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On estimates in Hardy spaces for the Stokes flow in a half space

Yoshikazu Giga[1,⋆], **Shin'ya Matsui**[2,⋆⋆], **Yasuyuki Shimizu**[1,⋆]

[1] Department of Mathematics, Hokkaido University, Sapporo 060, Japan
[2] Faculty of Buisiness Adminstration and Information Science, Hokkaido Information University, Nishi-Nopporo, Ebetshu 069, Japan

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0. Introduction

We consider the Stokes equation

$$
\begin{aligned}
u_t - \Delta u + \nabla p = 0, \text{div}\, u = 0 \text{ in } \Omega \times (0,\infty),\\
u = u_0 \text{ at } t = 0,\\
u = 0 \text{ on } \partial\Omega \times (0,\infty)
\end{aligned}
\tag{0.1}
$$

in a domain Ω in $\mathbb{R}^n (n \geq 2)$ with smooth boundary. Here $u = (u^1, \dots, u^n)$ are unknown velocity field and p is unknown pressure field. Initial data u_0 is assumed to satisfy a *compatibility condition* : div $u_0 = 0$ in Ω and the normal component of u_0 equals zero on $\partial\Omega$. This system is a typical parabolic equation and it has several properties resembling the heat equation.

If $\Omega = \mathbb{R}^n$, u is reduced to a solution of the heat equation with initial data u_0 because there is no boundary condition. For example, a regularity-decay estimate

$$
\|\nabla u(t)\|_p \leq C t^{-1/2} \|u_0\|_p \text{ for } t > 0 \tag{0.2}
$$

holds for all $1 \leq p \leq \infty$ with C independent of t and u_0, where $\|f(t)\|_p := \left(\int_\Omega |f(t,x)|^p dx\right)^{1/p}$ and ∇ denotes the gradient in the space variables. If $p = 2$, the estimate (0.2) is still valid for any domain. Indeed, since the

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Stokes operator A is self–adjoint and nonnegative, the operator A generates an analytic semigroup e^{-tA}. This yields

$$\|A^{1/2}e^{-tA}u_0\|_2 \le Ct^{-1/2}\|u_0\|_2.$$

Since $u = e^{-tA}u_0$ and $\|A^{1/2}u\|_2 = \|\nabla u\|_2$, (0.2) follows for $p = 2$.(See Borchers and Miyakawa [3] for applications.) For $1 < p < \infty$, (0.2) is valid for bounded domains (Giga [7]) and for a half space (Ukai [13]). The estimate (0.2) is also valid for exterior domain with $n \ge 3$, with extra restriction $1 < p < n$.(See Borchers and Miyakawa [2], Giga and Sohr [8], Iwashita [10].)

However, there was no result for $p = 1$ or $p = \infty$ where the boundary of Ω is not empty. The main difficulty lies in the fact that the projection associated with the Helmholtz decomposition is not bounded in L^1 type spaces, because it involves singular integral operators such as the Riesz operators. Nevertheless in this paper, we prove (0.2) for $p = 1$ where Ω is a half space $\mathbb{R}^n_+ = \{x = (x_1, \cdots, x_n); x_n > 0\}$.

Theorem 0.1. *Let u be the solution of the Stokes equation (0.1) in $\Omega = \mathbb{R}^n_+$ with initial data $u_0 \in L^1(\mathbb{R}^n)$, which satisfy the compatibility condition. Then there is a constant C independent of u_0 such that*

$$\|\nabla u(t)\|_1 \le Ct^{-1/2}\|u_0\|_1 \tag{0.3}$$

for all $t > 0$.

This is rather surprising since we do not expect $\|u(t)\|_1 \le C\|u_0\|_1$ for $\Omega = \mathbb{R}^n_+$. Actually, the estimate (0.3) follows from a stronger estimate:

Theorem 0.2. *Under the same hypothesis as in Theorem 0.1, there is a constant C' independent of u_0 such that*

$$\|\nabla u(t)\|_{\mathcal{H}^1(\mathbb{R}^n_+)} \le C't^{-1/2}\|u_0\|_1 \tag{0.4}$$

for all $t > 0$.

Here

$$\|f\|_{\mathcal{H}^1(\mathbb{R}^n_+)} = \inf\{\|\tilde{f}\|_{\mathcal{H}^1(\mathbb{R}^n)}; \tilde{f} \in \mathcal{H}^1(\mathbb{R}^n),\ \tilde{f}|_{\mathbb{R}^n_+} \equiv f\},$$

where $\mathcal{H}^1(\mathbb{R}^n)$ is the Hardy space in $\mathbb{R}^n$ defined later.

Combining the Sobolev inequality with (0.3), we have

$$\|u(t)\|_{n/(n-1)} \le C_0t^{-1/2}\|u_0\|_1 \tag{0.5}$$

with C_0 independent of $t > 0$ and u_0. This has been already proved by Borchers and Miyakawa [1] where a general $L^p - L^q$ estimate

$$\|u(t)\|_p \le C_0t^{-\alpha}\|u_0\|_q$$

with $\alpha = (n/2)(1/q - 1/p)$ has been proved for all $1 \leq q < p \leq \infty$ where $\Omega = \mathbb{R}^n_+$. Their method does not depend on (0.3). For $1 < q < p < \infty$, such estimate has been proved by Ukai [13]. There is an extensive literature on $L^p - L^q$ estimates for exterior domains Ω ($n \geq 3$) (e.g. Giga and Sohr [9], Borchers and Miyakawa [2], Iwashita [10], Chen [4]) but the case $q = 1$ and $p = \infty$ is included only in Chen [4] for $n = 3$.

To show (0.4), we recall the solution formula obtained by Ukai [13]. The solution is represented by the Gauss kernel and various Riesz operators. It is known by Carpio [4] that the solution $u = G_t * u_0$ of the heat equation with initial data $u_0 \in L^1(\mathbb{R}^n)$ satisfies

$$\|\nabla u(t)\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq C_1 t^{-1/2} \|u_0\|_1 \tag{0.6}$$

where G_t is the Gauss kernel. If the solution of (0.1) were represented only by G_t and a Riesz operator in $\mathbb{R}^n$, (0.6) could yield (0.4) since the Riesz operator is bounded in $\mathcal{H}^1$. Unfortunately, the formula contains the Riesz operator in tangential variables $x' = (x_1, \ldots, x_{n-1})$ to $\partial\mathbb{R}^n_+$, and therefore it is not clear that such operators are bounded in $\mathcal{H}^1(\mathbb{R}^n)$. To overcome this difficulty, we rewrite Ukai's formula so that ∇u does not contain tangential Riesz operators using the operator Λ whose symbol equals $|\xi'|$, where $(\xi', \xi_n) = \xi \in \mathbb{R}^n$. Because of this, we need to prove

$$\|\Lambda u(t)\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq C_2 t^{-1/2} \|u_0\|_1 \tag{0.7}$$

in addition to (0.6). Although there are several extra technical difficulties, this is a rough idea for the proof of (0.4).

1. The solution formula

In this section we recall the solution formula for (0.1) obtained by Ukai [13] for later use.

First, we establish conventions of notations. For an n-dimensional vector a, we denote its tangential component $(a_1, \ldots, a_{n-1})$ by $a' \in \mathbb{R}^{n-1}$, so that $a = (a', a_n)$. We set $\partial_j = \partial/\partial x_j$ and let $\nabla' = (\partial_1, \cdots, \partial_{n-1})$. Hereafter, C denotes a positive constant which may differ from one occasion to another.

Let $\mathcal{F}$ be the Fourier transform in $\mathbb{R}^n$:

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx,$$

and let $\hat{f}$ be the Fourier tranform of f in the tangential space:

$$\hat{f}(\xi', x_n) = \int_{\mathbb{R}^{n-1}} e^{-ix'\cdot\xi'} f(x', x_n)dx'.$$

The Riesz operators R_j $(j = 1, \ldots, n)$, S_j $(j = 1, \ldots, n-1)$, and the operator Λ are defined by

$$\begin{aligned} \mathcal{F}(R_j f)(\xi) &= \tfrac{i\xi_j}{|\xi|}\mathcal{F}f(\xi), \\ \mathcal{F}(S_j f)(\xi) &= \tfrac{i\xi_j}{|\xi'|}\mathcal{F}f(\xi), \\ \mathcal{F}(\Lambda f)(\xi) &= |\xi'|\mathcal{F}f(\xi). \end{aligned}$$

We set $R' = (R_1, \ldots, R_{n-1})$, $S = (S_1, \ldots, S_{n-1})$ and define U by

$$Uf = rR' \cdot S(R' \cdot S + R_n)e,$$

where r is the restriction operator from $\mathbb{R}^n$ to $\mathbb{R}^n_+$, and e is the extension operator from $\mathbb{R}^n_+$ onto $\mathbb{R}^n$ with value 0, that is,

$$ef = \begin{cases} f & \text{for } x_n \geq 0, \\ 0 & \text{for } x_n < 0. \end{cases}$$

We also define the operators $E(t)$ and $F(t)$ by

$$\begin{aligned} [E(t)f](x) &= \textstyle\int_{\mathbb{R}^n_+} \{G_t(x-y) - G_t(x'-y', x_n+y_n)\}\, f(y)dy, \\ [F(t)f](x) &= \textstyle\int_{\mathbb{R}^n_+} \{G_t(x-y) + G_t(x'-y', x_n+y_n)\}\, f(y)dy, \end{aligned}$$

where G_t is the Gauss kernel $G_t(x) = (4\pi t)^{-n/2}e^{-|x|^2/4t}$. Note that $E(t)f$ (resp. $F(t)f$) is the solution to the heat equation in $\mathbb{R}^n_+$ with Dirichlet (resp. Neumann) data:

$$\begin{aligned} z_t - \Delta z &= 0 \text{ in } \mathbb{R}^n_+ \times (0,T), \\ z|_{t=0} &= f, \\ z|_{x_n=0} &\equiv 0. \text{ (resp. } \partial_n z|_{x_n=0} = 0.) \end{aligned}$$

We recall the formula obtained by Ukai.

Theorem 1.1(Ukai). *The solution to (0.1) can be expressed as*

$$u^n = UE(t)V_1u_0, \tag{1.1a}$$

$$u' = E(t)V_2u_0 - SUE(t)V_1u_0, \tag{1.1b}$$

where $V_1u_0 = -S \cdot u_0' + u_0^n$ *and* $V_2u_0 = u_0' + Su_0^n$.

We give a formal proof of Theorem 1.1 for the reader's convenience. By (0.1), we get $\Delta p = 0$ in $\mathbb{R}^n_+$. Applying the tangential Fourier transform, the equation $\Delta p = 0$ is reduced to an ordinary differential equation $(\partial_n^2 - |\xi'|^2)\hat{p} = 0$. Assuming that p is bounded, we get $(\partial_n + |\xi'|)\hat{p} = 0$. We set $v^n = (\partial_n + \Lambda)u^n$ and $v' = V_2u = u' + Su^n$. Then v satisfies $v_t -$

$\Delta v = 0$, $v^n|_{t=0} = \Lambda V_1 u_0$, $v'|_{t=0} = V_2 u_0$, and $v|_{x_n=0} = 0$. Thus v solves the heat equation in $\mathbb{R}^n_+$ with zero Dirichlet data. Solving for v with some manipulations we get (1.1).

To solve our problem, we rewrite the formula (1.1). Note that the vector field u in (1.1) is given as a restriction $r\bar{u}$ of a vector field $\bar{u} = (\bar{u}', \bar{u}_n)$ of the form

$$\bar{u}^n = R' \cdot S(R' \cdot S + R_n) eE(t) V_1 u_0, \tag{1.2a}$$

$$\bar{u}' = E(t) V_2 u_0 - SR' \cdot S(R' \cdot S + R_n) eE(t) V_1 u_0. \tag{1.2b}$$

Lemma 1.2. *Let j be an integer with $1 \le j \le n$. Assume that* $\operatorname{div} u_0 = 0$ *in $\mathbb{R}^n_+$ when $j = n$. Then the first space derivative of $\bar{u}$ are expressed as*

$$\begin{aligned} \partial_j \bar{u}^n = -R_j \{ & R' \cdot \Lambda eE(t) u_0' - R_n \nabla' \cdot eE(t) u_0' \\ & + R' \cdot \nabla' eE(t) u_0^n + R_n \Lambda eE(t) u_0^n \}, \end{aligned} \tag{1.3a}$$

$$\begin{aligned} \partial_j \bar{u}' = \partial_j & E(t) u_0' + w_j \\ & + R_j \{ R' (\nabla' \cdot eE(t) u_0') - R_n \nabla' (\nabla' \Lambda^{-1} \cdot eE(t) u_0') \\ & - R' \Lambda eE(t) u_0^n + R_n \nabla' eE(t) u_0^n \}, \end{aligned} \tag{1.3b}$$

where

$$w_j = \begin{cases} \partial_j \nabla' \Lambda^{-1} E(t) u_0^n & \text{for } 1 \le j \le n-1, \\ -\nabla' (\nabla' \cdot \Lambda^{-1} F(t) u_0') & \text{for } j = n. \end{cases} \tag{1.4}$$

Proof. To show (1.3), it is convenient to use the Fourier transformation by $\partial_j \bar{u}$ in (1.2). Note that the operators S_j and $eE(t)$ commute. Then we get

$$\begin{aligned} \mathcal{F}(\partial_j \bar{u}^n) &= i\xi_j \frac{i\xi'}{|\xi|} \cdot \frac{i\xi'}{|\xi'|} \left(\frac{i\xi'}{|\xi|} \cdot \frac{i\xi'}{|\xi'|} + \frac{i\xi_n}{|\xi|} \right) \\ &\qquad \times \left(-\frac{i\xi'}{|\xi'|} \cdot \mathcal{F}(eE(t)u_0') + \mathcal{F}(eE(t)u_0^n) \right) \\ &= -\frac{i\xi_j}{|\xi|} \Bigg\{ \left(\frac{i\xi'}{|\xi|} |\xi'| - \frac{i\xi_n}{|\xi|} i\xi' \right) \cdot \mathcal{F}(eE(t)u_0') \\ &\qquad + \left(\frac{i\xi'}{|\xi|} \cdot i\xi' + \frac{i\xi_n}{|\xi|} |\xi'| \right) \mathcal{F}(eE(t)u_0^n) \Bigg\}, \end{aligned}$$

$$\begin{aligned}\mathcal{F}(\partial_j \bar{u}') &= i\xi_j \left(\mathcal{F}(E(t)u_0') + \frac{i\xi'}{|\xi'|}\mathcal{F}(E(t)u_0^n) \right) \\ &\quad -i\xi_j \frac{i\xi'}{|\xi'|} \left(\frac{i\xi'}{|\xi|} \cdot \frac{i\xi'}{|\xi'|} \right) \left(\frac{i\xi'}{|\xi|} \cdot \frac{i\xi'}{|\xi'|} + \frac{i\xi_n}{|\xi|} \right) \\ &\quad \times \left(-\frac{i\xi'}{|\xi'|} \cdot \mathcal{F}(eE(t)u_0') + \mathcal{F}(eE(t)u_0^n) \right) \\ &= i\xi_j \mathcal{F}(eE(t)u_0') - \frac{\xi_j \xi'}{|\xi'|}\mathcal{F}(eE(t)u_0^n) \\ &\quad + \frac{i\xi_j}{|\xi|} \Bigg\{ \frac{i\xi'}{|\xi|}\xi' \cdot \mathcal{F}(eE(t)u_0) - \frac{i\xi_n}{|\xi|} i\xi' \left(i\xi' \cdot \frac{1}{|\xi'|}\mathcal{F}(eE(t)u_0') \right) \\ &\quad - \left(\frac{i\xi'}{|\xi|}|\xi'| - \frac{i\xi_n}{|\xi|} i\xi' \right) \mathcal{F}(eE(t)u_0^n) \Bigg\}.\end{aligned}$$

By the inverse Fourier transform the first identity implies (1.3a). To show (1.3b), we must handle the term $i\xi_j(i\xi'/|\xi'|)\mathcal{F}[E(t)u_0^n]$. By the inverse Fourier transform this term is transformed to $\partial_j \nabla' \Lambda^{-1} E(t)u_0^n$. For $1 \le j \le n-1$ this equals to w_j. For $j = n$ we invoke the assumption div $u_0 = 0$ so that $\partial_n u_0^n = -\nabla' \cdot u_0'$:

$$\begin{aligned}\partial_n \nabla' \Lambda^{-1} E(t)u_0^n &= \partial_n \nabla' \Lambda^{-1} \int_{\mathbb{R}^n_+} \{G_t(x-y) \\ &\qquad -G_t(x'-y', x_n + y_n)\} u_0^n(y)dy \\ &= \nabla' \Lambda^{-1} \int_{\mathbb{R}^n_+} \Bigg\{ \frac{\partial}{\partial x_n} G_t(x-y) \\ &\qquad - \frac{\partial}{\partial x_n} G_t(x'-y', x_n+y_n) \Bigg\} u_0^n(y)dy \\ &= \nabla' \Lambda^{-1} \int_{\mathbb{R}^n_+} \Bigg\{ -\frac{x_n - y_n}{2t} G_t(x-y) \\ &\qquad + \frac{x_n + y_n}{2t} G_t(x'-y', x_n+y_n) \Bigg\} u_0^n(y)dy \\ &= \nabla' \Lambda^{-1} \Bigg\{ \int_{\mathbb{R}^{n-1}} \Big[\{-G_t(x-y) \\ &\qquad -G_t(x'-y', x_n+y_n)\} u_0^n(y) \Big]_{y_n=0}^{y_n=+\infty} dy' \\ &\qquad + \int_{\mathbb{R}^n_+} \{G_t(x-y)\end{aligned}$$

$$
\begin{aligned}
&\quad +G_t(x'-y', x_n+y_n)\big\} \, \partial_n u_0^n(y)dy \Bigg\} \\
&= -\nabla' \Lambda^{-1} \int_{\mathbb{R}^n_+} \{G_t(x-y) \\
&\quad +G_t(x'-y', x_n+y_n)\big\} \, \nabla' \cdot u_0'(y)dy \\
&= -\nabla'(\nabla' \cdot \Lambda^{-1} F(t) u_0') = w_n. \quad \square
\end{aligned}
$$

2. Proof of theorem

To prove Theorem 0.1, we need to estimate the right hand side of (1.3) in $L^1(\mathbb{R}^n)$. In this section we estimate these terms in the Hardy space $\mathcal{H}^1$ instead of L^1, which is the subspace of L^1. We recall the definition of the Hardy space $\mathcal{H}^1$. Note that the following definition is one of many equivalent definitions of the Hardy space. (See Fefferman and Stein [6].)

Definition 2.1. A function $f \in L^1(\mathbb{R}^n)$ belongs to the Hardy space $\mathcal{H}^1 = \mathcal{H}^1(\mathbb{R}^n)$ if

$$f^*(x) = \sup_{s>0} |G_s * f(x)| \in L^1(\mathbb{R}^n),$$

where the symbol $*$ denotes the convolution with respect to the space variable x. The norm of $f \in \mathcal{H}^1(\mathbb{R}^n)$ is defined by

$$\|f\|_{\mathcal{H}^1} := \|f^*\|_{L^1(\mathbb{R}^n)}$$

Here, we remark that a L^1 function f belongs to $\mathcal{H}^1$ if and only if its Riesz transform $R_j f$ belongs to $L^1(\mathbb{R}^n)$ for all j, and that

$$\|f\|_{\mathcal{H}^1} \cong \|f\|_{L^1(\mathbb{R}^n)} + \sum_{j=1}^{n} \|R_j f\|_{L^1(\mathbb{R}^n)} \quad \text{(equivalent norm)}.$$

For the convenience, we denote the operator norm of R_j in $\mathcal{H}^1$ by $|||\cdot|||_{\mathcal{H}^1}$.

To estimate (1.3) in $\mathcal{H}^1$, we require the following lemma.

Lemma 2.2. *Let K be an integral operator of form*

$$Kf(x) = \int_{\mathbb{R}^n} k(x,y) f(y)dy \quad \textit{for } x \in \mathbb{R}^n. \tag{2.1}$$

If the kernel $k(x,y)$ satisfies that

$$\sup_{y \in \mathbb{R}^n} \|k(\cdot, y)\|_{\mathcal{H}^1} = k_0 < \infty,$$

then K is a bounded operator from $L^1(\mathbb{R}^n)$ to $\mathcal{H}^1(\mathbb{R}^n)$, i.e.

$$\|Kf\|_{\mathcal{H}^1} \le k_0 \|f\|_{L^1(\mathbb{R}^n)}. \tag{2.2}$$

Proof. By definition of $\mathcal{H}^1$,

$$\begin{aligned}(Kf)^*(x) &= \sup_{s>0} \left| \int_{\mathbb{R}^n} G_s(x-z) \int_{\mathbb{R}^n} k(z,y) f(y) dy dz \right| \\ &\le \sup_{s>0} \left| \int_{\mathbb{R}^n} f(y) \int_{\mathbb{R}^n} G_s(x-z) k(z,y) dz dy \right| \\ &\le \int_{\mathbb{R}^n} |f(y)| \left\{ \sup_{s>0} \left| \int_{\mathbb{R}^n} G_s(x-z) k(z,y) dz \right| \right\} dy.\end{aligned} \tag{2.3}$$

Integrating (2.3) with respect to x, we get

$$\begin{aligned}\|Kf\|_{\mathcal{H}^1} &\le \int_{\mathbb{R}^n} |f(y)| \|k(\cdot,y)\|_{\mathcal{H}^1} dy \\ &\le k_0 \|f\|_{L^1(\mathbb{R}^n)}. \quad \square\end{aligned}$$

We next show several pointwise estimates on the heat kernel.

Lemma 2.3. *Assume that real parameters l and m satisfy $0 \le l \le n$ and $m \ge 0$. Then there exists a constant $C = C_{l,m}$ which does not depend on $x \in \mathbb{R}^n$ and $t \ge 0$ such that*

$$|\partial_j G_t(x)| \le C t^{(l+m-n-1)/2} |x'|^{-l} |x_n|^{-m} \tag{2.4a}$$

for $1 \le j \le n$ with $n \ge 2$,

$$|\partial_j \partial_k \Lambda^{-1} G_t(x)| \le C t^{(l+m-n-1)/2} |x'|^{-l} |x_n|^{-m} \tag{2.4b}$$

for $1 \le j,k \le n-1$ with $n \ge 3$,

$$|\Lambda G_t(x)| \le C t^{(l+m-n-1)/2} |x'|^{-l} |x_n|^{-m} \tag{2.4c}$$

with $n \ge 2$.

In (2.4a) the restriction $l \le n$ is unnecessary.

Proof. We first prove (2.1a). Since $\partial_j G_t(x) = -(x_j/2t) G_t(x)$ and $e^{-|x|^2/4t} \le C|t^{-1/2}x|^{-\alpha}$ for $\alpha \ge 0$, we have

$$\begin{aligned}\partial_j G_t(x) &= -\frac{x_j}{2t} G_t(x) \\ &= -\frac{x_j}{2t^{n/2+1}} e^{-|x'|^2/4t} e^{-|x_n|^2/4t} \\ &\le C t^{(l+m-n-1)/2} |x'|^{-l} |x_n|^{-m}.\end{aligned} \tag{2.5}$$

We next show (2.4b). Note that Λ^{-1} is equal to $(-\Delta')^{-1/2} = \left(\sum_{k=1}^{n-1} \partial_k^2 \right)^{-1/2}$, so the integral kernel of Λ^{-1} is $c_n |x'|^{-n+2}$ for $n \ge 3$, where c_n is some positive constant. Therefore we have

$$\partial_j \partial_k \Lambda^{-1} G_t(x) = c_n \partial_j \partial_k \int_{\mathbb{R}^{n-1}} |x'-y'|^{-n+2} G_t(y', x_n) dy'. \tag{2.6}$$

Set $x = t^{1/2}z$ to get

$$\partial_{x_j}\partial_{x_k}\Lambda^{-1}G_t(x) = t^{-(n+1)/2}\partial_{z_j}\partial_{z_k}\Lambda^{-1}G_1(z).$$

So it is sufficient to show (2.4b) for $t = 1$, i.e.

$$|\partial_j\partial_k\Lambda^{-1}G_1(z)| \le C|z'|^{-l}|z_n|^{-m}. \tag{2.7}$$

In fact, if (2.7) is valid, then we have

$$\begin{aligned}|\partial_{x_j}\partial_{x_k}\Lambda^{-1}G_t(x)| &= t^{-(n+1)/2}|\partial_{z_j}\partial_{z_k}\Lambda^{-1}G_1(z)|\\ &\le Ct^{-(n+1)/2}|z'|^{-l}|z_n|^{-m}\\ &= Ct^{(l+m-n-1)/2}|x'|^{-l}|x_n|^{-m}\end{aligned}$$

for any $t > 0$.

Let ψ_1 be a smooth function in $\mathbb{R}^{n-1}$ such that $0 \le \psi_1 \le 1$, supp $\psi \subset \{|z'| \le 1\}$, and $\psi_1|_{|z'|<1/2} \equiv 1$. Set $\psi_2 = 1 - \psi_1$. Then

(2.8)
$$\begin{aligned}\partial_j\partial_k\Lambda^{-1}G_1(z) = \tfrac{C}{(4\pi)^{n/2}}e^{-z_n^2/4}\Big\{&\partial_j\partial_k\textstyle\int_{\mathbb{R}^{n-1}}\frac{\psi_1(z'-y')}{|z'-y'|^{n-2}}e^{-|y'|^2/4}dy'\\ &+ \partial_j\partial_k\textstyle\int_{\mathbb{R}^{n-1}}\frac{\psi_2(z'-y')}{|z'-y'|^{n-2}}e^{-|y'|^2/4}dy'\Big\}\\ = Ce^{-z_n^2/4}\{I_1(z') + I_2(z')\}.&\end{aligned}$$

The estimate of the term I_1: We have

$$\begin{aligned}I_1(z') &= \partial_j\partial_k\textstyle\int_{|y'|\le 1}\frac{\psi_1(y')}{|y'|^{n-2}}e^{-|z'-y'|^2/4}dy'\\ &= \textstyle\int_{|y'|\le 1}\frac{\psi_1(y')}{|y'|^{n-2}}K_{j,k}(z'-y')dy,\end{aligned} \tag{2.9}$$

where

$$K_{j,k}(z') = \left(\frac{z_jz_k}{4} - \frac{\delta_{j,k}}{2}\right)e^{-|z'|^2/4}$$

and $\delta_{j,k}$ is Kronecker's delta. Recalling $|z' - y'| \le |z'| + 1$ and $|z' - y'|^2 \ge |z'|^2/2 - 1$ holds for $|y'| \le 1$, we get

$$\begin{aligned}|K_{j,k}(z'-y')| &\le \left\{\tfrac{(|z'|+1)^2}{4} + \tfrac{1}{2}\right\}e^{-(|z'|^2-2)/8}\\ &= \tfrac{e^{1/4}}{4}\left\{(|z'|+1)^2 + 2\right\}e^{-|z'|^2/8}\\ &\le C|z'|^{-l}.\end{aligned}$$

Hence we have

$$\begin{aligned}|I_1(z')| &\le C\textstyle\int_{|y'|\le 1}\frac{1}{|y'|^{n-2}}|z'|^{-l}dy'\\ &\le C|z'|^{-l}\end{aligned}$$

The estimate of the term I_2: We have

$$
\begin{aligned}
I_2(z') &= \int_{\mathbb{R}^{n-1}} \tfrac{(\partial_j\partial_k\psi_2)(z'-y')}{|z'-y'|^{n-2}} e^{-|y'|^2/4} dy' \\
&\quad -(n-2)\Big\{\int_{\mathbb{R}^{n-1}} (\partial_j\psi_2)(z'-y') \tfrac{z_k-y_k}{|z'-y'|^n} e^{-|y'|^2/4} dy' \\
&\quad + \int_{\mathbb{R}^{n-1}} (\partial_k\psi_2)(x'-y') \tfrac{z_j-y_j}{|z'-y'|^n} e^{-|y'|^2/4} dy'\Big\} \\
&\quad + \int_{\mathbb{R}^{n-1}} \psi_2(z'-y') L_{j,k}(z'-y') e^{-|y'|^2/4} dy' \\
&= J_1(z') - (n-1)J_2(z') + J_3(z'),
\end{aligned} \tag{2.10}
$$

where

$$
L_{j,k}(z') = (n-2)\left\{ n\frac{x_j x_k}{|z'|^{n+2}} - \frac{\delta_{j,k}}{|z'|^n} \right\}.
$$

Since the support of $\partial_j\psi_2$ and $\partial_j\partial_k\psi_2$ are included in $1/2 \le |z| \le 1$, the estimates of J_1 and J_2 can be obtained like as the estimate of I_1:

$$
\begin{aligned}
|J_1(z')| &= \left| \int_{1/2\le|y'|\le 1} \frac{(\partial_j\partial_k\psi_2)(y')}{|y'|^{n-2}} e^{-|z'-y'|^2/4} dy' \right| \\
&\le \|\nabla^2\psi_2\|_{L^\infty} \int_{1/2\le|y'|\le 1} \frac{1}{|y'|^{n-2}} e^{-(|z'|^2-2)/8} dy' \\
&\le C|z'|^{-l},
\end{aligned} \tag{2.11}
$$

$$
\begin{aligned}
|J_2(z')| &\le \|\nabla\psi_2\|_{L^\infty} \int_{1/2\le|y'|\le 1} \frac{1}{|y'|^{n-1}} e^{-(|z'|^2-2)/8} dy' \\
&\le C|z'|^{-l}.
\end{aligned} \tag{2.12}
$$

To estimate the term J_3, we use the inequality $|z'|^l \le C_l(|z'-y'|^l + |y'|^l)$. Since $|L_{j,k}(z')| \le \frac{C}{|z'-y'|^{n+1}}$, we get

$$
\begin{aligned}
|J_3(z')| &\le C|z'|^{-l} \int_{|z'-y'|\ge 1/2} \left(\tfrac{|z'-y'|^l}{|z'-y'|^n} + \tfrac{|y'|^l}{|z'-y'|^n} \right) e^{-|y'|^2/4t} dy' \\
&\le C|z'|^{-l} \int_{|z'-y'|\ge 1/2} (2^{l-n} + 2^n|y'|^l) e^{-|y'|^2/4} dy' \\
&= C|z'|^{-l}.
\end{aligned} \tag{2.13}
$$

Combining the estimate (2.11), (2.12), and (2.13), we get $|I_2(z')| \le C|z'|^{-l}$ and

$$
\begin{aligned}
|\partial_j\partial_k \Lambda^{-1} G_1(z)| &\le C e^{-x_n^2/4} |z'|^{-l} \\
&\le C_{l,m} |z'|^{-l} |z_n|^{-m}.
\end{aligned} \tag{2.14}
$$

This proves (2.7) for $n \ge 3$.

The estimate (2.4c) for $n \ge 3$ is easily obtained by the fact that Λ is equal to $(-\Delta')\Lambda^{-1} = -(\partial_1^2 + \cdots \partial_{n-1}^2)\Lambda^{-1}$ and by applying (2.4b).

Finally, we show (2.4c) for $n = 2$. Note that Λ is equal to $|\partial_1| = \partial_1 S_1$. So we have

$$\begin{aligned} \Lambda G_t(x) &= \partial_1 S_1 G_t(x) \\ &= \partial_1 \frac{1}{\pi} \lim_{\epsilon \downarrow 0} \int_{|y_1|>\epsilon} \frac{1}{y_1} G_t(x_1 - y_1, x_2) dy_1. \end{aligned} \tag{2.15}$$

(See Torchinsky [12], p.266.) Integrating by parts we get

$$\begin{aligned} \int_{|y_1|>\epsilon} \tfrac{1}{y_1} G_t(x_1 - y_1, x_2) dy_1 &= \Big[\log|y_1| G_t(x_1 - y_1, x_2)\Big]_{\epsilon}^{\infty} \\ &+ \Big[\log|y_1| G_t(x_1 - y_1, x_2)\Big]_{-\infty}^{-\epsilon} \\ &- \int_{|y_1|>\epsilon} \log|y_1| \partial_{y_1} G_t(x_1 - y_1, x_2) dy_1 \\ &= \log\epsilon \big(G_t(x_1+\epsilon, x_2) - G_t(x_1 - \epsilon, x_2)\big) \\ &+ \int_{|y_1|>\epsilon} \log|y_1| \tfrac{x_1 - y_1}{2t} G_t(x_1 - y_1, x_2) dy_1. \end{aligned}$$

Sending $\epsilon \downarrow 0$, we get

$$\Lambda G_t(x) = \frac{1}{\pi} \partial_1 \int_{-\infty}^{\infty} \log|y_1| \frac{x_1 - y_1}{2t} G_t(x_1 - y_1, x_2) dy_1. \tag{2.16}$$

Set $x = t^{1/2} z$ and $y = t^{1/2} w$. Then we have

$$\begin{aligned} (\Lambda G_t)(x) &= \tfrac{1}{\pi} t^{-1/2} \partial_{z_1} \int_{-\infty}^{\infty} (\log|w_1| + \log t^{1/2}) \tfrac{z_1 - w_1}{2t^{1/2}} \\ &\quad \times t^{-1} G_1(z_1 - w_1, w_2) t^{1/2} dw_1 \\ &= t^{-3/2} (\Lambda G_1)(z). \end{aligned}$$

So it is sufficient to show (2.4c) for $t = 1$.

$$\begin{aligned} \Lambda G_1(z) &= \frac{1}{\pi} \frac{1}{4\pi} e^{-z_2^2/4} \partial_1 \Big\{ \int_{|y_1|<1} \log|y_1| \frac{z_1 - y_1}{2} e^{-(z_1 - y_1)^2/4} dy_1 \\ &\quad + \int_{|y_1|>1} \log|y_1| \frac{z_1 - y_1}{2} e^{-(z_1 - y_1)^2/4} dy_1 \Big\} \\ &= \frac{1}{4\pi^2} e^{-z_2^2/4} (I_1(z_1) + I_2(z_1)). \end{aligned} \tag{2.17}$$

The estimate of I_1: We have

$$I_1(z_1) = \int_{-1}^{1} \log|y_1| \tfrac{1}{2} \Big(1 - \tfrac{|z_1 - y_1|^2}{2}\Big) e^{-(z_1 - y_1)^2/4} dy_1.$$

As the same suggestion to (2.11), we obtain

$$\begin{aligned} |I_1(z_1)| &\le \tfrac{1}{2} \int_{-1}^{1} |\log|y_1|| \Big(1 + \tfrac{(|z_1|+1)^2}{4}\Big) e^{-\frac{|z_1|^2}{8} + \frac{1}{4}} dy_1 \\ &\le C(1 + |z_1|^2) e^{-|z_1|^2/8}. \end{aligned} \tag{2.18}$$

The estimate of I_2: The method is similar to the case $n \geq 3$. Integrating by parts,

$$I_2(z_1) = \partial_1 \Big\{ \Big[\log |y_1| e^{-(z_1-y_1)^2/4} \Big]_1^{+\infty} + \Big[\log |y_1| e^{-(z_1-y_1)^2/4} \Big]_{-\infty}^{-1} - \int_{|y_1|>1} \frac{1}{y_1} e^{-(z_1-y_1)^2/4} dy_1 \Big\}$$
$$= \int_{|y_1|>1} \frac{1}{y_1} \frac{z_1-y_1}{2} e^{-(z_1-y_1)^2/4} dy_1$$
$$= e^{-(z_1+1)^2/4} - e^{-(z_1-1)^2/4} + \int_{|y_1|>1} \frac{1}{y_1^2} e^{-(z_1-y_1)^2/4} dy_1.$$

We set $w_1 = z_1 - y_1$ and obtain

$$I_2(z_1) = e^{-(z_1+1)^2/4} - e^{-(z_1-1)^2/4} + \int_{|z_1-w_1|>1} \frac{1}{(z_1-w_1)^2} e^{-w_1^2/4} dw_1.$$

Using $|z_1|^l \leq C(|z_1 - w_1|^l + |w_1|^l)$, we obtain
(2.19)

$$|I_2(z_1)| \leq |e^{-(z_1+1)^2/4}| + |e^{-(z_1-1)^2/4}| + C \int_{|z_1-w_1|>1} \frac{1}{|z_1|^l} \left(|z_1 - w_1|^{l-2} + \frac{|w_1|^l}{|z_1-w_1|^2} \right) e^{-|w_1|^2/4} dw_1 \leq C|z|^{-l}$$

since $l \leq 2$ so that $|z_1 - w_1|^{l-2} \leq 1$. Combining the estimate (2.18) and (2.19),we obtain (2.4c) for $n = 2$. $\square$

We are now ready to show the key lemma for the main theorem.

Lemma 2.4. *Assume a function $a = a(x)$ is in $L^1(\mathbb{R}^n_+)$. Then*

(2.20a) $\|\partial_j E(t)a\|_{\mathcal{H}^1} \leq Ct^{-1/2} \|a\|_{L^1(\mathbb{R}^n_+)}$ *for* $1 \leq j \leq n$,

(2.20b) $\|\partial_j \partial_k \Lambda^{-1} eE(t)a\|_{\mathcal{H}^1} \leq Ct^{-1/2} \|a\|_{L^1(\mathbb{R}^n_+)}$ *for* $1 \leq j, k \leq n-1$,

(2.20c) $\|\Lambda eE(t)a\|_{\mathcal{H}^1} \leq Ct^{-1/2} \|a\|_{L^1(\mathbb{R}^n_+)}$,

(2.20d) $\|\partial_j \partial_k \Lambda^{-1} F(t)a\|_{\mathcal{H}^1} \leq Ct^{-1/2} \|a\|_{L^1(\mathbb{R}^n_+)}$ *for* $1 \leq j, k \leq n-1$.

Proof. To show (2.20a,b,c), we extend the function $a(x)$ from $\mathbb{R}^n_+$ onto $\mathbb{R}^n$ with $a(x', x_n) = -a(x', -x_n)$ for $x_n < 0$. Then

$$[E(t)a](x) = G_t * a(x) = \int_{\mathbb{R}^n} G_t(x-y)a(y)dy,$$
$$[eE(t)a](x) = \theta(x_n)[E(t)a](x),$$

where θ is the Heaviside function, i.e.

$$\theta(x_n) = \begin{cases} 1 & \text{for } x_n \geq 0, \\ 0 & \text{for } x_n < 0. \end{cases}$$

Since $G_s * (\partial_j G_t)(x) = \partial_j G_{s+t}(x)$, the estimate (2.4a) implies

$$|G_s * (\partial_j G_t)(x)| \leq C(s+t)^{(l+m-n-1)/2} |x'|^{-l} |x_n|^{-m}$$

for any nonnegative l and m. Thus, for $0 \leq l + m \leq n+1$ we have

$$(\partial_j G_t)^*(x) \leq C t^{(l+m-n-1)/2} |x'|^{-l} |x'|^{-m}.$$

Therefore we obtain

$$\|\partial_j G_t\|_{\mathcal{H}^1} \leq \sum_{k=1}^{4} C_{l,m} t^{(l+m-n-1)/2} \int_{\Omega_k} |x'|^{-l} |x_n|^{-m} dx, \tag{2.21}$$

where $\Omega_1 = \{|x'| \leq t^{1/2}, |x_n| \leq t^{1/2}\}$, $\Omega_2 = \{|x'| > t^{1/2}, |x_n| \leq t^{1/2}\}$, $\Omega_3 = \{|x'| \leq t^{1/2}, |x_n| > t^{1/2}\}$ and $\Omega_4 = \{|x'| > t^{1/2}, |x_n| > t^{1/2}\}$. For each integration of (2.21), we take suitable l and m such that $l = m = 0$ in Ω_1, $l = n$, $m = 0$ in Ω_2, $l = 0$, $m = 2$ in Ω_3 and $l = n - 1/2$, $m = 3/2$ in Ω_4. We thus observe that the right hand side of (2.21) is estimated from above by a constant times $t^{-1/2}$. Thus (2.20a) is obtained. The estimate is obtained by Carpio [3, Lemma 2.1] but the proof contains misprint in [3, p.457 line 4], so we gave the proof.

To prove (2.20b), we put $k(x,y) = \partial_j \partial_k \Lambda^{-1} \theta(x_n) G_t(x-y)$. Then

$$\begin{aligned} (G_s * k(\cdot,y)(x)) &= \int_{\mathbb{R}^n} G_s(z-x) k(z,y) dz \\ &= \frac{1}{(4\pi s)^{n/2}} \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^{n-1}} e^{-|z'-x'|^2/4s} \partial_j \partial_k \Lambda^{-1} e^{-|z'-y'|^2/4t} dz' \times \\ &\qquad \int_0^{+\infty} e^{-|z_n - x_n|^2/4s} e^{-|z_n - y_n|^2/4t} dz_n. \end{aligned} \tag{2.22}$$

Since the integrand in the last integral in (2.22) is nonnegative, we get

$$|(G_s * k(\cdot,y)(x)| \leq |\partial_j \partial_k \Lambda^{-1} G_{s+t}(x)|.$$

By (2.4b) a calculation similar to the one to derive (2.21) yields

$$\sup_y \|k(\cdot,y)\|_{\mathcal{H}^1} \leq C t^{-1/2}$$

for $n \geq 3$ and for $n = 2$ with $j = k = 1$. Applying Lemma 2.2 we get (2.20b,c). Note that (2.20b) agrees with (2.20c) if $n = 2$.

The estimate (2.20d) is obtained in the same way as above but this time we have to extend $a(x)$ as an even function in x_n, i.e. $a(x', x_n) = a(x', -x_n)$ for $x_n < 0$. $\square$

We are now ready to prove Theorem 0.2. By Lemma 1.2 and Lemma 2.4,

$$\begin{aligned}\|\partial_j \bar{u}_n\|_{\mathcal{H}^1} \leq & |||R_j|||_{\mathcal{H}^1} \Big\{ \textstyle\sum_{k=1}^{n-1} |||R_k|||_{\mathcal{H}^1} (\|\Lambda eE(t)u_0^k\|_{\mathcal{H}^1} \\ & + \|\partial_k eE(t)u_0^n\|_{\mathcal{H}^1}) \\ & + |||R_n|||_{\mathcal{H}^1} (\|\nabla \cdot eE(t)u_0'\|_{\mathcal{H}^1} + \|\Lambda eE(t)u_0^n\|_{\mathcal{H}^1}) \Big\} \\ \leq & \, Ct^{-1/2} \|u_0\|_{L^1(\mathbb{R}^n_+)}, \\ \|\partial_j \bar{u}'\|_{\mathcal{H}^1} \leq & \, Ct^{-1/2} \|u_0\|_{L^1(\mathbb{R}^n_+)}.\end{aligned}$$

Since $u = \bar{u}|_{\mathbb{R}^n_+}$, we now get

$$\|\nabla u\|_{L^1(\mathbb{R}^n_+)} \leq \|\nabla u\|_{\mathcal{H}^1(\mathbb{R}^n_+)} \leq \|\nabla \bar{u}\|_{\mathcal{H}^1} \leq Ct^{-1/2} \|u_0\|_{L^1(\mathbb{R}^n_+)}.$$

The proof is complete.

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Note added in proof. Recently, the third author proved (0.2) for $p = \infty$ for the Stokes flow in a half space.