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Theorems of Barth-Lefschetz type and Morse theory on the space of paths

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0. Introduction

In the 1920's Lefschetz [Le] stated the following theorem now known as the Lefschetz theorem on hyperplane sections. Let $N \subset \mathbb{P}^v$ be a connected complex submanifold of complex dimension n. Let H be a hyperplane and $N \cap H$ a nonsingular hyperplane section. Then the relative cohomology groups satisfy:

$$H^j(N, N \cap H; \mathbb{C}) = 0, \quad j \leq n-1.$$

Fifty years later Barth [B] generalized Lefschetz's theorem: Let $M, N \subset \mathbb{P}^v$ be complex submanifolds of complex dimensions m, n, respectively. If M and N meet properly, then,

$$H^j(N, N \cap M; \mathbb{C}) = 0, \quad j \leq \min(n+m-v, 2m-v+1).$$

Generalizations of Barth's results to homotopy groups were first obtained by Larsen [La] and Barth-Larsen [B-L] and later by Sommese and Fulton-Lazarsfeld [F-L]. In particular they prove a "connectedness" theorem for closed local complete intersections $M, N \subset \mathbb{P}^v$ of complex dimensions m, n, respectively. For such varieties it is shown that the relative homotopy groups satisfy:

$$\pi_j(N, N \cap M) = 0, \quad j \leq \min(n+m-v, 2m-v+1).$$

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Sommese [S1][S2] generalized this result to submanifolds of certain homogeneous complex manifolds. Finally in 1987 Okonek [O] generalized these homotopy results to include varieties with singularities. For a more complete survey of this topic we refer the reader to [F-L].

In 1961 T. Frankel [F] proved a "connectedness" theorem for complex submanifolds of a Kähler manifold of positive holomorphic sectional curvature. Let V be a complete Kähler manifold of positive holomorphic sectional curvature and of complex dimension v. Let $M, N \subset V$ be compact complex submanifolds of dimensions m and n, respectively. Frankel proved that if $m + n \geq v$ then M and N must intersect. Apparently it was not seen how to apply Frankel's technique to the more general "connectedness" results described above. However the relevance of Frankel's work to these results was noted by Fulton in [Fu].

In this paper we show that a variant of Frankel's argument together with Morse theory on a space of paths leads to an elegant proof of homotopy connectedness theorems for complex submanifolds of certain Kähler manifolds of non-negative holomorphic bisectional curvature. In particular, we prove:

Theorem 0.1. *Let V be a Kähler manifold. Suppose that $M, N \subset V$ are complex submanifolds of complex dimensions m, n, respectively, such that M is compact and N is a closed subset of V. Let*

$$\imath_* : \pi_j(N, N \cap M) \to \pi_j(V, M),$$

be the homomorphism induced by the inclusion.

(i) If $V = \mathbb{P}^v$ then $\imath_*$ is an isomorphism for $j \leq n + m - v$ and is a surjection for $j = n + m - v + 1$.
(ii) If $V =$ Gr$(p, p + q; \mathbb{C})$ then $\imath_*$ is an isomorphism for $j \leq n + m - 2pq + (p + q - 1)$ and is a surjection for $j = n + m - 2pq + (p + q)$.
(iii) If $V =$ Gr$(2, p + 2; \mathbb{R})$ then $\imath_*$ is an isomorphism for $j \leq n + m - p$ and is a surjection for $j = n + m - p + 1$.

Theorem 0.1 in the case (i) that $V = \mathbb{P}^v$ is due to Fulton-Lazarsfeld [F-L, Theorem 9.6]. Their result is actually somewhat more general.

Corollary 0.2. *Suppose V, M and N satisfy the same hypotheses as in Theorem 0.1. Morover,*

(i) if $V = \mathbb{P}^v$ set $\ell = v$.
(ii) if $V =$ Gr$(p, p + q; \mathbb{C})$ set $v = pq$ and $\ell = p + q - 1$.
(iii) if $V =$ Gr$(2, p + 2; \mathbb{R})$ set $v = p$ and $\ell = p$.

In each of these three cases, we have:

(0.1) *If* $j \leq 2m - v - (v - \ell) + 1$ *then* $\pi_j(V, M) = 0$.

$$\begin{aligned} &\textit{If } j \leq \min(2m - v - (v - \ell) + 1,\ n + m - v - (v - \ell)) \\ (0.2) \qquad &\quad \textit{then } \pi_j(N, N \cap M) = 0. \end{aligned}$$

In the case (i) that $V = \mathbb{P}^v$ (0.1) is due to Larsen [La]. Sommese [S2] proved both statements of the corollary in cases (i) and (ii). He also has similar results when V is a simple abelian variety.

The proof of Theorem 0.1 relies on the computation of the index of a critical point of the energy on a suitable space of paths. This is done in Sect. 2. In Sect. 1 we outline the results we require from Morse theory on path spaces. The results are taken from Milnor [M] with some modifications. In Sect. 3 we combine the results of the previous sections to derive the connectedness theorem.

1. Morse theory

Let V be a complete Riemannian manifold and let M and N be submanifolds (intersecting or not) with M compact and N a closed subset of V. We let $\mathcal{P}(V; M, N)$ denote the set of C^k paths $\gamma : [0, 1] \to V$ such that $\gamma(0) \in M$ and $\gamma(1) \in N$. The energy E of the path defines a function on $\mathcal{P}(V; M, N)$ given by:

$$E(\gamma) = \int_0^1 |\dot{\gamma}(t)|^2 dt.$$

We are interested in studying the topology of $\mathcal{P}(V; M, N)$ via the Morse theory of the function E. There are two approaches to this study. In one the space of paths is given the structure of a Hilbert manifold and Morse theory on Hilbert manifolds is applied to the energy E. This is the approach developed in detail by Palais [P]. An older approach to this problem, originating with M. Morse, approximates the path space by finite-dimensional manifolds and employs techniques from finite-dimensional Morse theory. This is the approach described by Milnor in [M]. For ease of exposition we will follow the latter approach.

In [M], Milnor studies a special case of the above problem, namely, the topology of the path space $\mathcal{P}(V; p, q)$ where p and q are points in V. While our problem is more general it turns out the results we require are stated in [M]. Moreover, the proofs given in [M] apply to the general case with only minor changes that can easily be made by the reader. Accordingly, in this section, we will describe the general set-up, state the results we will need and give the appropriate references to [M].

We begin by defining the path space. A *piecewise smooth path* from M to N is a map $\gamma : [0, 1] \to V$ such that:

(i) there is a subdivision $0 = t_0 < t_1 < \cdots < t_k = 1$ of $[0, 1]$ such that each $\gamma|_{[t_{i-1}, t_i]}$ is smooth.

(ii) $\gamma(0) \in M$ and $\gamma(1) \in N$.

The set of all piecewise smooth paths from M to N in V will be denoted by $\Omega(V; M, N)$ or simply Ω.

The set $\Omega(V; M, N)$ can be topologized as follows: Let ρ denote the Riemannian distance function on V. Let $\gamma_1, \gamma_2 \in \Omega(V; M, N)$. Define the distance $d(\gamma_1, \gamma_2)$ by:

$$d(\gamma_1, \gamma_2) = \max_{0 \leq t \leq 1} \rho(\gamma_1(t), \gamma_2(t)) + \int_0^1 (|\dot{\gamma}_1(t)| - |\dot{\gamma}_2(t)|)^2 dt.$$

Note that $\dot{\gamma}_1$ and $\dot{\gamma}_2$ are not defined at finitely many points in $[0, 1]$, however the integral is defined. This metric induces the required topology. The energy of a path

$$E(\gamma) = \int_0^1 |\dot{\gamma}(t)|^2 dt$$

defines a continuous map $\Omega(V; M, N) \to \mathbb{R}$.

Define the *tangent space* of Ω at γ, $T_\gamma \Omega$, to be the vector space of piecewise smooth vector fields W along γ such that $W(0) \in T_{\gamma(0)} M$ and $W(1) \in T_{\gamma(1)} N$. A standard computation shows that the first variation of E in the direction $W \in T_\gamma \Omega$, denoted $E_*(W)$, is given by:

$$\frac{1}{2} E_*(W) = \langle W, \dot{\gamma} \rangle \Big|_0^1 - \sum_t \langle W(t), \Delta_t \dot{\gamma} \rangle - \int_0^1 \langle W, \frac{D\dot{\gamma}}{dt} \rangle dt,$$

where $\Delta_t \dot{\gamma} = \dot{\gamma}(t^+) - \dot{\gamma}(t^-) =$ the discontinuity of $\dot{\gamma}$ at t. It follows that γ is a critical point of E if:

(i) γ is a smooth geodesic.
(ii) γ is normal to M and N at $\gamma(0)$ and $\gamma(1)$, respectively.

Let $W_1, W_2 \in T_\gamma \Omega$. If γ is a critical point of E then the second variation of E along γ, denoted $E_{**}(W_1, W_2)$, is given by:

$$\begin{aligned} \frac{1}{2} E_{**}(W_1, W_2) = &- \sum_t \langle W_2(t), \Delta_t \frac{DW_1}{dt} \rangle \\ &- \int_0^1 \langle W_2, \frac{D^2 W_1}{dt^2} + R(\dot{\gamma}, W_1)\dot{\gamma} \rangle dt. \end{aligned}$$

Let Ω_c denote the closed subset $E^{-1}([0, c]) \subset \Omega$ and let $\mathring{\Omega}_c$ denote the open subset $E^{-1}([0, c))$. Following Milnor we construct a finite dimensional approximation to Ω_c: Choose some subdivision $0 = t_0 < t_1 < \cdots < t_k = 1$ of $[0, 1]$. Let $\Omega(t_0, \ldots, t_k)$ be the subspace of Ω consisting of paths $\gamma : [0, 1] \to V$ such that:

(i) $\gamma(0) \in M$ and $\gamma(1) \in N$
(ii) $\gamma|_{[t_{i-1},t_i]}$ is a geodesic for each $i = 1, \dots, k$.

Define the subspaces:

$$\Omega_c(t_0, \dots t_k) = \Omega_c \cap \Omega(t_0, \dots, t_k)$$

$$\overset{\circ}{\Omega}_c(t_0, \dots t_k) = \overset{\circ}{\Omega}_c \cap \Omega(t_0, \dots, t_k).$$

Theorem 1.1. *Let V be a complete Riemannian manifold and let M and N be submanifolds with M compact and N a closed subset of V. Let c be a fixed positive number such that $\Omega_c \neq \phi$. Then for all sufficiently fine subdivisions $0 = t_0 < t_1 < \cdots < t_k = 1$ of $[0,1]$ the set $\overset{\circ}{\Omega}_c(t_0, \dots t_k)$ can be given the structure of a smooth finite dimensional manifold.*

Proof. [M] Sect. 16. □

Denote the manifold of broken geodesics $\overset{\circ}{\Omega}_c(t_0, \dots t_k)$ by B. Let

$$E|_B : B \to \mathbb{R}$$

denote the restriction to B of the energy function $E : \Omega \to \mathbb{R}$.

Theorem 1.2. *$E|_B : B \to \mathbb{R}$ is a smooth map. For each $a < c$ the set $B_a = (E|_B)^{-1}([0,a])$ is compact and is a deformation retract of the set Ω_a. The critical points of $E|_B$ are precisely the same as the critical points of E in $\overset{\circ}{\Omega}_c$, that is, the smooth geodesics from M to N intersecting M and N orthogonally and with energy less than c. The index of the hessian of $E|_B$ at each such critical point γ is equal to the index of E_{**} at γ.*

Proof. [M] Sect. 14 and Sect. 16. □

Now suppose that every nontrivial critical point γ of E on Ω has index $\lambda > \lambda_0 \geq 0$. We remark that this implies that $N \cap M \neq \phi$. Since otherwise there exists a nontrivial minimizing geodesic from M to N and the index of such a geodesic must be zero. It follows that if every nontrivial critical point γ on Ω has index $\lambda > \lambda_0 \geq 0$ then the space Ω_0 of minimal (i.e., trivial) geodesics can be identified with the subspace $N \cap M \subset \Omega$.

Theorem 1.3. *Suppose N intersects M transversally and that every nontrivial critical point of E on Ω has index $\lambda > \lambda_0 \geq 0$. Then the relative homotopy groups $\pi_j(\Omega, \Omega_0)$ are zero for $0 \leq j \leq \lambda_0$.*

The proof of the theorem requires a lemma about functions on finite-dimensional manifolds: Let X be a smooth manifold and $f : X \to \mathbb{R}$ be a smooth real-valued function with minimum value 0 such that each $X_c = f^{-1}([0,c])$ is compact.

Lemma 1.4. *If the set X_0 of minimal points has a neighborhood U with a retraction $r : U \to X_0$ and if every critical point in $X \setminus X_0$ has index $> \lambda_0$ then*

$$\pi_j(X, X_0) = 0 \quad \textit{for} \quad 0 \le j \le \lambda_0.$$

Proof. [M] Sect. 22. □

Proof of the theorem: It suffices to prove that

$$\pi_j(\overset{\circ}{\Omega}_c, \Omega_0) = 0 \qquad 0 \le j \le \lambda_0$$

for arbitrarily large values of c. By Theorem 1.2 $\overset{\circ}{\Omega}_c$ contains a smooth manifold $\overset{\circ}{\Omega}_c(t_0, t_1, \ldots, t_k)$ as a deformation retract. Also by Theorem 1.2, the energy function $E : \Omega \to \mathbb{R}$ when restricted to $\overset{\circ}{\Omega}_c(t_0, \ldots, t_k)$ has the property that every nontrivial critical point in $\overset{\circ}{\Omega}_c(t_0, \ldots, t_k)$ has index $\lambda > \lambda_0$. The space of minimal geodesics $\Omega_0 \simeq M \cap N$ is contained in $\overset{\circ}{\Omega}_c(t_0, \ldots, t_k)$. To apply the lemma it only remains to show that there is a neighborhood $U \subset \overset{\circ}{\Omega}_c(t_0, \ldots, t_k)$ of Ω_0 and a retraction $r : U \to \Omega_0$.

Consider the neighborhoods $\overset{\circ}{\Omega}_\varepsilon(t_0, \ldots, t_k)$ of Ω_0 for $\varepsilon > 0$. We claim there is an $\varepsilon_0 > 0$ such that E has no nontrivial critical points in $\overset{\circ}{\Omega}_{\varepsilon_0}(t_0, \cdots, t_k)$. To prove this, suppose the contrary. Then there is a sequence $\{\gamma_i\}$ of critical points of E with $E(\gamma_i) < \varepsilon_i$ and $\varepsilon_i \downarrow 0$. In particular the γ_i are smooth geodesics with $\gamma_i(0) \in M, \ \gamma_i(1) \in N$ and such that the image of γ_i intersects both M and N orthogonally. Since $E(\gamma_i) \downarrow 0$ and M is compact the image of the γ_i converges to a point $x \in N \cap M$. Let A denote a geodesically convex neighborhood of x. By rechoosing the sequence $\{\gamma_i\}$ we can suppose that each γ_i lies in A and moreover it is the unique geodesic lying in A joining its endpoints. Successively rescale A. In the limit we have an n-plane, T_xN, ($\dim N = n$) and and an m-plane, T_xM, ($\dim M = m$) intersecting tranversally. There are sequences, $y_i \in T_xM$ and $z_i \in T_xN$, with $y_i \to 0, \ z_i \to 0$ and straight lines L_i joining y_i to z_i. Moreover the L_i intersect T_xN and T_xM orthogonally. Clearly this latter condition is impossible, proving the claim. Let $U = \overset{\circ}{\Omega}_{\varepsilon_0}(t_0, \ldots, t_k)$. The retraction $r : U \to \Omega_0$ is given by following the gradient flow lines of E on $\overset{\circ}{\Omega}_{\varepsilon_0}(t_0, \ldots, t_k)$. □

Let V be a complete Kähler manifold. Let $M, N \subset V$ be complex submanifolds of complex dimensions m, n, respectively and suppose that M is compact and and N is a closed subset of V. To prove the analog of Theorem 1.3 we do not need to assume that M and N intersect transversally.

Theorem 1.5. *Let V be a complete Kähler manifold. Let $M, N \subset V$ be complex submanifolds and suppose that M is compact and N is a closed subset of V. If every nontrivial critical point of E on Ω has index $\lambda > \lambda_0 \geq 0$ then the relative homotopy groups $\pi_j(\Omega, \Omega_0)$ are zero for $0 \leq j \leq \lambda_0$.*

Proof. The proof is the same as the proof of Theorem 1.3 except that in the limit after rescaling the convex neighborhood A we have a complex n-plane, $T_x N$, intersecting an analytic variety (M rescaled). However the same contradiction results, proving the theorem. □

2. The index of a critical point

Let V be a complete Kähler manifold of complex dimension v, with complex structure J and Levi-Civita connection ∇. Let M and N be complex submanifolds of complex dimensions m and n, respectively. We continue to denote, by $\Omega(V; M, N) = \Omega$, the space of paths $\gamma : [0, 1] \to V$ constrained by the requirements that $\gamma(0) \in M$ and $\gamma(1) \in N$. Consider the energy of a path

$$E(\gamma) = \int_0^1 |\dot{\gamma}|^2 dt$$

as a function on Ω. As shown in Sect. 1 γ is a critical point of E if:

(i) γ is a smooth geodesic
(ii) γ is normal to M and N at $\gamma(0)$ and $\gamma(1)$, respectively.

Let $W_1, W_2 \in T_\gamma \Omega$. If γ is a critical point of E then we rewrite the second variation of E along γ by:

$$\begin{aligned} \frac{1}{2} E_{**}(W_1, W_2) &= \langle \nabla_{W_1} W_2, \dot{\gamma} \rangle \Big|_0^1 + \int_0^1 \langle \nabla_{\dot{\gamma}} W_1, \nabla_{\dot{\gamma}} W_2 \rangle dt \\ &\quad - \int_0^1 \langle R(\dot{\gamma}, W_1)\dot{\gamma}, W_2 \rangle dt. \end{aligned} \tag{2.1}$$

Suppose that γ is a nontrivial critical point and that $W(0)$ is a vector in $T_{\gamma(0)} M$. Parallel translate $W(0)$ along γ to construct a vector field $W = W(t)$ along γ. Of course, $W(1)$ need not be tangent to N at $\gamma(1)$ so W is not necessarily an element of $T_\gamma \Omega$. However formally we have:

$$\frac{1}{2} E_{**}(W, W) = \langle \nabla_W W, \dot{\gamma} \rangle \Big|_0^1 - \int_0^1 \langle R(\dot{\gamma}, W)\dot{\gamma}, W \rangle dt. \tag{2.2}$$

V is Kähler so JW is also parallel along γ. M is complex so $JW(0) \in T_{\gamma(0)} M$. Thus we also have:

$$\frac{1}{2} E_{**}(JW, JW) = \langle \nabla_{JW} JW, \dot{\gamma} \rangle \Big|_0^1 - \int_0^1 \langle R(\dot{\gamma}, JW)\dot{\gamma}, JW \rangle dt. \tag{2.3}$$

Adding (2.2) and (2.3) and using $\nabla_{JW} JW = -\nabla_W W$ we have:

$$\begin{aligned} &\frac{1}{2}E_{**}(W,W) + \frac{1}{2}E_{**}(JW,JW) = \\ &\quad -\int_0^1 (\langle R(\dot\gamma,W)\dot\gamma,W\rangle + \langle R(\dot\gamma,JW)\dot\gamma,JW\rangle)dt. \end{aligned} \tag{2.4}$$

Using the symmetries of the curvature tensor we have:

$$\langle R(\dot\gamma,W)\dot\gamma,W\rangle + \langle R(\dot\gamma,JW)\dot\gamma,JW\rangle = \langle R(\dot\gamma,J\dot\gamma)W,JW\rangle. \tag{2.5}$$

This expression is the holomorphic bisectional curvature of the complex lines $\dot\gamma \wedge J\dot\gamma$ and $W \wedge JW$.

Let $\{W_1(0),\dots,W_m(0),JW_1(0),\dots,JW_m(0)\}$ be an orthonormal framing of $T_{\gamma(0)}M$. For each $i = 1,\dots,m$, parallel translate $W_i(0)$ along γ to construct parallel vector fields $\{W_1,\dots,W_m,JW_1,\dots,JW_m\}$ along γ. Note that the vectors $W_i(1), JW_i(1)$ are perpendicular to both $\dot\gamma(1)$ and $J\dot\gamma(1)$. Thus the vector space

$$S = \text{span}\{W_1(1),\dots,W_m(1),JW_1(1),\dots,JW_m(1)\}$$

is a complex m-dimensional space lying in a complex $(v-1)$-dimensional subspace of $T_{\gamma(1)}V$. It follows that the subspace $S \cap T_{\gamma(1)}N$ has complex dimension at least equal to

$$k = m + n - (v-1).$$

Moreover, the vector fields $\{W, JW\}$ with $W(1), JW(1) \in S \cap T_{\gamma(1)}N$ are parallel and lie in $T_\gamma \Omega$.

Theorem 2.1. *Suppose that V is a Kähler manifold of positive holomorphic bisectional curvature, that M and N are complex submanifolds and that γ is a nontrivial critical point of the energy on $\Omega(V;M,N)$. Then,*

$$\textit{index}(\gamma) \geq m + n - (v-1).$$

Proof. There are at least k pairs $\{W, JW\}$ that are parallel along γ and lie in $T_\gamma\Omega$. For each such pair, using the curvature assumption, (2.4) and (2.5) we have:

$$\begin{aligned} &E_{**}(W,W) + E_{**}(JW,JW) = \\ &\quad -2\int_0^1 \langle R(\dot\gamma,J\dot\gamma)W,JW\rangle dt < 0. \end{aligned}$$

The result follows. □

Consider next the case where V is a Kähler manifold of non-negative holomorphic bisectional curvature. Fix $x \in V$ and let $X \wedge JX$ be a complex line in T_xV. Let $\mathcal{C}(x, X \wedge JX)$ be the cone:

$$\mathcal{C}(x, X \wedge JX) = \{Y \in T_xV : \langle R(X, JX)Y, JY\rangle > 0\}.$$

Note that $\mathcal{C}$ is a complex cone; if $Y \in \mathcal{C}$ then $JY \in \mathcal{C}$.

Definition. Let $\ell(x, X \wedge JX)$ denote the maximal number of orthogonal pairs (Y, JY) in $\mathcal{C}(x, X \wedge JX)$. Set:

(i) $\ell(x) = \inf\limits_{X \wedge JX} \ell(x, X \wedge JX)$

(ii) $\ell = \inf\limits_{x \in V} \ell(x)$.

We say that ℓ is the *complex positivity* of V.

Remark. If V is a hermitian symmetric space then $\ell(x) = \ell$ for every $x \in V$. If V is a hermitian symmetric space such that the isotropy subgroup acts transitively on complex lines in T_xV then $\ell(x) = \ell(x, X \wedge JX)$ for any complex line $X \wedge JX$ in T_xV. In particular this is true if V is the complex Grassmann manifold or the complex quadric.

Proposition 2.2. *If $V = \prod_{i=1}^{t} V_i$ and ℓ_i denotes the complex positivity of V_i then the complex positivity of V is $\ell = \min_i \ell_i$.*

Proof. Clear. □

Let $\mathrm{Gr}(p, p+q; \mathbb{C})$ denote the complex Grassmann manifold of complex p-planes in $\mathbb{C}^{p+q}$.

Lemma 2.3. *If $V = Gr(p, p+q; \mathbb{C})$ then for any $x \in Gr(p, p+q; \mathbb{C})$ and any complex line $X \wedge JX$ through x:*

(i) The cone $\mathcal{C}(x, X \wedge JX)$ is a complex subspace of complex dimension $\ell(x, X \wedge JX)$.

(ii) $\ell(x, X \wedge JX) = \ell = p + q - 1$.

Proof. Let (ω_{AB}), $\quad 1 \le A, B \le p+q$, be the Maurer-Cartan one-form of the group $U(p+q)$. The one-forms $\{\omega_{i\alpha}\}$, $\;1 \le i \le p$, $\;p+1 \le \alpha \le p+q$, give a unitary coframe for the Kähler metric on $\mathrm{Gr}(p, p+q; \mathbb{C})$ considered as a hermitian symmetric space. With respect to this coframe the curvature two-form is given by:

$$\Omega_{i\alpha, j\beta} = -\delta_{\alpha\beta} \sum_{\gamma=p+1}^{p+q} \omega_{i\gamma} \wedge \bar{\omega}_{j\gamma} - \delta_{ij} \sum_{k=1}^{p} \omega_{k\alpha} \wedge \bar{\omega}_{k\beta}. \tag{2.6}$$

In particular,

$$\Omega_{i\alpha,i\alpha} = -\sum_{\gamma=p+1}^{p+q} \omega_{i\gamma} \wedge \bar{\omega}_{i\gamma} - \sum_{k=1}^{p} \omega_{k\alpha} \wedge \bar{\omega}_{k\alpha}. \tag{2.7}$$

Now suppose that the vectors Y_1, Y_2 are orthogonal and lie in $\mathcal{C}(x, X \wedge JX)$. Let $a, b \in \mathbb{R}$ and consider,

$$\begin{aligned} \langle R(X,JX)(aY_1 + bY_2), J(aY_1 + bY_2)\rangle &= a^2 \langle R(X,JX)Y_1, JY_1\rangle \\ &+2ab\langle R(X,JX)Y_2, JY_1\rangle + b^2 \langle R(X,JX)Y_2, JY_2\rangle. \end{aligned}$$

By (2.7) the middle term of the right hand side vanishes and it follows that $aY_1 + bY_2 \in \mathcal{C}(x, X \wedge JX)$.

The equality $\ell = p + q - 1$ follows immediately from (2.7). □

Now suppose V is the complex quadric. As a symmetric space V can be identified with the real Grassmann manifold $\mathrm{Gr}(2, p+2; \mathbb{R})$. Using the same reasoning as above we have,

Lemma 2.4. *If $V = Gr(2, p+2; \mathbb{R})$ then for any $x \in Gr(2, p+2; \mathbb{R})$ and any complex line $X \wedge JX$ through x:*

(i) *The cone $\mathcal{C}(x, X \wedge JX)$ is a complex subspace of complex dimension $\ell(x, X \wedge JX)$.*
(ii) $\ell(x, X \wedge JX) = \ell = p$.

Theorem 2.5. *Suppose that V is a complete Kähler manifold of non-negative holomorphic bisectional curvature. Further suppose that for any $x \in V$ and any complex line $X \wedge JX$ through x, the cone $\mathcal{C}(x, X \wedge JX)$ is a complex subspace of complex dimension $\ell(x, X \wedge JX) \geq \ell$. Let M and N be complex submanifolds of complex dimensions m and n, respectively, and γ be a nontrivial critical point of energy on $\Omega(V; M, N)$. Then*

$$\mathit{index}(\gamma) \geq m + n - (v-1) - (v - \ell).$$

Proof. The argument in the proof of Theorem 2.1 shows that if $W, JW \in S \cap T_{\gamma(1)}N$ then

$$E_{**}(W,W) + E_{**}(JW,JW) = -2\int_0^1 \langle R(\dot\gamma, J\dot\gamma)W, JW\rangle \leq 0.$$

To get strict inequality we want

$$\langle R(\dot\gamma, J(\dot\gamma))W, JW\rangle > 0$$

at $\gamma(0)$. This is insured by requiring that:

$$W(0) \wedge JW(0) \in \mathcal{C}(\gamma(0), \dot\gamma \wedge J\dot\gamma).$$

The result follows. □

Corollary 2.6. *Suppose* $V = \prod_i^t Gr(p_i, p_i + q_i; \mathbb{C})$. *Let* M *and* N *be complex submanifolds of complex dimensions* m *and* n*, respectively. Let* γ *be a nontrivial critical point of the energy on* $\Omega(V; M, N)$ *then,*

$$\textit{index}(\gamma) \geq m + n + \min_i(p_i + q_i - 1) - 2\prod_i^t p_i q_i + 1.$$

Proof. The result follows from Proposition 2.2, Theorem 2.5 and Lemma 2.3. $\square$

A similar result holds for products of complex quadrics. We leave the exact formulation to the reader.

3. Applications

In this section we apply Morse theory to the path spaces $\Omega(V; M, N)$ and derive versions of the theorems of Lefschetz, Barth, Sommese, Fulton-Lazarsfeld, etc.

Let V be a complete Kähler manifold of non-negative holomorphic bisectional curvature and of complex dimension v. Suppose that for any $x \in V$ and any complex line $X \wedge JX$ through x, the cone $\mathcal{C}(x, X \wedge JX)$ is a complex subspace of complex dimension $\ell(x, X \wedge JX) \geq \ell$. Let $M, N \subset V$ be complex submanifolds of complex dimensions m, n, respectively and suppose that M is compact and N is a closed subset of V. We consider the path space $\Omega(V; M, N) = \Omega$ as described in Sect. 1.

Theorem 3.1. *Suppose that,*

$$\lambda_0 = n + m - v - (v - \ell) \geq 0.$$

Then relative homotopy groups $\pi_j(\Omega, N \cap M)$ *are zero for* $0 \leq j \leq \lambda_0$.

Proof. The theorem follows from Theorem 1.5 and Theorem 2.5. $\square$

Theorem 3.1 and the long exact homotopy sequence of the pair $(\Omega, N \cap M)$ imply that the homomorphism induced by the inclusion:

$$\imath_* : \pi_j(N \cap M) \to \pi_j(\Omega) \qquad (3.1)$$

is an isomorphism when $j < n + m - v - (v - \ell)$ and is a surjection when $j = n + m - v - (v - \ell)$.

Consider the fibration:

$$\begin{array}{ccc} \Omega(V;M,x) & \longrightarrow & \Omega(V;M,N) \\ & & \downarrow e \\ & & N \end{array} \tag{3.2}$$

where e is the evaluation map $e : \gamma \mapsto \gamma(1)$ and $x \in N$. It is well-known that the homotopy groups of the fiber $\Omega(V;M,x)$ satisfy:

$$\pi_j(\Omega(V;M,x)) \simeq \pi_{j+1}(V,M), \tag{3.3}$$

for all j. The long exact homotopy sequence of the fibration is:

$$\begin{aligned} &\cdots \longrightarrow \pi_{j+1}(N) \longrightarrow \pi_j(\Omega(V;M,x)) \longrightarrow \pi_j(\Omega) \\ &\xrightarrow{e_*} \pi_j(N) \longrightarrow \pi_{j-1}(\Omega(V;M,x)) \longrightarrow \cdots \end{aligned} \tag{3.4}$$

Thus, using (3.3), the long exact sequence (3.4) becomes:

$$\begin{aligned} &\cdots \to \pi_{j+1}(N) \to \pi_{j+1}(V,M) \to \pi_j(\Omega) \to \pi_j(N) \\ &\to \pi_j(V,M) \to \cdots \end{aligned} \tag{3.5}$$

We have:

Theorem 3.2. *Let V be a complete Kähler manifold of non-negative holomorphic bisectional curvature. Suppose that for any $x \in V$ and any complex line $X \wedge JX$ through x, the cone $\mathcal{C}(x, X \wedge JX)$ is a complex subspace of complex dimension $\ell(x, X \wedge JX) \geq \ell$. Let $M, N \subset V$ be complex submanifolds of complex dimensions m, n, respectively, such that M is compact and N is a closed subset of V. Then the homomorphism induced by the inclusion*

$$\imath_* : \pi_j(N, N \cap M) \to \pi_j(V, M)$$

is an isomorphism for $j \leq n + m - v - (v - \ell)$ and is a surjection for $j = n + m - v - (v - \ell) + 1$.

Proof. For $\lambda_0 = n + m - v - (v - \ell)$ consider the diagram:

$$\begin{array}{ccccccccc} \pi_{\lambda_0+1}(N) & \to & \pi_{\lambda_0+1}(V,M) & \to & \pi_{\lambda_0}(\Omega) & \to & \pi_{\lambda_0}(N) & \to & \pi_{\lambda_0}(V,M) \\ \uparrow\simeq & & \uparrow & & \uparrow \text{onto} & & \uparrow\simeq & & \uparrow \\ \pi_{\lambda_0+1}(N) & \to & \pi_{\lambda_0+1}(N, N\cap M) & \to & \pi_{\lambda_0}(N\cap M) & \to & \pi_{\lambda_0}(N) & \to & \pi_{\lambda_0}(N, N\cap M) \end{array}$$

The vertical arrows are induced by inclusion. The top row is the long exact sequence (3.5). The bottom row is the long exact sequence of the pair $(N, N \cap M)$. The result follows using Theorem 3.1 and the commutivity of the diagram. $\square$

Corollary 3.3. *Under the same hypotheses as in Theorem 3.2, if*

$$j \leq 2m - v - (v - \ell) + 1$$

then

$$\pi_j(V, M) = 0.$$

Proof. Apply Theorem 3.2 to the case $N = M$. $\square$

Corollary 3.4. *Under the same hypothesis as in Theorem 3.2, if*

$$j \leq \min(2m - v - (v - \ell) + 1,\ n + m - v - (v - \ell))$$

then

$$\pi_j(N, N \cap M) = 0.$$

Proof. Follows from Corollary 3.3 and Theorem 3.2. $\square$

The statements of Theorem 3.2 and its corollaries apply, in particular, to:

(i) $V = \mathbb{P}^v$ with $\ell = v$.
(ii) $V = \mathrm{Gr}(p, p + q; \mathbb{C})$ with $v = pq$ and $\ell = p + q - 1$.
(iii) $V = \mathrm{Gr}(2, p + 2; \mathbb{R})$ with $v = p$ and $\ell = p$.

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