Math. Z. 228, 83–120 (1998)

Mathematische Zeitschrift

Asymptotically self-similar global solutions of the nonlinear Schrödinger and heat equations

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Received 15 May 1996; in final form 21 October 1996

1 Introduction

In this paper we study global solutions, including self-similar solutions, of the initial value problem for the following nonlinear Schrödinger equation

$$iu_t + \triangle u = \gamma |u|^\alpha u, \tag{1.1}$$
$$u(0,x) = \varphi(x) \tag{1.2}$$

Here, $u = u(t,x)$ is a complex valued function defined on $[0,\infty) \times \mathbb{R}^N$, γ is a real number, $\alpha > 0$, and the initial condition $\varphi(x)$ is a complex valued function defined on $\mathbb{R}^N$. Also, at the end of the paper, we will extend some of our results to include the nonlinear heat equation analogous to (1.1)

There is a well known principle which has frequently been used to prove existence of global solutions of nonlinear equations. Suppose the set of solutions of some nonlinear equation is invariant under a certain group of transformations. For example, the set of solutions of (1.1) is invariant under the transformations $u \mapsto u_\lambda$, where $u_\lambda(t,x) = \lambda^{\frac{2}{\alpha}} u(\lambda^2 t, \lambda x)$, for all $\lambda > 0$. Suppose next that some norm $\|\cdot\|$, defined on a space of initial values φ, is invariant with respect to this group of transformations (restricted to spatial functions). In our situation, this means that $\|\varphi_\lambda\| = \|\varphi\|$, for all $\lambda > 0$, where $\varphi_\lambda(x) = \lambda^{\frac{2}{\alpha}}\varphi(\lambda x)$. Under these circumstances, one can often prove that initial data φ for which $\|\varphi\|$ is sufficiently small give rise to global solutions of the nonlinear equation.

To our knowledge, this idea was first discovered by T. Kato and H. Fujita [9, 24], for the Navier-Stokes system, who proved that data small in

$H^{\frac{1}{2}}(\mathbb{R}^3)$ yield global solutions. The second author [32] used this idea to prove that data small with respect to either the $L^{\frac{N\alpha}{2}}(\mathbb{R}^N)$ norm, if $\dfrac{N\alpha}{2} > 1$, or the norm $\displaystyle\int_0^\infty \|e^{t\triangle}\varphi\|_{L^\infty}^\alpha \, dt$ give rise to global solutions of the nonlinear heat equation analogous to (1.1). Kato [21] then showed that solutions of the Navier-Stokes equations with data small in $L^N(\mathbb{R}^N)$ are global. Y. Giga and T. Miyakawa [14] and Y. Giga [10] extended these arguments to more general nonlinear heat equations and to fractional power spaces in L^r in the case of the Navier-Stokes equation. Developing this idea further, Y. Giga and T. Miyakawa [15] subsequently proved global existence of solutions to the Navier-Stokes system, expressed in terms of the vorticity, with initial vorticity small in certain Morrey spaces of measures. In [4], the authors applied this principle to (1.1) to show global existence of solutions for data small with respect to fractional homogeneous Sobolev norms (see T. Kato [23] and H. Pecher [27] for recent related results); and in [6], we again applied this principle to equation (1.1), using a norm on functions φ defined in terms of a time integral of $||S(t)\varphi||_{L^{\alpha+2}}$, where $S(t)$ is the linear Schrödinger group. As a corollary, we showed, for certain values of α, that sufficiently oscillatory data (defined in a somewhat restricted sense: see Corollary 2.5 in [6]) give rise to global solutions. More recently, this principle was applied by M. Cannone [1] and M. Cannone and F. Planchon [2] to the Navier-Stokes system using Besov norms. In particular, as proved for (1.1), highly oscillatory initial data (in a more general context than defined in [6]) give rise to global solutions. Moreover, [1, 2, 15] all include results on the existence of self-similar solutions. Indeed, both the Morrey spaces used by Y. Giga and T. Miyakawa and the Besov spaces used by M. Cannone and F. Planchon are sufficiently weak to include homogeneous data, for which the resulting global solutions are necessarily self-similar. One advantage of this method of investigating self-similar solutions is that one easily obtains self-similar solutions which are not radially symmetric. These ideas have been applied to a general nonlinear heat equation by F. Ribaud [29]. In this brief historical survey, we have focused on the Navier-Stokes equation, parabolic equations and the nonlinear Schrödinger equation. We hope that authors of similar results for other nonlinear equations will forgive the omission.

One purpose of this paper is to prove the existence of global solutions, including self-similar solutions, to the nonlinear Schrödinger equation (1.1) using norms analagous to those used by Cannone and Planchon. There is, however, a serious difficulty in applying these ideas to (1.1). Besov norms for data $\varphi(x)$ are equivalent to weighted norms of the type $|||\varphi||| = \sup\limits_{t>0} t^\beta \|e^{t\triangle}\varphi\|_{L^p}$, and it is these latter norms which are well suited to prove global existence. For (1.1), as we shall see, it is relatively easy to prove global existence of solutions with data φ which are small with respect to a certain norm $|||\varphi||| = \sup_{t>0} t^{\beta}\|S(t)\varphi\|_{L^p}$, where instead of the heat semigroup $e^{t\triangle}$, the Schrödinger group $S(t)$ is used. Since these norms do not seem to have any well known equivalent forms, it is more difficult to determine which functions are finite with respect to this norm. Indeed, to obtain self-similar solutions, one needs homogeneous initial data. A significant portion of this paper is therefore devoted to explicitly calculating $S(t)\varphi$ for functions of the type $\varphi(x) = |x|^{-p}$.

Once self-similar and other global solutions are proved to exist by this method, it is important to compare them to solutions in $H^1(\mathbb{R}^N)$, known to exist since the work of Ginibre and Velo [16]. It turns out, at least for α in a certain range, that a class of global H^1 solutions are asymptotically self-similar. In other words, the difference between such a solution and one of the self-similar solutions constructed below tends to zero in $L^{\alpha+2}(\mathbb{R}^N)$ more rapidly than either of them do separately.

Many of the results in this paper concerning equation (1.1) are valid for the range of α given by

$$\alpha_0 < \alpha < \frac{4}{N-2}, \tag{1.3}$$

where α_0 is the positive root of the equation

$$N\alpha^2 + (N-2)\alpha - 4 = 0.$$

Throughout the entire paper, the hypothesis (1.3) is to be interpreted as $\alpha_0 < \alpha < \infty$ if $N = 1$ or 2.

The power α_0 in the study of equation (1.1) was first encountered by Strauss [31]. One way to understand α_0 is that it is the value of α for which $\dfrac{N\alpha}{2} = \dfrac{\alpha+2}{\alpha+1}$. The equality of these two numbers is "significant" since the $L^{\frac{N\alpha}{2}}(\mathbb{R}^N)$ norm is invariant under the dilations $\varphi_\lambda(x) = \lambda^{\frac{2}{\alpha}}\varphi(\lambda x)$ mentioned above, and the norm $L^{\frac{\alpha+2}{\alpha+1}}(\mathbb{R}^N)$ is the dual of the norm $L^{\alpha+2}(\mathbb{R}^N)$, which appears in the energy of (regular) solutions of (1.1). Also, $L^{\frac{\alpha+2}{\alpha+1}}(\mathbb{R}^N)$ is the image of $L^{\alpha+2}(\mathbb{R}^N)$ under the nonlinear map $u \mapsto |u|^\alpha u$. In fact, the condition (1.3) is equivalent to the condition

$$\frac{\alpha+2}{\alpha+1} < \frac{N\alpha}{2} < \alpha + 2.$$

(If the reader finds this explanation of α_0 a bit far fetched, we invite him/her to find a better one.)

The Schrödinger group $S(t) = e^{it\triangle}$ and the heat semigroup $e^{t\triangle}$ are both part of the analytic semigroup defined by convolution $e^{z\triangle}\varphi = G_z \star \varphi$, where

$$G_z(x) = (4\pi z)^{-\frac{N}{2}} e^{-\frac{|x|^2}{4z}},$$

is defined for all nonzero complex numbers z with $\operatorname{Re} z \geq 0$. Equivalently, these operators can be defined via the Fourier transform as

$$e^{z\triangle}\varphi = \mathcal{F}^{-1}\left(e^{-4z\pi^2|\cdot|^2}\mathcal{F}\varphi\right),$$

for all complex z with $\operatorname{Re} z \geq 0$. It follows that for each $t \in \mathbb{R}$, $S(t)$ is a continuous map on the space of tempered distributions $\mathcal{S}'(\mathbb{R}^N)$ and that for each $\varphi \in \mathcal{S}'(\mathbb{R}^N)$, the map $t \mapsto S(t)\varphi$ is continuous from $\mathbb{R}$ into $\mathcal{S}'(\mathbb{R}^N)$.

Our approach to self-similar solutions of (1.1) is via the corresponding integral equation (2.1) below. Previous work [20, 25, 26] on self-similar solutions of (1.1) has been based on an analysis of the ordinary differential equation verified by the profile of the self-similar solution. The focus has been to study the asymptotic behavior (in space) of this profile in order to determine the regularity of the self-similar solution. Our work has a very different orientation and does not immediately recover these previous results.

Clearly, the methods we use here apply equally well, if not more easily, to the nonlinear heat equation corresponding to (1.1). While this has been done to some extent [29], it has not yet been proved that a class of global solutions are asymptotic to the self-similar solutions constructed by this method.

The outline of the rest of this paper is as follows. In the next section we prove the basic global existence theorem (Theorem 2.1) and show how it relates to some of the previously known results for equation (1.1). In Sect. 3 we study the action of the linear Schrödinger group $S(t)$ on homogeneous functions of the type $\varphi(x) = |x|^{-p}$ and $\varphi(x) = \omega(x)|x|^{-p}$, where $0 < \operatorname{Re} p < N$ and ω is homogeneous of degree 0. It turns out (for certain ω) that for all $t > 0$, $S(t)\varphi$ is C^∞ and belongs to $L^r(\mathbb{R}^N)$ for large r (Corollary 3.4 and Propositions 3.7 and 3.9). Moreover, we show in Sect. 4 (Proposition 4.3) that these functions can verify the hypotheses of Theorem 2.1, thereby giving rise to global, self-similar solutions of (1.1), both with and without radially symmetry. Also in Sect. 4, we show that certain H^1 solutions are asymptotically self-similar (Propositions 4.7 and 4.8). The H^1 initial values φ which lead to this behavior decay like $|x|^{-\frac{2}{\alpha}}$ as $|x| \to \infty$, where α is such that $\varphi \in H^1(\mathbb{R}^N)$, but $\varphi \notin L^{\frac{\alpha+2}{\alpha+1}}(\mathbb{R}^N)$), i.e. $\alpha_0 < \alpha < 4/N$. In Sect. 5, we establish an analogue of Theorem 2.1 for an equation related to (1.1) by the pseudo-conformal transformation. By constructing asymptotically self-similar H^1 solutions of that equation, and by applying the inverse pseudo-conformal transformation, we obtain solutions of (1.1) on $(-\infty, 0)$ which have an asymptotically self-similar blow up behavior at 0. However, these solutions of (1.1) are not H^1 solutions (see Theorem 5.7 and Remark 5.10 below). Finally, in Sect. 6, we outline how these methods can be applied to the nonlinear heat equation. In particular, if $\alpha > \dfrac{2}{N}$, we show that a large class of global solutions are asymptotically self-similar. As in the case of the nonlinear Schrödinger equation, the initial values giving rise to such behavior decay like $|x|^{-\frac{2}{\alpha}}$ as $|x| \to \infty$. These results differ from those of Escobedo and Kavian [7] and Escobedo, Kavian and Matano [8], who study asymptotically self-similar solutions of the nonlinear heat equation when $\alpha < \dfrac{2}{N}$. Finally, we mention that Planchon [28] has recently proved, by methods rather different from those in the current article, that a large class of solutions of the Navier-Stokes system are asymptotically self-similar.

2 Existence of global solutions

In this section we prove the existence of global solutions to the integral equation corresponding to (1.1), i.e.

$$u(t) = S(t)\varphi - i\gamma \int_0^t S(t-s)\left(|u(s)|^\alpha u(s)\right)\, ds, \tag{2.1}$$

where $S(t)$ is the linear Schrödinger group (Theorem 2.1 below). The proof uses only the most basic properties of the linear Schrödinger group $S(t)$, i.e.

$$\|S(t)\varphi\|_{L^{\alpha+2}} \leq |4\pi t|^{-\frac{N\alpha}{2(\alpha+2)}} \|\varphi\|_{L^{\frac{\alpha+2}{\alpha+1}}}.$$

The calculation used is not so different from the calculation near the end of Sect. 5 in [32]. Indeed, the simplicity of the proof is somewhat remarkable given the long history of very technical proofs of local and global existence of solutions of (1.1) and (2.1), [16, 17, 22, 3, 4]. The relationship of Theorem 2.1 to previous results will be explained below in Remark 2.2 and Proposition 2.3. We note that the results in this paper apply, as do the results in [6], to equation (2.1) with γ being of either sign.

Before stating Theorem 2.1, we need one more definition. If α verifies (1.3), let

$$\beta = \frac{4-(N-2)\alpha}{2\alpha(\alpha+2)}. \tag{2.2}$$

It follows easily that

$$\beta(\alpha+1) < 1, \quad \frac{N\alpha}{2(\alpha+2)} < 1, \tag{2.3}$$

and

$$\beta + 1 - \frac{N\alpha}{2(\alpha+2)} - \beta(\alpha+1) = 0. \tag{2.4}$$

One verifies that $\sup_{t>0} t^\beta \|S(t)\varphi\|_{L^{\alpha+2}}$ is invariant under the transformations $\varphi_\lambda(x) = \lambda^{\frac{2}{\alpha}}\varphi(\lambda x)$, (see formula (3.6) in Sect. 3), though this fact is not explicitly needed for the proof of Theorem 2.1.

Theorem 2.1. *Suppose (1.3), and let β be given by (2.2). Suppose further that $\rho > 0$ and $M > 0$ satisfy the inequality*

$$\rho + KM^{\alpha+1} \le M,$$

where $K = K(\alpha, N, \gamma)$ is given explicitly below by (2.12). Let φ be a tempered distribution such that

$$\sup_{t>0} |t|^\beta \|S(t)\varphi\|_{L^{\alpha+2}} \le \rho. \tag{2.5}$$

It follows that there exists a unique positively global (i.e. defined for all $t \ge 0$) solution u of (2.1) such that

$$\sup_{t>0} |t|^\beta \|u(t)\|_{L^{\alpha+2}} \le M. \tag{2.6}$$

Furthermore,

(a) $u(t) - S(t)\varphi \in C([0,\infty), H^{-\frac{N\alpha}{2(\alpha+2)}}(\mathbb{R}^N))$*;*
(b) $\lim_{t\downarrow 0} u(t) = \varphi$ *as tempered distributions.*

Suppose φ and ψ verify (2.5) and let u and v be respectively the solutions of (2.1) satisfying (2.6) with initial values φ and ψ. It follows that

$$\sup_{t>0} |t|^\beta \|u(t) - v(t)\|_{L^{\alpha+2}} \le (1 - KM^\alpha)^{-1} \sup_{t>0} |t|^\beta \|S(t)(\varphi - \psi)\|_{L^{\alpha+2}}.$$

If, in addition, $S(t)(\varphi - \psi)$ has the stronger decay property

$$\sup_{t>0} |t|^\beta (1+|t|)^\delta \|S(t)(\varphi - \psi)\|_{L^{\alpha+2}} < \infty, \tag{2.7}$$

for some $\delta > 0$ such that $\beta(\alpha+1) + \delta < 1$, and if $K'M^\alpha < 1$, where K' is given by (2.13) below, then

$$\begin{aligned} &\sup_{t>0} |t|^\beta (1+|t|)^\delta \|u(t) - v(t)\|_{L^{\alpha+2}} \\ &\quad \le (1 - K'M^\alpha)^{-1} \sup_{t>0} |t|^\beta (1+|t|)^\delta \|S(t)(\varphi - \psi)\|_{L^{\alpha+2}}. \end{aligned} \tag{2.8}$$

Remarks 2.2.

(a) The estimate (2.5), interpreted as a decay rate for large $|t|$, is slower than the decay rate of $\|S(t)\varphi\|_{L^{\alpha+2}}$ if $\varphi \in L^{\frac{\alpha+2}{\alpha+1}}(\mathbb{R}^N)$, i.e. $\beta < \dfrac{N\alpha}{2(\alpha+2)}$. The last part of the theorem imposes additional decay properties only on the difference $\varphi - \psi$, not on the initial values separately. As an example, suppose φ and ψ verify (2.5) and that $\varphi - \psi \in L^{\frac{\alpha+2}{\alpha+1}}(\mathbb{R}^N)$. It follows that

$$\|S(t)(\varphi - \psi)\|_{L^{\alpha+2}} \leq (4\pi t)^{-\frac{N\alpha}{2(\alpha+2)}} \|\varphi - \psi\|_{L^{\frac{\alpha+2}{\alpha+1}}}.$$

In particular, (2.7) is satisfied with $\delta = \delta_0 = \dfrac{N\alpha}{2(\alpha+2)} - \beta$. (In fact, $\delta_0 = \sigma$ in Proposition 2.3 (d).) One easily checks that

$$\beta(\alpha+1) + \delta_0 = \beta\alpha + (\beta + \delta_0) = 1.$$

Therefore, (2.7) and (2.8) hold for all δ with $0 \leq \delta < \delta_0$. In other words, for large t,

$$\|u(t) - v(t)\|_{L^{\alpha+2}} \leq C_\varepsilon t^{-\frac{N\alpha}{2(\alpha+2)}+\varepsilon}, \tag{2.9}$$

for all $\varepsilon > 0$.

(b) If $\sup\limits_{t<0} |t|^\beta \|S(t)\varphi\|_{L^{\alpha+2}} \leq \rho$, instead of (2.5), the same conclusions hold for negatively global solutions, i.e. solutions defined for all $t \leq 0$.

(c) If the estimate (2.5) is verified only on the finite interval $(0, T]$, respectively $[-T, 0)$, the proof of Theorem 2.1 will show that the same conclusions hold for solutions defined on $[0, T]$, respectively $[-T, 0]$. In particular, uniqueness is a local property. For example, if $\varphi \in H^1(\mathbb{R}^N)$, then for all $t \in \mathbb{R}$, $\|S(t)\varphi\|_{L^{\alpha+2}} \leq C\|S(t)\varphi\|_{H^1} = C\|\varphi\|_{H^1}$. Thus, for sufficiently small $T > 0$, $\sup\limits_{t\in[-T,T]} |t|^\beta \|S(t)\|_{L^{\alpha+2}} \leq \rho$, and so by Theorem 2.1, there exists a unique solution $u(t)$ of (2.1) on $[-T, T]$ such that

$$\sup_{t\in[-T,T]} |t|^\beta \|u(t)\|_{L^{\alpha+2}} \leq M.$$

On the other hand, since $\alpha < \dfrac{4}{N-2}$, there is a "classical" (in the sense that it is well known) solution $u_1(t)$ of (2.1),

$$u_1 \in C([-T_1, T_1], H^1(\mathbb{R}^N)),$$

for some $T_1 > 0$. Clearly then, (after possibly choosing $T > 0$ a bit smaller),

$$\sup_{t\in[-T,T]} |t|^\beta \|u_1(t)\|_{L^{\alpha+2}} \leq M.$$

It follows that the "weak" solution constructed by Theorem 2.1 coincides with the classical solution on $[-T,T]$. As another consequence of "local uniqueness", let φ verify (2.5) and let u be the (positively) global solution constructed by Theorem 2.1. In addition, suppose for some $t_0 > 0$ that $\psi = u(t_0)$ verifies $\sup\limits_{t\in[0,T]} t^\beta \|S(t)\psi\|_{L^{\alpha+2}} \le \rho$. (This is not a priori true since the only information we know from Theorem 2.1 is that $\psi = u(t_0) \in L^{\alpha+2}(\mathbb{R}^N)$.) Let $v(t)$ be the solution of (2.1) on $[0,T]$ with initial value ψ. It follows that $v(t) = u(t+t_0)$ for small $t \ge 0$ since $t^\beta\|u(t+t_0)\|_{L^{\alpha+2}} \le M$ for sufficiently small t. (Again, a similar remark holds if $\sup\limits_{t\in[-T,0]} |t|^\beta \|S(t)\psi\|_{L^{\alpha+2}} \le \rho$.)

(d) Since $\alpha < \dfrac{4}{N-2}$, it follows that $\dfrac{N\alpha}{2(\alpha+2)} < 1$, and so property (a) after (2.6) implies that $u(t) - S(t)\varphi \in C([0,\infty), H^{-1}(\mathbb{R}^N))$.

(e) Theorem 2.1, while providing new cases where global solutions of (2.1) exist, does not include most of the standard results of local and global existence. First of all, the standard H^1 theory is valid for $0 < \alpha < \dfrac{4}{N-2}$, without the lower limit α_0 . Also, while Theorem 2.1 does imply local existence and uniqueness for $\varphi \in H^1(\mathbb{R}^N)$ (remark (c) above), there is no mechanism for continuing solutions which verify a priori estimates. Thus, the known global existence results based on Sobolev inequalities and energy conservation do not follow from Theorem 2.1. Finally, Theorem 2.1 does not even include all the previous results of global existence based on smallness of a certain norm, as in [4], for example.

(f) The last part of Theorem 2.1 was inspired by some of the arguments of Kato [21].

Proof of Theorem 2.1. Let X be the set of Bochner measurable functions $u : (0,\infty) \to L^{\alpha+2}(\mathbb{R}^N)$ such that $\sup\limits_{t>0} t^\beta \|u(t)\|_{L^{\alpha+2}}$ is finite. We denote by X_M the set of $u \in X$ such that

$$\sup_{t>0} t^\beta \|u(t)\|_{L^{\alpha+2}} \le M.$$

Endowed with the metric, $d(u,v) = \sup\limits_{t>0} t^\beta \|u(t) - v(t)\|_{L^{\alpha+2}}$, X_M is a complete metric space. We will show that the mapping defined formally by

$$\mathcal{P}_\varphi u(t) = S(t)\varphi - i\gamma \int_0^t S(t-s)\left(|u(s)|^\alpha u(s)\right)\, ds, \tag{2.10}$$

is a strict contraction on X_M.

Note that if $\varphi \in \mathcal{S}'(\mathbb{R}^N)$, then $S(t)\varphi \in C(\mathbb{R}, \mathcal{S}'(\mathbb{R}^N))$. Therefore, if φ verifies (2.5), then $S(t)\varphi : (0,\infty) \to L^{\alpha+2}(\mathbb{R}^N)$ is weakly continuous, hence Bochner measurable.

We observe first that if $u \in X$, then by the Sobolev embedding theorem

$$\begin{aligned}
\|S(t-s)\left(|u(s)|^\alpha u(s)\right)\|_{H^{-\frac{N\alpha}{2(\alpha+2)}}} &= \|\,|u(s)|^\alpha u(s)\|_{H^{-\frac{N\alpha}{2(\alpha+2)}}} \\
\leq C\|\,|u(s)|^\alpha u(s)\|_{L^{\frac{\alpha+2}{\alpha+1}}} &= C\|u(s)\|_{L^{\alpha+2}}^{\alpha+1} \\
\leq C s^{-\beta(\alpha+1)}
\end{aligned}$$

It follows from (2.3) that

$$\int_0^t S(t-s)(|u(s)|^\alpha u(s))\,ds$$

is in $C([0,\infty); H^{-\frac{N\alpha}{2(\alpha+2)}}(\mathbb{R}^N))$, taking on the value 0 at $t=0$. Thus, the right hand side of (2.10) can be interpreted as a continuous function into the space of tempered distributions, with initial value φ.

Next, suppose that φ and ψ verify (2.5) and that u and v are in X_M. It follows that

$$\begin{gathered}
t^\beta \|\mathcal{P}_\varphi u(t) - \mathcal{P}_\psi v(t)\|_{L^{\alpha+2}} \leq \\
t^\beta \|S(t)\varphi - S(t)\psi\|_{L^{\alpha+2}} + \\
t^\beta |\gamma| \int_0^t \|S(t-s)\left(|u(s)|^\alpha u(s) - |v(s)|^\alpha v(s)\right)\|_{L^{\alpha+2}}\,ds.
\end{gathered}$$

Since

$$\begin{aligned}
&\|S(t-s)\left(|u(s)|^\alpha u(s) - |v(s)|^\alpha v(s)\right)\|_{L^{\alpha+2}} \\
&\leq (4\pi(t-s))^{-\frac{N\alpha}{2(\alpha+2)}} \|\,|u(s)|^\alpha u(s) - |v(s)|^\alpha v(s)\|_{L^{\frac{\alpha+2}{\alpha+1}}} \\
&\leq (\alpha+1)(4\pi(t-s))^{-\frac{N\alpha}{2(\alpha+2)}} \\
&\qquad \left(\|u(s)\|_{L^{\alpha+2}}^\alpha + \|v(s)\|_{L^{\alpha+2}}^\alpha\right) \|u(s) - v(s)\|_{L^{\alpha+2}} \\
&\leq 2(\alpha+1)(4\pi(t-s))^{-\frac{N\alpha}{2(\alpha+2)}} s^{-\beta(\alpha+1)} M^\alpha d(u,v),
\end{aligned}$$

we obtain

$$t^\beta \|\mathcal{P}_\varphi u(t) - \mathcal{P}_\psi v(t)\|_{L^{\alpha+2}} \leq t^\beta \|S(t)\varphi - S(t)\psi\|_{L^{\alpha+2}} + K M^\alpha d(u,v), \tag{2.11}$$

where

$$\begin{aligned}
K &= K(\alpha, N, \gamma) \\
&= 2|\gamma|(\alpha+1)(4\pi)^{-\frac{N\alpha}{2(\alpha+2)}} B\left(1 - \frac{N\alpha}{2(\alpha+2)}, 1 - \beta(\alpha+1)\right),
\end{aligned} \tag{2.12}$$

where $B(\cdot,\cdot)$ is the beta function. (Properties (2.3) and (2.4) have been used in the above calculation.)

Setting $\psi = 0$ in (2.11), we see that

$$\sup_{t>0} t^{\beta}\|\mathcal{P}_{\varphi}u(t)\|_{L^{\alpha+2}} < \rho + KM^{\alpha+1} \leq M.$$

Thus, $\mathcal{P}_{\varphi}$ maps X_M into itself. Next, setting $\psi = \varphi$ in (2.11), we see that

$$d(\mathcal{P}_{\varphi}u - \mathcal{P}_{\varphi}v) < KM^{\alpha}d(u,v).$$

Since $KM^{\alpha} < 1$, it follows that $\mathcal{P}_{\varphi}$ is a strict contraction on X_M , and so has a unique fixed point. This proves the first part of the theorem, including statements (a) and (b).

To prove the continuous dependence result, it suffices to observe that (2.11) implies

$$d(u,v) \leq \sup_{t>0} t^{\beta}\|S(t)\varphi - S(t)\psi\|_{L^{\alpha+2}} + KM^{\alpha}d(u,v).$$

To prove the stronger decay estimate, we modify the calculation (2.11) as follows:

$$\begin{aligned}t^{\beta}(1+t)^{\delta}\|u(t)-v(t)\|_{L^{\alpha+2}} &\leq t^{\beta}(1+t)^{\delta}\|S(t)\varphi - S(t)\psi\|_{L^{\alpha+2}} + \\ &t^{\beta}(1+t)^{\delta}|\gamma|\int_0^t \|S(t-s)\left(|u(s)|^{\alpha}u(s) - |v(s)|^{\alpha}v(s)\right)\|_{L^{\alpha+2}}ds.\end{aligned}$$

Since

$$\begin{aligned}&(1+t)^{\delta}\|S(t-s)\left(|u(s)|^{\alpha}u(s) - |v(s)^{\alpha}v(s)|\right)\|_{L^{\alpha+2}} \leq \\ &2(\alpha+1)\left(\frac{1+t}{1+s}\right)^{\delta}(4\pi(t-s))^{-\frac{N\alpha}{2(\alpha+2)}}s^{-\beta(\alpha+1)}M^{\alpha} \\ &\sup_{0<\tau\leq t}\tau^{\beta}(1+\tau)^{\delta}\|u(\tau)-v(\tau)\|_{L^{\alpha+2}},\end{aligned}$$

and $\left(\frac{1+t}{1+s}\right)^{\delta} \leq \left(\frac{t}{s}\right)^{\delta}$, we deduce

$$\begin{aligned}&t^{\beta}(1+t)^{\delta}\|u(t)-v(t)\|_{L^{\alpha+2}} \\ &\leq t^{\beta}(1+t)^{\delta}\|S(t)\varphi - S(t)\psi\|_{L^{\alpha+2}} + \\ &K'M^{\alpha}\sup_{0<\tau\leq t}\tau^{\beta}(1+\tau)^{\delta}\|u(\tau)-v(\tau)\|_{L^{\alpha+2}},\end{aligned}$$

where

$$\begin{aligned}K' &= K'(\alpha,N,\gamma,\delta) \qquad (2.13) \\ &= 2|\gamma|(\alpha+1)(4\pi)^{-\frac{N\alpha}{2(\alpha+2)}}B\left(1-\frac{N\alpha}{2(\alpha+2)}, 1-\beta(\alpha+1)-\delta\right).\end{aligned}$$

This completes the proof. $\square$

Proposition 2.3. *Let α, β, ρ and M be as in the statement of Theorem 2.1.*

(a) *If ξ is a tempered distribution such that $\limsup_{t\downarrow 0} t^\beta \|S(t)\xi\|_{L^{\alpha+2}} < \rho$, then $\varphi = e^{\frac{ic|x|^2}{4}}\xi$ verifies (2.5) for all sufficiently large $c > 0$. This applies in particular if $\xi \in H^1(\mathbb{R}^N)$.*

(b) *Let $\xi \in L^{\frac{\alpha+2}{\alpha+1}}(\mathbb{R}^N)$. There exists $T > 0$ such that if $\tau > T$ and $\varphi = S(\tau)\xi$, then φ verifies (2.5).*

(c) *If φ and its Fourier transform are in $L^{\frac{\alpha+2}{\alpha+1}}(\mathbb{R}^N)$, then $\sup_{t\in\mathbb{R}} |t|^\beta \|S(t)\varphi\|_{L^{\alpha+2}} < \infty$. In particular, a sufficiently small multiple of φ satisfies (2.5), as well as (2.5) with $t > 0$ replaced by $t < 0$.*

(d) *If $(-\triangle)^{\frac{\sigma}{2}}\varphi \in L^q(\mathbb{R}^N)$, where*

$$\sigma = \frac{N\alpha^2 + (N-2)\alpha - 4}{2\alpha(\alpha+2)} \quad \textit{and} \quad \frac{N}{q} = \frac{N\alpha^2 + (N+2)\alpha + 4}{2\alpha(\alpha+2)},$$

then $\sup_{t\in\mathbb{R}} |t|^\beta \|S(t)\varphi\|_{L^{\alpha+2}} \le C\|(-\triangle)^{\frac{\sigma}{2}}\varphi\|_{L^q}$. In particular, if $\|(-\triangle)^{\frac{\sigma}{2}}\varphi\|_{L^q}$ is sufficiently small, then φ satisfies (2.5), as well as (2.5) with $t > 0$ replaced by $t < 0$. (Remark: as is the norm $\|\varphi\|_{L^{\frac{N\alpha}{2}}}$, the norm $\|(-\triangle)^{\frac{\sigma}{2}}\varphi\|_{L^q}$ is invariant under the dilations $\varphi_\lambda(x) = \lambda^{\frac{2}{\alpha}}\varphi(\lambda x)$. Moreover, $\|\varphi\|_{L^{\frac{N\alpha}{2}}} \le C\|(-\triangle)^{\frac{\sigma}{2}}\varphi\|_{L^q}$ since $\dfrac{1}{q} - \dfrac{2}{N\alpha} = \dfrac{\sigma}{N}$.)

(e) *If the tempered distribution φ is such that $\limsup_{t\downarrow 0} t^\beta \|S(t)\varphi\|_{L^{\alpha+2}} < \infty$, then $\varphi \in W^{-2,\alpha+2}(\mathbb{R}^N)$.*

(f) *Suppose $\varphi \in H^1(\mathbb{R}^N)$ satisfies (2.5). The solution of (2.1) constructed in Theorem 2.1 coincides for all $t \ge 0$ with the "classical" H^1 solution, which is therefore global.*

(g) *Let φ be a tempered distribution verifying (2.5), and let $u(t)$ be the solution of equation (2.1) constructed in Theorem 2.1. If $u(t_0) \in H^1(\mathbb{R}^N)$ for some $t_0 > 0$, then the same is true for all $t > 0$ and u is a "classical" H^1 solution on $(0,\infty)$. In particular, $\varphi \in L^2(\mathbb{R}^N)$. Finally, if $\limsup_{t\downarrow 0} t^\beta \|u(t)\|_{L^{\alpha+2}}$ is sufficiently small, then $\varphi \in H^1(\mathbb{R}^N)$. (If $\alpha < \dfrac{4}{N}$ or if $\gamma \ge 0$, this is always verified.)*

Proof. (a) One can easily check, using the explicit representation of $S(t)$ as a convolution operator, that

$$[S(t)\varphi](x) = (1+ct)^{-\frac{N}{2}} e^{\frac{ic|x|^2}{4(1+ct)}} \left[S\left(\frac{t}{1+ct}\right)\xi\right]\left(\frac{x}{1+ct}\right).$$

Setting $\tau = \dfrac{t}{1+ct}$, one readily computes that

$$\sup_{t>0} t^\beta \|S(t)\varphi\|_{L^{\alpha+2}} = \sup_{0<\tau<\frac{1}{c}} (1-c\tau)^\sigma \tau^\beta \|S(\tau)\xi\|_{L^{\alpha+2}},$$

where

$$\sigma = \frac{N\alpha^2 + (N-2)\alpha - 4}{2\alpha(\alpha+2)} > 0,$$

(positive since $\alpha > \alpha_0$). This proves (a).

(b) This is a simple consequence of

$$t^\beta \|S(t)\varphi\|_{L^{\alpha+2}} = t^\beta \|S(t+\tau)\xi\|_{L^{\alpha+2}} \leq Ct^\beta (t+\tau)^{-\frac{N\alpha}{2(\alpha+2)}} \|\xi\|_{L^{\frac{\alpha+2}{\alpha+1}}},$$

and the fact that $\beta < \dfrac{N\alpha}{2(\alpha+2)}$. (Statements (a) and (b) are analogous to Corollary 2.5 in [6].)

(c) Since $\varphi \in L^{\frac{\alpha+2}{\alpha+1}}(\mathbb{R}^N)$, it follows that

$$\|S(t)\varphi\|_{L^{\alpha+2}} \leq |4\pi t|^{-\frac{N\alpha}{2(\alpha+2)}} \|\varphi\|_{L^{\frac{\alpha+2}{\alpha+1}}}.$$

Since $\mathcal{F}\varphi \in L^{\frac{\alpha+2}{\alpha+1}}(\mathbb{R}^N)$, it follows that

$$\begin{aligned} \|S(t)\varphi\|_{L^{\alpha+2}} &= \left\| \mathcal{F}^{-1} \left(e^{-4it\pi^2|\cdot|^2} \mathcal{F}\varphi \right) \right\|_{L^{\alpha+2}} \\ &\leq C\|e^{-4it\pi^2|\cdot|^2} \mathcal{F}\varphi\|_{L^{\frac{\alpha+2}{\alpha+1}}} = C\|\mathcal{F}\varphi\|_{L^{\frac{\alpha+2}{\alpha+1}}}. \end{aligned}$$

The result (c) now easily follows since $0 < \beta < \dfrac{N\alpha}{2(\alpha+2)}$.

(d) The indices σ and q verify

$$\frac{N}{q} - \frac{N}{2} = \beta, \quad \frac{1}{q'} - \frac{1}{\alpha+2} = \frac{\sigma}{N}, \quad \frac{\alpha+2}{\alpha+1} < q < 2, \quad 2 < q' < \alpha+2.$$

Therefore,

$$\begin{aligned} \|S(t)\varphi\|_{L^{\alpha+2}} &= \|(-\triangle)^{-\frac{\sigma}{2}} S(t)(-\triangle)^{\frac{\sigma}{2}}\varphi\|_{L^{\alpha+2}} \\ &\leq C\|S(t)(-\triangle)^{\frac{\sigma}{2}}\varphi\|_{L^{q'}} \leq C|t|^{-\beta}\|(-\triangle)^{\frac{\sigma}{2}}\varphi\|_{L^q}, \end{aligned}$$

which establishes the result.

(e) Using the Fourier transform representation of $S(t)$, one readily verifies that

$$i\triangle \int_0^t S(\tau)\varphi\, d\tau = S(t)\varphi - \varphi.$$

Since $\beta < 1$, the assumption $\sup_{t \downarrow 0} t^\beta \|S(t)\varphi\|_{L^{\alpha+2}} < \infty$ implies that

$$\psi = \int_0^t S(\tau)\varphi \, d\tau \in L^{\alpha+2}(\mathbb{R}^N),$$

for $t > 0$ sufficiently small. Thus, $\varphi = S(t)\varphi - i\triangle\psi \in W^{-2,\alpha+2}(\mathbb{R}^N)$.

(f) Suppose $\varphi \in H^1(\mathbb{R}^N)$. Let $I \subset [0,\infty)$ be the largest interval containing 0 such that the two solutions coincide on I. The arguments in Remark 2.2 (c) show that I is nontrivial and that I is an open subset of $[0,\infty)$. Thus, I is of the form $[0,T^*)$. Suppose that $T^* < \infty$. Conservation of energy (for the classical solution) and the estimate (2.6) imply that $\|\nabla u(t)\|_{L^2}$ remains bounded as $t \uparrow T^*$. Thus, the classical solution can be continued up to and past T^*. Since both the classical solution and the solution constructed by Theorem 2.1 are in $C([0,T^*], H^{-\frac{N\alpha}{2(\alpha+2)}}(\mathbb{R}^N))$, they must agree at T^*, i.e. $T^* \in I$. This contradiction proves that $I = [0,\infty)$.

(g) That $u(t)$ is a classical H^1 solution on $(0,\infty)$ follows from Remark 2.2 (c) and conservation of energy as in the preceding argument. Since $\|u(t)\|_{L^2}$ is a constant for H^1 solutions, it follows that $\varphi \in L^2(\mathbb{R}^N)$. By Theorem 1.1 in [4], there exists $C > 0$ such that if $\varphi \notin H^1(\mathbb{R}^N)$, then $t^{\frac{1}{\alpha} - \frac{N-2}{4}} \|\nabla u(t)\|_{L^2} \geq C$ for small $t > 0$. Energy conservation then implies that $t^\beta \|u(t)\|_{L^{\alpha+2}} \geq C_1$, which completes the proof. □

3 Homogeneous data and the linear Schrödinger group

In this section we study the action of the linear Schrödinger group $S(t)$ on homogeneous functions. The main results, given in Corollary 3.4 and Propositions 3.7 and 3.9, are that for a wide class of homogeneous functions φ, $S(t)\varphi$ (for all $t > 0$) is in $C^\infty(\mathbb{R}^N)$ and belongs to $L^r(\mathbb{R}^N)$ for large enough r. This is true in spite of the fact that φ itself belongs to no Lebesgue space on $\mathbb{R}^N$. The proofs of these results depend on explicit calculations with the gamma function and analytic continuation arguments.

We begin by establishing notation, recalling some well-known facts, and making some definitions. The gamma function satisfies the following relation

$$c^{-z}\Gamma(z) = \int_0^\infty e^{-ct} t^{z-1} \, dt, \tag{3.1}$$

valid for $c > 0$ and $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$. Also, if Ω denotes the domain of the standard branch of the logarithm, i.e.

$$\Omega = \{z \in \mathbb{C};\ z \text{ is not a negative real number or } 0\},$$

then for a fixed complex number p, the function $f(z) = z^p = e^{p \log z}$ is analytic in Ω. Note that if $r > 0$, then $(rz)^p = r^p z^p$ for all $z \in \Omega$. Also, $|r^p| = r^{\mathrm{Re}\, p}$ if $r > 0$.

Another function that plays a central role in the analysis is given by

$$H(y; a, b) = \int_0^1 e^{iyr} r^{a-1} (1-r)^{b-1}\, dr, \tag{3.2}$$

where $a, b \in \mathbb{C}$ with $\mathrm{Re}\, a > 0$ and $\mathrm{Re}\, b > 0$, and $y \in \mathbb{R}$ (or $\mathbb{C}$). Note that $H(y; a, b)$ is separately analytic as a function of y, a, and b in the domains just specified. In addition, if $y \in \mathbb{R}$, then

$$|H(y; a, b)| \le H(0; \mathrm{Re}\, a, \mathrm{Re}\, b) = B(\mathrm{Re}\, a, \mathrm{Re}\, b) = \frac{\Gamma(\mathrm{Re}\, a)\Gamma(\mathrm{Re}\, b)}{\Gamma(\mathrm{Re}\,(a+b))}, \tag{3.3}$$

where $B(\cdot, \cdot)$ is the beta function.

To fix notation, we let $D_\lambda = D_{\lambda,p}$ be the dilation operator

$$D_\lambda \varphi(x) = D_{\lambda,p} \varphi(x) = \lambda^p \varphi(\lambda x). \tag{3.4}$$

where $\lambda > 0$ and p is a fixed complex power such that $0 < \mathrm{Re}\, p < N$. It is easy to check that:

$$\|D_{\lambda,p}\varphi\|_{L^r} = \lambda^{\mathrm{Re}\, p - \frac{N}{r}} \|\varphi\|_{L^r},$$

and that

$$S(t) = D_\lambda S(\lambda^2 t) D_{\frac{1}{\lambda}}, \tag{3.5}$$

$$t^{\frac{\mathrm{Re}\, p}{2} - \frac{N}{2r}} \|S(t) D_\lambda \varphi\|_{L^r} = (\lambda^2 t)^{\frac{\mathrm{Re}\, p}{2} - \frac{N}{2r}} \|S(\lambda^2 t)\varphi\|_{L^r}, \tag{3.6}$$

for all $\lambda > 0$.

We are interested in studying the action of the Schrödinger group $S(t)$ on functions φ which are fixed by D_λ i.e. such that $D_\lambda \varphi(x) = D_{\lambda,p}\varphi(x) = \varphi(x)$ for all $\lambda > 0$. Such functions will be called *p-homogeneous*, (rather than "homogeneous of degree $-p$" since p is not necessarily real). For example, $\varphi(x) = |x|^{-p}$ is a p-homogeneous function, as are all functions of the form $\varphi(x) = \omega(x)|x|^{-p}$, where $\omega(x)$ is homogeneous of degree 0. If φ is p-homogeneous, we see that

$$S(t)\varphi = D_{\frac{1}{\sqrt{t}}} S(1)\varphi, \tag{3.7}$$

for all $t > 0$. If in addition $S(1)\varphi \in L^r(\mathbb{R}^N)$ for some r, then the same is true for $S(t)\varphi$ for all $t > 0$ and

$$\|S(t)\varphi\|_{L^r} = t^{\frac{N}{2r} - \frac{\mathrm{Re}\, p}{2}} \|S(1)\varphi\|_{L^r}. \tag{3.8}$$

It turns out that $\|S(1)\varphi\|_{L^r}$ is finite for a large class of p-homogeneous functions φ, for large enough values of r. This motivates the following definition.

Definition 3.1. *The p-homogeneous function φ, where $0 < \operatorname{Re} p < N$, is SC-regular if $S(t)\varphi \in L^r(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ for all $t > 0$ and all r such that*

$$r > \max\left\{\frac{N}{\operatorname{Re} p}, \frac{N}{N - \operatorname{Re} p}\right\}.$$

For such r, $\|S(t)\varphi\|_{L^r}$ clearly verifies formula (3.8). If in addition, $S(t)\varphi \in C^\infty(\mathbb{R}^N)$ for all $t > 0$, then φ is SC$^\infty$-regular.

Remark 3.2. If a p-homogeneous function φ is SC$^\infty$-regular, then it follows from formula (3.7) that $S(t)\varphi \in C^\infty((0,\infty) \times \mathbb{R}^N)$.

The main results of this section are that $|x|^{-p}$, and more generally $P_k(x)|x|^{-p-k}$, $0 < \operatorname{Re} p < N$, are SC$^\infty$-regular p-homogeneous functions, where P_k is a homogeneous harmonic polynomial of degree k. (See Corollary 3.4 and Propositions 3.7 and 3.9 below.) It follows that the set of homogeneous functions $\omega(x)$ of degree 0 such that $\varphi(x) = \omega(x)|x|^{-p}$ is an SC$^\infty$-regular p-homogeneous function is dense as a subset of $L^2(S^{N-1})$. Indeed, they include all linear combination of functions of the form $P_k(x)|x|^{-k}$, where P_k is a homogeneous harmonic polynomial of degree k. At the end of this section we give some results concerning the action of $S(t)$ separately on the part of $|x|^{-p}$ near the origin and on the part of $|x|^{-p}$ "near infinity."

Our analysis begins with the simplest p-homogeneous function.

Proposition 3.3. *Let $\varphi(x) = |x|^{-p}$ where $0 < \operatorname{Re} p < N$. For $t > 0$ and $x \in \mathbb{R}^N$,*

$$[S(t)\varphi](x) = (4it)^{-\frac{p}{2}} \Gamma(p/2)^{-1} H\left(\frac{|x|^2}{4t}; \frac{p}{2}, \frac{N-p}{2}\right), \tag{3.9}$$

where the function H is defined by (3.2).

Proof. The basic idea is to express $|x|^{-p}$ using the gamma function, then change variables so that the Gauss kernel appears in the integral. It will then be possible to apply the operator $e^{z\triangle}$.

By formula (3.1), if $x \neq 0$

$$\begin{aligned}
|x|^{-p} &= \Gamma(p/2)^{-1} \textstyle\int_0^\infty e^{-|x|^2 t} t^{\frac{p}{2}-1}\, dt \\
&= 4^{-\frac{p}{2}} \Gamma(p/2)^{-1} \textstyle\int_0^\infty e^{-\frac{|x|^2}{4s}} s^{-\frac{p}{2}-1}\, ds \\
&= 4^{-\frac{p}{2}} (4\pi)^{\frac{N}{2}} \Gamma(p/2)^{-1} \textstyle\int_0^\infty G_s(x) s^{\frac{N}{2}-\frac{p}{2}-1}\, ds.
\end{aligned}$$

This integral, in addition to being absolutely convergent for each $x \neq 0$, is an absolutely convergent Bochner integral in $L^1(\mathbb{R}^N) + C_0(\mathbb{R}^N)$. In other words

$$\varphi = 4^{-\frac{p}{2}}(4\pi)^{\frac{N}{2}}\Gamma(p/2)^{-1}\int_0^\infty G_s(\cdot)s^{\frac{N}{2}-\frac{p}{2}-1}\,ds.$$

Next, we apply the heat semigroup, $e^{t\triangle}$ for $t>0$, which gives

$$e^{t\triangle}\varphi = 4^{-\frac{p}{2}}(4\pi)^{\frac{N}{2}}\Gamma(p/2)^{-1}\int_0^\infty G_{s+t}(\cdot)s^{\frac{N}{2}-\frac{p}{2}-1}\,ds.$$

This integral now is absolutely convergent in $C_0(\mathbb{R}^N)$, where pointwise evaluation is a bounded linear functional. Making the change of variables $r=\dfrac{t}{s+t}$, we see that for all $x\in\mathbb{R}^N$

$$\begin{aligned}(e^{t\triangle}\varphi)(x) &= 4^{-\frac{p}{2}}(4\pi)^{\frac{N}{2}}\Gamma(p/2)^{-1}\textstyle\int_0^1 G_{\frac{t}{r}}(x)\left(\frac{t-tr}{r}\right)^{\frac{N}{2}-\frac{p}{2}-1}\frac{t}{r^2}\,dr\\ &= (4t)^{-\frac{p}{2}}(4\pi t)^{\frac{N}{2}}\Gamma(p/2)^{-1}\textstyle\int_0^1 G_{\frac{t}{r}}(x)r^{\frac{p}{2}-\frac{N}{2}-1}(1-r)^{\frac{N}{2}-\frac{p}{2}-1}\,dr\\ &= (4t)^{-\frac{p}{2}}\Gamma(p/2)^{-1}\textstyle\int_0^1 e^{-\frac{r|x|^2}{4t}}r^{\frac{p}{2}-1}(1-r)^{\frac{N}{2}-\frac{p}{2}-1}\,dr.\end{aligned} \tag{3.10}$$

We next claim that formula (3.10) is valid not only for $t>0$, but for all $t\in\mathbb{C}$ with $\operatorname{Re}t>0$. Indeed, if $\eta\in\mathcal{S}(\mathbb{R}^N)$, then $\langle e^{t\triangle}\varphi,\eta\rangle$ is an analytic function of t on the open half plane $\operatorname{Re}t>0$, and continuous on the closed half plane $\operatorname{Re}t\ge 0$. Next, if we integrate the right side of (3.10) against $\eta(x)$ over $\mathbb{R}^N$, the result is also an analytic function of t on the right half plane $\operatorname{Re}t>0$, continuous at least on the closed half plane with $t=0$ removed. By the identity theorem, these two functions are equal on the open half plane. By continuity, they are equal also for $t=i\tau$, $\tau\in\mathbb{R}$, $\tau\ne 0$. Since η is an arbitrary Schwartz function, (3.10), as an identity between two tempered distributions, has been proved for all complex $t\ne 0$ with $\operatorname{Re}t\ge 0$. This establishes the proposition. □

Corollary 3.4. *Under the hypotheses of Proposition 3.3, it follows that*

(a) *$S(t)\varphi$ is C^∞ on $\mathbb{R}^N$ for all $t>0$;*
(b) *$S(t)\varphi\in L^\infty(\mathbb{R}^N)$ for all $t>0$, and $\|S(t)\varphi\|_{L^\infty}\le C(p)t^{-\frac{\operatorname{Re}p}{2}}$. If in addition $p\in\mathbb{R}$, then*

$$\|S(t)\varphi\|_{L^\infty} = |S(t)\varphi(0)| = (4t)^{-\frac{p}{2}}\frac{\Gamma\left(\frac{N-p}{2}\right)}{\Gamma\left(\frac{N}{2}\right)}.$$

Lemma 3.5. *If $y>0$, $\operatorname{Re}a>0$ and $\operatorname{Re}b>0$, and if n and m are nonnegative integers such that*

$$n+2>\operatorname{Re}a \text{ and } m+2>\operatorname{Re}b,$$

then

$$H(y;a,b)=y^{-a}\sum_{k=0}^{m}C_k(a,b)e^{\frac{(a+k)\pi i}{2}}y^{-k}+C_{m+1}(a,b)y^{-a-m-1}\times$$
$$\frac{m+1}{\Gamma(m+2-b)}\int_0^\infty\int_0^1(1-s)^m\left(-i-\frac{st}{y}\right)^{-a-m-1}ds\,e^{-t}t^{m+1-b}dt$$
$$+e^{iy}y^{-b}\sum_{k=0}^{n}C_k(b,a)e^{-\frac{(b+k)\pi i}{2}}y^{-k}+C_{n+1}(b,a)e^{iy}y^{-b-n-1}\times$$
$$\frac{n+1}{\Gamma(n+2-a)}\int_0^\infty\int_0^1(1-s)^n\left(i-\frac{st}{y}\right)^{-b-n-1}ds\,e^{-t}t^{n+1-a}\,dt, \quad (3.11)$$

where

$$\begin{aligned}C_k(a,b)&=\frac{\Gamma(a+k)}{k!}\frac{\Gamma(k+1-b)}{\Gamma(1-b)} \qquad (3.12)\\ &=\frac{\Gamma(a)}{k!}a(a+1)\cdots(a+k-1)(1-b)(2-b)\cdots(k-b).\end{aligned}$$

Remark 3.6. If b is a positive integer, then $C_k(a,b)=0$ for $k\geq b$, and the coefficient of the first integral term

$$\frac{C_{m+1}(a,b)}{\Gamma(m+2-b)}=\frac{\Gamma(a+m-1)}{(m+1)!\Gamma(1-b)}$$

is zero since $\Gamma(z)$ has poles at $0,-1,-2,\ldots$ Thus, the first part of the expansion has precisely b terms and no integral remainder. Similarly, if a is a positive integer, then the second part of the expansion has precisely a terms and no integral remainder.

Proof of Lemma 3.5. For the moment, we assume that

$$0<\operatorname{Re}a<1, \quad 0<\operatorname{Re}b<1.$$

Using formula (3.1) twice, first with $c=r$, $z=1-a$, and then with $c=1-r$, $z=1-b$, we rewrite formula (3.2) as follows.

$$\begin{aligned}&\Gamma(1-a)\Gamma(1-b)H(y;a,b)\\ &\quad=\textstyle\int_0^\infty\int_0^\infty\int_0^1 e^{iyr}e^{-rs}s^{-a}e^{-(1-r)t}t^{-b}\,dr\,ds\,dt\\ &\quad=\textstyle\int_0^\infty\int_0^\infty\int_0^1 e^{(iy-s+t)r}\,dr\,s^{-a}e^{-t}t^{-b}\,ds\,dt\\ &\quad=\textstyle\int_0^\infty\int_0^\infty \frac{e^{iy-s+t}-1}{iy-s+t}s^{-a}e^{-t}t^{-b}\,ds\,dt\\ &\quad=\textstyle\int_0^\infty\int_0^\infty \frac{s^{-a}}{-iy-t+s}e^{-t}t^{-b}\,ds\,dt+e^{iy}\int_0^\infty\int_0^\infty\frac{t^{-b}}{iy-s+t}e^{-s}s^{-a}\,dt\,ds\\ &\quad=\textstyle\int_0^\infty\int_0^\infty \frac{s^{-a}}{-iy-t+s}\,ds\,e^{-t}t^{-b}\,dt+e^{iy}\int_0^\infty\int_0^\infty\frac{s^{-b}}{iy-t+s}\,ds\,e^{-t}t^{-a}\,dt.\end{aligned}$$
$$(3.13)$$

We therefore consider the integral

$$f(w) = \int_0^\infty \frac{s^{-a}}{w+s}\, ds, \tag{3.14}$$

where $w \in \Omega$ (the domain of the standard branch of the logarithm), and $0 < \operatorname{Re} a < 1$. It is known (by changing variables in the beta function) that $f(1) = \Gamma(1-a)\Gamma(a)$. Next, if w is a positive real number, we set $s = wt$; and so

$$f(w) = \int_0^\infty \frac{(wt)^{-a}}{w+wt} w\, dt = w^{-a}\Gamma(1-a)\Gamma(a), \quad w > 0. \tag{3.15}$$

Since $f(w)$ and $w^{-a} = e^{-a\log w}$ are both holomorphic in Ω, (3.15) is true for all $w \in \Omega$. Substituting (3.15) back into (3.13), with $w = \pm iy - t$, we see that

$$H(y;a,b) = \\ \tfrac{\Gamma(a)}{\Gamma(1-b)} \int_0^\infty (-iy-t)^{-a} e^{-t} t^{-b}\, dt + \tfrac{\Gamma(b)}{\Gamma(1-a)} e^{iy} \int_0^\infty (iy-t)^{-b} e^{-t} t^{-a}\, dt. \tag{3.16}$$

The next step is to replace $(-iy-t)^{-a}$ and $(iy-t)^{-b}$ in (3.16) by their finite Taylor formulas around $t = 0$ with integral remainder terms. If $f(t) = (-iy-t)^{-a}$ and $g(t) = (iy-t)^{-b}$, then

$$f^{(k)}(t) = a(a+1)\cdots(a+k-1)(-iy-t)^{-a-k},$$
$$g^{(k)}(t) = b(b+1)\cdots(b+k-1)(-iy-t)^{-b-k}.$$

Since,

$$f(t) = \sum_{k=0}^{m} \frac{1}{k!} f^{(k)}(0) t^k + \frac{1}{m!} \int_0^t (t-s)^m f^{(m+1)}(s)\, ds,$$

and similarly for $g(t)$, we see that

$$\begin{aligned} H(y;a,b) &= \frac{\Gamma(a)}{\Gamma(1-b)} \sum_{k=0}^{m} \frac{f^{(k)}(0)}{k!} \int_0^\infty e^{-t} t^{-b+k}\, dt \\ &+ \frac{\Gamma(a)}{\Gamma(1-b)} \int_0^\infty \frac{1}{m!} \int_0^t (t-s)^m f^{(m+1)}(s)\, ds\, e^{-t} t^{-b}\, dt \\ &+ \frac{\Gamma(b)}{\Gamma(1-a)} e^{iy} \sum_{k=0}^{n} \frac{g^{(k)}(0)}{k!} \int_0^\infty e^{-t} t^{-a+k}\, dt \\ &+ \frac{\Gamma(b)}{\Gamma(1-a)} e^{iy} \int_0^\infty \frac{1}{n!} \int_0^t (t-s)^n g^{(n+1)}(s)\, ds\, e^{-t} t^{-a}\, dt; \end{aligned}$$

and so

$$
\begin{aligned}
&H(y;a,b) = \\
&\quad \frac{\Gamma(a)}{\Gamma(1-b)} \sum_{k=0}^{m} \frac{a(a+1)\cdots(a+k-1)}{k!}(-iy)^{-a-k}\Gamma(k+1-b) \\
&\qquad\qquad + \frac{\Gamma(a)}{\Gamma(1-b)} \int_0^\infty \frac{1}{m!} \int_0^t (t-s)^m a(a+1)\cdots \\
&\quad (a+m)(-iy-s)^{-a-m-1}\, ds\, e^{-t} t^{-b}\, dt \\
&\qquad + \frac{\Gamma(b)}{\Gamma(1-a)} e^{iy} \sum_{k=0}^{n} \frac{b(b+1)\cdots(b+k-1)}{k!}(iy)^{-b-k}\Gamma(k+1-a) \\
&\qquad\quad + \frac{\Gamma(b)}{\Gamma(1-a)} e^{iy} \int_0^\infty \frac{1}{n!} \int_0^t (t-s)^n b(b+1)\cdots \\
&\qquad\qquad (b+n)(iy-s)^{-b-n-1}\, ds\, e^{-t} t^{-a}\, dt.
\end{aligned}
$$

Furthermore, since $y > 0$,

$$
\begin{aligned}
&\int_0^\infty \int_0^t (t-s)^m (-iy-s)^{-a-m-1}\, ds\, e^{-t} t^{-b}\, dt \\
&= y^{-a-m-1} \int_0^\infty \int_0^1 (1-s)^m \left(-i - \frac{st}{y}\right)^{-a-m-1} ds\, e^{-t} t^{m+1-b}\, dt;
\end{aligned}
$$

and so we obtain the formulation (3.11)–(3.12).

Formula (3.11) has been proved only for $y > 0$, $0 < \operatorname{Re} a < 1$, and $0 < \operatorname{Re} b < 1$. On the other hand, the right hand side is an analytic function in a for $0 < \operatorname{Re} a < n+2$, with $y > 0$ and b ($0 < \operatorname{Re} b < m+2$) fixed, and also an analytic function in b for $0 < \operatorname{Re} b < m+2$, with $y > 0$ and a ($0 < \operatorname{Re} a < n+2$) fixed. (Recall that the $1/\Gamma(z)$ is an entire function.) It follows that (3.11) is true for all $y > 0$, and all a and b in the region stated in the lemma. □

Proposition 3.7. *Let $\varphi(x) = |x|^{-p}$ where $0 < \operatorname{Re} p < N$. $S(t)\varphi \in L^r(\mathbb{R}^N)$ for all $t > 0$ and all r such that*

$$
r > \max\left\{\frac{N}{\operatorname{Re} p}, \frac{N}{N - \operatorname{Re} p}\right\};
$$

and $\|S(t)\varphi\|_{L^r}$ verifies formula (3.8). Moreover, $S(t)\varphi(x)$ is given by the explicit formula (3.17) below for $x \neq 0$.

Remark 3.8. This result along with Corollary 3.4 shows that $\varphi(x) = |x|^{-p}$ where $0 < \operatorname{Re} p < N$ is an SC$^\infty$-regular p-homogeneous function.

Proof. Applying the asymptotic expression from Lemma 3.5 to the formula (3.9) in Proposition 3.3, we see that, under the combined hypotheses of the proposition and the lemma, with $a = p/2$ and $b = (N-p)/2$, if $\tau > 0$ and $x \neq 0$, then (still denoting $\varphi(x) = |x|^{-p}$)

$$\begin{aligned}
[S(\tau)\varphi](x) = |x|^{-p} \sum_{k=0}^{m} A_k(a,b) e^{\frac{k\pi i}{2}} \left(\frac{|x|^2}{4\tau}\right)^{-k} + \\
A_{m+1}(a,b)|x|^{-p} \left(\frac{|x|^2}{4\tau}\right)^{-m-1} \frac{(m+1)e^{-\frac{a\pi i}{2}}}{\Gamma(m+2-b)} \times \\
\int_0^\infty \int_0^1 (1-s)^m \left(-i - \frac{4\tau st}{|x|^2}\right)^{-a-m-1} ds\, e^{-t} t^{m+1-b}\, dt + \\
e^{\frac{i|x|^2}{4\tau}} |x|^{-N+p} (4\tau)^{\frac{N}{2}-p} \sum_{k=0}^{n} B_k(b,a) e^{-\frac{(N+2k)\pi i}{4}} \left(\frac{|x|^2}{4\tau}\right)^{-k} + \\
e^{\frac{i|x|^2}{4\tau}} |x|^{-N+p} (4\tau)^{\frac{N}{2}-p} B_{n+1}(b,a) \left(\frac{|x|^2}{4\tau}\right)^{-n-1} \frac{(n+1)e^{-\frac{a\pi i}{2}}}{\Gamma(n+2-a)} \times \\
\int_0^\infty \int_0^1 (1-s)^n \left(i - \frac{4\tau st}{|x|^2}\right)^{-b-n-1} ds\, e^{-t} t^{n+1-a}\, dt, \quad (3.17)
\end{aligned}$$

where

$$\begin{aligned}
A_k(a,b) &= \frac{C_k(a,b)}{\Gamma(a)} = \frac{\Gamma(a+k)}{\Gamma(a)k!} \frac{\Gamma(k+1-b)}{\Gamma(1-b)} \qquad (3.18) \\
&= \frac{1}{k!} a(a+1)\cdots(a+k-1)(1-b)(2-b)\cdots(k-b),
\end{aligned}$$

and

$$\begin{aligned}
B_k(b,a) &= \frac{C_k(b,a)}{\Gamma(a)} = \frac{\Gamma(b+k)}{\Gamma(a)k!} \frac{\Gamma(k+1-a)}{\Gamma(1-a)} \qquad (3.19) \\
&= \frac{\Gamma(b)}{\Gamma(a)k!} b(b+1)\cdots(b+k-1)(1-a)(2-a)\cdots(k-a).
\end{aligned}$$

Note that $A_0(a,b) = 1$.

By Corollary 3.4, $S(t)\varphi \in C^\infty(\mathbb{R}^N)$ for $t > 0$. Thus, to determine whether $S(t)\varphi \in L^r(\mathbb{R}^N)$, it suffices to consider $|x|$ large. Proposition 3.7 now follows immmediately from formula (3.17). □

Proposition 3.9. *The function given by $\varphi(x) = \omega(x)|x|^{-p}$, where $0 < \operatorname{Re} p < N$, and $\omega(x) = P_k(x)|x|^{-k}$, P_k being a homogeneous harmonic polynomial of degree k, is an SC$^\infty$-regular p-homogeneous function.*

Proof. It follows from the representation of the Schrödinger group via the Fourier transform,

$$S(t)\varphi = e^{it\triangle}\varphi = \mathcal{F}^{-1}\left(e^{-4it\pi^2|\cdot|^2}\mathcal{F}\varphi\right),$$

and the considerations in Sect. 3.2 of Chapter 3 of Stein [30] that

$$S(t)[P_k\,|\cdot|^{-p-k}] = P_k[S_{N+2k}(t)\,|\cdot|^{-p-k}], \tag{3.20}$$

where on the right side of (3.20) $|\cdot|^{-p-k}$ is interpreted as a tempered distribution on $\mathbb{R}^{N+2k}$ and $S_{N+2k}(t)$ is the Schrödinger group on $\mathbb{R}^{N+2k}$. The resulting radially symmetric function $S_{N+2k}(t)\,|\cdot|^{-p-k}$ is then re-interpreted as a function on $\mathbb{R}^N$. The result then follows from Corollary 3.4 and formula (3.17) with N replaced by $N+2k$. □

Proposition 3.10. *Let φ be an SC-regular p-homogeneous function, with $0 < \operatorname{Re} p < N$. Let q and r be dual exponents such that*

$$r > \max\left\{\frac{N}{\operatorname{Re} p}, \frac{N}{N-\operatorname{Re} p}\right\}.$$

Set $f = S(1)\varphi$, so $f \in L^r(\mathbb{R}^N)$.

Suppose $\varphi = \varphi_1 + \varphi_2$, where $\varphi_1 \in L^q(\mathbb{R}^N)$. It follows that $S(t)\varphi_2 \in L^r(\mathbb{R}^N)$ for all $t > 0$ and

$$\|f - D_{\sqrt{t}}S(t)\varphi_2\|_{L^r} \le Ct^{-\frac{N}{2q}+\frac{\operatorname{Re} p}{2}}\|\varphi_1\|_{L^q}, \quad \forall t > 0, \tag{3.21}$$

which converges to 0 as $t \to \infty$. If $\varphi_1 \in L^1(\mathbb{R}^N)$, then $t^{\frac{p}{2}}S(t)\varphi_2 \in C(\mathbb{R}^N)$ for all $t > 0$ and converges to $f(0)$ uniformly on compact subsets of $\mathbb{R}^N$ as $t \to \infty$. Finally, suppose in addition φ is SC$^\infty$-regular and $\varphi_1 \in L^1(\mathbb{R}^N)$ has compact support. It follows that $S(t)\varphi_2 \in C^\infty(\mathbb{R}^N)$ for all $t > 0$.

Proof. The condition on r implies that $r \ge 2$. Thus, on the one hand,

$$\|S(t)\varphi_1\|_{L^r} \le Ct^{-\frac{N(r-2)}{2r}}\|\varphi_1\|_{L^q},$$

for all $t > 0$. On the other hand, by (3.7)

$$\begin{aligned}\|S(t)\varphi_1\|_{L^r} &= \|S(t)(\varphi - \varphi_2)\|_{L^r}\\ = \|D_{\frac{1}{\sqrt{t}}}S(1)\varphi - S(t)\varphi_2]\|_{L^r} &= t^{\frac{N}{2r}-\frac{\operatorname{Re} p}{2}}\|f - D_{\sqrt{t}}S(t)\varphi_2\|_{L^r}.\end{aligned}$$

These two estimates immediately give (3.21).

Suppose next that $\varphi_1 \in L^1(\mathbb{R}^N)$. It follows that $S(t)\varphi_1 \in C(\mathbb{R}^N)$ for all $t > 0$. Since φ is SC-regular, it is also true that $S(t)\varphi_2 \in C(\mathbb{R}^N)$ for all $t > 0$. Since dilating the spatial variable only does not change the L^∞ norm, (3.21) implies that

$$\left\| f\left(\frac{\cdot}{\sqrt{t}}\right) - t^{\frac{p}{2}} S(t)\varphi_2 \right\|_{L^\infty} \leq C t^{-\frac{N}{2}+\frac{\operatorname{Re} p}{2}} \|\varphi_1\|_{L^1},$$

from which it follows that $t^{\frac{p}{2}} S(t)\varphi_2$ converges to $f(0)$ uniformly on compact sets.

Finally, if $\varphi_1 \in L^1(\mathbb{R}^N)$ has compact support, then for all $t > 0$, $S(t)\varphi_1$ is C^∞. Since φ is SC$^\infty$-regular, the same is true for $S(t)\varphi$, and therefore also for $S(t)\varphi_2$. This proves the proposition. □

Intuitively, φ_1 contains the singular part of φ near the origin and φ_2 contains the slowly decaying part of φ for large $|x|$. Indeed, $\varphi \in L^q(\{|x| < 1\})$ if and only if $q < \dfrac{N}{\operatorname{Re} p}$, i.e. $r > \dfrac{N}{N - \operatorname{Re} p}$. Thus, the above proposition can be applied with $\varphi_1 = \eta\varphi + \psi$, where η is any L^∞ function with compact support and ψ is in $L^q(\mathbb{R}^N)$ for all

$$q < \max\left\{ \frac{N}{\operatorname{Re} p}, \frac{N}{N - \operatorname{Re} p} \right\}.$$

In particular, data which decay enough like $|x|^{-p}$ (or another SC-regular p-homogeneous function) as $|x| \to \infty$, give rise to solutions of the linear Schrödinger equation which are asymptotic (in time) to the solutions with pure homogeneous data.

The final result of this section (Proposition 3.11 below) shows, at least to some extent, that the oscillating part of the development (3.17) is due to the singularity of $|x|^{-p}$ near the origin.

Notation. To simplify the reading and printing of what follows, we denote by Σ_1, R_1, Σ_2, and R_2 the four terms on the right side of formula (3.17). In other words, Σ_1 and Σ_2 are the two finite sums, Σ_1 beginning with $|x|^{-p}$, and R_1 and R_2 are the two integral remainder terms. Of course $\Sigma_1 = \Sigma_1(t,x)$, etc. Thus, (3.17) reads simply

$$[S(t)\varphi](x) = \Sigma_1(t,x) + R_1(t,x) + \Sigma_2(t,x) + R_2(t,x).$$

Proposition 3.11. *Let $\varphi(x) = |x|^{-p}$, $0 < \operatorname{Re} p < N$; and let η be a C^∞ cut-off function, i.e. identically 1 in a neighborhood of the origin and of compact support. If $\operatorname{Re} p > \dfrac{N}{2}$, then*

$$S(t)(\eta\varphi) - (1-\eta)(\Sigma_2 + R_2) \in H^\infty(\mathbb{R}^N),$$

for all $t > 0$.

Proof. Let η be a C^∞ cut-off function. Clearly then

$$S(t)\varphi = S(t)(\eta\varphi) + S(t)[(1-\eta)\varphi], \tag{3.22}$$
$$S(t)\varphi = \eta S(t)\varphi + (1-\eta)S(t)\varphi. \tag{3.23}$$

Since $S(t)\varphi$ is C^∞ on $\mathbb{R}^N$, and since $\eta\varphi$ is in $L^1(\mathbb{R}^N)$ with compact support, it follows that each term in (3.22) and (3.23) is C^∞ on $\mathbb{R}^N$, with $\eta S(t)\varphi$ being of compact support.

Also, $(1-\eta)\varphi$, $(1-\eta)\Sigma_1$ and $(1-\eta)R_1$ (for a fixed t) are all in $W^{\infty,r}(\mathbb{R}^N)$ for all $r > \dfrac{N}{\operatorname{Re} p}$. We write

$$\begin{aligned} S(t)[\eta\varphi] &- (1-\eta)(\Sigma_2 + R_2) \\ &= S(t)\varphi - S(t)[(1-\eta)\varphi] - (1-\eta)(\Sigma_2 + R_2) \\ &= \eta S(t)\varphi + (1-\eta)S(t)\varphi - S(t)[(1-\eta)\varphi] - (1-\eta)(\Sigma_2 + R_2) \\ &= \eta S(t)\varphi + (1-\eta)(\Sigma_1 + R_1) - S(t)[(1-\eta)\varphi]. \end{aligned}$$

If $\operatorname{Re} p > \dfrac{N}{2}$, then $(1-\eta)\varphi$, $(1-\eta)\Sigma_1$ and $(1-\eta)R_1$ (for a fixed t) are all in $H^\infty(\mathbb{R}^N)$, as must be $S(t)[(1-\eta)\varphi]$, since $S(t)$ preserves $H^\infty(\mathbb{R}^N)$. Since $\eta S(t)\varphi$ is C^∞ with compact support, this proves the proposition. □

Conjecture: $S(t)[\eta\varphi] - (1-\eta)(\Sigma_2 + R_2) \in \mathcal{S}(\mathbb{R}^N)$, $S(t)[(1-\eta)\varphi] - (1-\eta)(\Sigma_1 + R_1) \in \mathcal{S}(\mathbb{R}^N)$.

4 Self-similar solutions

We recall the notion of self-similar solutions.

Definition 4.1. *A solution $u(t,x)$ of (2.1) is self-similar if for some p with $\operatorname{Re} p = \dfrac{2}{\alpha}$, $u(t,x) = \lambda^p u(\lambda^2 t, \lambda x)$ for all $\lambda > 0$. Note that a self similar solution verifies*

$$u(t,x) = t^{-\frac{p}{2}} f\left(\frac{x}{\sqrt{t}}\right), \tag{4.1}$$

where $f = u(1,\cdot)$.

Remark 4.2. Note that if u is a self-similar solution, then

$$\|u(t)\|_{L^r} = t^{\frac{N}{2r} - \frac{1}{\alpha}} \|f\|_{L^r},$$

for every $1 \le r \le \infty$ such that $f \in L^r(\mathbb{R}^N)$. In particular, if $f \in L^2(\mathbb{R}^N)$, then

$$\|u(t)\|_{L^2} = t^{\frac{N}{4} - \frac{1}{\alpha}} \|f\|_{L^2}.$$

This implies that, except in the case $\alpha = \dfrac{4}{N}$, a self-similar solution **cannot** be a classical H^1 solution, since H^1 solutions satisfy the conservation of charge.

Proposition 4.3. *Assume (1.3), and suppose* $\operatorname{Re} p = \dfrac{2}{\alpha}$. *If $\varphi(x)$ is a finite linear combination of functions of the form $P_k(x)|x|^{-p-k}$, where P_k is a homogeneous harmonic polynomial of degree k (including $k = 0$), then $\|S(1)\varphi\|_{L^{\alpha+2}}$ is finite and*

$$t^{\beta}\|S(t)\varphi\|_{L^{\alpha+2}} = \|S(1)\varphi\|_{L^{\alpha+2}}, \quad \forall t > 0, \tag{4.2}$$

where β is given by (2.2). If, in addition, $\|S(1)\varphi\|_{L^{\alpha+2}}$ is sufficiently small, there exists a self-similar solution u of (2.1) with initial value φ, having all the properties described in Theorem 2.1.

Proof. It follows from the conditions on α and p that

(i) $0 < \operatorname{Re} p < N$,
(ii) $\alpha + 2 > \max\left\{\dfrac{N}{\operatorname{Re} p}, \dfrac{N}{N - \operatorname{Re} p}\right\}$.

(Statement (i) is true since $\alpha_0 > \dfrac{2}{N}$.) The finiteness of $\|S(1)\varphi\|_{L^{\alpha+2}}$ therefore follows from Propositions 3.7 and 3.9; and formula (4.2) is the same as formula (3.8) with $r = \alpha + 2$, $\operatorname{Re} p = \dfrac{2}{\alpha}$, and β given by (2.2).

The fact that the solution u with initial value φ is self-similar is a consequence of the uniqueness provision. Since $\lambda^p\varphi(\lambda x) = \varphi(x)$ for all $\lambda > 0$, the functions $\lambda^p u(\lambda^2 t, \lambda x)$ are all solutions of (2.1) with the same initial value φ and all verifying (2.6). □

Remark 4.4. One may even allow an infinite sum $\varphi = \sum \varphi_m$, where each φ_m is as in the statement of Proposition 4.3. One need only impose two conditions: that the sum $\varphi = \sum \varphi_m$ converges in the sense of tempered distributions and that $\sum \|S(1)\varphi_m\|_{L^{\alpha+2}} < \infty$. This gives a very wide class of self-similar solutions. It would of course be quite interesting to characterize the set of tempered distributions (homogeneous or otherwise) such that $\sup\limits_{t>0} t^{\beta}\|S(t)\varphi\|_{L^{\alpha+2}} < \infty$.

Remark 4.5. While we do not know how smooth these self-similar solutions are, we can at least note that $u \in C((0,\infty), L^{\alpha+2}(\mathbb{R}^N))$. Indeed, since $u \in L^{\infty}_{\text{loc}}((0,\infty), L^{\alpha+2}(\mathbb{R}^N))$, it follows from (4.1) that $f \in L^{\alpha+2}(\mathbb{R}^N)$. The operator $s \mapsto f(s\,\cdot\,)$ being continuous $(0,\infty) \to L^{\alpha+2}(\mathbb{R}^N)$, continuity follows from (4.1).

Lemma 4.6. *Suppose φ verifies $\sup\limits_{t>0} t^\beta \|S(t)\varphi\|_{L^{\alpha+2}} < \infty$, where α verifies (1.3). If $\varphi = \varphi_1 + \varphi_2$ where $\varphi_1 \in L^{\frac{\alpha+2}{\alpha+1}}(\mathbb{R}^N)$ and $\varphi_2 \in H^1(\mathbb{R}^N)$, then $\sup\limits_{t>0} t^\beta \|S(t)\varphi_2\|_{L^{\alpha+2}} < \infty$.*

Proof. Since $\varphi_2 \in H^1(\mathbb{R}^N)$ and $\alpha < \dfrac{4}{N-2}$, we see that

$$\|S(t)\varphi_2\|_{L^{\alpha+2}} \le C\|S(t)\varphi_2\|_{H^1} \le C\|\varphi_2\|_{H^1} < \infty.$$

In particular,

$$\lim_{t\downarrow 0} t^\beta \|S(t)\varphi_2\|_{L^{\alpha+2}} < \infty.$$

Next,

$$\begin{aligned} t^\beta \|S(t)\varphi_2\|_{L^{\alpha+2}} &= t^\beta \|S(t)(\varphi - \varphi_1)\|_{L^{\alpha+2}} \\ &\le t^\beta \|S(t)\varphi\|_{L^{\alpha+2}} + t^\beta \|S(t)\varphi_1\|_{L^{\alpha+2}} \\ &\le C + Ct^\beta t^{-\frac{N\alpha}{2(\alpha+2)}} \|\varphi_1\|_{L^{\frac{\alpha+2}{\alpha+1}}} = C + Ct^{-\sigma}\|\varphi_1\|_{L^{\frac{\alpha+2}{\alpha+1}}}, \end{aligned}$$

where σ is given in Proposition 2.3 (d); $\sigma > 0$ since $\alpha > \alpha_0$. Thus

$$\limsup_{t\to\infty} t^\beta \|S(t)\varphi_2\|_{L^{\alpha+2}} < \infty.$$

This proves the lemma. □

Proposition 4.7. *Suppose (1.3) and that φ is a p-homogeneous function, with $\operatorname{Re} p = \dfrac{2}{\alpha}$, such that*

(i) $\sup\limits_{t>0} t^\beta \|S(t)\varphi\|_{L^{\alpha+2}} < \infty$;

(ii) $\varphi = \varphi_1 + \varphi_2$ *where* $\varphi_1 \in L^{\frac{\alpha+2}{\alpha+1}}(\mathbb{R}^N)$ *and* $\varphi_2 \in H^1(\mathbb{R}^N)$.

After multiplying φ by a sufficiently small constant so that φ and φ_2 verify the formula (2.5) with M small enough to apply the last part of Theorem 2.1, let $v(t)$ be the global H^1 solution with initial value φ_2, let $u(t,x) = t^{-\frac{p}{2}} f\left(\dfrac{x}{\sqrt{t}}\right)$ be the self-similar solution with initial value φ, and set

$$w(t,x) = v(t,x) - t^{-\frac{p}{2}} f\left(\frac{x}{\sqrt{t}}\right).$$

It follows that for all $\varepsilon > 0$,

$$\|w(t)\|_{L^{\alpha+2}} = O(t^{-\frac{N\alpha}{2(\alpha+2)}+\varepsilon}),$$

and

$$\|f - t^{\frac{p}{2}} v(t, x\sqrt{t})\|_{L^{\alpha+2}} \le C_\varepsilon t^{-\frac{N(\alpha+1)}{2(\alpha+2)} + \frac{1}{\alpha} + \varepsilon},$$

as $t \to \infty$. Both converge to 0 as $t \to \infty$ if ε is sufficiently small.

Proof. This is an immediate consequence of Lemma 4.6, Theorem 2.1, and formula (2.9) in Remark 2.2 (a). Also, compare formula (3.21). □

Proposition 4.8. *If $\alpha_0 < \alpha < \dfrac{4}{N}$, the decomposition $\varphi = \varphi_1 + \varphi_2$ as described in Lemma 4.6 can be realized with φ as in Proposition 4.3 and $\varphi_1 = \eta\varphi + \psi$, where η and ψ are such that*

(i) $\eta \in W^{1,\infty}(\mathbb{R}^N)$,
(ii) $\eta(x) = 1$ *for x in a neighborhood of $x = 0$,*
(iii) $\eta(x)|x|^{-\frac{2}{\alpha}} \in L^{\frac{\alpha+2}{\alpha+1}}(\mathbb{R}^N)$,
(iv) $\psi \in L^{\frac{\alpha+2}{\alpha+1}}(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$.

For example, η could be a C^1 cut-off function with compact support.

Proof. This is trivial to verify. Unfortunately, to verify that $(1-\eta)\varphi \in H^1(\mathbb{R}^N)$, one has to impose the additional restriction $\alpha < \dfrac{4}{N}$. □

Remark 4.9. The results of this section show that if $\alpha_0 < \alpha < \dfrac{4}{N}$, then a wide class of initial values in $H^1(\mathbb{R}^N)$ which are "asymptotically homogeneous" in $\mathbb{R}^N$ (i.e. the functions φ_2 in Proposition 4.8), give rise to global, asymptotically (in time) self-similar solutions of equation (2.1). Note that for such solutions, the known scattering theories do **not** apply. Indeed, the H^1 scattering theory does not apply because $\alpha < 4/N$; and the scattering theory in $X = H^1 \cap L^2(|x|^2 dx)$ does not apply beacuse $\varphi_2 \notin X$.

5 Self-similar blow up

In Sect. 4, we constructed self-similar solutions of (1.1), and also a class of classical H^1 solutions that are asymptotically self-similar as $t \to \infty$. However, self-similar solutions can also describe blow up behavior, as in the case of certain nonlinear heat equations (see for example Y. Giga and R.V. Kohn [11, 12, 13] and M.A. Herrero and J.J.L. Velázquez [18, 19]). In this section, we exhibit a class of negatively global solutions of (1.1) which are asymptotically self-similar at the "blow up" time $t = 0$ (see Theorem 5.7 below). To accomplish this, we apply the pseudo-conformal transformation, which interchanges the behaviors at $t = 0$ and at $t = \infty$. More precisely, consider a solution u of the equation (1.1) on $(-\infty, 0)$, and set

$$v(s, y) = s^{-\frac{N}{2}} e^{i\frac{|y|^2}{4s}} u\left(-\frac{1}{s}, \frac{y}{s}\right), \tag{5.1}$$

for $y \in \mathbb{R}^N$ and $s > 0$. It follows (formally) that v satisfies the (non-autonomous) nonlinear Schrödinger equation

$$iv_s + \triangle v = \gamma s^{\frac{N\alpha-4}{2}} |v|^\alpha v, \tag{5.2}$$

on $(0, \infty)$. We write the Cauchy problem for the equation (5.2) in the integral form

$$v(s) = S(s)\psi - i\gamma \int_0^s S(s-\tau)\tau^{\frac{N\alpha-4}{2}} (|v(\tau)|^\alpha v(\tau))\, d\tau. \tag{5.3}$$

For (5.3), we have the following analogue of Theorem 2.1.

Theorem 5.1. *Suppose (1.3), and let σ be given by*

$$\sigma = \frac{N\alpha^2 + (N-2)\alpha - 4}{2\alpha(\alpha+2)}. \tag{5.4}$$

Suppose further that $\rho > 0$ and $M > 0$ satisfy the inequality

$$\rho + \widetilde{K} M^{\alpha+1} \le M,$$

where $\widetilde{K} = \widetilde{K}(\alpha, N, \gamma)$ is given by

$$\widetilde{K} = 2|\gamma|(\alpha+1)(4\pi)^{-\frac{N\alpha}{2(\alpha+2)}} B\left(1 - \frac{N\alpha}{2(\alpha+2)}, \frac{N\alpha}{2} - 1 - \sigma(\alpha+1)\right).$$

Let ψ be a tempered distribution such that

$$\sup_{s>0} |s|^\sigma \|S(s)\psi\|_{L^{\alpha+2}} \le \rho. \tag{5.5}$$

It follows that there exists a unique positively global (i.e. defined for all $s \ge 0$) solution v of (5.3) such that

$$\sup_{s>0} |s|^\sigma \|v(s)\|_{L^{\alpha+2}} \le M. \tag{5.6}$$

Furthermore,

(a) $v(s) - S(s)\psi \in C([0,\infty), H^{-\frac{N\alpha}{2(\alpha+2)}}(\mathbb{R}^N))$;
(b) $\lim_{s \downarrow 0} v(s) = \psi$ *as tempered distributions.*

Suppose ψ and $\widetilde{\psi}$ verify (5.5) and let v and $\widetilde{v}$ be respectively the solutions of (5.3) satisfying (5.6) with initial values ψ and $\widetilde{\psi}$. It follows that

$$\sup_{s>0} |s|^\sigma \|v(s) - \widetilde{v}(s)\|_{L^{\alpha+2}} \le (1 - \widetilde{K} M^\alpha)^{-1} \sup_{s>0} |s|^\sigma \|S(s)(\psi - \widetilde{\psi})\|_{L^{\alpha+2}}.$$

If, in addition, $S(s)(\psi - \widetilde{\psi})$ has the stronger decay property

$$\sup_{s>0} |s|^\sigma (1+|s|)^\delta \|S(s)(\psi - \widetilde{\psi})\|_{L^{\alpha+2}} < \infty, \tag{5.7}$$

for some $\delta > 0$ such that $\sigma(\alpha+1) + \delta < \dfrac{N\alpha}{2} - 1$, and if $\widetilde{K}' M^\alpha < 1$, where $\widetilde{K}'$ is given by

$$\widetilde{K}' = 2|\gamma|(\alpha+1)(4\pi)^{-\frac{N\alpha}{2(\alpha+2)}} B\left(1 - \frac{N\alpha}{2(\alpha+2)}, \frac{N\alpha}{2} - 1 - \sigma(\alpha+1) - \delta\right),$$

then

$$\begin{aligned} &\sup_{s>0} |s|^\sigma (1+|s|)^\delta \|v(s) - \widetilde{v}(s)\|_{L^{\alpha+2}} \\ &\le (1 - \widetilde{K}' M^\alpha)^{-1} \sup_{s>0} |s|^\sigma (1+|s|)^\delta \|S(s)(\psi - \widetilde{\psi})\|_{L^{\alpha+2}}. \end{aligned} \tag{5.8}$$

Proof. The proof is identical to the proof of Theorem 2.1, replacing β by σ. □

Remark 5.2. The value of σ given by (5.4) is the same as defined in Proposition 2.3 (d). Moreover, $\beta + \sigma = \dfrac{N\alpha}{2(\alpha+2)}$, where β is given by (2.2). Indeed, β and σ play analagous (and dual) roles in Sects. 2 and 4, on the one hand, and Sect. 5 on the other hand. As another example, the limiting value of δ as used in Remark 2.2 (a) is precisely σ. In the analogous remark for equation (5.3), which we invite the reader to formulate, the limiting value of δ is precisely β.

Remark 5.3. As is the case for equation (1.1), the set of solutions of equation (5.2) is invariant under a group of dilations; and this allows one to define self-similar solutions (see Proposition 5.5 below). These dilations, when restricted to spatial functions (i.e. initial values) are precisely $D_{\lambda,q}$, where $\mathrm{Re}\, q = N - \dfrac{2}{\alpha}$ and $\lambda > 0$. (Recall formula (3.4).) The norm defined by the left side of (5.5) is invariant with respect to these transformations. Moreover, the Lebesgue norm left invariant by these transformation is $L^{\frac{N\alpha}{N\alpha-2}}(\mathbb{R}^N)$, which is the dual of $L^{\frac{N\alpha}{2}}(\mathbb{R}^N)$, the invariant Lebesgue norm for equation (1.1).

Remark 5.4. It follows from Theorem 3.4, p. 90 of [6] that the initial value problem for (5.2) is locally well-posed in the space $H^1(\mathbb{R}^N)$. Moreover (by Theorem 3.8, p. 91 of [6]), we have the energy identity

$$\frac{d}{ds} E(s) = s^{\frac{N\alpha-6}{2}} \frac{N\alpha - 4}{2} \frac{\gamma}{\alpha+2} \|v(s)\|_{L^{\alpha+2}}^{\alpha+2},$$

where

$$E(s) = \frac{1}{2}\|\nabla v(s)\|_{L^2}^2 + \frac{\gamma}{\alpha+2} s^{\frac{N\alpha-4}{2}} \|v(s)\|_{L^{\alpha+2}}^{\alpha+2},$$

and conservation of charge

$$\frac{d}{ds}\|v(s)\|_{L^2} = 0$$

(Note that the energy identity is established in [6] for solutions in $H^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N, |x|^2 dx)$, but that it holds for H^1 solutions by the continuous dependence property (iv) of Theorem 3.4 of [6].) It follows in particular that property (f) of Proposition 2.3 holds as well for the equation (5.2). Therefore, if $\psi \in H^1(\mathbb{R}^N)$ satisfies (5.5), then the solution of (5.3) constructed in Theorem 5.1 coincides for all $s \geq 0$ with the "classical" H^1 solution, which is therefore global.

Let now $\widetilde{\psi}(y) = |y|^{-q}$ (or more generally a finite combination of functions of the form $P_k(y)|y|^{-q-k}$, where P_k is a homogeneous harmonic polynomial of degree $k \geq 0$), where $q \in \mathbb{C}$ verifies

$$\operatorname{Re} q = N - \frac{2}{\alpha}.$$

By Proposition 3.9 $\widetilde{\psi}$ is an SC$^\infty$-regular q-homogeneous function (see Definition 3.1). Condition (1.3) implies that

$$\alpha + 2 \geq \max\left\{\frac{N}{\operatorname{Re} q}, \frac{N}{N - \operatorname{Re} q}\right\}.$$

In particular, by formula (3.8) with $r = \alpha + 2$ (and p replaced by q), we see that

$$\sup_{s>0} s^\sigma \|S(s)\widetilde{\psi}\|_{L^{\alpha+2}} = \|S(1)\widetilde{\psi}\|_{L^{\alpha+2}} < \infty;$$

and so we may apply Theorem 5.1 with $\psi = c\widetilde{\psi}$ where c is a sufficiently small constant. The following proposition is now proved by the same arguments as in the proof of Proposition 4.3.

Proposition 5.5. *Assume (1.3) and let $\widetilde{\psi}$ be as above. If c is small enough, then there exists a solution v of (5.3) having all the properties described in Theorem 5.1. Moreover, v is self-similar, i.e.*

$$v(s, y) \equiv \lambda^q v(\lambda^2 s, \lambda y),$$

for all $\lambda > 0$. In particular,

$$v(s, y) \equiv s^{-\frac{q}{2}} f\left(\frac{y}{\sqrt{s}}\right),$$

where $f(\cdot) = v(1, \cdot)$.

Let $\widetilde{\psi}$ be as above, and let η be a C^∞ cut-off function, i.e. identically 1 in a neighborhood of the origin and of compact support. One verifies easily that $(1-\eta)\widetilde{\psi} \in H^1(\mathbb{R}^N)$ provided $\alpha > \dfrac{4}{N}$. In addition, $\eta\widetilde{\psi} \in L^{\frac{\alpha+2}{\alpha+1}}(\mathbb{R}^N)$ because $\alpha < \dfrac{4}{N-2}$. Therefore, adapting the proofs of Lemma 4.6 and Propositions 4.7 and 4.8, we obtain the following result.

Proposition 5.6. *Suppose* $\dfrac{4}{N} < \alpha < \dfrac{4}{N-2}$. *Let* $\widetilde{\psi}$ *be as in Proposition 5.5 and* η *a cut-off function as described above. Let* $\psi = c\widetilde{\psi}$, $\psi_1 = \eta\psi$, *and* $\psi_2 = (1-\eta)\psi$. *If* c *is small enough, then* ψ, ψ_1 *and* ψ_2 *all verify the assumptions of the first part of Theorem 5.1, and* ψ *and* ψ_2 *verify the assumptions of the last part of Theorem 5.1. If we denote by* v, v_1 *and* v_2 *the corresponding solutions of (5.3), then* v *is self-similar as described in Proposition 5.5, and for all* $\varepsilon > 0$,

$$\left\| v_2(s,\cdot) - s^{-\frac{q}{2}} f\left(\frac{\cdot}{\sqrt{s}}\right) \right\|_{L^{\alpha+2}} = O\left(s^{-\frac{N\alpha}{2(\alpha+2)}+\varepsilon}\right), \tag{5.9}$$

as $s \to \infty$, *where* $f(\cdot) = v(1,\cdot)$. *Finally,* v_2 *is a "classical"* H^1 *solution of (5.2).*

The following theorem now follows by expressing the above results, via the inverse pseudo-conformal transformation, in terms of solutions of equation (1.1).

Theorem 5.7. *Suppose* $\dfrac{4}{N} < \alpha < \dfrac{4}{N-2}$, *and let* v *and* v_2 *be as in Proposition 5.6. Let* u *be defined by*

$$u(t,x) = (-t)^{-\frac{N}{2}} e^{i\frac{|x|^2}{4t}} v\left(-\frac{1}{t}, -\frac{x}{t}\right), \tag{5.10}$$

for $x \in \mathbb{R}^N$ *and* $t < 0$, *and let* u_2 *be defined similarly in terms of* v_2. *Then both* u *and* u_2 *are solutions of (1.1) on* $(-\infty, 0)$ *in the sense of Theorem 2.1. Moreover,* u *is self similar, i.e.*

$$u(t,x) = (-t)^{-\frac{p}{2}} g\left(\frac{x}{\sqrt{-t}}\right),$$

with $p = N - q$ *and* $g(x) = e^{-i\frac{|x|^2}{4}} v(1,x)$. *Furthermore, for any* $\varepsilon > 0$,

$$\|(-t)^{\frac{p}{2}} u_2(t, x\sqrt{-t}) - g(x)\|_{L^{\alpha+2}} = O\left((-t)^{\frac{4-(N-2)\alpha}{2\alpha(\alpha+2)}-\varepsilon}\right), \tag{5.11}$$

as $t \uparrow 0$, which converges to 0 as $t \uparrow 0$ for ε sufficiently small. In addition, $u_2 \in C((-\infty,0), L^r(\mathbb{R}^N))$ for any $r \in \left[2, \dfrac{2N}{N-2}\right]$, $\|u_2(t)\|_{L^2}$ is constant, and $u_2 \in C((-\infty,0), H^1_{\mathrm{loc}}(\mathbb{R}^N))$.

Remark 5.8. Of course, the construction of self-similar solutions u of (1.1) by formula (5.10) in Theorem 5.7 is valid for the full range $\alpha_0 < \alpha < \dfrac{4}{N-2}$. It is not clear if these are the same self-similar solutions - up to complex conjugation - described in Proposition 4.3.

In the proof of Theorem 5.7, we will use the following lemma.

Lemma 5.9. *Assume (1.3), and let σ and β be given respectively by (5.4) and (2.2). Let $\psi \in \mathcal{S}'(\mathbb{R}^N)$ be such that $\sup\limits_{s>0} s^\sigma \|S(s)\psi\|_{L^{\alpha+2}} < \infty$, and let $v \in L^\infty_{\mathrm{loc}}((0,\infty), L^{\alpha+2}(\mathbb{R}^N))$ be such that $\sup\limits_{s>0} s^\sigma \|v(s)\|_{L^{\alpha+2}} < \infty$. Suppose that v satisfies equation (5.3) for $s > 0$, and let u be defined by (5.10) for $x \in \mathbb{R}^N$ and $t < 0$. Then the following conclusions hold.*

(i) $(-t)^\beta u(t) \in L^\infty((-\infty,0), L^{\alpha+2}(\mathbb{R}^N))$;
(ii) *there exists* $\varphi \in \mathcal{S}'(\mathbb{R}^N)$ *such that* $u(t) \underset{t\uparrow 0}{\longrightarrow} \varphi$ *in* $\mathcal{S}'(\mathbb{R}^N)$;
(iii) $\sup\limits_{t<0} (-t)^\beta \|S(t)\varphi\|_{L^{\alpha+2}} < \infty$;
(iv) u *satisfies the equation (2.1) on* $(-\infty,0)$.

Proof. An elementary calculation shows that

$$(-t)^\beta \|u(t)\|_{L^{\alpha+2}} = \left(-\frac{1}{t}\right)^\sigma \left\| v\left(-\frac{1}{t}\right)\right\|_{L^{\alpha+2}},$$

and (i) follows.

We claim that

$$v(s) = S(s-\tau)v(\tau) - i\gamma \int_\tau^s S(s-\mu)\mu^{\frac{N\alpha-4}{2}} |v(\mu)|^\alpha v(\mu)\, d\mu, \quad (5.12)$$

for all $\tau, s > 0$. Note first that (5.12) makes sense. Indeed, $v(\tau) \in L^{\alpha+2}(\mathbb{R}^N) \hookrightarrow \mathcal{S}'(\mathbb{R}^N)$, so that $s \mapsto S(s-\tau)v(\tau)$ belongs to $C(\mathbb{R}, \mathcal{S}'(\mathbb{R}^N))$. Furthermore, $\mu^{\frac{N\alpha-4}{2}} |v(\mu)|^\alpha v(\mu) \in L^\infty_{\mathrm{loc}}((0,\infty), L^{\frac{\alpha+2}{\alpha+1}}(\mathbb{R}^N))$. Note that $L^{\frac{\alpha+2}{\alpha+1}}(\mathbb{R}^N) \hookrightarrow H^{-1}(\mathbb{R}^N)$ and that $(S(s))_{s\in\mathbb{R}}$ is a group of isometries in $H^\ell(\mathbb{R}^N)$ for every $\ell \in \mathbb{R}$. Therefore, the integral on the right hand side of (5.12) belongs to $C((0,\infty), H^{-1}(\mathbb{R}^N))$ as a function of s. We now establish (5.12). Set $w(s) = v(s) - S(s)\psi$. It follows from (5.3) that

$$w(s) = -i\gamma \int_0^s S(s-\mu)\mu^{\frac{N\alpha-4}{2}} |v(\mu)|^\alpha v(\mu)\, d\mu,$$

for all $s > 0$. Since $\mu^{\frac{N\alpha-4}{2}}|v(\mu)|^\alpha v(\mu) \in L^1((0,T), L^{\frac{\alpha+2}{\alpha+1}}(\mathbb{R}^N))$ and $L^{\frac{\alpha+2}{\alpha+1}}(\mathbb{R}^N) \hookrightarrow H^{-1}(\mathbb{R}^N)$, it follows from standard semigroup theory that $w \in C([0,\infty), H^{-1}(\mathbb{R}^N))$ and that

$$w(s) = S(s-\tau)w(\tau) - i\gamma \int_\tau^s S(s-\mu)\mu^{\frac{N\alpha-4}{2}}|v(\mu)|^\alpha v(\mu)\, d\mu.$$

Adding $S(s)\psi = S(s-\tau)S(\tau)\psi$ to both sides of the above identity yields (5.12).

Consider now the dilation operator D_λ and the multiplier M_b defined by

$$(D_\lambda \omega)(x) = \lambda^{\frac{N}{2}}\omega(\lambda x), \quad (M_b\omega)(x) = e^{i\frac{b|x|^2}{4}}\omega(x).$$

We have

$$u\left(-\frac{1}{s}\right) = M_{-s}D_s v(s). \tag{5.13}$$

On the other hand, it follows from [5, formulas (3.2) and (3.3)], that

$$M_{-s}D_s S(s-\tau) = S\left(-\frac{1}{s}+\frac{1}{\tau}\right)M_{-\tau}D_\tau, \tag{5.14}$$

for $s,\tau > 0$. By applying $M_{-s}D_s$ to both sides of (5.12), and by using (5.14) then (5.13), we obtain

$$\begin{aligned}
&u\left(-\tfrac{1}{s}\right)\\
&= S\left(-\tfrac{1}{s}+\tfrac{1}{\tau}\right)u\left(-\tfrac{1}{\tau}\right)\\
&\quad -i\gamma\int_\tau^s S\left(-\tfrac{1}{s}+\tfrac{1}{\mu}\right)\mu^{\frac{N\alpha-4}{2}}M_{-\mu}D_\mu(|v(\mu)|^\alpha v(\mu))\, d\mu\\
&= S\left(-\tfrac{1}{s}+\tfrac{1}{\tau}\right)u\left(-\tfrac{1}{\tau}\right)\\
&\quad -i\gamma\int_\tau^s S\left(-\tfrac{1}{s}+\tfrac{1}{\mu}\right)\mu^{-2}\left|u\left(-\tfrac{1}{\mu}\right)\right|^\alpha u\left(-\tfrac{1}{\mu}\right)\, d\mu.
\end{aligned}$$

Setting $t = -\dfrac{1}{s}$, $\theta = -\dfrac{1}{\tau}$, and making the change of variables $\mu = -\dfrac{1}{\eta}$, we obtain

$$u(t) = S(t-\theta)u(\theta) - i\gamma\int_\theta^t S(t-\eta)|u(\eta)|^\alpha u(\eta)\, d\eta, \tag{5.15}$$

for all $t,\eta < 0$.

Fix $t < 0$. Since $|u|^\alpha u \in L^1((0,T), L^{\frac{\alpha+2}{\alpha+1}}(\mathbb{R}^N)) \subset L^1((0,T), H^{-1}(\mathbb{R}^N))$ for all $T > 0$, the integral in the right hand side of (5.15) converges to

$$\int_0^t S(t-\eta)|u(\eta)|^\alpha u(\eta)\, d\eta,$$

in $H^{-1}(\mathbb{R}^N)$ as $\theta \uparrow 0$. On the other hand, $u(t) \in L^{\alpha+2}(\mathbb{R}^N)$ is fixed; and so, by letting $\theta \uparrow 0$ in (5.15), we see that $S(t-\theta)u(\theta)$ has a limit ω in

$H^{-1}(\mathbb{R}^N) + L^{\alpha+2}(\mathbb{R}^N)$ as $\theta \uparrow 0$. In particular, $\omega \in \mathcal{S}'(\mathbb{R}^N)$, and we may set $\varphi = S(-t)\omega \in \mathcal{S}'(\mathbb{R}^N)$. It follows that $S(t-\theta)u(\theta) \underset{\theta\uparrow 0}{\longrightarrow} S(t)\varphi$ in $\mathcal{S}'(\mathbb{R}^N)$. Therefore, letting $\theta \uparrow 0$ in (5.15), we obtain that u satisfies the equation (2.1) on $(-\infty, 0)$. This proves (ii) and (iv). Finally, (iii) follows from (i) and the estimates used in the proof of (2.11). This completes the proof. □

Proof of Theorem 5.7. The property that u and u_2 are solutions of (1.1) on $(-\infty, 0)$ in the sense of Theorem 2.1 follows from Lemma 5.9. Since $v(s,y) \equiv s^{-\frac{q}{2}} f\left(\frac{y}{\sqrt{s}}\right)$ by Proposition 5.5, we have

$$u(t,x) = (-t)^{-\frac{N}{2}}(-t)^{\frac{q}{2}} e^{i\frac{|x|^2}{4t}} f\left(\frac{x}{\sqrt{-t}}\right) = (-t)^{-\frac{p}{2}} g\left(\frac{x}{\sqrt{-t}}\right),$$

with p and g defined above (note that $\operatorname{Re} p = \dfrac{2}{\alpha}$). Writing (5.9) in terms of u and u_2, we obtain

$$O\left(s^{-\frac{N\alpha}{2(\alpha+2)}+\varepsilon}\right)^{\alpha+2} = \int_{\mathbb{R}^N} \left| s^{-\frac{N}{2}} u_2\left(-\frac{1}{s}, \frac{y}{s}\right) - s^{-\frac{q}{2}} g\left(\frac{y}{\sqrt{s}}\right)\right|^{\alpha+2} dy.$$

Setting $x = \dfrac{y}{\sqrt{s}}$ and $t = -\dfrac{1}{s}$, we deduce

$$\begin{aligned} &O\left((-t)^{\frac{N\alpha}{2(\alpha+2)}-\varepsilon}\right)^{\alpha+2} \\ &= (-t)^{-\frac{N}{2}} \int_{\mathbb{R}^N} \left|(-t)^{\frac{N}{2}} u_2(t, x\sqrt{-t}) - (-t)^{\frac{q}{2}} g(x)\right|^{\alpha+2} dx \\ &= (-t)^{-\frac{N}{2}+\frac{(\alpha+2)\operatorname{Re} q}{2}} \int_{\mathbb{R}^N} \left|(-t)^{\frac{N-q}{2}} u_2(t, x\sqrt{-t}) - g(x)\right|^{\alpha+2} dx; \end{aligned}$$

and so,

$$\begin{aligned} &\|(-t)^{\frac{N-q}{2}} u_2(t, x\sqrt{-t}) - g(x)\|_{L^{\alpha+2}} \\ &= O\left((-t)^{\frac{N\alpha}{2(\alpha+2)}+\frac{N}{2(\alpha+2)}-\frac{\operatorname{Re} q}{2}-\varepsilon}\right) = O\left((-t)^{\frac{4-(N-2)\alpha}{2\alpha(\alpha+2)}-\varepsilon}\right), \end{aligned}$$

which is estimate (5.11). Finally, the regularity properties of u_2 follow easily from the regularity of v_2 described in Proposition 5.6. □

Remark 5.10. It follows from (5.11) that $\|(-t)^{\frac{p}{2}} u_2(t, x\sqrt{-t})\|_{L^{\alpha+2}} \to \|g\|_{L^{\alpha+2}}$ as $t \uparrow 0$. Therefore,

$$\|u_2(t)\|_{L^{\alpha+2}} \approx (-t)^{-\frac{4-(N-2)\alpha}{2\alpha(\alpha+2)}} \|g\|_{L^{\alpha+2}},$$

which blows up as $t \uparrow 0$.

However, the relationship of these solutions to blow up of solutions in H^1 is not clear. Indeed, the solution u_2 constructed in Theorem 5.7 is definitely not in H^1. If it were, then v_2 (which is an H^1 solution) would be a solution in $X = H^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N, |x|^2 dx)$, which is impossible since its initial value ψ_2 is not in X. Moreover, the above result is true regardless of the sign of γ, and in particular if $\gamma > 0$ (or even $\gamma = 0$), in which case no H^1 solution blows up. It is nonetheless intriguing that, in the case $\dfrac{4}{N} < \alpha < \dfrac{4}{N-2}$ and $\gamma < 0$, when there do exist H^1 solutions of (1.1) which blow up in finite time, we can exhibit asymptotically self-similar blowing up solutions which just fail to be in H^1.

6 The nonlinear heat equation

In this section, we consider solutions of the integral equation

$$u(t) = e^{t\triangle}\varphi - \gamma \int_0^t e^{(t-s)\triangle} \left(|u(s)|^\alpha u(s) \right) \, ds \tag{6.1}$$

where $\gamma \in \mathbb{R}$ and $\alpha > \dfrac{2}{N}$. Given such an α, one can always choose q such that

$$1 < \frac{q}{\alpha+1} < \frac{N\alpha}{2} < q. \tag{6.2}$$

While q is not uniquely determined by (6.2), we consider q fixed once and for all. All the results below are valid with any value of q verifying (6.2). Next, we set

$$\beta = \frac{1}{\alpha} - \frac{N}{2q}. \tag{6.3}$$

One verifies easily that

$$\beta(\alpha+1) < 1, \quad \frac{N\alpha}{2q} < 1, \quad \beta + 1 - \frac{N\alpha}{2q} - \beta(\alpha+1) = 0. \tag{6.4}$$

The proof of the following theorem parallels almost exactly the proof of Theorem 2.1. The only additional features needed are the fact that $(e^{t\triangle})_{t\geq 0}$ is a contraction C_0 semigroup on all $L^r(\mathbb{R}^N)$, $1 \leq r < \infty$ and the better smoothing properties of the heat semigroup,

$$\|e^{t\triangle}\varphi\|_{L^p} \leq (4\pi t)^{-\frac{N}{2}\left(\frac{1}{r}-\frac{1}{p}\right)} \|\varphi\|_{L^r},$$

whenever $1 \leq r \leq p \leq \infty$ and $t > 0$. The first part of the theorem was already proved by F. Ribaud [29], Theorem 4.5.2.

Theorem 6.1. *Let $\alpha > \dfrac{2}{N}$ and let q and β verify (6.2) and (6.3). Suppose further that $\rho > 0$ and $M > 0$ satisfy the inequality*

$$\rho + KM^{\alpha+1} \leq M,$$

where $K = K(\alpha, N, \gamma, q)$ can be explicitly computed. Let φ be a tempered distribution such that

$$\sup_{t>0} t^{\beta} \|e^{t\triangle}\varphi\|_{L^q} \leq \rho. \tag{6.5}$$

It follows that there exists a unique positively global (i.e. defined for all $t \geq 0$) solution u of (6.1) such that

$$\sup_{t>0} t^{\beta} \|u(t)\|_{L^q} \leq M. \tag{6.6}$$

Furthermore,

(a) $u(t) - e^{t\triangle}\varphi \in C([0,\infty), L^{\frac{q}{\alpha+1}}(\mathbb{R}^N))$*, taking the value 0 at $t = 0$;*
(b) $\lim_{t\downarrow 0} u(t) = \varphi$*, as tempered distributions.*

Suppose φ and ψ verify (6.5) and let u and v be respectively the solutions of (6.1) satisfying (6.6) with the initial values φ and ψ. It follows that

$$\sup_{t>0} t^{\beta} \|u(t) - v(t)\|_{L^q} \leq (1 - KM^{\alpha})^{-1} \sup_{t>0} t^{\beta} \|e^{t\triangle}(\varphi - \psi)\|_{L^q}.$$

If, in addition, $e^{t\triangle}(\varphi - \psi)$ has the stronger decay property

$$\sup_{t>0} t^{\beta}(1+t)^{\delta} \|e^{t\triangle}(\varphi - \psi)\|_{L^q} < \infty \tag{6.7}$$

for some $\delta > 0$ such that $\beta(\alpha+1) + \delta < 1$, and with M perhaps smaller, then

$$\sup_{t>0} t^{\beta}(1+t)^{\delta} \|u(t) - v(t)\|_{L^q} \leq C \sup_{t>0} t^{\beta}(1+t)^{\delta} \|e^{t\triangle}(\varphi - \psi)\|_{L^q}. \tag{6.8}$$

As in Remark 2.2 (a), if we suppose that $\varphi - \psi \in L^{\frac{q}{\alpha+1}}(\mathbb{R}^N)$, then (6.7) is verified with $\delta = \delta_0$, where

$$\delta_0 = \frac{N(\alpha+1)}{2q} - \frac{1}{\alpha}.$$

Since $\beta(\alpha+1) + \delta_0 = 1$, it follows that (6.8) holds for all $\delta \in (0, \delta_0)$.

Next, we need to identify those homogeneous functions φ such that $\sup_{t>0} t^{\beta} \|e^{t\triangle}\varphi\|_{L^q}$ is finite, since these initial data give rise to self-similar solutions of (6.1). This question has been extensively studied [1, 2, 29]. In particular, F. Ribaud ([29], Theorem 4.5.3) has proved that if $\varphi(x) =$

$\omega(x)|x|^{-\frac{2}{\alpha}}$, where ω is homogeneous of degree 0 and $\omega \in L^\infty(S^{N-1})$ with small enough norm, then the solution of (6.1) with the initial value φ is self-similar.

In fact, we can improve this with the following simple remarks. Let $\varphi(x)$ be as above, where $\omega \in L^q(S^{N-1})$. Set $\varphi_1 = \eta\varphi$ where η is an L^∞ (or smoother) cut-off function (identically 1 near the origin and of compact support), and write $\varphi = \varphi_1 + \varphi_2$. We can draw the following conclusions.

(a) $\varphi_1 \in L^r(\mathbb{R}^N)$ for all $1 \le r < \dfrac{N\alpha}{2}$, and so $e^{t\triangle}\varphi_1 \in L^q(\mathbb{R}^N)$ for all $t > 0$.
(b) $\varphi_2 \in L^q(\mathbb{R}^N)$, and so $e^{t\triangle}\varphi_2 \in L^q(\mathbb{R}^N)$ for all $t > 0$.
(c) $e^{t\triangle}\varphi = e^{t\triangle}\varphi_1 + e^{t\triangle}\varphi_2 \in L^q(\mathbb{R}^N)$ for all $t > 0$, and so $\sup\limits_{t>0} t^\beta \|e^{t\triangle}\varphi\|_{L^q}$ is finite (by dilation properties $t^\beta\|e^{t\triangle}\varphi\|_{L^q}$ does not depend on $t > 0$).
(d) $\sup\limits_{t>0} t^\beta \|e^{t\triangle}\varphi_2\|_{L^q}$ is finite (by an argument similar to the proof of Lemma 4.6).

(The first part of each of statements (a) and (b) follows easily by integrating with polar coordinates.) The following theorem is now straightforward to prove by the same arguments as in Sect. 4.

Theorem 6.2. *Let $\varphi = \varphi_1 + \varphi_2$ as above, except that we multiply φ by a sufficiently small constant. Let $u(t,x) = t^{-\frac{1}{\alpha}} f\left(\dfrac{x}{\sqrt{t}}\right)$ be the self-similar solution of (6.1) with initial data φ, constructed by Theorem 6.1. Let $v(t,x)$ be the global solution of (6.1) with initial data φ_2, constructed by Theorem 6.1 (which corresponds to the “classical” solution of (6.1) since $\varphi_2 \in L^q(\mathbb{R}^N)$ and $q > \dfrac{N\alpha}{2}$, see Weissler [33]). Set $w(t,x) = v(t,x) - t^{-\frac{1}{\alpha}} f\left(\dfrac{x}{\sqrt{t}}\right)$. It follows that for all $\varepsilon > 0$,*

$$\|w(t)\|_{L^q} \le O(t^{-\frac{N\alpha}{2q}+\varepsilon})$$

and

$$\|f - t^{\frac{1}{\alpha}} v(t, x\sqrt{t})\|_{L^q} \le C_\varepsilon t^{-\frac{N(\alpha+1)}{2q}+\frac{1}{\alpha}+\varepsilon},$$

as $t \to \infty$. Both converge to 0 as $t \to \infty$ if ε is sufficiently small.

It is clear that the decomposition $\varphi = \varphi_1 + \varphi_2$ in Theorem 6.2 is not the most general for which the conclusion is true (cf. Sect. 4).

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