \in J$.

(i) Suppose that the condition (L$^\infty$) is satisfied and that the operator $S(\lambda) = -\Delta + \partial_{\xi_0} b^\infty(\cdot,0,\lambda) : X_2 \to Y_2$ *is an isomorphism for all* $\lambda \in J$. *Then* J *is an admissible interval for the operator* $F : \mathbb{R}\times X_p \to Y_p$ *defined by (5) and the conclusion of Theorem 5 is valid in this context at any point* $\lambda_0 \in J$ *such that* $\dim\ker D_u F(\lambda_0,0)$ *is odd and the condition (19) is satisfied with* $\lambda = \lambda_0$.

(ii) If the condition (LL) is satisfied with $\partial_{\xi_0} b(x,0,\lambda) \equiv V(x) - \lambda$, *then* $J \subset (-\infty,\omega)$ *where* $\omega \equiv \liminf_{|x|\to\infty} V(x)$ *and* J *is an admissible interval for the operator* $F : \mathbb{R}\times X_p \to Y_p$ *defined by (5). Furthermore, for every eigenvalue* $\lambda_0 \in J$ *of odd multiplicity of* $-\Delta + V : X_2 \subset Y_2 \to Y_2$ *the conclusion of Theorem 5 is valid in this context.*

Remark 4.2 Under the hypotheses of part (ii), there exists a continuous $N-$periodic function P such that $\lim_{|x|\to\infty}\{V(x) - P(x)\} = 0$ and $\partial_{\xi_0} b^\infty(x,0,\lambda) \equiv P(x) - \lambda$. Thus $\omega = \inf\left\{P(x) : x \in \mathbb{R}^N\right\}$ and so the condition (M$_\lambda$) implies that $\lambda \leq \omega$ for all $\lambda \in J$. But J is open so in fact $J \subset (-\infty,\omega)$. Furthermore, by Remark 3.4, the essential spectrum, σ_e, of the Schrödinger operator $-\Delta + V : X_2 \subset Y_2 \to Y_2$ is equal to the whole spectrum of the $N-$periodic Schrödinger operator $-\Delta + P : X_2 \subset Y_2 \to Y_2$ and so $J \subset \mathbb{R}\backslash\sigma_e$.

Proof. Using Theorem 5 for part (i) and Theorem 6 for part (ii), we need only prove that if $(\lambda, u) \in J \times X_p$ and $F^\infty(\lambda, u) = 0$ then $u = 0$.

Suppose that $(\lambda, u) \in J \times X_p$ and that

$$-\sum_{\alpha,\beta=1}^{N} a^\infty_{\alpha\beta}(x, u(x), \nabla u(x))\partial_\alpha \partial_\beta u(x) + b^\infty(x, u(x), \nabla u(x), \lambda) = 0$$

for almost all $x \in \mathbb{R}^N$. Recalling (9) and (10), we set $M = \sup\{u(x) : x \in \mathbb{R}^N\}$ and $\Omega = \{x \in \mathbb{R}^N : u(x) = M\}$. The continuity of u implies that Ω is a closed subset of $\mathbb{R}^N$. Suppose that $M > 0$. Since $u \in C^1(\mathbb{R}^N)$ and $\lim_{|x|\to\infty} u(x) = 0$, Ω is non-empty and bounded and there exists $x_0 \in \Omega$ such that $\nabla u(x_0) = 0$. (In fact, $\nabla u(x) = 0$ for all $x \in \Omega$.) Thus $b^\infty(x_0, M, 0, \lambda) > 0$ by assumption (M_λ) and so there exists $\varepsilon, \nu > 0$ such that $b^\infty(x, u(x), \nabla u(x), \lambda) > 0$ and

$$\sum_{\alpha,\beta=1}^{N} a^\infty_{\alpha\beta}(x, u(x), \nabla u(x))\eta_\alpha \eta_\beta \geq \nu\,|\eta|^2 \text{ for all } \eta \in \mathbb{R}^N$$

for all $x \in B(x_0, \varepsilon) = \{x : |x - x_0| < \varepsilon\}$. Hence

$$\sum_{\alpha,\beta=1}^{N} c_{\alpha\beta}(x)\partial_\alpha \partial_\beta u(x) > 0 \text{ for all } x \in B(x_0, \varepsilon)$$

where $c_{\alpha\beta}(x) = a^\infty_{\alpha\beta}(x, u(x), \nabla u(x))$ and it follows from the maximum principle, Theorem 9.6 of [13], that $u(x) = M$ for all $x \in B(x_0, \varepsilon)$. Hence if $M > 0$ we find that Ω is a non-empty subset of $\mathbb{R}^N$ which is both open and closed. But this implies that $\Omega = \mathbb{R}^N$, contradicting the fact that Ω is bounded. Hence $M \leq 0$, and a similar argument shows that $\inf\{u(x) : x \in \mathbb{R}^N\} \geq 0$. Thus $u = 0$ and the proof is complete. $\square$

4.2 Using variational identities

When the condition (M_λ) is not satisfied, an alternative is offered under the following conditions which ensure that all solutions of the equation $F^\infty(\lambda, u) = 0$, with $u \in X_p$ for some $p > N$, satisfy an integral identity of the type found by Pohozaev. Under appropriate conditions this can be used to show that $u = 0$. We refer to Sect. 5 and 6 of our paper [27] for these results.

(V) There exist two functions

$$Q = Q(\xi) \in C^3(\mathbb{R}^{N+1}) \text{ and } g = g(\xi_0, \lambda) \in C^1(\mathbb{R}^2)$$

such that

$$a^{\infty}_{\alpha\beta}(x,\xi) = \partial_{\xi_\alpha}\partial_{\xi_\beta}Q(\xi) \text{ and}$$
$$b^{\infty}(x,\xi,\lambda) = \partial_{\xi_0}Q(\xi) - \sum_{\alpha=1}^{N}\xi_\alpha\partial_{\xi_\alpha}\partial_{\xi_0}Q(\xi) + g(\xi_0,\lambda)$$

for all $x \in \mathbb{R}^N, \xi = (\xi_0, \xi_1, ..\xi_N)$ and $\lambda \in \mathbb{R}$. Furthermore,

$$Q(\xi_0, 0) = \partial_{\xi_0}Q(\xi_0, 0) = 0 \text{ for all } \xi_0 \in \mathbb{R}, \qquad (21)$$
$$\partial_{\xi_\alpha}Q(0) = 0 \text{ for } \alpha = 1, ..., N,$$

and there exists a continuous function $\nu : \mathbb{R}^{N+1} \to (0,\infty)$ such that

$$\sum_{\alpha,\beta=1}^{N}\partial_{\xi_\alpha}\partial_{\xi_\beta}Q(0)\eta_\alpha\eta_\beta \geq \nu(\xi)\left|\eta\right|^2$$

for all $\eta \in \mathbb{R}^N$ and $\xi \in \mathbb{R}^{N+1}$.

Remark 4.3 The condition (V) means that the equation $F^{\infty}(\lambda, u) = 0$ has the variational form

$$-\sum_{\alpha=1}^{N}\partial_\alpha\left\{\partial_{\xi_\alpha}Q\left(u(x),\nabla u(x)\right)\right\} + \partial_{\xi_0}Q\left(u(x),\nabla u(x)\right) + g(u(x),\lambda) = 0 \qquad (22)$$

associated with the formal Euler-Lagrange equation of the functional

$$\int_{\mathbb{R}^N}\left\{Q\left(u(x),\nabla u(x)\right) + \int_0^{u(x)} g(s,\lambda)ds\right\}dx.$$

Under the assumption (V) and the condition (23) introduced below we show in Theorem 5.2 of [27] that any solution, $u \in X_p$ for some $p \in (N,\infty)$, of the equation (22) satisfies the following energy identity,

$$\int_{\mathbb{R}^N}\sum_{\alpha=1}^{N}\partial_{\xi_\alpha}Q\left(u,\nabla u\right)\partial_\alpha u + \partial_{\xi_0}Q\left(u,\nabla u\right)u + g(u,\lambda)dx = 0,$$

and Pohozaev identity,

$$\int_{\mathbb{R}^N}\sum_{\alpha=1}^{N}\partial_{\xi_\alpha}Q\left(u,\nabla u\right)\partial_\alpha u dx =$$
$$N\int_{\mathbb{R}^N}\left\{Q\left(u(x),\nabla u(x)\right) + \int_0^{u(x)} g(s,\lambda)ds\right\}dx.$$

Remark 4.4 As is shown in Sect. 5 of [27], the properties of Q required in (21) involve no real restriction. If they are not satisfied, they can be recovered by replacing Q by

$$\widetilde{Q}(\xi) = Q(\xi) - Q(\xi_0, 0) - \sum_{\alpha=1}^{N} \partial_{\xi_\alpha} Q(0)\xi_\alpha$$

and g by

$$\widetilde{g}(\xi_0, \lambda) = g(\xi_0, \lambda) + \partial_{\xi_0} Q(\xi_0, 0),$$

since $\widetilde{Q}$ and $\widetilde{g}$ generate the same functions $a^\infty_{\alpha\beta}$ and b^∞as Q and g.

Remark 4.5 If the conditions (B),(A) and (V) are satisfied then so is (L$^\infty$) with $A^\infty_{\alpha\beta} = \partial_{\xi_\alpha}\partial_{\xi_\beta} Q(0)$. Thus we can suppose that

$$D_u F^\infty(\lambda, 0) = -\Delta + \partial_{\xi_0} b^\infty(\cdot, 0, \lambda)$$

where $\partial_{\xi_0} b^\infty(\cdot, 0, \lambda)$ is equal to the constant $\partial_{\xi_0} g(0, \lambda)$. If, in addition, the condition (LL) is satisfied with $\partial_{\xi_0} b(x, 0, \lambda) \equiv V(x) - \lambda$, then $V(\infty) = \lim_{|x|\to\infty} V(x)$ exists and $\partial_{\xi_0} b^\infty(\cdot, 0, \lambda) \equiv V(\infty) - \lambda$.

Theorem 8 *Let the conditions (B), (E), (A) and (V) be satisfied and let $p \in (N, \infty)$. Consider an open interval J such that,*

$$(a) \qquad g(0, \lambda) = 0 \text{ and } \partial_{\xi_0} g(0, \lambda) > 0 \text{ for all } \lambda \in J, \tag{23}$$

(b) there exists $a \in \mathbb{R}$ such that

$$NQ(\xi) \geq (a+1) \sum_{\alpha=1}^{N} \xi_\alpha \partial_{\xi_\alpha} Q(\xi) + a\xi_0 \partial_{\xi_0} Q(\xi) \text{ for all } \xi \in \mathbb{R}^{N+1} \tag{24}$$

and

$$N \int_0^s g(t, \lambda)dt \geq ag(s, \lambda)s \text{ for all } (s, \lambda) \in \mathbb{R} \times J. \tag{25}$$

(i) The following properties hold:
The operator $S(\lambda) = -\Delta + \partial_{\xi_0} b^\infty(\cdot, 0, \lambda) : X_2 \to Y_2$ is an isomorphism for all $\lambda \in J$ and J is an admissible interval for the operator $F : \mathbb{R} \times X_p \to Y_p$ defined by (5). The conclusion of Theorem 5 is valid in this context at any point $\lambda_0 \in J$ such that $\dim\ker D_u F(\lambda_0, 0)$ is odd and the condition (19) is satisfied with $\lambda = \lambda_0$.

(ii) *If in addition the condition (LL) is satisfied with* $\partial_{\xi_0} b(x,0,\lambda) \equiv V(x) - \lambda$*, then the condition (23) is satisfied if and only if* $\lambda < V(\infty) = \lim_{|x|\to\infty} V(x)$*. If so,* $J = (-\infty, V(\infty))$ *is an admissible interval for the operator* $F : \mathbb{R} \times X_p \to Y_p$ *defined by (5) provided that (24) and (25) are satisfied, and the conclusion of Theorem 5 is valid in this context at any eigenvalue* $\lambda_0 \in J$ *of* $-\Delta + V : X_2 \subset Y_2 \to Y_2$ *which has odd multiplicity.*

Remark 4.6 The condition (V) restricts the applicability of this result to cases where the differential operator F^∞ has no explicit dependence on the variable x. In particular, the condition (23) means that $\partial_{\xi_0} b^\infty(x,0,\lambda) = \partial_{\xi_0} g(0,\lambda) > 0$ for all $x \in \mathbb{R}$. Since the spectrum of $-\Delta : X_2 \subset Y_2 \to Y_2$ is the interval $[0,\infty)$, it follows that $-\Delta + \partial_{\xi_0} b^\infty(\cdot,0,\lambda) : X_2 \to Y_2$ is an isomorphism whenever (23) holds. When (LL) is satisfied the condition (23) becomes $V(\infty) - \lambda > 0$. In this case, the essential spectrum, σ_e, of the Schrödinger operator $-\Delta + V : X_2 \subset Y_2 \to Y_2$ is the interval $[V(\infty), \infty)$ and so $\mathbb{R} \backslash \sigma_e$ is an admissible interval.

Proof. The conditions (B),(E),(A) and (V) imply that the condition (L$^\infty$) is also satisfied. Furthermore, by the above remark, $-\Delta + \partial_{\xi_0} b^\infty(\cdot,0,\lambda) : X_2 \to Y_2$ is an isomorphism for all $\lambda \in J$. Thus, using Theorem 5 for part (i) and Theorem 6 for part (ii), we need only show that if $(\lambda, u) \in J \times X_p$ and $F^\infty(\lambda, u) = 0$ then $u = 0$. This follows from Corollary 6.1 in [27]. $\square$

4.3 Using asymptotic linearity

In Theorems 7(ii) and 8(ii) the admissible interval lies below the essential spectrum of the linearization about the trivial solution. We now present a situation where there is global bifurcation in gaps of the essential spectrum of this linearization.

Theorem 9 *Let the conditions (B), (E), (A) and (LL) be satisfied and let* $p \in (N, \infty)$*. Suppose that there is an* $N-$*periodic function* $P \in C(\mathbb{R}^N)$ *such that*

$$\lim_{|x|\to\infty} \{V(x) - P(x)\} = 0 \tag{26}$$

and that

$$a^\infty_{\alpha\beta}(x,\xi) = \delta_{\alpha\beta} \text{ and } b^\infty(x,\xi,\lambda) = \{P(x) - \lambda\}\, \xi_0 \tag{27}$$

for all $(x,\xi,\lambda) \in \mathbb{R}^N \times \mathbb{R}^{N+2}$ *with* $\xi = (\xi_0, \xi_1, .., \xi_N)$*. Consider an open interval* $J \subset \mathbb{R}\backslash\sigma_e$ *and an eigenvalue* $\lambda_0 \in J$ *of odd multiplicity of* $-\Delta + V : X_2 \subset Y_2 \to Y_2$*. Then* J *is an admissible interval for the operator* $F : \mathbb{R} \times X_p \to Y_p$ *defined by (5) and the conclusion of Theorem 5 is valid in this context.*

Remark 4.7 By Remark 3.4, the essential spectrum, σ_e, of the operator $-\Delta + V : X_2 \subset Y_2 \to Y_2$ is equal to the entire spectrum, Σ, of the periodic Schrödinger operator, $-\Delta + P : X_2 \subset Y_2 \to Y_2$.

Proof. Using Theorem 6 we need only show that if $(\lambda, u) \in J \times X_p$ and $F^{\infty}(\lambda, u) = 0$ then $u = 0$.

Suppose that $(\lambda, u) \in J \times X_p$ and that

$$-\Delta u + \{P - \lambda\} u = 0.$$

It follows from Theorem 3 that $u \in X_2$.

However, as is well-known ([2] Theorem 5.4 of Chapter 3), the spectrum Σ of the periodic Schrödinger operator $-\Delta + P : X_2 \subset Y_2 \to Y_2$, contains no eigenvalues and so $u = 0$. $\square$

5 Examples

The following examples illustrate the use of the general results.

Example 1 For $m \geq 1/2$, consider the equation

$$- div \left\{ \left(1 + |\nabla u(x)|^2\right)^{m-1} \nabla u(x) \right\} + u(x) \left\{ V(x) + r(x) \left| u(x) \right|^{\tau} \right\} \left\{ 1 + |\nabla u(x)|^2 \right\}^{\gamma} - \lambda u(x) = 0$$

where $\tau > 0, \gamma \geq 0$ and $V, r \in C(\mathbb{R}^N)$ with

$$\lim_{|x| \to \infty} V(x) = V(\infty) \text{ and } \lim_{|x| \to \infty} r(x) = r(\infty)$$

for some constants $V(\infty), r(\infty) \in \mathbb{R}$.
Setting

$$a_{\alpha\beta}(x, \xi) = [1 + |\overline{\xi}|^2]^{m-1} \left\{ \delta_{\alpha\beta} + 2(m-1) \frac{\xi_\alpha \xi_\beta}{1 + |\overline{\xi}|^2} \right\}$$

and

$$b(x, \xi, \lambda) = \xi_0 \left\{ V(x) + r(x) \left| \xi_0 \right|^{\tau} \right\} \left\{ 1 + |\overline{\xi}|^2 \right\}^{\gamma} - \lambda \xi_0$$

for $(x, \xi, \lambda) \in \mathbb{R}^N \times \mathbb{R}^{N+2}$ with $\xi = (\xi_0, \overline{\xi})$ where $\xi_0 \in \mathbb{R}$ and $\overline{\xi} \in \mathbb{R}^N$, it is easy to see that the conditions (B), (E), (A) and (LL) are all satisfied with

$$D_u F(\lambda, 0) v = -\Delta v + [V - \lambda] v$$

and

$$F^{\infty}(\lambda, u) = -div\left\{\left(1+|\nabla u|^2\right)^{m-1}\nabla u\right\} + u\left\{V(\infty)+r(\infty)|u(x)|^{\tau}\right\}\left\{1+|\nabla u|^2\right\}^{\gamma} - \lambda u$$

for $u, v \in X_p$.
The restriction $m \geq 1/2$ is required to ensure that the ellipticity condition (E) is satisfied.

The essential spectrum, σ_e, of the operator $-\Delta + V : X_2 \subset Y_2 \to Y_2$ is the interval $[V(\infty), \infty)$ and the interval $J = \mathbb{R} \backslash \sigma_e$ contains an eigenvalue of odd multiplicity if and only if $\Lambda < V(\infty)$ where

$$\Lambda = \inf\left\{\frac{\int_{\mathbb{R}^N} |\nabla u|^2 + V u^2}{\int_{\mathbb{R}^N} u^2} : u \in X_2 \text{ and } u \neq 0\right\}.$$

Indeed Λ is a simple eigenvalue in this case. (See Theorem 3.4 of [2], for example.)
Since

$$b^{\infty}(x, s, 0, \lambda)s = [V(\infty) - \lambda + r(\infty)|s|^{\tau}]s^2$$

we see that the condition (M_λ) is satisfied provided that

$$\lambda < V(\infty) \text{ and } r(\infty) \geq 0.$$

If $r(\infty) < 0$, there are no values of λ at which the condition (M_λ) is satisfied. However, when $\gamma = 0$, the condition (V) is satisfied with

$$Q(\xi) = \frac{1}{2m}\left\{\left(1+|\bar{\xi}|^2\right)^{m} - 1\right\}$$

and

$$g(\xi_0, \lambda) = [V(\infty) - \lambda]\xi_0 + r(\infty)|\xi_0|^{\tau}\xi_0.$$

It is easy to check that the inequality (24) is true for any constant

$$a \leq \begin{cases} \frac{N}{2} - 1 \text{ if } 1/2 \leq m \leq 1 \\ \frac{N}{2m} - 1 \text{ if } m > 1 \end{cases}$$

whereas, if $\lambda < V(\infty)$ and $r(\infty) < 0$, the inequality (25) holds for any $a \in [\frac{N}{\tau+2}, \frac{N}{2}]$. It follows that a constant a can be chosen so that (24) and (25) hold simultaneously provided that

$$\lambda < V(\infty), r(\infty) < 0 \text{ and } \begin{cases} N \geq 3 \text{ and } \frac{1}{\tau+2} \leq \frac{1}{2} - \frac{1}{N} \text{ if } 1/2 \leq m \leq 1 \\ N > 2m \text{ and } \frac{1}{\tau+2} \leq \frac{1}{2m} - \frac{1}{N} \text{ if } m > 1. \end{cases}$$

Also $\partial_{\xi_0} b^\infty(x,0,\lambda) \equiv V(\infty) - \lambda$ and the condition (23) is satisfied if and only if $\lambda < V(\infty)$.
Having verified these properties of Example 1, we can now draw the following conclusions.

Using Theorem 7(ii) we see that $J = (-\infty, V(\infty))$ is an admissible interval provided that

$$\lambda < V(\infty) \text{ and } r(\infty) \geq 0.$$

If

$$\gamma = 0, \lambda < V(\infty) \text{ and } r(\infty) < 0,$$

Theorem 8(ii) shows that $J = (-\infty, V(\infty))$ is an admissible interval provided that

$$\begin{cases} N \geq 3 \text{ and } \frac{1}{\tau+2} \leq \frac{1}{2} - \frac{1}{N} \text{ if } 1/2 \leq m \leq 1 \\ N > 2m \text{ and } \frac{1}{\tau+2} \leq \frac{1}{2m} - \frac{1}{N} \text{ if } m > 1. \end{cases}$$

In either of these situations there is global bifurcation in the sense of Theorem 2 from every eigenvalue of odd multiplicity of $-\Delta + V : X_2 \subset Y_2 \to Y_2$ in the interval $J = (-\infty, V(\infty))$.

Case 1 Setting $m = 1$ and $\gamma = 0$ in the above example we obtain the semilinear nonlinear Schrödinger equation

$$-\Delta u(x) + V(x)u(x) + r(x)\,|u(x)|^\tau\, u(x) - \lambda u(x) = 0$$

which can be treated by the discussion in [16]. The results in [16] require that $\beta > -\infty$ where

$$\beta = \inf_{C \geq 0} \beta(C) \text{ and } \beta(C) = \lim_{R\to\infty} \inf_{|x|\geq R \text{ and } |s|\leq C} \{V(x) + r(x)\,|s|^\tau\}.$$

If $r(\infty) \geq 0$, then $\beta = V(\infty)$ and we recover the same conclusion as in [16]. If $r(\infty) < 0$, then $\beta = -\infty$ and the approach used in [16] fails. However, the discussion in Example 1 using Theorem 8(ii) shows that $J = (-\infty, V(\infty))$ is still an admissible interval when $r(\infty) < 0$ provided that we are in the super-critical case where $N \geq 3$ and $\tau \geq \frac{4}{N-2}$.

Case 2 Setting $m = 1/2$ in the above example we obtain a nonlinear perturbation of the mean curvature equation

$$-\,div\left\{\frac{\nabla u(x)}{\sqrt{\left(1+|\nabla u(x)|^2\right)}}\right\} +$$
$$u(x)\{V(x) + r(x)\,|u(x)|^\tau\}\left\{1 + |\nabla u(x)|^2\right\}^\gamma - \lambda u(x) = 0.$$

As in Case 1, the interval $J = (-\infty, V(\infty))$ is an admissible interval provided that

$$\text{either } r(\infty) \geq 0$$
$$\text{or } r(\infty) < 0, \gamma = 0, N \geq 3 \text{ and } \tau \geq \frac{4}{N-2}.$$

Example 2 Consider the equation

$$-\Delta u(x) + [P(x) + q(x) - \lambda]\, u(x) + r(x)B(u(x), \nabla u(x), \lambda) = 0$$

where $P, q, r \in C(\mathbb{R}^N)$ with P an $N-$periodic function and

$$\lim_{|x|\to\infty} q(x) = \lim_{|x|\to\infty} r(x) = 0.$$

Also,

$$B \in C^1(\mathbb{R}^{N+2}) \text{ with } B(0, \lambda) = 0 \text{ and } \nabla_\xi B(0, \lambda) = 0$$

for all $\lambda \in \mathbb{R}$.

The conditions (B), (E), (A) and (LL) are clearly satisfied with

$$D_u F(\lambda, 0)v = -\Delta v + [P + q - \lambda]v$$

and

$$F^\infty(\lambda, u) = -\Delta u + [P - \lambda]u.$$

Thus the conditions (26) and (27) are satisfied.

Let Σ denote the spectrum of the periodic Schrödinger operator $-\Delta + P : X_2 \subset Y_2 \to Y_2$ and consider an open interval $J \subset \mathbb{R}\backslash\Sigma$. By Theorem 9 there is global bifurcation in the sense of Theorem 2 from every eigenvalue of odd multiplicity of $-\Delta + [P + q] : X_2 \subset Y_2 \to Y_2$ in the interval J.

Remark 5.1 Criteria ensuring the existence of eigenvalues in spectral gaps for perturbations of a periodic Schrödinger operator can be found in [19] and [14]. In the case $N = 1$, we can obtain much more precise information about branches of solutions.

Example 3 Consider the differential equation

$$-u''(x) + [P(x) + q(x) - \lambda]\, u(x) + r(x)C(u(x), u'(x))u(x) = 0$$

where $P, q, r \in C(\mathbb{R})$ and

$$C \in C^1(\mathbb{R}^2) \text{ with } C(0) = 0.$$

Furthermore, P is periodic, $q \not\equiv 0$ does not change sign,

$$\lim_{|x|\to\infty} q(x) = \lim_{|x|\to\infty} r(x) = 0,$$
$$\int_{-\infty}^{\infty} x^2 |q(x)| \, dx < \infty \text{ and } \int_{-\infty}^{\infty} |xr(x)| \, dx < \infty.$$

Except in some rare and completely determined cases, [30] page 175, the spectrum, Σ, of the operator $-u'' + Pu : X_2 \subset Y_2 \to Y_2$ consists of a countable number of disjoint closed intervals and

$$\mathbb{R}\backslash\Sigma = (-\infty, b_0) \cup_{i=1}^{\infty} (a_i, b_i)$$

where the sequences, $\{a_i\}_{i=1}^{\infty}$ and $\{b_i\}_{i=0}^{\infty}$, are such that

$$-\infty < b_0, \lim_{i\to\infty} b_i = \infty \text{ and } b_i < a_{i+1} < b_{i+1} \text{ for all } i = 0, 1..$$

By Theorem 2.2 of [38], there exists i_0 such that, for all $i \geq i_0$, the operator $-u'' + [P + q]u : X_2 \subset Y_2 \to Y_2$ has exactly one eigenvalue, λ_i, in the interval $J_i = (a_i, b_i)$. Furthermore λ_i is a simple eigenvalue and, if $\int_{-\infty}^{\infty} |q(x)| \left(1 + x^2\right) dx$ is small enough we even have $i_0 = 1$.

For any $p \in (1, \infty)$ and $i \geq i_0$, it follows from Theorem 9 that J_i is an admissible interval for $F : \mathbb{R} \times X_p \to Y_p$ and that there is global bifurcation at λ_i. Furthermore the possibility (ii) in Theorem 2 cannot occur for the component, C_i, of solutions bifurcating from λ_i.

Remark 5.2 In defining the components of solutions in Theorem 2 we restricted our attention to an admissible interval. This is because the degree theory which underlies our whole approach is only available in such intervals. However Example 3 shows that, in general, we should not expect to be able to continue branches of solutions beyond the admissible intervals. Indeed in Example 3 there are no solutions of the problem outside the admissible intervals. To see this we argue as follows. If $(\lambda, u) \in \mathbb{R} \times X_p$ and $F(\lambda, u) = 0$, we have

$$-u'' + [P + U - \lambda] \, u = 0$$

where

$$U(x) = q(x) + r(x)C(u(x), u'(x)).$$

Since $P + U \in L^{\infty}(\mathbb{R})$ it follows from Theorem 3 that $u \in X_2$. But $C(u(\cdot), u'(\cdot)) \in L^{\infty}(\mathbb{R})$ and so $xU(x) \in L^1(\mathbb{R})$. It now follows from the results in [15] that $u = 0$ if $\lambda \in \Sigma$.

If the condition $\int_{-\infty}^{\infty} |xr(x)| \, dx < \infty$ is relaxed to $r \in L^1(\mathbb{R})$, the above discussion shows that $U \in L^1(\mathbb{R})$. The results in [15] now show that $u = 0$ if λ belongs to the interior of Σ.

6 Exponential decay of solutions

In our paper [27] we have investigated the exponential decay of solutions of rather general quasilinear second order equations. Using the assumptions (A), (B) and (L^∞) introduced in the present article these conditions for exponential decay can be expressed rather simply and we see that they are particularly relevant for the situations discussed in Theorems 7 and 8.

Theorem 10 *Let the conditions (B), (A) and (L^∞) be satisfied and suppose that $F(\lambda, u) = 0$ where $u \in X_p$ for some $p \in (N, \infty)$ and $F : \mathbb{R} \times X_p \to Y_p$ is the operator defined by (5). Let ρ denote the spectral radius of the positive definite matrix $\left[A^\infty_{\alpha\beta}\right]$ appearing in (L^∞) and set*

$$\delta(\lambda) = \lim \inf_{|x|\to\infty} \partial_{\xi_0} b(x, 0, \lambda).$$

If $\delta(\lambda) > 0$, then

$$\lim_{|x|\to\infty} e^{\mu|x|} u(x) = 0$$

for any $\mu < \sqrt{\frac{\delta(\lambda)}{\rho}}$.

Remark 6.1 We always have $\rho > 0$ but, after $D_u F^\infty(\lambda, 0)$ has been reduced to the form $-\Delta + \partial_{\xi_0} b(x, 0, \lambda)$ by a linear change of variable, we obtain $\rho = 1$. Note also that

$$\delta(\lambda) = \inf_{x\in\mathbb{R}} \partial_{\xi_0} b^\infty(x, 0, \lambda),$$

so, as one might expect, the estimate for the decay rate is determined by the limit operator $F^\infty(\lambda, \cdot)$.

Proof. This is a special case of Theorem 2.1 of [27]. Indeed, by (A) and (L^∞), the condition (2.5') of [27] is satisfied. Referring to Remark 2.1 of [27], we set

$$c_j(x, \xi, \lambda) = \int_0^1 \partial_{\xi_j} b(x, t\xi, \lambda) dt \text{ for } j = 0, 1, .., N$$

and we observe that by (B),(A) and (L^∞),

$$\lim \inf_{|x|\to\infty} c_0(x, 0, \lambda) = \delta(\lambda)$$

whereas

$$\lim_{|x|\to\infty} c_j(x, 0, \lambda) = 0 \text{ for } j = 1, .., N.$$

Similarly, using $\rho(x)$ to denote the spectral radius of the matrix $[a_{\alpha\beta}(x,0)]$, we find that

$$\lim_{|x|\to\infty} \rho(x) = \rho.$$

In the notation of Sect. 2 of [27] we now have $\delta^\infty = \delta(\lambda), \rho^\infty = \rho$ and $c^\infty = 0$. The result now follows immediately from Theorem 2.1 of [27]. □

An essential requirement in the above result is that $\delta(\lambda) > 0$ and this is satisfied by all the solutions on the components bifurcating from λ_0 in some of the special cases discussed in Sect. 4.

Consider first the situation covered by Theorem 7. We see that $\delta(\lambda) \geq 0$ for all $\lambda \in J$. However, under the more stringent conditions required for part (ii) we see that $\delta(\lambda) = \omega - \lambda$ and so $\delta(\lambda) > 0$ for all $\lambda \in J = (-\infty, \omega)$. Thus all solutions with $\lambda \in J$ decay exponentially. (Recall that (LL) implies (L^∞).)

Suppose now that the hypotheses of Theorem 7(i) are satisfied. Then, for all $\lambda \in J$, $\partial_{\xi_0} b^\infty(x,0,\lambda) = \partial_{\xi_0} g(0,\lambda) > 0$ for all $x \in \mathbb{R}^N$ and so $\delta(\lambda) = \partial_{\xi_0} g(0,\lambda) > 0$. Recalling that the hypotheses of Theorem 8(i) ensure that (L^∞) also holds we see that all solutions with $\lambda \in J$ decay exponentially.

The situation treated is Sect. 4.3 is completely different and we may have $\delta(\lambda) < 0$ for all solutions in the admissible interval J for the operator F under the hypotheses of Theorem 9. In fact the main interest of that result is that it covers cases where $\inf J > \inf \sigma_e \geq \inf P$ and so $\delta(\lambda) = \inf P - \lambda < 0$ for $\lambda \in J$. However solutions with $\lambda \in J$ may still decay exponentially even in this case. To illustrate this, consider Example 2 in Sect. 5 with the additional requirement that

$$B(\xi,\lambda) = C(\xi,\lambda)\xi_0$$

where $C \in C^1(\mathbb{R}^{N+2})$ and $C(0,\lambda) = 0$ for all $\lambda \in \mathbb{R}$. Suppose that (λ, u) is a solution with $u \in X_p$ for some $p \in (N,\infty)$ and $\lambda \in J \subset \mathbb{R}\backslash\Sigma$ where $\inf J > \inf \Sigma$. Then $\delta(\lambda) < 0$. Nonetheless, setting

$$U(x) = q(x) + r(x)C(u(x), \nabla u(x), \lambda),$$

we see that u satisfies the linear Schrödinger equation

$$-\Delta u + [P + U - \lambda]\, u = 0$$

where $\lim_{|x|\to\infty} U(x) = 0$. Thus the essential spectrum of the self-adjoint operator $-\Delta + (P+U)$ is Σ. It follows from Theorem 3(iii) with $V = P+U$ that $u \in X_2$ and then from Theorem C.3.4 of [31] (see also Proposition 6(3) in [26]) that u decays exponentially. So far we do not know if exponential decay occurs for solutions with $\lambda \in J$ in the general context of Theorem 9.

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