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On the blow-up rate and the blow-up set of breaking waves for a shallow water equation

Adrian Constantin, Joachim Escher

[1] Institute for Mathematics, University Zürich, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland

[2] Fachbereich 17, University Kassel, Heinrich-Plett-Strasse 40, D-34132 Kassel, Germany (e-mail: escher@mathematik.uni-kassel.de)

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Abstract. We consider the problem of the development of singularities for classical solutions to a new periodic shallow water equation. Blow-up can occur only in the form of wave-breaking, i.e. the solution remains bounded but its slope becomes unbounded in finite time. A quite detailed description of the wave-breaking phenomenon is given: there is at least a point (in general depending on time) where the slope becomes infinite exactly at breaking time. The precise blow-up rate is established and for a large class of initial data we also determine the blow-up set.

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1. Introduction

In this paper we consider the problem of the development of singularities for classical solutions to the initial value problem

$$
(1) \qquad \begin{cases} u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \qquad t > 0, \quad x \in \mathbb{R}, \\ u(0,x) = u_0(x), \qquad x \in \mathbb{R}, \\ u(t, x+1) = u(t,x), \qquad t \geq 0, \quad x \in \mathbb{R}, \end{cases}
$$

where u_0 is a given periodic initial value.

This quasilinear partial differential equation is a model for wave motion on shallow water cf. [12], $u(t,x)$ representing the water's free surface above a flat bottom. Equation (1) was abstractly derived as a bi-Hamiltonian generalization of the famous Korteweg-de Vries (KdV) equation [28]. Recently,

R. Camassa and D. Holm gave a physical derivation of (1) and found that the equation has solitons: two solitary waves keep their shape and size after interaction while the ultimate position of each wave is affected only with a phase shift by the nonlinear interaction (see [19]). In [13] it was conjectured that wave-breaking occurs for certain initial profiles of (1) - for results on wave-breaking for solutions to (1) we refer to [15], [18], [19]. Equation (1) is integrable in the sense of an infinite-dimensional Hamiltonian system (for this aspect we refer to [21], [16]).

The KdV equation and Whitham's equation (see [45]) are well-known models for wave-motion on shallow water with a flat bottom. However, whereas the KdV equation admits soliton interaction for its solitary waves [23], [24], [25], [39], the KdV equation does not model wave-breaking: it is shown in [7] that solutions exist globally for initial data in $L^2(\mathbb{S})$ - see also [35] (here $\mathbb{S}$ is the unit circle). On the other hand, wave-breaking has been observed for certain solutions to Whitham's equation (for a formal discussion we refer to [45] - see [18] for a rigorous proof) but the numerical calculations carried out [23] do not support any strong claim that soliton interaction holds for this model. An equation modelling both soliton interaction and wave-breaking on shallow water is an important problem in the theory of water waves [45], [31]. This explains the numerous papers devoted recently to the study of equation (1) - see [1], [2], [8]-[22], [28]-[33], [37], [38], [42], [43].

Let us also mention that the above described model for wave-motion on shallow water arises in an entirely different context, namely, the partial differential equation (1) is a re-expression of the geodesic flow in the group $\mathcal{D}^3(\mathbb{S})$ of orientation-preserving diffeomorphisms of the circle. We will use this connection with infinite dimensional geometry to study in detail the wave-breaking phenomena.

We shortly present now the geometrical picture. Let

$$\mathcal{D}^3(\mathbb{S}) = \left\{ \begin{array}{c} \eta : \mathbb{S} \to \mathbb{S}\,,\ \eta \text{ bijective and orientation-preserving,} \\ \text{and } \eta, \eta^{-1} \in H^3(\mathbb{S}) \end{array} \right\}$$

be the group of orientation-preserving diffeomorphisms of the circle modelled on the Sobolev space $H^3(\mathbb{S})$. We say $f \in \mathcal{H}^3(\mathbb{S}, \mathbb{S})$ if for every $x \in \mathbb{S}$ and any chart (O, ϕ) containing x and any chart (O', ψ) of $f(x)$, the map $\psi \circ f \circ \phi^{-1} : \phi(O) \to \mathbb{R}$ is in $H^3(\phi(O), \mathbb{R})$. Since $\mathcal{D}^3(\mathbb{S})$ is open in $\mathcal{H}^3(\mathbb{S}, \mathbb{S})$, we conclude that $\mathcal{D}^3(\mathbb{S})$ is also an infinite dimensional manifold, cf. [27], which locally, around each of its points η, looks like a Hilbert space. Furthermore, $\mathcal{D}^3(\mathbb{S})$ can be given a group structure with multiplication being the composition of the maps. $\mathcal{D}^3(\mathbb{R})$ is not precisely a Lie group (right translation $R_\eta(\phi) := \phi \circ \eta$ is C^∞ but left translation and inversion are only C^0) but it shares some important Lie group properties.

If $\mathcal{H}^3(\mathbb{S})$ is the vector space of all $H^3(\mathbb{S})$-vector fields, any tangent vector X_η to $\mathcal{D}^3(\mathbb{S})$ at η is of the form $X \circ \eta$ with some $X \in \mathcal{H}^3(\mathbb{S})$. For a given $X \in \mathcal{H}^3(\mathbb{S})$, let $X^R(\eta) = X \circ \eta$ denote the right-invariant vector field on $\mathcal{D}^3(\mathbb{S})$ whose value at the identity e is X. $\mathcal{H}^3(\mathbb{S})$ can be thought of as the Lie algebra of $\mathcal{D}^3(\mathbb{S})$. The Lie algebra bracket of $\mathcal{H}^3(\mathbb{S})$ is given by

$$\left(\mathcal{L}_{X^R} Y^R\right)(\eta) = [X, Y] \circ \eta, \quad X, Y \in \mathcal{H}^3(\mathbb{S}),\, \eta \in \mathcal{D}^3(\mathbb{S}),$$

where $[X, Y]$ is the Lie bracket of vector fields on $\mathbb{S}$.

Consider now the $H^1(\mathbb{S})$-metric

$$\langle f, g\rangle_{H^1(\mathbb{S})} = \int_{\mathbb{S}} f(x)g(x)\,dx + \int_{\mathbb{S}} f'(x)g'(x)\,dx, \quad f, g \in H^1(\mathbb{S}), \tag{2}$$

on $T_e\mathcal{D}^3(\mathbb{S}) \simeq \mathcal{H}^3(\mathbb{S})$. We can define a metric on all $T\mathcal{D}^3(\mathbb{S})$ by right translation, i.e. for $V, W \in T_\eta\mathcal{D}^3(\mathbb{S})$,

$$\langle V, W\rangle_{H^1(\mathbb{S})} := \left\langle V \circ \eta^{-1}, W \circ \eta^{-1}\right\rangle_{H^1(\mathbb{S})}.$$

This metric is right-invariant by definition and makes $\mathcal{D}^3(\mathbb{S})$ into a weak Riemannian manifold, i.e. the topology induced by this metric is weaker than the topology of $\mathcal{D}^3(\mathbb{S})$.

(1) is a re-expression of the geodesic flow in the group $\mathcal{D}^3(\mathbb{S})$ with the above described (right-invariant) metric: if v solves (1) and if $q = q(t, x)$ solves

$$q_t = u(t, q), \qquad t > 0, \quad x \in \mathbb{S},$$

with $q(0, x) = x$ on $\mathbb{S}$, then the curve $q(t, \cdot) : t \geq 0$ is a geodesic issuing from the identity in $\mathcal{D}^3(\mathbb{S})$. Conversely, if $q(t, \cdot) : t \geq 0$ is a geodesic, then $v = q_t \circ q^{-1}$ solves (1) up to breakdown of the geodesic flow, cf. [21] and [40]; for the explicit computations we refer to [36]. This aspect of (1) resembles to the situation for Euler's equation of hydrodynamics [6], [26], [27].

As mentioned above, wave-breaking occurs for certain initial profiles of (1). In this paper we give a quite detailed description of this phenomenon. We first prove that the maximal existence time $T > 0$ of a classical solution to (1) is finite if and only if the slope of the solution becomes unbounded in finite time (while the solution remains bounded). This is classically referred to as wave breaking (see [45]). In contrast to other blow-up results for hyperbolic partial differential equations (see [3], [4], [34], [44]) where few can be said about the way blow-up occurs, we will see that the model (1) offers a very nice picture of the wave-breaking phenomena. Namely, if $T < \infty$, we have

$$\lim_{t\to T}\left(\min_{x\in\mathbb{S}}\{u_x(t, x)\}\right) = -\infty$$

and the exact blow-up rate is

$$\lim_{t\to T}\Big((T-t)\,\min_{x\in\mathbb{S}}\{u_x(t,x)\}\Big)=-2.$$

Further, we will show that for a large class of odd initial data the blow-up set consists of the three points $\{0,\frac{1}{2},1\}$. More precisely, we have

$$u_x(t,0)=u_x\left(t,\frac{1}{2}\right)=u_x(t,1)\to-\infty\quad\text{as}\quad t\to T<\infty$$

while

$$\sup_{(t,x)\in[0,T)\times\mathbb{S}}|u(t,x)|<\infty$$

and

$$\sup_{t\in[0,T)}|u_x(t,x)|<\infty,\quad x\in\left(0,\frac{1}{2}\right)\cup\left(\frac{1}{2},1\right).$$

Formally, the wave breaks at a discrete set of points and elsewhere we do not oberseve anything “bad” at breaking time!

2. The blow-up rate

This section is devoted to a general discussion of wave-breaking for solutions to (1).

Let us first recall

Theorem A [17] *Given* $u_0\in H^3(\mathbb{S})$*, there exists a maximal* $T=T(u_0)>0$ *and a unique strong solution*

$$u=u(\cdot,u_0)\in C([0,T);H^3(\mathbb{S}))\cap C^1([0,T);H^2(\mathbb{S}))$$

to problem (1)*. The solution depends continuously on the initial data, i.e., the mapping* $u_0\mapsto u(\cdot,u_0)$ *is continuous from* $H^3(\mathbb{S})$ *to* $C([0,T);H^3(\mathbb{S}))\cap C^1([0,T);H^2(\mathbb{S}))$ *and if* $T<\infty$ *we have* $\lim_{t\to T}|u(t,\cdot)|_{H^3(\mathbb{S})}=\infty$*. Moreover,* $\int_{\mathbb{S}}[u^2(t,x)+u_x^2(t,x)]\,dx$ *is conserved on* $[0,T)$*. For* $u_0\in H^4(\mathbb{S})$ *the solution* u *posseses the additional regularity*

$$u\in C([0,T);H^4(\mathbb{S}))\;\cap\;C^1([0,T);H^3(\mathbb{S})).$$

A solution u to (1) on some interval $[0,t_0)$ with $t_0>0$ is called **classical** if it satisfies the partial differential equation pointwise on $[0,t_0)\times\mathbb{S}$. By Sobolev’s imbedding theorem and Theorem A, any $u_0\in H^4(\mathbb{S})$ yields a classical solution of (1) on its maximal existence interval.

Throughout this paper we always take $u_0 \in H^3(\mathbb{S})$ and we consider the problem of the development of singularities for strong solutions to (1). In view of Theorem A and the above observation, it is easy to see that all results will also hold for classical solutions, provided the initial data $u_0 \in H^4(\mathbb{S})$.

By Sobolev's imbedding theorem and the conservation law from Theorem A we deduce that the solution u satisfies

$$\sup_{t\in[0,T)\times\mathbb{S}} |u(t,x)| < \infty. \tag{3}$$

It is convenient to introduce the potential $y(t,x)$ associated to a solution $u(t,x)$ of (1), defined by $y := u - u_{xx}$. Problem (1) is equivalent to

$$\begin{cases} y_t = -2yu_x - y_x u, & t > 0, \quad x \in \mathbb{R}, \\ y(0,x) = y_0(x), & x \in \mathbb{R}, \\ y(t,x+1) = y(t,x), & t \geq 0, \quad x \in \mathbb{R}, \end{cases} \tag{4}$$

as an evolution equation in $C([0,T);H^1(\mathbb{S})) \cap C^1([0,T);L_2(\mathbb{S}))$.

Theorem 1 *The maximal existence time T is finite if and only if the slope of the solution becomes unbounded from below in finite time.*

Proof. By Theorem A and Sobolev's imbedding theorem it is clear that if the slope of the solution becomes unbounded from below in finite time, then $T < \infty$.

Let $T < \infty$ and assume that for some constant $K > 0$ we have

$$u_x(t,x) \geq -K, \quad (t,x) \in [0,T) \times \mathbb{S}.$$

Using (4) and integration by parts we find that

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{S}} [y(t,x)]^2 \, dx &= -3 \int_{\mathbb{S}} u_x(t,x) \, [y(t,x)]^2 \, dx \\ &\leq 3K \int_{\mathbb{S}} [y(t,x)]^2 \, dx, \quad t \in (0,T). \end{aligned}$$

Gronwall's inequality yields

$$\int_{\mathbb{S}} [y(t,x)]^2 \, dx \leq e^{3Kt} \int_{\mathbb{S}} [y_0(x)]^2 \, dx, \quad t \in (0,T). \tag{5}$$

Let us now approximate $u_0 \in H^3(\mathbb{S})$ in the space $H^3(\mathbb{S})$ with a sequence $u_0^n \in H^4(\mathbb{S})$, $n \geq 1$. We denote by u^n the solution of (1) with initial data u_0^n, defined on the maximal interval of existence $[0,T_n)$ given by Theorem A, and let $y^n := u^n - u_{xx}^n$.

The additional regularity of u^n (ensured by Theorem A) enables us to differentiate the first equation of (1), which leads to

$$\frac{d}{dt}\int_{\mathbb{S}}[y_x^n(t,x)]^2\,dx = -5\int_{\mathbb{S}} u_x^n(t,x)\,[y_x^n(t,x)]^2\,dx \tag{6}$$
$$-4\int_{\mathbb{S}} u^n(t,x)\,y^n(t,x)\,y_x^n(t,x)\,dx,$$
$$t\in(0,T_n).$$

As above, we have

$$\frac{d}{dt}\int_{\mathbb{S}}[y^n(t,x)]^2\,dx = -3\int_{\mathbb{S}} u_x^n(t,x)\,[y^n(t,x)]^2\,dx, \quad t\in(0,T_n). \tag{7}$$

We first claim that there is a sequence $n_k\to\infty$ with

$$\inf_{t\in[0,T_{n_k})}\Big[\min_{x\in\mathbb{S}} u_x^{n_k}(t,x)\Big] = -\infty. \tag{8}$$

Indeed, if this does not hold, we find that for $n\geq 1$ large enough

$$\inf_{t\in[0,T_n)}\Big[\min_{x\in\mathbb{S}} u_x^n(t,x)\Big] > -\infty$$

and by (6)-(7), taking into account (3), we would obtain

$$\frac{d}{dt}\int_{\mathbb{S}}\Big([y^n(t,x)]^2+[y_x^n(t,x)]^2\Big)\,dx$$
$$\leq K_n\int_{\mathbb{S}}\big([y^n(t,x)]^2+[y_x^n(t,x)]^2\big)\,dx, \quad t\in(0,T_n),$$

for some $K_n>0$. But in this case Gronwall's inequality gives

$$\int_{\mathbb{S}}\Big([y^n(t,x)]^2+[y_x^n(t,x)]^2\Big)\,dx \leq e^{K_n t}\int_{\mathbb{S}}\Big([y_0^n(t,x)]^2$$
$$+[y_{0,x}^n(t,x)]^2\Big)\,dx, \quad t\in(0,T_n).$$

In view of Theorem A, this implies that $T_n=\infty$ for all $n\geq 1$ large enough which is in contradiction to the continuous dependence on initial data (we assumed $T<\infty$).

Therefore relation (8) holds and we obtain

$$\sup_{t\in[0,T_{n_k})} |u_x^{n_k}(t,\cdot)|_{L^\infty(\mathbb{S})} = \infty,$$

which on its turn implies

$$\sup_{t\in[0,T_{n_k})} |u^{n_k}(t,\cdot)|_{H^2(\mathbb{S})} = \infty.$$

Taking now into account the relation $y^{n_k} = u^{n_k} - u^{n_k}_{xx}$, we find

$$\sup_{t\in[0,T_{n_k})} |y^{n_k}(t,\cdot)|_{L^2(\mathbb{S})} = \infty.$$

The previous relation and (5) can not hold simultaneously in view of the continuous dependence on initial data. The obtained contradiction shows that our assumption (on the boundedness from below of the slope of the solution) is false. The proof is complete. □

Let us now derive a useful equivalent form of equation (1).

The operator $(I-\partial_x^2)^{-1}$ acting on $L^2(\mathbb{S})$ has the following representation

$$\left[(I-\partial_x^2)^{-1} f\right](x) = \int_{\mathbb{S}} G(x-\xi)\, f(\xi)\, d\xi, \quad f \in L^2(\mathbb{S}),$$

with the Green's function

$$G(x) = \frac{\cosh\left(x - [x] - \frac{1}{2}\right)}{2\sinh\left(\frac{1}{2}\right)}, \quad x \in \mathbb{S}.$$

Equation (1) can also be written as

$$\left(I-\partial_x^2\right)(u_t + uu_x) = -\partial_x\left(u^2 + \frac{1}{2}u_x^2\right).$$

We obtain from here that

$$u_t + uu_x - \partial_x\left(G * \left(u^2 + \frac{1}{2}u_x^2\right)\right) = 0,$$

where $*$ stands for convolution with respect to the spatial variable. By differentiation we obtain

$$\begin{aligned} u_{tx} + uu_{xx} + u_x^2 &= -\partial_x^2\left(G * \left[u^2 + \frac{1}{2}u_x^2\right]\right) \\ &= u^2 + \frac{1}{2}u_x^2 - G * \left[u^2 + \frac{1}{2}u_x^2\right] \end{aligned}$$

so that

$$u_{tx} + uu_{xx} = u^2 - \frac{1}{2}u_x^2 - G * \left[u^2 + \frac{1}{2}u_x^2\right] \tag{9}$$

in the space $C([0,T); H^1(\mathbb{S}))$.

The following result plays an important role in our further investigations:

Lemma 1 [18] *Let* $t_0 > 0$ *and* $v \in C^1([0,t_0); H^2(\mathbb{S}))$. *Then for every* $t \in [0,t_0)$ *there exists at least one point* $\xi(t) \in \mathbb{S}$ *with*

$$m(t) := \min_{x\in\mathbb{S}} \left[v_x(t,x)\right] = v_x(t,\xi(t)),$$

and the function m *is almost everywhere differentiable on* $(0, t_0)$ *with*

$$\frac{dm}{dt}(t) = v_{tx}(t, \xi(t)) \qquad a.e.\ on\ (0, t_0).$$

We use the previous lemma to give more insight into the blow-up mechanism for the wave-breaking solutions to problem (1).

Theorem 2 *Let* $u_0 \in H^3(\mathbb{S})$ *and let* $T > 0$ *be the maximal existence time of the corresponding solution to* (1). *If* T *is finite, we have*

$$\lim_{t \to T} (T - t) \min_{x \in \mathbb{S}} u_x(t, x) = -2.$$

Proof. We already know by Theorem 1 that

$$\liminf_{t \to T} \min_{x \in \mathbb{S}} u_x(t, x) = -\infty.$$

Define now $m(t) := \min_{x \in \mathbb{S}} [u_x(t, x)]$, $t \in [0, T)$, and let $\xi(t) \in \mathbb{S}$ be a point where this minimum is attained. Clearly $u_{xx}(t, \xi(t)) = 0$ since $u(t, \cdot) \in H^3(\mathbb{S}) \subset C^2(\mathbb{S})$. Evaluating (9) at $\xi(t)$ and using Lemma 1, we obtain the relation

(10)
$$\frac{dm}{dt} + \frac{1}{2} m^2 = u^2(t, \xi(t)) - \left[G * \left(u^2 + \frac{1}{2} u_x^2 \right) \right] (t, \xi(t)) \quad \text{a.e. on (0,T)}.$$

By Young's inequality we have for $t \in [0, T)$ that

$$\begin{aligned} |G * \left(u^2 + \frac{1}{2} u_x^2 \right) (t, \cdot)|_{L^\infty(\mathbb{S})} &\leq |G|_{L^\infty(\mathbb{S})} \, |u^2(t, \cdot) \\ &\quad + \frac{1}{2} u_x^2(t, \cdot)|_{L^1(\mathbb{S})} \\ &\leq \frac{\cosh\left(\frac{1}{2}\right)}{2 \sinh\left(\frac{1}{2}\right)} \, |u(t, \cdot)|^2_{H^1(\mathbb{S})} \\ &= \frac{\cosh\left(\frac{1}{2}\right)}{2 \sinh\left(\frac{1}{2}\right)} \, |u_0|^2_{H^1(\mathbb{S})}, \end{aligned}$$

if we take into account the conservation law from Theorem A.

Recalling that by (3) the solution is bounded on $[0, T) \times \mathbb{S}$, we find a constant $K > 0$ such that

$$\left| u^2(t, \xi(t)) - \left[G * \left(u^2 + \frac{1}{2} u_x^2 \right) \right] (t, \xi(t)) \right| \leq K, \quad t \in [0, T).$$

From (10) and the previous relation we obtain

(11)
$$-K \leq \frac{dm}{dt} + \frac{1}{2} m^2 \leq K \quad \text{a.e. on (0,T)}.$$

Let $\varepsilon \in (0, \frac{1}{2})$. Since $\liminf_{t\to T} m(t) = -\infty$ by Theorem 1, there is some $t_0 \in (0,T)$ with $m(t_0) < 0$ and $m^2(t_0) > \frac{K}{\varepsilon}$. Let us first prove that

$$m^2(t) > \frac{K}{\varepsilon}, \quad t \in [t_0, T). \tag{12}$$

Since m is locally Lipschitz (it belongs to $W^{1,\infty}_{loc}(\mathbb{R})$ by Lemma 1) there is some $\delta > 0$ such that

$$m^2(t) > \frac{K}{\varepsilon}, \quad t \in (t_0, t_0+\delta).$$

Pick $\delta > 0$ maximal with this property. If $\delta < T - t_0$ we would have $m^2(t_0+\delta) = \frac{K}{\varepsilon}$ while

$$\frac{dm}{dt} \le -\frac{1}{2}m^2 + K < -\frac{1}{2}m^2 + \varepsilon m^2 < 0 \quad \text{a.e. on } (t_0, t_0+\delta).$$

Being locally Lipschitz, the function m is absolutely continuous and therefore we would obtain by integrating the previous relation on $[t_0, t_0+\delta]$ that

$$m(t_0+\delta) \le m(t_0) < 0$$

which on its turn would yield

$$m^2(t_0+\delta) \ge m^2(t_0) > \frac{K}{\varepsilon}.$$

The obtained contradiction completes the proof of relation (12).

A combination of (11) and (12) enables us to infer

$$\frac{1}{2} + \varepsilon \ge -\frac{\frac{dm}{dt}}{m^2} \ge \frac{1}{2} - \varepsilon \quad \text{a.e. on (0,T).} \tag{13}$$

Since m is locally Lipschitz on $[0,T)$ and (12) holds, it is easy to check that $\frac{1}{m}$ is locally Lipschitz on (t_0,T). Differentiating the relation $m(t) \cdot \frac{1}{m(t)} = 1$, $t \in (t_0,T)$, we get

$$\frac{d}{dt}\left(\frac{1}{m}\right) = -\frac{\frac{dm}{dt}}{m^2} \quad \text{a.e. on } (t_0,T),$$

with $\frac{1}{m}$ absolutely continuous on (t_0,T). For $t \in (t_0,T)$, integrate (13) on (t,T) to obtain

$$\left(\frac{1}{2}+\varepsilon\right)(T-t) \ge -\frac{1}{m(t)} \ge \left(\frac{1}{2}-\varepsilon\right)(T-t), \quad t \in (t_0,T),$$

that is,

$$\frac{1}{\frac{1}{2}+\varepsilon} \le -m(t)\,(T-t) \le \frac{1}{\frac{1}{2}-\varepsilon},, \quad t \in (t_0,T).$$

By the arbitraryness of $\varepsilon \in (0, \frac{1}{2})$ the statement of Theorem 2 follows. □

We conclude this section by presenting sufficient conditions on the initial profile to ensure wave breaking for the corresponding solution to (1).

Theorem B [17] *Assume* $u_0 \in H^3(\mathbb{S}),\ u_0 \not\equiv 0$, *satisfies*

$$\int_{\mathbb{S}} u_0(x)\,dx = 0 \quad \textit{or} \quad \int_{\mathbb{S}} [u_0^3(x) + u_0(x)u_{0,x}^2(x)]\,dx = 0.$$

Then the maximal existence time of the corresponding solution to (1) *is finite.*

Theorem C [20] *Assume* $u_0 \in C^\infty(\mathbb{S})$ *is such that*

$$\min_{x\in\mathbb{S}}[u_0'(x)] + \max_{x\in\mathbb{S}}[u_0'(x)] \le -2\sqrt{3}\,|u_0|_{H^1(\mathbb{S})}.$$

Then the maximal existence time of the corresponding solution to (1) *is finite.*

3. The blow-up set

In this section we determine for a large class of initial data the exact blow-up set for the corresponding wave-breaking solution to the initial value problem (1).

To a strong solution u to (1) with initial data $u_0 \in H^3(\mathbb{R})$ and with maximal existence time $T > 0$ (given by Theorem A) we associate the differential equation

$$(14) \qquad \begin{cases} q_t = u(t,q), & t \in (0,T), \\ q(0,x) = x, & x \in \mathbb{R}. \end{cases}$$

As already mentioned in the Introduction, (14) is the equation for the geodesic in $\mathcal{D}^3(\mathbb{S})$ issuing from the identity in the direction u_0.

Lemma 2 *Let* $u_0 \in H^3(\mathbb{S})$ *and let* $T > 0$ *be the maximal existence time of the corresponding solution to* (1). *Then the system* (14) *has a unique solution* $q \in C^1([0,T) \times \mathbb{R}, \mathbb{R})$. *Moreover, for each fixed* $t \in [0,T)$, *the map* $q(t,\cdot)$ *in an increasing diffeomorphism of* $\mathbb{R}$ *with* $q_x(t,x) > 0$ *for* $(t,x) \in [0,T) \times \mathbb{R}$.

Proof. For fixed $x \in \mathbb{R}$ we deal with an ordinary differential equation. By Sobolev's imbedding theorem we have that $u \in C^1([0,T) \times \mathbb{R}, \mathbb{R})$ and by (3) we know that u is bounded on $[0,T) \times \mathbb{R}$. Therefore classical results in the theory of ordinary differential equations (cf. [5]) yield the first assertion.

From (14) we obtain the system

$$(15)\qquad \begin{cases} \frac{d}{dt}\, q_x = u_x(t,q)\, q_x, & t \in (0,T), \\ q_x(0,x) = 1, & x \in \mathbb{R}. \end{cases}$$

The solution to (15) is given by

$$(16)\qquad q_x(t,x) = e^{\int_0^t u_x(s,q(s,x))\, ds}, \quad (t,x) \in [0,T) \times \mathbb{R}.$$

For every fixed $t \in (0,T)$ we have by Sobolev's imbedding theorem (and the periodicity in the spatial variable) that

$$\sup_{(s,x)\in[0,t]\times\mathbb{R}} |u_x(s,x)| < \infty.$$

Combining this with (16) we find a constant $K > 0$ such that

$$q_x(t,x) \geq e^{-tK}, \quad (t,x) \in [0,T) \times \mathbb{R}.$$

The proof is complete. □

Lemma 3 *Let* $u_0 \in H^3(\mathbb{S})$ *and let* $T > 0$ *be the maximal existence time of the corresponding solution to* (1). *If* $y := u - u_{xx}$ *is the associated potential, we have*

$$y(t, q(t,x))\, q_x^2(t,x) = y_0(x), \quad (t,x) \in [0,T) \times \mathbb{R}.$$

Proof. Differentiate the left-hand side with respect to time and take into account (4) and (15)-(16). □

Lemma 4 *Assume* $u_0 \in H^3(\mathbb{S})$ *is odd and let* $T > 0$ *be the maximal existence time of the corresponding solution to* (1). *Then* $q(t,\xi) = \xi$ *for* $\xi \in \{0, \frac{1}{2}, 1\}$ *and* $t \in [0,T)$.

Proof. An odd initial profile for (1) develops into a wave $u(t,\cdot)$ which is odd at any time $t \in [0,T)$, cf. [15].

We will prove the statement for $\xi = \frac{1}{2}$. Since $u(t,\cdot)$ is odd, we see that $u(t,\frac{1}{2}) = 0$, $t \in [0,T)$. From (14) we have that

$$\begin{cases} \frac{d}{dt}\, q(t,\frac{1}{2}) = u(t, q(t,\frac{1}{2})), & t \in (0,T), \\ q(0,\frac{1}{2}) = \frac{1}{2}. \end{cases}$$

Since $u(t,\frac{1}{2}) = 0$ for $t \in [0,T)$ we conclude by uniqueness that the constant function $\frac{1}{2}$ is the solution of the above differential equation, that is, $q(t,\frac{1}{2}) = \frac{1}{2}$ for $t \in [0,T)$.

The remaining cases can be treated similarly. □

Lemma 5 *Assume $u_0 \in H^3(\mathbb{S})$ is odd and let $T > 0$ be the maximal existence time of the corresponding solution to* (1)*. For $\frac{1}{2} \le x \le 1$ we have the representation formulas*

(17)
$$\begin{aligned} u(t,x) = & \tfrac{1}{\sinh(\frac{1}{2})}\, \sinh(1-x)\, \int_{\frac{1}{2}}^{x} \sinh\left(\xi - \tfrac{1}{2}\right)\, y(t,\xi)\, d\xi \\ & + \tfrac{1}{\sinh(\frac{1}{2})}\, \sinh\left(x - \tfrac{1}{2}\right)\, \int_{x}^{1} \sinh(1-\xi)\, y(t,\xi)\, d\xi, \quad t \in [0,T), \end{aligned}$$

whereas

(18)
$$\begin{aligned} u_x(t,x) = & -\tfrac{1}{\sinh(\frac{1}{2})}\, \cosh(1-x)\, \int_{\frac{1}{2}}^{x} \sinh\left(\xi - \tfrac{1}{2}\right)\, y(t,\xi)\, d\xi \\ & + \tfrac{1}{\sinh(\frac{1}{2})}\, \cosh\left(x - \tfrac{1}{2}\right) \int_{x}^{1} \sinh(1-\xi)\, y(t,\xi)\, d\xi, \quad t \in [0,T). \end{aligned}$$

The representations on $[0, \frac{1}{2}]$ are obtained by reflection in $x = \frac{1}{2}$ if we recall that $u(t,\cdot)$ is odd for $t \in [0,T)$.

Proof. From Section 2 we know that

$$u(t,x) = \int_{\mathbb{S}} G(x-\xi)\, y(t,\xi)\, d\xi, \quad (t,x) \in [0,T) \times \mathbb{S},$$

where $G(x) := \frac{\cosh(x - [x] - \frac{1}{2})}{2\sinh(\frac{1}{2})},\ x \in \mathbb{S}$.

Fix $(t,x) \in [0,T) \times [\frac{1}{2}, 1]$ and let $\sigma := \frac{1}{4\sinh(\frac{1}{2})}$. We have

$$\begin{aligned} u(t,x) = & \sigma\, e^{x} \int_0^x e^{-\xi - \frac{1}{2}} y(t,\xi)\, d\xi + \sigma\, e^{-x} \int_0^x e^{\xi + \frac{1}{2}} y(t,\xi)\, d\xi \\ & + \sigma\, e^{x} \int_x^1 e^{-\xi + \frac{1}{2}} y(t,\xi)\, d\xi + \sigma\, e^{-x} \int_x^1 e^{\xi - \frac{1}{2}} y(t,\xi)\, d\xi. \end{aligned}$$

As before, since $u_0(\cdot)$ is odd, we have for $t \in [0,T)$ that $u(t,\cdot)$ and $y(t,\cdot)$ are odd too. Combining this with the spatial periodicity of both $u(t,\cdot)$ and $y(t,\cdot)$ as $t \in [0,T)$, and changing variables ($\xi \mapsto 1-\xi$) we arrive to the identities

$$\begin{aligned} & e^{x} \int_0^{1-x} e^{-\xi - \frac{1}{2}} y(t,\xi)\, d\xi + e^{x} \int_x^1 e^{-\xi + \frac{1}{2}} y(t,\xi)\, d\xi \\ & \quad = 2e^{x - \frac{1}{2}} \int_x^1 \sinh(1-\xi)\, y(t,\xi)\, d\xi, \end{aligned}$$

$$\begin{aligned} & e^{-x} \int_0^{1-x} e^{\xi + \frac{1}{2}} y(t,\xi)\, d\xi + e^{-x} \int_x^1 e^{\xi - \frac{1}{2}} y(t,\xi)\, d\xi \\ & \quad = 2e^{-x + \frac{1}{2}} \int_x^1 \sinh(\xi - 1)\, y(t,\xi)\, d\xi, \end{aligned}$$

$$e^x \int_{1-x}^{\frac{1}{2}} e^{-\xi-\frac{1}{2}} y(t,\xi)\, d\xi + e^x \int_{\frac{1}{2}}^{x} e^{-\xi-\frac{1}{2}} y(t,\xi)\, d\xi$$
$$= 2e^{x-1} \int_{\frac{1}{2}}^{x} \sinh\left(\frac{1}{2} - \xi\right) y(t,\xi)\, d\xi,$$

$$e^{-x} \int_{1-x}^{\frac{1}{2}} e^{\xi+\frac{1}{2}} y(t,\xi)\, d\xi + e^{-x} \int_{\frac{1}{2}}^{x} e^{\xi+\frac{1}{2}} y(t,\xi)\, d\xi$$
$$= 2e^{1-x} \int_{\frac{1}{2}}^{x} \sinh\left(\xi - \frac{1}{2}\right) y(t,\xi)\, d\xi.$$

The sum of the right-hand sides in the fist two identities is

$$4\sinh\left(x - \frac{1}{2}\right) \int_x^1 \sinh(1-\xi)\, y(t,\xi)\, d\xi$$

whereas the sum of the right-hand sides of the last two identities is

$$4\sinh(1-x) \int_{\frac{1}{2}}^{x} \sinh\left(\xi - \frac{1}{2}\right) y(t,\xi)\, d\xi.$$

Therefore, adding these four identities side by side yields the formula for $u(t,x)$. The formula for $u_x(t,x)$ is obtained now by differentiation. □

Theorem 3 *Assume* $y_0 \in H^1(\mathbb{S})$ *is odd,* $y_0 \not\equiv 0$*, and* $y_0(x) \geq 0$ *on* $[0, \frac{1}{2}]$ *whereas* $y_0(x) \leq 0$ *on* $[\frac{1}{2}, 1]$*. Then the corresponding solution to* (1) *blows-up in finite time* T*. We have that*

$$u_x(t,0) = u_x\left(t, \frac{1}{2}\right) = u_x(t,1) \to -\infty \quad \textit{as} \quad t \to T < \infty \tag{19}$$

while

$$\sup_{(t,x)\in[0,T)\times\mathbb{S}} |u(t,x)| < \infty$$

and

$$\sup_{t\in[0,T)} |u_x(t,x)| < \infty, \quad x \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right). \tag{20}$$

Proof. If y_0 is odd, then clearly $u_0 \in H^3(\mathbb{S})$ is odd and by Theorem B we know that $T < \infty$. Observe that $u(t,\cdot)$ is odd for $t \in [0,T)$, cf. [15], and the solution $u(t,x)$ is bounded on $[0,T) \times \mathbb{S}$ if we recall (3).

By Lemmas 2-4 we obtain for $t \in [0,T)$ that

$$\begin{cases} y(t,x) & \geq 0, \quad x \in \left[0, \frac{1}{2}\right], \\ y(t,x) & \leq 0, \quad x \in \left[\frac{1}{2}, 1\right]. \end{cases} \tag{21}$$

Let us now denote

$$f(t,x) := \frac{1}{\sinh\left(\frac{1}{2}\right)}\ \sinh(1-x)\int_{\frac{1}{2}}^{x} \sinh\left(\xi - \frac{1}{2}\right)\ y(t,\xi)\,d\xi,$$
$$(t,x) \in [0,T) \times \left[\frac{1}{2}, 1\right],$$

$$g(t,x) := \frac{1}{\sinh\left(\frac{1}{2}\right)}\ \sinh\left(x - \frac{1}{2}\right)\int_{x}^{1} \sinh(1-\xi)\, y(t,\xi)\,d\xi,$$
$$(t,x) \in [0,T) \times \left[\frac{1}{2}, 1\right].$$

From (21) we see that

$$f(t,x) \le 0 \quad \text{and} \quad g(t,x) \le 0 \quad \text{for} \quad (t,x) \in [0,T) \times [\frac{1}{2}, 1].$$

If $K := \sup_{(t,x)\in[0,T)\times\mathbb{S}} |u(t,x)|$, we obtain from the previous relation and (17) that

$$|f(t,x)| + |g(t,x)| \le K, \quad (t,x) \in [0,T) \times \left[\frac{1}{2}, 1\right]. \tag{22}$$

On the other hand, comparing the representation formulas (17) and (18), we obtain the identity

$$u_x(t,x) = -\frac{\cosh(1-x)}{\sinh(1-x)}\, f(t,x) + \frac{\cosh\left(x-\frac{1}{2}\right)}{\sinh\left(x-\frac{1}{2}\right)}\, g(t,x),$$
$$t \in [0,T),\ x \in \left(\frac{1}{2}, 1\right).$$

By the previous relation and (22) we conclude that

$$|u_x(t,x)| \le \left(\frac{\cosh(1-x)}{\sinh(1-x)} + \frac{\cosh\left(x-\frac{1}{2}\right)}{\sinh\left(x-\frac{1}{2}\right)}\right) K,$$
$$t \in [0,T),\ x \in \left(\frac{1}{2}, 1\right).$$

Together with the observation that $u_x(t,\cdot)$ is even for $t \in [0,T)$, this relation proves (20).

Since $u(t,\cdot)$ is periodic and odd for all $t \in [0,T)$, it is easy to deduce the relation

$$u_x(t,0) = u_x\left(t, \frac{1}{2}\right) = u_x(t,1), \quad t \in [0,T).$$

By Theorem 2 we know that

$$\min_{x\in\mathbb{S}}[u_x(t,x)] \to -\infty \quad \text{as} \quad t \to T$$

and therefore (19) is forced by (20). $\square$

Remark a) The same statement is true if we consider in Theorem 3 odd initial potentials $y_0 \in H^1(\mathbb{S})$, $y_0 \not\equiv 0$, and $y_0(x) \le 0$ on $[0, \frac{1}{2}]$ whereas $y_0(x) \ge 0$ on $[\frac{1}{2}, 1]$. To see this, it is enough to perform a shift in y_0 by half a period.

b) In the class of initial potentials considered in Theorem 3, one can find functions which are identical zero on some proper nondegenerated subinterval of $[0, 1]$ containing $\frac{1}{2}$. Nevertheless, of this continuum of zeros, $\frac{1}{2}$ is the only one in the blow-up set! $\square$

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