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# $L^p$ estimates on convex domains of finite type

**Bert Fischer**

Mathematik, Bergische Universität-GHS Wuppertal, Gaußstr. 20, 42097 Wuppertal
(e-mail: fischer@math.uni-wuppertal.de)

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## 1 Introduction

This paper continues the investigation of convex finite type domains by means of explicite integral formulas which started with [DiFo] and [DiFiFo]. In [DiFo] Diederich and Fornæss constructed smooth support functions for convex domains of finite type and proved that these support functions satisfy some nice estimates on the given domain. In [DiFiFo] the authors used these support functions to construct some $\bar{\partial}$-solving Cauchy-Fantappié kernels. After proving some additional estimates for the support functions and their Leray decomposition they could prove that the solutions given by these kernels satisfy the best possible Hölder estimates.

These results have also been used in [DiMa] to improve some theorem of [BrChDu] about the zero sets of functions of the Nevanlinna class in convex domains of finite type.

In this paper we construct some $\bar{\partial}$-solving integral operators that satisfy the best possible estimates with respect to $L^p$ norms. More precisely we prove the following theorem.

**Theorem 1.1** *Let $D \subset\subset \mathbb{C}^n$ be a linearly convex domain with $C^\infty$-smooth boundary of finite type $m$. We denote by $L^p_{(0,r)}(D)$ the Banach space of $(0,r)$-forms whose coefficients belong to $L^p(D)$ by $\Lambda^\alpha_{(0,r)}(D)$ the Banach space of $(0,r)$-forms whose coefficients are uniformly Hölder continuous of order $\alpha$ on $D$ and by $BMO_{(0,r)}(D)$ the space of $(0,r)$-forms with BMO-coefficients.*

*Then there are bounded linear operators $T_r$ such that $\bar\partial T_r f = f$ for all $f \in L^p_{(0,r+1)}(D)$ with $\bar\partial f = 0$ and these operators satisfy the following estimates.*

*(i) For $1 \le p < mn+2$ we have $||T_r f||_{L^q} \le C_p ||f||_{L^p}$ for $1/q = 1/p - 1/(mn+2)$.*
*(ii) For $p = mn+2$ we have $||T_r f||_{BMO} \le C||f||_{L^p}$*
*(iii) For $mn+2 < p$ we have $||T_r f||_{\Lambda^\alpha} \le C_p ||f||_{L^p}$ for $\alpha = 1/m - (n+2/m)/p$.*

In [ChKrMa] it was shown that the gain of regularity which is given in our theorem is the best possible in the case of complex ellipsoids, that is for domains of the form $|z_1|^{m_1} + \ldots + |z_n|^{m_n} < 1$ with $m = \max m_i$. Since all complex ellipsoids are also convex domains of finite type our result is optimal in the sense that there exists a domain and a $\bar\partial$ closed form with $L^p$-coefficients that does not admit any solution with estimates better than stated above.

A result similar to Theorem 1.1 has recently been obtained in a paper by A. Cumenge [Cu], where she constructs solution operators with the help of the Bergman kernel, and uses certain estimates for the Bergman kernel, which are given in [Mc2] to prove the required estimates.

In this paper we will make use of the support functions defined in [DiFo] and of some of the estimates proved in [DiFiFo]. It would have been the easiest just to use the same Cauchy-Fantappié kernels and only make the necessary modifications (transform the boundary integral into a volume integral) to be able to apply these operators also to $L^p$-forms. In fact it is quite easy to see that such an operator satisfies the first estimate of our theorem for all $q$ such that $1/q > 1/p - 1/(mn+2)$. However, this operator seems to be not good enough to get exactly the best possible estimates. So finally we construct some integral operators of Berndtsson-Andersson type (see also [DiMa]) which contain certain weights which are suitable for our purpose.

This article is organised in the following way: In Sect. 2 we briefly recall the definition of the support function from [DiFo] and the definition of the Leray decomposition from [DiFiFo]. Then we prove some first estimate for a modified support function and define a $\bar\partial$ solving weighted kernel of Berndtsson-Andersson type. We also formulate two lemmas which give some sufficient conditions for the estimates of Theorem 1.1. In Sect. 3 we recall some well known facts about convex domains of finite type and review some estimates for the support function and their decomposition which have already been proved in [DiFiFo]. Then we prove several estimates for the ingredients of our kernel. In Sect. 4 we give an auxiliary integral estimate and finally prove the two lemmas which have been given in Sect. 2.

## 2 Solution operators

Let $D = \{\varrho < 0\}$ be a convex domain with $C^\infty$-smooth boundary of finite type $m$. It was shown in [Mc2] that the defining function $\varrho$ can be chosen in such a way that there exists a neighbourhood $U$ of $\partial D$ such that $|\nabla\varrho(\zeta)| > \frac{1}{2}$ for all $\zeta \in U$ and all the domains $D_\zeta := \{z : \varrho(z) < \varrho(\zeta)\}$ are convex domains of finite type $m$ as well. Let us assume that the defining function already has this property. If $n_\zeta$ is the unit outer normal vector at the point $\zeta$ on the hypersurface $\{z : \varrho(z) = \varrho(\zeta)\}$ then we can find a family $\Phi(\zeta)$ of unitary transformations such that $\Phi(\zeta)n_\zeta = (1, 0, \dots, 0)$ for all $\zeta \in U$. As in [DiFo] and [DiFiFo] we define

$$\varrho_\zeta(w) := \varrho(\zeta + \bar{\Phi}^T(\zeta)w),$$

$$\tilde{S}_\zeta(w) := 3w_1 + Kw_1^2 - c\sum_{j=2}^{m} M^{2^j}\sigma_j \sum_{\substack{|\alpha|=j\\ \alpha_1=0}} \frac{1}{\alpha!}\frac{\partial^j r_\zeta}{\partial w^\alpha}(0)w^\alpha$$

for $K, M > 0$ suitably large, $c > 0$ suitably small (all independent of $\zeta$), and put

$$\tilde{S}(z,\zeta) := S_\zeta(\Phi(\zeta)(z-\zeta)).$$

At this point we have to mention that in [DiFo] the support function and the estimates have been given only for $\zeta \in \partial D$. However it is easy to see that all the results remain true at least for all $\zeta$ in some (possible smaller) neighbourhood $U$ of $\partial D$. Now wherever $S$ is defined we can construct a Leray decomposition in the following way. We just define

$$\tilde{Q}_\zeta^1(w) := 3 + Kw_1$$

and for $k > 1$

$$\tilde{Q}_\zeta^k(w) := -c\sum_{j=2}^{m} M^{2^j}\sigma_j \sum_{\substack{|\alpha|=j\\ \alpha_1=0,\, \alpha_k>0}} \frac{\alpha_k}{j\alpha!}\frac{\partial^j r_\zeta}{\partial w^\alpha}(0)\frac{w^\alpha}{w_k}$$

and set

$$\tilde{Q}(z,\zeta) := \Phi^T(\zeta)Q_\zeta(\Phi(\zeta)(z-\zeta)).$$

Since we want to define $Q$ for all $\zeta$ we choose two neighbourhoods $bD \subset\subset U_1 \subset\subset U_2 \subset\subset U$ of the boundary and a smooth cut off function $0 \le \chi \le 1$ such that $\chi(\zeta) = 1$ for $\zeta \in U_1$ and $\chi(\zeta) = 0$ for $\zeta \in D \backslash U_2$. Using this we can define

$$\hat{Q}(z,\zeta) = \chi(\zeta)\tilde{Q}(z,\zeta)$$

Before we can construct our solution operator we have to prove the following lemma.

**Lemma 2.1** *There exists a constant $C_1$ such that for all $z, \zeta \in D$ we have*

$$\mathit{Re}\left(\langle \hat{Q}(z,\zeta), z-\zeta\rangle + C_1\varrho(\zeta)\right) \lesssim \varrho(z) + \varrho(\zeta) - |z-\zeta|^m.$$

*Proof.* First, if $\zeta \in D\backslash U_1$ and therefore $\varrho(\zeta) < -c$ then the inequality will always be satisfied if only $C_1$ is large enough. For $\zeta \in U_1$ the term on the left hand side becomes $\mathrm{Re}\,\tilde{S}(z,\zeta) + C_1\varrho(\zeta)$. Let us write $z = \mu n_\zeta + \lambda v$, where $v$ is some complex tangential vector at $\zeta$ on the hypersurface $\{z : \varrho(z) = \varrho(\zeta)\}$. By Theorem 2.3 from [DiFo] we have

$$\mathrm{Re}\,\tilde{S}(z,\zeta) + C_1\varrho(\zeta) \leq \mathrm{Re}\,\mu - K(\mathrm{Im}\,\mu)^2 - c\sum |a_{\alpha\beta}(\zeta,v)|\,|\lambda|^j + C_1\varrho(\zeta).$$

Since the domain $D_\zeta$ is also of finite type $m$ the sum can be estimated from below by $c|\lambda|^m$. Together with the first two terms this gives an estimate by $c|z-\zeta|^m$. Moreover we have $\mathrm{Re}\,\mu \leq C'\varrho(z) - C'\varrho(\zeta)$. Thus we get the desired result if we choose $C_1$ large enough and larger than $C'$. □

Now we define

$$s_j(z,\zeta) := (\bar{z}_j - \bar{\zeta}_j)d\zeta_j,$$
$$Q_j(z,\zeta) := \frac{\hat{Q}_j(z,\zeta)}{C_1\varrho(\zeta)}d\zeta_j$$

with the constant $C_1$ from Lemma 2.1 and $G(z) := z^{-N}$. For convenience we also introduce the notation $S(z,\zeta) = \langle \hat{Q}(z,\zeta), z-\zeta\rangle + C_1\varrho(\zeta)$ Using these ingredients the Berndtsson-Andersson kernel becomes

$$K := \sum_{k=0}^{n-1} c_{nk}G^{(k)}\left(1 + \langle Q(z,\zeta), z-\zeta\rangle\right)\frac{s\wedge(\bar{\partial}Q)^k\wedge(\bar{\partial}s)^{n-1-k}}{\langle s(\zeta,z),\zeta - z\rangle^{n-k}}$$

Note that due to Lemma 2.1 we have

$$\mathrm{Re}\,(1 + \langle Q(z,\zeta), z-\zeta\rangle) = \mathrm{Re}\,\frac{S(z,\zeta)}{C_1\varrho(\zeta)} \gtrsim \frac{\varrho(\zeta)}{C_1\varrho(\zeta)} > 0$$

that $G$ is holomorphic on the set $\mathrm{Re}\,(z) > 0$ and that $G(1) = 1$. Now we can also introduce the notation $K_r(z,\zeta)$ for the part of the kernel which is of degree $(0,r)$ with respect to $z$ and define

$$T_r f(z) := \int_{\zeta\in D} f(\zeta)\wedge K_r(z,\zeta)$$

Since due to the weight function the kernel vanishes for $\zeta \in bD$ the integral operators $T_r$ are indeed solution operators in $D$ (see [BeAn] for more details).

In order to prove the estimates of Theorem 1.1 it is enough to show the following two lemmas.

**Lemma 2.2** *Let $r = (mn+2)/(mn+1)$ and let $\varepsilon$ be an arbitrary small constant. Then the kernel satisfies the following estimates*

$$\int_{z\in D} |K(z,\zeta)|^r d\sigma_{2n} \le C \tag{1}$$

$$\int_{z\in D} |\varrho(z)|^{-\varepsilon}|K(z,\zeta)|^r d\sigma_{2n} \le C_\varepsilon |\varrho(\zeta)|^{-\varepsilon} \tag{2}$$

$$\int_{\zeta\in D} |\varrho(\zeta)|^{-\varepsilon}|K(z,\zeta)|^r d\sigma_{2n} \le C_\varepsilon |\varrho(z)|^{-\varepsilon} \tag{3}$$

**Lemma 2.3** *Let $r = (mn+2)/(mn+1)$ and for $p > mn+2$ let $p'$ be the dual exponent to $p$ and $\alpha = 1/m - (n+2/m)/p$. Then the kernel satisfies the following estimates*

$$\int_{\zeta\in D} |\nabla_z K(z,\zeta)|^r d\sigma_{2n} \le C(\varrho(z))^{-r} \tag{4}$$

$$\int_{\zeta\in D} |\nabla_z K(z,\zeta)|^{p'} d\sigma_{2n} \le C(\varrho(z))^{p'(\alpha-1)} \tag{5}$$

In fact for $p = 1$ the first statement of Theorem 1.1 just follows from (1) by means of Hölder inequality. To prove Theorem 1.1 (i) for $p > 1$ we have to use (2), (3) and some standard argumentation which can be found for instance in [Ra] and [McSt].

In order to prove the other two statements of Theorem 1.1 we want to make use of the well known Hardy-Littlewood lemma (for the BMO-version see for instance [McSt])

**Proposition 2.4** *Let $g \in C^1(D)$. If for some $\alpha$ with $0 < \alpha < 1$ there exists a constant $C$ such that $|\nabla g(z)| \le C\mathrm{dist}\,(z, bD)^{\alpha-1}$ then $g$ belongs to $\Lambda_\alpha(D)$. If there exist a constant $C$ such that $|\nabla g(z)| \le C\mathrm{dist}\,(z, bD)^{-1}$ then $g$ belongs to $BMO(D)$.*

Now the last two results of our theorem follow from (4) and (5) by means of Hölder inequality.

Before we can prove Lemma 2.2 and Lemma 2.3, which will be done in Sect. 4, we need some estimates for the ingredients of the kernels. In particular we need estimates for $Q$, $dQ$ and $\nabla_z dQ$ and we need estimates for the weight which can be reduced to certain estimates for $S$. In order to get all these estimates we have to explore the special geometry of our class of domains, which will be done in the next section.

## 3 Basic estimates

Let $D = \{\varrho < 0\} \subset \mathbb{C}^n$ be a bounded convex domain with $\mathcal{C}^\infty$-boundary of finite type $m$. As above we assume that the domains $D_\zeta := \{z : \varrho(z) <$

$\varrho(\zeta)\}$ are also convex and of finite type $m$ for all $\zeta \in U$. For $\zeta \in U$ and $\varepsilon < \varepsilon_0$ we define some sort of complex directional boundary distances by

$$\tau(\zeta, v, \varepsilon) := \max\{c : |\varrho(\zeta + \lambda v) - \varrho(\zeta)| < \varepsilon \quad \text{for all} \quad \lambda \in \mathbb{C},\, |\lambda| < c\}.$$

For a fixed point $\zeta$ and a fixed radius $\varepsilon$ we define the $\varepsilon$-extremal basis $(v_1, \ldots, v_n)$ centred at $\zeta$ as in [Mc2]. If it is important to mention the dependence on $\zeta$ and $\varepsilon$ of the coordinates with respect to this basis, we denote their components by $z_{k,\zeta,\varepsilon}$. Let $v_k$ be a unit vector in the $z_{k,\zeta,\varepsilon}$-direction and write $\tau_k(\zeta, \varepsilon) := \tau(\zeta, v_k, \varepsilon)$. We can now define the polydiscs

$$AP_\varepsilon(\zeta) := \{z \in \mathbb{C}^n : |z_{k,\zeta,\varepsilon}| \le A\tau_k(\zeta, \varepsilon) \forall k\}.$$

(Note that the factor $A$ in front means blowing up the polydisc around its centre and not just multiplying each point by $A$.)

Using these polydiscs we define the pseudo distance

$$d(z, \zeta) := \inf\{\varepsilon : z \in P_\varepsilon(\zeta)\}.$$

The following statements can be found in the literature (see for instance [Mc1], [Mc2], [BrNaWa], [BrChDu]):

**Proposition 3.1** *(i) There exists a constant $C > 1$ (independent of $\zeta$ and $\varepsilon$) such that*

$$CP_{\varepsilon/2}(\zeta) \supset \tfrac{1}{2} P_\varepsilon(\zeta) \quad \textit{for all} \quad \zeta,\, \varepsilon$$

*(ii) The pseudo distance $d(z, \zeta)$ satisfies the properties*

$$d(z, \zeta) \approx d(\zeta, z),$$
$$d(z, \zeta) \lesssim d(z, w) + d(w, \zeta).$$

*(iii) We have $\tau_1(\zeta, \varepsilon) \approx \varepsilon$ and $\varepsilon^{\frac{1}{2}} \lesssim \tau_n(\zeta, \varepsilon) \le \ldots \le \tau_2(\zeta, \varepsilon) \lesssim \varepsilon^{\frac{1}{m}}$. For $z \in P_\varepsilon(\zeta)$ we have $|z - \zeta| \lesssim \varepsilon^{\frac{1}{m}}$ and $z \notin P_\varepsilon(\zeta)$ implies $|z - \zeta| \gtrsim \varepsilon$.*

*(iv) Let $w$ be any orthonormal coordinate system centred at $z$ and let $v_j$ be the unit vector in the $w_j$-direction. Then we have*

$$\left| \frac{\partial^{|\alpha+\beta|} \varrho(z)}{\partial w^\alpha \partial \bar{w}^\beta} \right| \lesssim \frac{\varepsilon}{\prod_j \tau(z, v_j, \varepsilon)^{\alpha_j + \beta_j}}$$

*for all multiindices $\alpha$ and $\beta$ with $|\alpha + \beta| \ge 1$.*

Using the polydiscs defined above we also introduce the following polyannuli

$$P^i_\varepsilon(\zeta) := CP_{2^i\varepsilon}(\zeta) \backslash \tfrac{1}{2} P_{2^i\varepsilon}(\zeta).$$

Note that we now have $d(z,\zeta) \approx 2^i\varepsilon$ for all $z \in P^i_\varepsilon(\zeta)$. Due to Proposition 3.1 (i) the constant $C$ can be chosen in such a way that we have

$$\bigcup_{i=0}^{\infty} P_\varepsilon^{-i}(\zeta) \supset P_\varepsilon(\zeta)\backslash\{\zeta\}$$

and

$$\bigcup_{i=0}^{\infty} P_\varepsilon^{i}(\zeta) \supset P_{\varepsilon_0}(\zeta)\backslash P_\varepsilon(\zeta).$$

In fact the last covering is finite for every fixed $\varepsilon$ but the number of the polyannuli which are involved tends to infinity if $\varepsilon$ tends to zero.

Now we want to recall some of the estimates which have been proved in [DiFiFo]. We begin with an estimate for our modified support function $S$.

**Lemma 3.2** *For all $z$ and $\zeta$ in $U$ and $\varepsilon < \varepsilon_0$ we have*

$$|S(z,\zeta)| \gtrsim \varepsilon$$

*for $\zeta \in P^0_\varepsilon(z)$ or $z \in P^0_\varepsilon(\zeta)$.*

*Proof.* The proof of this statement is not exactly the same as the proof of Lemma 4.2 in [DiFiFo]. But it can be proved in the same way and is even a little bit easier. So we omit the details here. □

We now come to the estimates for the components of $Q$, $\bar\partial Q$ and $\nabla_z Q$. First let us fix a point $z_0 \in U$ and choose a small number $\varepsilon$. Now we want to write all forms with respect to the $\varepsilon$-extremal coordinates at $z_0$, which we denote by $w^*$. We choose a unitary transformation $\Phi^*$ such that $w^* = \Phi^*(\zeta - z_0)$. If we define

$$Q^*(w^*) := \bar\Phi^* Q(z_0, z_0 + (\bar\Phi^*)^T w^*)$$

then we have $\sum_i Q_i(z_0,\zeta)d\zeta_i = \sum_k Q^*_k(w^*)dw^*_k$ and

$$\bar\partial Q = \sum_{lk} \frac{\partial}{\partial \bar w^*_l} Q^*_k(w^*) d\bar w^*_l \wedge dw^*_k.$$

**Lemma 3.3** *For all $w^*$ with $|w^*_j| < \tau_j(z_0,\varepsilon)$ we have*

$$|Q^*_k(w^*)| \lesssim \frac{\varepsilon}{\tau_k(z_0,\varepsilon)}$$

$$\left|\frac{\partial}{\partial z_j} Q^*_k(w^*)\right| \lesssim \frac{\varepsilon}{\tau_k(z_0,\varepsilon)}$$

$$\left|\frac{\partial}{\partial \bar w^*_j} Q^*_k(w^*)\right| \lesssim \frac{\varepsilon}{\tau_j(z_0,\varepsilon)\tau_k(z_0,\varepsilon)}$$

*and the involved constants are independent of $z_0$ and $\varepsilon$.*

*Proof.* This lemma can be proved in the same way as Lemma 5.1 in [DiFiFo]. The necessary minor modifications are left to the reader. □

We also have to consider the case that $\zeta_0$ is a fixed point in $U$ and $z$ varies in some $P_\varepsilon(\zeta_0)$. Here we want to write everything with respect to the $\varepsilon$-extremal coordinates at $\zeta_0$. As above we choose an appropriate unitary transformation $\Phi^*$ and define $w^* = \Phi^*(\zeta - \zeta_0)$ and $w_* = \Phi^*(z - \zeta_0)$. If we define

$$Q^*(w_*) := \bar{\Phi}^* Q(\zeta_0 + (\bar{\Phi}^*)^T w_*, \zeta_0)$$

then we have $\sum_i Q_i(z, \zeta_0) d\zeta_i = \sum_k Q_k^*(w_*) dw_k^*$ and

$$\bar{\partial} Q = \sum_{lk} \frac{\partial}{\partial \bar{w}_l^*} Q_k^*(w_*) d\bar{w}_l^* \wedge dw_k^*.$$

**Lemma 3.4** *For all $w_*$ with $|w_{*j}| < \tau_j(\zeta_0, \varepsilon)$ we have*

$$|Q_k^*(w_*)| \lesssim \frac{\varepsilon}{\tau_k(\zeta_0, \varepsilon)}$$

$$\left| \frac{\partial}{\partial z_j} Q_k^*(w_*) \right| \lesssim \frac{\varepsilon}{\tau_k(\zeta_0, \varepsilon)}$$

$$\left| \frac{\partial}{\partial \bar{w}_j^*} Q_k^*(w_*) \right| \lesssim \frac{\varepsilon}{\tau_j(\zeta_0, \varepsilon) \tau_k(\zeta_0, \varepsilon)}$$

*and the involved constants are independent of $\zeta_0$ and $\varepsilon$.*

*Proof.* Using Lemma 2.1 from [DiFiFo] we can simply assume that the coordinates $w$ in the definition of $Q$ are already the $\varepsilon$-extremal coordinates at $\zeta_0$. Then the rest of the proof is again the same as above. □

As a consequence of the last two lemmas we get the following result.

**Lemma 3.5** *For $z_0$ in $U$ fixed, $\varepsilon < \varepsilon_0$ and $\zeta \in P_\varepsilon(z_0)$ we have*

$$|(\bar{\partial} Q)^k(z_0, \zeta)| \lesssim \left( \frac{\varepsilon^k}{\varrho(\zeta)^k} + \frac{\varepsilon^{k+1}}{\varrho(\zeta)^{k+1}} \right) \tau_1(z_0, \varepsilon)^{-2} \prod_{j=n-k+2}^{n} \tau_j(z_0, \varepsilon)^{-2} \tag{6}$$

$$\begin{aligned} |\nabla_z (\bar{\partial} Q)^k(z_0, \zeta)| \lesssim & \left( \frac{\varepsilon^{k-1}}{\varrho(\zeta)^{k-1}} + \frac{\varepsilon^k}{\varrho(\zeta)^k} \right) \\ & \times \left( \frac{1}{\varrho(\zeta)} + \frac{\varepsilon}{\varrho(\zeta) \tau_{n-k+2}(z_0, \varepsilon)^2} \right) \\ & \times \tau_1(z_0, \varepsilon)^{-2} \prod_{j=n-k+3}^{n} \tau_j(z_0, \varepsilon)^{-2} \end{aligned} \tag{7}$$

*and an analog statement to (6) is also true for $\zeta_0$ in $U$ fixed and $z \in P_\varepsilon(\zeta_0)$.*

*Proof.* By definition we have

$$Q(z,\zeta) = \sum_j \frac{\chi(\zeta)}{C_1\varrho(\zeta)}\tilde{Q}_j(z,\zeta)d\zeta_j$$

and therefore

$$\bar\partial Q(z,\zeta) = \sum_{jk}\left(\frac{\partial}{\partial\bar\zeta_k}\left(\frac{\chi(\zeta)}{C_1\varrho(\zeta)}\right)\tilde{Q}_j(z,\zeta) + \frac{\chi(\zeta)}{C_1\varrho(\zeta)}\frac{\partial}{\partial\bar\zeta_k}\tilde{Q}_j(z,\zeta)\right)d\bar\zeta_k\wedge d\zeta_j$$

Computing the $k$th exterior product we get

$$(\bar\partial Q)^k = \frac{\chi^k}{C_1^k\varrho^k}(\bar\partial\tilde{Q})^k + c_k\frac{1}{C_1\varrho}\left(\bar\partial\chi - \frac{\chi}{C_1\varrho}\bar\partial\varrho\right)\wedge\tilde{Q}\wedge\frac{\chi^{k-1}}{C_1^{k-1}\varrho^{k-1}}(\bar\partial\tilde{Q})^{k-1} \tag{8}$$

In order to prove (6) we just have to write this equation with respect to the $\varepsilon$-extremal coordinates at $z_0$. Then we can use Lemma 3.3 and see that the first term can be estimated by

$$\frac{1}{\varrho(\zeta)^k}\frac{\varepsilon^k}{\prod_{j=1}^k\tau_{\mu_j}(z_0,\varepsilon)\tau_{\nu_j}(z_0,\varepsilon)}$$

where the $\mu_j$ must be pairwise different and the $\nu_j$ must be pairwise different. So every index may appear at most twice. Since we have $\tau_1(z,\varepsilon)\le\tau_n(z,\varepsilon)\le\ldots\le\tau_2(z,\varepsilon)$ the first term can also be estimated by

$$\frac{\varepsilon^k}{\varrho(\zeta)^k}\tau_1(z_0,\varepsilon)^{-2}\prod_{j=n-k+2}^{n}\tau_j(z_0,\varepsilon)^{-2}$$

To estimate the second term we again use Lemma 3.3, the fact that $\bar\partial\chi$ is bounded (and therefore $\lesssim 1/\tau_j$) and the fact that $\partial\varrho/\partial\bar\zeta_k\lesssim\varepsilon/\tau_k$ (see Proposition 3.1 (iv). We get an estimate by

$$\frac{1}{\varrho(\zeta)^k}\left(\frac{1}{\tau_\mu(z_0,\varepsilon)}+\frac{\varepsilon}{\varrho(\zeta)\tau_\mu(z_0,\varepsilon)}\right)\frac{\varepsilon}{\tau_\nu(z_0,\varepsilon)}\frac{\varepsilon^{k-1}}{\prod_{j=1}^{k-1}\tau_{\mu_j}(z_0,\varepsilon)\tau_{\nu_j}(z_0,\varepsilon)}$$

which can again be estimated by the term on the right hand side of (6). In order to prove an analog formula for fixed $\zeta_0$ we do exactly the same, namely we write (8) with respect to the $\varepsilon$-extremal coordinates at $\zeta_0$ and than make use of Lemma 3.4.

To prove (7) we first have to apply $\nabla_z$ to (8). We get

$$\begin{aligned}\nabla_z(\bar\partial Q)^k &= c\frac{\chi^k}{\varrho^k}(\bar\partial\tilde Q)^{k-1}\wedge(\nabla_z\bar\partial\tilde Q)\\ &\quad +c'\frac{\chi^{k-1}}{\varrho^k}\left(\bar\partial\chi-\frac{\chi}{\varrho}\bar\partial\varrho\right)\wedge(\nabla_z\tilde Q)\wedge(\bar\partial\tilde Q)^{k-1}\\ &\quad +c''\frac{\chi^{k-1}}{\varrho^k}\left(\bar\partial\chi-\frac{\chi}{\varrho}\bar\partial\varrho\right)\wedge\tilde Q\wedge(\bar\partial\tilde Q)^{k-2}\wedge(\nabla_z\bar\partial\tilde Q)\end{aligned}$$

Using again Lemma 3.3 we see that the first term can be estimated by

$$\frac{1}{\varrho(\zeta)^k}\frac{\varepsilon^{k-1}}{\prod_{j=1}^{k-1}\tau_{\mu_j}(z_0,\varepsilon)\tau_{\nu_j}(z_0,\varepsilon)} \lesssim \frac{\varepsilon^{k-1}}{\varrho(\zeta)^k}\tau_1(z_0,\varepsilon)^{-2}\prod_{j=n-k+3}^{n}\tau_j(z_0,\varepsilon)^{-2}$$

For the second term we get

$$\begin{aligned}&\frac{1}{\varrho(\zeta)^k}\left(\frac{1}{\tau_\mu(z_0,\varepsilon)}+\frac{\varepsilon}{\varrho(\zeta)\tau_\mu(z_0,\varepsilon)}\right)\frac{\varepsilon}{\tau_\nu(z_0,\varepsilon)}\frac{\varepsilon^{k-1}}{\prod_{j=1}^{k-1}\tau_{\mu_j}(z_0,\varepsilon)\tau_{\nu_j}(z_0,\varepsilon)}\\ &\lesssim\left(\frac{\varepsilon^k}{\varrho(\zeta)^k}+\frac{\varepsilon^{k+1}}{\varrho(\zeta)^{k+1}}\right)\tau_1(z_0,\varepsilon)^{-2}\prod_{j=n-k+2}^{n}\tau_j(z_0,\varepsilon)^{-2}\end{aligned}$$

And the last term gives

$$\begin{aligned}&\frac{1}{\varrho(\zeta)^k}\left(\frac{1}{\tau_\mu(z_0,\varepsilon)}+\frac{\varepsilon}{\varrho(\zeta)\tau_\mu(z_0,\varepsilon)}\right)\frac{\varepsilon}{\tau_\nu(z_0,\varepsilon)}\frac{\varepsilon^{k-2}}{\prod_{j=1}^{k-2}\tau_{\mu_j}(z_0,\varepsilon)\tau_{\nu_j}(z_0,\varepsilon)}\\ &\lesssim\left(\frac{\varepsilon^{k-1}}{\varrho(\zeta)^k}+\frac{\varepsilon^{k}}{\varrho(\zeta)^{k+1}}\right)\tau_1(z_0,\varepsilon)^{-2}\prod_{j=n-k+3}^{n}\tau_j(z_0,\varepsilon)^{-2}\end{aligned}$$

which altogether give the estimate (7) of our lemma. □

## 4 Integral estimates

In this section we will finally give the proofs of Lemma 2.2 and Lemma 2.3. First let us mention that for $k=0$ our kernel becomes the well known Martinelli-Bochner kernel with some additional weight. Since by Lemma 2.1 the weight is bounded and all necessary properties are already known for the Martinelli-Bochner kernel we can restrict our attention to the case $1\le k\le n-1$. In this case the kernel contains at least one factor $\chi$ and therefore vanishes if $\zeta$ does not belong to $U_2$. Since the only singularity is of the form $|z-\zeta|^{-d}$ it is also clear that everything is bounded if $|z-\zeta|>\varepsilon_0$. So it is enough to consider the case that $z$ and $\zeta$ are in $U$ and $\zeta\in P_{\varepsilon_0}(z)$

or $z \in P_{\varepsilon_0}(\zeta)$ respectively. For fixed $\zeta_0$ we will define $\varrho := |\varrho(\zeta_0)|$ and then split the polydisc into the two parts $P_\varrho(\zeta_0)$ and and $P_{\varepsilon_0}(\zeta_0)\backslash P_\varrho(\zeta_0)$. Remember that the first set can be covered by $\bigcup_{i=0}^\infty P_\varrho^{-i}(\zeta_0)$ and the second set is covered by $\bigcup_{i=0}^\infty P_\varrho^{i}(\zeta_0)$. So basicaly we have to deal with domains of the form $P^0_{c\varrho}(\zeta_0)$. We have the following lemma

**Lemma 4.1** *Let $c$ be an arbitrary constant and let $\varepsilon$ and $\varepsilon'$ be small constants. Further let $1 \leq p \leq (mn+2)/(mn+1)$ and define*

$$\delta(k,p) := (2-2p) + (k-1)(1-p) + (2n-2k-1)(1-p)/m + 1/m$$
$$\alpha(p) := \frac{1}{m} - (p-1)\frac{mn+1}{m}$$

*Then we have $\alpha(p) \geq 0$ and for $1 \leq k \leq n-1$ we get*

$$\alpha(p) \leq \delta(k,p) \leq \frac{1}{m} \tag{9}$$

*Moreover we have*

$$\int_{z \in P^0_{c\varrho}(\zeta_0)} \times \frac{d\sigma_{2n}}{|\varrho(z)|^\varepsilon \tau_1(\zeta_0, c\varrho)^{(2-\varepsilon')p} \prod_{j=n-k+2}^n \tau_j^{2p}(\zeta_0, c\varrho)|z-\zeta_0|^{p(2n-2k-1)}}$$
$$\lesssim (c\varrho)^{\varepsilon' p - \varepsilon}(c\varrho)^{\delta(k,p)} \tag{10}$$

*and a similar statement is true if we integrate with respect to $\zeta \in P^0_{c|\varrho(z_0)|}(z_0)$.*

*Proof.* The estimates for $\delta(k,p)$ and $\alpha(p)$ are simple straight forward consequences of the assumptions on $p$ and $k$. To prove the integral estimate (10) we make use of the $(c\varrho)$-extremal coordinates at $\zeta_0$. First integrating with respect to the $z_1$ direction and using the fact that $|\varrho(z)| \gtrsim \operatorname{Re} z_1$ we get

$$\int_{z \in P^0_{c\varrho}(\zeta_0)} \times \frac{d\sigma_{2n}}{|\varrho(z)|^\varepsilon \tau_1(\zeta_0, c\varrho)^{(2-\varepsilon')p} \prod_{j=n-k+2}^n \tau_j^{2p}(\zeta_0, c\varrho)|z-\zeta_0|^{p(2n-2k-1)}}$$
$$\lesssim \tau_1(\zeta_0, c\varrho)^{2-(2-\varepsilon')p-\varepsilon}$$
$$\times \int_{z' \in P^0_{c\varrho}(\zeta_0)} \frac{d\sigma_{2n-2}}{\prod_{j=n-k+2}^n \tau_j^{2p}(\zeta_0, c\varrho)|z-\zeta_0|^{p(2n-2k-1)}}$$
$$\lesssim (c\varrho)^{\varepsilon' p - \varepsilon}(c\varrho)^{2-2p}$$

$$\times \int_{z' \in P^0_{c\varrho}(\zeta_0)} \frac{d\sigma_{2n-2}}{\prod_{j=n-k+2}^n \tau_j^{2p}(\zeta_0, c\varrho)|z-\zeta_0|^{p(2n-2k-1)}}$$

Now we integrate with respect to the coordinates $z_{n-k+2}, \dots, z_n$ and use the fact that $\tau_{n-k+2}(\zeta_0, c\varrho) \geq \cdots \geq \tau_n(\zeta_0, c\varrho) \geq (c\varrho)^{1/2}$. We get

$$\lesssim (c\varrho)^{\varepsilon' p-\varepsilon}(c\varrho)^{2-2p}(c\varrho)^{(k-1)(2-2p)/2} \int_{z'' \in P^0_{c\varrho}(\zeta_0)} \frac{d\sigma_{2n-2k}}{|z-\zeta_0|^{p(2n-2k-1)}}$$

Finally we can use polar coordinates and the fact that $\tau_{n-k+1}(\zeta_0, c\varrho) \leq \cdots \leq \tau_2(\zeta_0, c\varrho) \leq (c\varrho)^{1/m}$ and get

$$\lesssim (c\varrho)^{\varepsilon' p-\varepsilon}(c\varrho)^{2-2p}(c\varrho)^{(k-1)(2-2p)/2} \int_0^{(c\varrho)^(1/m)} \frac{t^{2n-2k-1}dt}{t^{p(2n-2k-1)}}$$

$$\lesssim (c\varrho)^{\varepsilon' p-\varepsilon}(c\varrho)^{2-2p}(c\varrho)^{(k-1)(2-2p)/2}(c\varrho)^{[(2n-2k-1)(1-p)+1]/m}$$

$$\lesssim (c\varrho)^{\varepsilon' p-\varepsilon}(c\varrho)^{\delta(k,p)}$$

which completes the proof of (10). □

*Proof of Lemma 2.2.* We start with the proof of (2). Fix $\zeta_0 \in U$ and let $\varrho := |\varrho(\zeta_0)|$. First we consider integration over the set $P_\varrho(\zeta_0)$ which can be covered by $\bigcup_{i=0}^\infty P_\varrho^{-i}(\zeta_0)$. We fix a constant $\delta$ with $\varepsilon/r < \delta < 1$. Since by Lemma 2.1 the quotient $\varrho(\zeta)/S(z,\zeta)$ is bounded, it follows from Lemma 3.2 that for $z \in P_\varrho^{-i}(\zeta_0)$ we have

$$\left(\frac{S(z,\zeta_0)}{C_1\varrho(\zeta_0)}\right)^{-N-k} \lesssim \left(\frac{\varrho}{2^{-i}\varrho}\right)^{k-\delta}$$

Lemma 3.5 implies that for $z \in P_\varrho^{-i}(\zeta_0)$ we have

$$(\bar\partial Q)^k \lesssim \frac{(2^{-i}\varrho)^k}{\varrho^k \tau_1^2(\zeta_0, 2^{-i}\varrho)\prod_{j=n-k+2}^n \tau_j^2(\zeta_0, 2^{-i}\varrho)}$$

Then the integral under consideration can be estimated as follows

$$\int_{z \in P_\varrho^{-i}(\zeta_0)} |\varrho(z)|^{-\varepsilon}|K(z,\zeta_0)|^r d\sigma_{2n}$$

$$\lesssim \varrho^{-\delta r} \int_{z \in P_\varrho^{-i}(\zeta_0)} \frac{1}{|\varrho(z)|^\varepsilon} \left( \left(\frac{\varrho}{2^{-i}\varrho}\right)^{k-\delta} \right.$$

$$\left. \times \frac{(2^{-i}\varrho)^{k-\delta}(2^{-i}\varrho)^\delta}{\varrho^{k-\delta}\tau_1^2(\zeta_0, 2^{-i}\varrho)\prod_{j=n-k+2}^n \tau_j^2(\zeta_0, 2^{-i}\varrho)|z-\zeta_0|^{2n-2k-1}} \right)^r d\sigma_{2n}$$

$$\lesssim \varrho^{-\delta r} \int_{z \in P_\varrho^{-i}(\zeta_0)}$$

$$\times \frac{d\sigma_{2n}}{|\varrho(z)|^\varepsilon \tau_1(\zeta_0, 2^{-i}\varrho)^{(2-\delta)r} \prod_{j=n-k+2}^n \tau_j^{2r}(\zeta_0, 2^{-i}\varrho)|z-\zeta_0|^{r(2n-2k-1)}}$$

where we also used the fact that $\tau_1(\zeta_0, 2^{-i}\varrho) \approx 2^{-i}\varrho$. Applying Lemma 4.1 we get

$$\int_{z \in P_\varrho^{-i}(\zeta_0)} |\varrho(z)|^{-\varepsilon} |K(z,\zeta_0)|^r d\sigma_{2n}$$
$$\lesssim \varrho^{-\delta r}(2^{-i}\varrho)^{\delta r - \varepsilon}(2^{-i}\varrho)^{\delta(k,r)} \lesssim 2^{-i(\delta r - \varepsilon)}|\varrho(\zeta_0)|^{-\varepsilon}(2^{-i}\varrho)^{\delta(k,r)}$$

Since $2^{-i}\varrho \lesssim 1$ (independent of $\zeta_0$) $\delta(k,p) \geq 0$ and $\delta r - \varepsilon > 0$ this also implies

$$\int_{z \in P_\varrho(\zeta_0)} |\varrho(z)|^{-\varepsilon} |K(z,\zeta_0)|^r d\sigma_{2n} \lesssim |\varrho(\zeta_0)|^{-\varepsilon}$$

To estimate the integral over $P_{\varepsilon_0}(\zeta_0) \backslash P_\varrho(\zeta_0)$ we use the covering by $\bigcup_{i=0}^\infty P_\varrho^i(\zeta_0)$ and for $z \in P_\varrho^i(\zeta_0)$ we have by Lemma 3.2

$$\left(\frac{S(z,\zeta_0)}{C_1\varrho(\zeta_0)}\right)^{-N-k} \lesssim \left(\frac{\varrho}{2^{-i}\varrho}\right)^{N+k} \lesssim 2^{-iN}\left(\frac{\varrho}{2^{-i}\varrho}\right)^k$$

For $(\bar\partial Q)^k$ we use the same estimate as above and then the integral becomes

$$\int_{z \in P_\varrho^i(\zeta_0)} |\varrho(z)|^{-\varepsilon} |K(z,\zeta_0)|^r d\sigma_{2n} \lesssim 2^{-iNr} \int_{z \in P_\varrho^i(\zeta_0)}$$
$$\times \frac{d\sigma_{2n}}{|\varrho(z)|^\varepsilon \tau_1(\zeta_0, 2^{-i}\varrho)^{2r} \prod_{j=n-k+2}^n \tau_j^{2r}(\zeta_0, 2^{-i}\varrho)|z-\zeta_0|^{r(2n-2k-1)}}$$

Applying Lemma 4.1 we get

$$\int_{z \in P_\varrho^i(\zeta_0)} |\varrho(z)|^{-\varepsilon} |K(z,\zeta_0)|^r d\sigma_{2n}$$
$$\lesssim 2^{-iNr}(2^i\varrho)^{-\varepsilon}(2^i\varrho)^{\delta(k,r)} \lesssim 2^{-i(Nr-\varepsilon)}|\varrho(\zeta_0)|^{-\varepsilon}(2^{-i}\varrho)^{\delta(k,r)}$$

Since $\varrho$ is bounded and $0 \leq \delta(k,p) \leq 1/m$ this implies

$$\int_{z \in P_{\varepsilon_0}(\zeta_0) \backslash P_\varrho(\zeta_0)} |\varrho(z)|^{-\varepsilon} |K(z,\zeta_0)|^r d\sigma_{2n} \lesssim |\varrho(\zeta_0)|^{-\varepsilon}$$

provided that $N > (\varepsilon + (1/m))/r$. This completes the proof of (2). Since all estimates are still valid for $\varepsilon = 0$ this also proves the first statement of Lemma 2.2.

It remains to show (3). For fixed $z_0$ we set $\varrho := |\varrho(z_0)|$. Again we split the domain of integration into the two parts $P_\varrho(z_0)$ and $P_{\varepsilon_0}(z_0)\backslash P_\varrho(z_0)$. On $P_\varrho(z_0)$ we use the covering $\bigcup_{i=0}^\infty P_\varrho^{-i}(z_0)$ and by Lemma 2.1 and Lemma 3.2 we see that for $\zeta \in P_\varrho^{-i}(z_0)$ we have the estimate

$$\left(\frac{S(z_0,\zeta)}{C_1\varrho(\zeta_0)}\right)^{-N-k} \lesssim \left(\frac{\varrho(\zeta)}{2^{-i}\varrho}\right)^{k-\delta}\left(\frac{\varrho(\zeta)}{\varrho}\right)^{\delta+\varepsilon/r}$$

where $\delta$ is a small positive constant. Estimating $(\bar\partial Q)^k$ as usually the integral becomes

$$\begin{aligned}
&\int_{\zeta\in P_\varrho^{-i}(z_0)} |\varrho(\zeta)|^{-\varepsilon}|K(z_0,\zeta)|^r d\sigma_{2n}\\
&\quad\lesssim \int_{\zeta\in P_\varrho^{-i}(z_0)} \frac{1}{|\varrho(\zeta)|^{\varepsilon}}\Bigg(\left(\frac{\varrho(\zeta)}{2^{-i}\varrho}\right)^{k-\delta}\left(\frac{\varrho(\zeta)}{\varrho}\right)^{\delta+\varepsilon/r}\\
&\quad\times\frac{(2^{-i}\varrho)^{k-\delta}(2^{-i}\varrho)^{\delta}}{\varrho(\zeta)^k\tau_1^2(z_0,2^{-i}\varrho)\prod_{j=n-k+2}^n\tau_j^2(z_0,2^{-i}\varrho)|z-\zeta_0|^{2n-2k-1}}\Bigg)^r d\sigma_{2n}\\
&\quad\lesssim \varrho^{-\delta r-\varepsilon}\int_{\zeta\in P_\varrho^{-i}(z_0)}\\
&\quad\times\frac{d\sigma_{2n}}{\tau_1(z_0,2^{-i}\varrho)^{(2-\delta)r}\prod_{j=n-k+2}^n\tau_j^{2r}(z_0,2^{-i}\varrho)|z-\zeta_0|^{r(2n-2k-1)}}
\end{aligned}$$

Applying Lemma 4.1 we get

$$\begin{aligned}
&\int_{\zeta\in P_\varrho^{-i}(z_0)} |\varrho(\zeta)|^{-\varepsilon}|K(z_0,\zeta)|^r d\sigma_{2n}\\
&\quad\lesssim \varrho^{-\delta r-\varepsilon}(2^{-i}\varrho)^{\delta r}(2^{-i}\varrho)^{\delta(k,r)} \lesssim 2^{-i\delta r}|\varrho(z_0)|^{-\varepsilon}(2^{-i}\varrho)^{\delta(k,r)}
\end{aligned}$$

which also implies

$$\int_{\zeta\in P_\varrho(z_0)} |\varrho(\zeta)|^{-\varepsilon}|K(z_0,\zeta)|^r d\sigma_{2n} \lesssim |\varrho(z_0)|^{-\varepsilon}$$

To estimate the integral over $P_{\varepsilon_0}(z_0)\backslash P_\varrho(z_0)$ we use the covering $\bigcup_{i=0}^\infty P_\varrho^i(z_0)$ and for $\zeta\in P_\varrho^i(z_0)$ we get from Lemma 3.2 the following estimate

$$\left(\frac{S(z_0,\zeta)}{C_1\varrho(\zeta_0)}\right)^{-N-k} \lesssim \left(\frac{\varrho(\zeta)}{2^{i}\varrho}\right)^{k+\varepsilon}\left(\frac{1}{2^{i}\varrho}\right)^{\delta(k,r)/r}$$

where we also used the fact that $\varrho(\zeta)$ is bounded. Estimating $(\bar\partial Q)^k$ as above, the integral becomes

$$\begin{aligned}&\int_{\zeta\in P^i_\varrho(z_0)} |\varrho(\zeta)|^{-\varepsilon}|K(z_0,\zeta)|^r d\sigma_{2n}\\ &\lesssim \int_{\zeta\in P^i_\varrho(z_0)} \frac{1}{|\varrho(\zeta)|^\varepsilon}\Bigg(\left(\frac{\varrho(\zeta)}{2^i\varrho}\right)^{k+\varepsilon}\left(\frac{1}{2^i\varrho}\right)^{\delta(k,r)/r}\\ &\times\frac{(2^i\varrho)^k}{\varrho(\zeta)^k\tau_1^2(z_0,2^i\varrho)\prod_{j=n-k+2}^n\tau_j^2(z_0,2^i\varrho)|z-\zeta_0|^{2n-2k-1}}\Bigg)^r d\sigma_{2n}\\ &\lesssim (2^i\varrho)^{-\varepsilon}(2^i\varrho)^{-\delta(k,r)}\int_{\zeta\in P^i_\varrho(z_0)}\\ &\times\frac{d\sigma_{2n}}{\tau_1^{2r}(z_0,2^i\varrho)\prod_{j=n-k+2}^n\tau_j^{2r}(z_0,2^i\varrho)|z-\zeta_0|^{r(2n-2k-1)}}\end{aligned}$$

Applying Lemma 4.1 we get

$$\int_{\zeta\in P^i_\varrho(z_0)} |\varrho(\zeta)|^{-\varepsilon}|K(z_0,\zeta)|^r d\sigma_{2n} \lesssim 2^{-i\varepsilon}|\varrho(z_0)|^{-\varepsilon}(2^i\varrho)^{-\delta(k)}(2^i\varrho)^{\delta(k)}$$

and therefore

$$\int_{\zeta\in P_\varrho(z_0)} |\varrho(\zeta)|^{-\varepsilon}|K(z_0,\zeta)|^r d\sigma_{2n} \lesssim |\varrho(z_0)|^{-\varepsilon}$$

which finally completes the proof of Lemma 2.2. □

*Proof of Lemma 2.3.* Lemma 3.5 already gives us an estimate for $\nabla_z$ $(\bar\partial Q)^k$. The next thing we have to investigate is the derivative of the weight function. Here we get

$$\begin{aligned}&\nabla_z(1+\langle Q(z,\zeta),z-\zeta\rangle)^{-N-k}\\ &=c_k(1+\langle Q(z,\zeta),z-\zeta\rangle)^{-N-k-1}\frac{\chi(\zeta)}{\varrho(\zeta)}\\ &\times\sum\left(\nabla\tilde Q_j(z,\zeta)(z_j-\zeta_j)+\tilde Q_j(z,\zeta)\right)\end{aligned}$$

Using the estimates from Lemma 3.5 and Lemma 3.2 we see that for $\zeta\in P^0_\varepsilon(z_0)$ this can be estimated by

$$\begin{aligned}&\left(\frac{\varrho(\zeta)}{S(z_0,\zeta)}\right)^{N+k+1}\left(\frac{1}{\varrho(\zeta)}\left(\frac{\varepsilon}{\tau_j(z_0,\varepsilon)}+\frac{\varepsilon}{\tau_j(z_0,\varepsilon)}|z-\zeta|\right)\right)\\ &\lesssim\left(\frac{\varrho(\zeta)}{S(z_0,\zeta)}\right)^{N+k}\frac{1}{\tau_j(z_0,\varepsilon)}\lesssim\left(\frac{\varrho(\zeta)}{S(z_0,\zeta)}\right)^{N+k}\frac{1}{\varepsilon}\end{aligned}$$

Also it is easy to see that $\nabla_z ds = 0$, $\nabla_z s \lesssim 1$ and $\nabla_z |z-\zeta| \lesssim 1$. Using all these estimates we get for $\zeta \in P^0_\varepsilon(z_0)$ that

$$\begin{aligned}
&\nabla_z \left( G^{(k)} \left(1 + \langle Q(z,\zeta), z-\zeta\rangle\right) \frac{s \wedge (\bar\partial Q)^k \wedge (\bar\partial s)^{n-1-k}}{\langle s(z,\zeta), \zeta - z\rangle^{n-k}} \right) \\
&\lesssim \left( \frac{\varrho(\zeta)}{S(z_0,\zeta)} \right)^{N+k} \\
&\times \frac{1}{\varepsilon} \frac{\varepsilon^k}{\varrho(\zeta)^k \tau_1^2(z_0,\varepsilon) \prod_{j=n-k+2}^n \tau_j^2(z_0,\varepsilon) |z-\zeta_0|^{2n-2k-1}} \\
&\quad + \left( \frac{\varrho(\zeta)}{S(z_0,\zeta)} \right)^{N+k} \left( \frac{\varepsilon^{k-1}}{\varrho(\zeta)^{k-1}} + \frac{\varepsilon^k}{\varrho(\zeta)^k} \right) \\
&\times \left( \frac{1}{\varrho(\zeta)} + \frac{\varepsilon}{\varrho(\zeta)\tau^2_{n-k+2}(z_0,\varepsilon)} \right) \\
&\times \frac{1}{\tau_1^2(z_0,\varepsilon) \prod_{j=n-k+3}^n \tau_j^2(z_0,\varepsilon) |z-\zeta_0|^{2n-2k-1}} \\
&\quad + \left( \frac{\varrho(\zeta)}{S(z_0,\zeta)} \right)^{N+k} \\
&\times \frac{\varepsilon^k}{\varrho(\zeta)^k \tau_1^2(z_0,\varepsilon) \prod_{j=n-k+2}^n \tau_j^2(z_0,\varepsilon) |z-\zeta_0|^{2n-2k}}
\end{aligned}$$

Since $k > 1$ and $|z-\zeta| \gtrsim \varepsilon$ for $\zeta \in P^0_\varepsilon(z)$ and using $\varrho(\zeta)/S(z,\zeta) \lesssim \varrho(\zeta)/\varepsilon$ for a couple of times this can be estimated by

$$\left( \frac{\varrho(\zeta)}{S(z_0,\zeta)} \right)^N \frac{\varepsilon}{\varrho(\zeta)^2} \frac{1}{\tau_1^2(z_0,\varepsilon) \prod_{j=n-k+2}^n \tau_j^2(z_0,\varepsilon) |z-\zeta_0|^{2n-2k-1}}$$

Now we fix $z_0$ and set $\varrho := |\varrho(z_0)|$. On $P_\varrho(z_0)$ we use the covering $\bigcup_{i=0}^\infty P_\varrho^{-i}(z_0)$ and for $\zeta \in P_\varrho^{-i}(z_0)$ Lemma 2.1 and Lemma 3.2 give us the estimate

$$\left( \frac{\varrho(\zeta)}{S(z_0,\zeta)} \right)^N \lesssim \left( \frac{\varrho(\zeta)}{\varrho} \right)^{1+\delta} \left( \frac{\varrho(\zeta)}{2^{-i}\varrho} \right)^{1-\delta}$$

where $\delta$ is a small positive constant. The integral under consideration can now be estimated by

$$\begin{aligned}
&\int_{\zeta \in P_\varrho^{-i}(z_0)} \left( \left( \frac{\varrho(\zeta)}{\varrho} \right)^{1+\delta} \left( \frac{\varrho(\zeta)}{2^{-i}\varrho} \right)^{1-\delta} \right. \\
&\left. \times \frac{(2^{-i}\varrho)^{1-\delta} (2^{-i}\varrho)^\delta}{\varrho(\zeta)^2 \tau_1^2(z_0,\varepsilon) \prod_{j=n-k+2}^n \tau_j^2(z_0,\varepsilon) |z-\zeta_0|^{2n-2k-1}} \right)^{p'} d\sigma_{2n}
\end{aligned}$$

$$\lesssim \varrho^{-p'-\delta p'} \int_{\zeta \in P_\varrho^{-i}(z_0)}$$
$$\times \frac{d\sigma_{2n}}{\tau_1(z_0,\varepsilon)^{(2-\delta)p'} \prod_{j=n-k+2}^{n} \tau_j^{2p'}(z_0,\varepsilon)|z-\zeta_0|^{p'(2n-2k-1)}}$$

Applying Lemma 4.1 we get

$$\int_{\zeta \in P_\varrho^{-i}(z_0)} |\nabla_z K(z_0,\zeta)|^{p'}$$
$$\lesssim \varrho^{-p'-\delta p'}(2^{-i}\varrho)^{\delta p'}(2^{-i}\varrho)^{\delta(k,p')} \lesssim 2^{-i\delta p'}|\varrho(z_0)|^{-p'}(2^{-i}\varrho)^{\alpha(p')}$$

Now we observe that $\alpha(p') = p'\alpha$ where $\alpha$ is as defined in Lemma 2.3. Since $p'(\delta+\alpha) > 0$ even for $\alpha = 0$ we get

$$\int_{\zeta \in P_\varrho(z_0)} |\nabla_z K(z_0,\zeta)|^{p'} \lesssim |\varrho(z_0)|^{p'(\alpha-1)}$$

To integrate over $P_{\varepsilon_0}(z_0)\backslash P_\varrho(z_0)$ we use the covering $\bigcup_{i=0}^{\infty} P_\varrho^i(z_0)$ and for $\zeta \in P_\varrho^i(z_0)$ Lemma 3.2 give us the estimate

$$\left(\frac{\varrho(\zeta)}{S(z_0,\zeta)}\right)^N \lesssim \left(\frac{\varrho(\zeta)}{2^i\varrho}\right)^2 \left(\frac{1}{2^i\varrho}\right)^{(\delta(k,p')-\alpha(p'))/p'}$$

Then the integral becomes

$$\int_{\zeta \in P_\varrho^i(z_0)} \left( \left(\frac{\varrho(\zeta)}{2^{-i}\varrho}\right)^2 \left(\frac{1}{2^{-i}\varrho}\right)^{(\delta(k,p')-\alpha(p'))/p'} \right.$$
$$\left. \times \frac{(2^{-i}\varrho)}{\varrho(\zeta)^2\tau_1^2(z_0,\varepsilon)\prod_{j=n-k+2}^{n}\tau_j^2(z_0,\varepsilon)|z-\zeta_0|^{2n-2k-1}} \right)^{p'} d\sigma_{2n}$$
$$\lesssim (2^i\varrho)^{-p'-\delta(k,p')+\alpha(p')}$$
$$\times \int_{\zeta \in P_\varrho^i(z_0)} \frac{d\sigma_{2n}}{\tau_1^{2p'}(z_0,\varepsilon)\prod_{j=n-k+2}^{n}\tau_j^{2p'}(z_0,\varepsilon)|z-\zeta_0|^{p'(2n-2k-1)}}$$

Applying Lemma 4.1 we get

$$\int_{\zeta \in P_\varrho^{-i}(z_0)} |\nabla_z K(z_0,\zeta)|^{p'} \lesssim (2^i\varrho)^{-p'-\delta(k,p')+\alpha(p')}(2^i\varrho)^{\delta(k,p')}$$

and since $\alpha(p') = p'\alpha < p'$ this implies

$$\int_{\zeta \in P_{\varepsilon_0}(z_0)\backslash P_\varrho(z_0)} |\nabla_z K(z_0,\zeta)|^{p'} \lesssim |\varrho(z_0)|^{p'(\alpha-1)}$$

which completes the proof of the lemma. □

## References


[BeAn] B. Berndtsson, M. Andersson: Henkin-Ramirez formulas with weight factors, Ann. Inst. Fourier **32** (1982), 91–110

[BrCa] J. Bruna, J. Castillo: Hölder and $L^p$-estimates for the $\bar{\partial}$ equation in some convex domains with real analytic boundary, Math. Ann. **269** (1984), 527–539

[BrChDu] J. Bruna, P. Charpentier, Y. Dupain: Zero varieties for the Nevanlinna class in convex domains of finite type in $\mathbb{C}^n$, Ann. Math. **147** (1998), 391–415

[BrNaWa] J. Bruna, A. Nagel, S. Wainger: Convex hypersurfaces and Fourier transform, Ann. Math. **127** (1988), 333–365

[ChKrMa] Z. Chen, S.G. Krantz, D. Ma: Optimal $L^p$ estimates for the $\bar{\partial}$- equation on complex ellipsoids in $\mathbb{C}^n$, Manusc. Math. **80** (1993), 131–149

[Cu] A. Cumenge: Sharp estimates for $\bar{\partial}$ on convex domains of finite type, to appear in Arkiv för Mat.

[DiFiFo] K. Diederich, B. Fischer, J.E. Fornæss: Hölder estimates on convex domains of finite type, Math. Z. **232** (1999), 43-61.

[DiFo] K. Diederich and J.E. Fornæss: Support functions for convex domains of finite type, Math. Z. **230** (1999), 145–164

[DiMa] K. Diederich and E. Mazzilli: Zero varieties for the Nevanlinna class on all convex domains of finite type, to appear.

[Kr] S.G. Krantz: Optimal Lipschitz and $L^p$ regularity for the equation $\bar{\partial}u = f$ on strongly pseudoconvex domains, Math. Ann. **219** (1976), 233–260

[Mc1] J. McNeal: Convex domains of finite type, J. Functional Anal. **108** (1992), 361–373.

[Mc2] J.D. McNeal: Estimates on the Bergman kernels of convex domains, Adv. in Math. **109** (1994), No. 1, 108–139

[McSt] J.D. McNeal, E.M. Stein: Mapping properties of the Bergman projection of convex domains of finite type, Duke Math. J. **73** (1994), 177–199

[Ra] R.M. Range: Holomorphic Functions and Integral Representations in Several Complex Variables, Springer-Verlag, New York, 1986