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The critical exponent of a reaction diffusion equation on some lie groups

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1 Introduction

We shall study the global existence and blow up of the following semilinear parabolic Cauchy problem

(1.1)
$$\begin{cases} Hu \equiv H_0 u + u^p = X_i^*(a_{ij}X_j u) - \partial_t u + u^p = 0 \text{ in } \mathbf{R}^n \times (0,\infty), \\ u(x,0) = u_0(x) \geqslant 0 \text{ in } \mathbf{R}^n, \end{cases}$$

where $a_{ij} = a_{ij}(x,t)$ are bounded measurable functions and the matrix $A = (a_{ij})$ is uniformly elliptic. $X_j, j = 1, ..., k$ are smooth vector fields in $\mathbf{R}^n$ and X_j^* is the formal adjoint of X_j.

Basic assumptions.

(i). We assume that X_j satisfy the Hörmander's condition for hypoellipticity i.e. the rank of the Lie algebra generated by X_1, ... X_p equals n. Given two points $x, y \in \mathbf{R}^n$, $d(x,y)$ will be the $(X_1, ..., X_p)$-control distance; $|x|$ denotes the Euclidean distance; $d(x) \equiv d(x,0)$; $B(x,r)$ will represent the metric ball $\{y \in \mathbf{R}^n | d(x,y) \leqslant r\}$. Properties of the distance, metric balls and the fundamental solutions of H_0 have been studied extensively in [NSW], [FeS], [KS1, 2] and [Sa];

(ii). There are positive constants Q, B, C and b such that

$$(1.2) \qquad |B(x,r)| = Br^Q; \quad \lim_{|x|\to\infty} d(x) = \infty; \quad \lim_{d(x)\to\infty} |x| = \infty;$$

(iii). G, the fundamental solution G of the linear operator H_0 in (1.1), satisfies

$$(1.3) \qquad \frac{1}{C(t-s)^{Q/2}} e^{-\frac{d(x,y)^2}{b(t-s)}} \leqslant G(x,t;y,s) \leqslant \frac{C}{(t-s)^{Q/2}} e^{-b\frac{d(x,y)^2}{t-s}},$$

for all $x, y \in \mathbf{R}^n$ and all $t > s$. $X_j G(.,t;y,s) \in L^2_{loc}(\mathbf{R}^n)$ for $j = 1, ..., k$ and $t > s$.

Equation (1.1) contains two important special cases. When $X_j = \partial_{x_j}$, $j = 1, ..., n$, the distance is just the Euclidean distance and (1.1) is just an uniformly parabolic equation on $\mathbf{R}^n \times (0, \infty)$. The well-known result by Aronson in [A] assures (1.3) is true for $Q = n$.

Another case is when $A = I$, $n = 2m + 1$, and

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial l}, \quad X_{m+j} = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial l}$$

when $j = 1, ..., m$. Here $(x_1, ..., x_m), (y_1, ..., y_m) \in \mathbf{R}^m$ and $l \in \mathbf{R}^1$. We all realize that $X_1, ..., X_{2m}$ is a basis for the Lie algebra of left invariant vector fields on the Heisenberg group $\mathbf{H}^m$, which is the Lie group whose underlying manifold is $\mathbf{R}^{2m+1}$ endowed with the group law

$$(v, w, l)(v', w', l') = (v + v', w + w', l + l' + 2(v'w - vw')),$$

where v, v', w, $w' \in \mathbf{R}^m$, l, $l' \in \mathbf{R}^1$. Then the equation $Hu = 0$ in (1.1) is the semilinear heat equation on $\mathbf{H}^m \times (0, \infty)$. It is well-known that (1.2) and (1.3) hold with $Q = 2m + 2$ which is the homogeneous dimension. Let $z = (v, w)$ then the distance function is $d((z, l), (0, 0)) = (|z|^4 + l^2)^{1/4}$.

In the Euclidean case problem (1.1) has been studied by many authors. In the famous paper [Fu], Fujita proved the following results: (a) when $A = I$, $1 < p < 1 + \frac{2}{n}$ and $u_0 > 0$, problem (1.1) possesses no global positive solutions;

(b) when $A = I$, $p > 1 + \frac{2}{n}$ and u_0 is smaller than a small Gaussian, then (1.1) has global positive solutions. So $1 + \frac{2}{n}$ is the critical exponent.

When A is no longer the identity matrix, Meier [Me] showed that a critical exponent also exists.

In many occasions people would like to see the existence of global positive solutions for a wider choice of initial values than those that decay exponentially. It is also desirable to know the range of the critical exponents when A is not just the identity matrix. In this paper we pick up the study in these matters.

In the general case, again we are mainly interested on the existence of global positive solutions. In recent years many authors have undertaken the research on linear subelliptic operators and their parabolic counter part. Our motivation for the study of the general semi-linear problem not only comes from a desire of generalization but also from the apparent need for new techniques. In fact the prevailing method in treating (1.1) relies on certain comparison results controlling solutions of (1.1) by those of some equations with constant coefficients. However this method does not seem to fit the new case because the intrinsic operators on the Heisenberg group and many other Lie groups are not of constant coefficients. The method we are using, which are based on some new estimates of heat kernels, are able to produce the following theorems which are new not only to the group case but also to the old Euclidean case.

Definition 1.1. *A function* $u = u(x,t)$ *such that* u, $X_1 u, ..., X_k u \in L^2_{loc}$ $(\mathbf{R}^n \times (0,\infty))$ *is called a solution of (1.1) if*

$$u(x,t) = \int_{\mathbf{R}^n} G(x,t;y,0)u_0(y)dy + \int_0^t \int_{\mathbf{R}^n} G(x,t;y,s)u^p(y,s)dyds$$

for all $(x,t) \in \mathbf{R}^n \times (0,\infty)$.

The main results of the paper are the next three theorems.

Theorem A. *(global existence) Suppose* $p > 1 + \frac{2}{Q}$*. There exists a constant* $b_0 > 0$*, such that for each nonnegative* $u_0 \in C^2(\mathbf{R}^n) \cap L^1(\mathbf{R}^n)$ *satisfying* $u_0(x) \leqslant \frac{b_0}{1+d(x)^Q}$ *for all* $x \in \mathbf{R}^n$*, there exists a positive and continuous solution of (1.1).*

Theorem B. *Suppose* $p > 3 + \frac{2}{Q}$*. There exists a constant* $b_0 > 0$*, such that for each nonnegative* $u_0 \in C^2(\mathbf{R}^n)$ *satisfying* $u_0(x) \leqslant \frac{b_0}{1+d(x)}$ *for all* $x \in \mathbf{R}^n$*, there exists a positive and continuous solution of (1.1).*

In the next theorem we present a blow up result.

Theorem C. *(a). Suppose* $X_j = \partial_{x_j}$*,* $j = 1, 2..., n$*. If* $p < 1 + \frac{2}{n}$*, then the only global non-negative solution of (1.1) is zero.*

(b). Suppose H_0 *in (1.1) is the heat operator on* $\mathbf{H}^m \times (0,\infty)$ *where* $\mathbf{H}^m$ *is the Heisenberg group. If* $p < 1 + \frac{2}{Q}$*, then the only global non-negative solution of (1.1) is zero. Here* $Q = 2m + 2$ *is the homogeneous dimension of* $\mathbf{H}^m$*.*

Remark 1.1. In the papers [LN] and [W], Lee, Ni and Wang studied (1.1) in the case when H_0 is the heat equation in $\mathbf{R}^n \times (0,\infty)$. They provided a nice condition on u_0 so that (1.1) has global positive solutions. However their methods seem to rely on the assumption that H_0 is the heat equation, which is of constant coefficients. It would be interesting to see whether their condition on u_0 would suffice for the general case we are studying here.

Remark 1.2. One may ask whether Theorem C holds for (1.1) in general. At this moment we do not know the anwser. We will explain the difficulty in Sect. 5. When $a = (a_{ij})$ are smooth and time-independent, part (a) of Theorem C was proved in [U].

Let us briefly discuss the method we are going to adopt. We will use the Schauder fixed point theorem to achieve existence. This requires us to obtain a number of new estimates involving the heat kernel. These estimates are presented in sections two and six. Theorem A, B and C will be proved in sections three, four and five respectively.

We conclude the introduction by listing notations that will be used frequently in the paper.

$G = G(x,t;y,s)$ will denote the fundamental solution of the linear operator H_0 in (1.1). For any $c > 0$, we write

$$G_c(x,t;y,s) \equiv \frac{1}{(t-s)^{Q/2}} \exp(-c\frac{d(x,y)^2}{t-s}), \qquad t > s. \tag{1.4}$$

Let u_0 be a positive function in $L^\infty(\mathbf{R}^n)$ and $a > 0$, we write

$$h_a(x,t) = \int_{\mathbf{R}^n} G_a(x,t;y,0)u_0(y)dy; \tag{1.5}$$

$$h(x,t) = \int_{\mathbf{R}^n} G(x,t;y,0)u_0(y)dy. \tag{1.6}$$

Given $V = V(x,t)$ and $c > 0$, we define

$$\begin{aligned} N_{c,\infty}(V) \equiv & \sup_{x,t} \int_{-\infty}^{t} \int_{\mathbf{R}^n} |V(y,s)| G_c(x,t;y,s) dy ds \\ & + \sup_{y,s} \int_{s}^{\infty} \int_{\mathbf{R}^n} |V(x,t)| G_c(x,t;y,s) dx dt. \end{aligned} \tag{1.7}$$

We note that the quantity $N_{c,\infty}(V)$ may be infinite for some V. However Proposition 6.1 below shows that $N_{c,\infty}(V) < \infty$ for all $c > 0$ if $|V(y,s)| \leqslant C/(1 + d(y)^\beta)$ for a $\beta > 2$. This fact, which seems unrelevant right know, will allow us to use the Schauder fixed point theorem via Lemma 6.1 in Sect. 6. Before going to the next section we remark that C will always be absolute constants that may change from line to line.

2 Preliminaries

In this section we provide two lemmas concerning $\int_{\mathbf{R}^n} G_a(x,t;y,0)u_0(y)dy$.

Lemma 2.1. *Given $a > 0$, let*

$$h_a(x,t) = \int_{\mathbf{R}^n} G_a(x,t;y,0)u_0(y)dy, \tag{2.1}$$

where u_0 is a bounded non-negative function. The following two statements are true.

(a). Given $p > 1$, there exists a constant $C(p)$ such that

$$h_a^p(x,t) \leqslant C(p)\, ||u_0||_{L^\infty}^{p-1}\, h_a(x,t), \tag{2.2}$$

for all $t > 0$.

(b). If $\lim_{d(x)\to\infty} u_0(x) = 0$, *then* $\lim_{d(x)\to\infty} h_a(x,t) = 0$ *uniformly with respect to* $t > 0$.

Proof. (a). Clearly

$$h_a(x,t) = \int_{\mathbf{R}^n} \frac{1}{t^{Q/2}} e^{-a\ d(x,y)^2/t}\ u_0(y)dy$$
$$= \int_{\mathbf{R}^n} \frac{1}{t^{Q/(2p)}} e^{-a\ d(x,y)^2/(pt)}\ u_0(y)\ \frac{1}{t^{Q/(2q)}} e^{-a\ d(x,y)^2/(qt)} dy,$$

where q is the conjugate of p. By Hölder's inequality and since

$$\int_{\mathbf{R}^n} \frac{1}{t^{Q/2}} e^{-a\ d(x,y)^2/t} dy \leqslant C,$$

we have

$$h_a^p(x,t) \leqslant \int_{\mathbf{R}^n} \frac{1}{t^{Q/2}} e^{-a\ d(x,y)^2/t}\ u_0^p(y)dy\ [\int_{\mathbf{R}^n} \frac{1}{t^{Q/2}} e^{-a\ d(x,y)^2/t} dy]^{p/q}$$
$$\leqslant C(p)\ ||u_0||_{L^\infty}^{p-1} \int_{\mathbf{R}^n} \frac{1}{t^{Q/2}} e^{-a\ d(x,y)^2/t}\ u_0(y)dy.$$

The last inequality implies (2.2). This proves (a).

Next we prove (b). For any $\delta > 0$, let $R > 0$ be such that $u_0(y) < \delta/2$ when $d(y) \geqslant R$. When $d(x) \geqslant 2R$ we have

$$h_a(x,t) = \int_{d(y)>R} \frac{1}{t^{Q/2}} e^{-a\ d(x,y)^2/t}\ u_0(y)dy$$
$$+ \int_{d(y)\leqslant R} \frac{1}{t^{Q/2}} e^{-a\ d(x,y)^2/t}\ u_0(y)dy$$
$$\leqslant \frac{\delta}{2} \int_{d(y)>R} \frac{1}{t^{Q/2}} e^{-a\ d(x,y)^2/t} dy$$
$$+ \int_{d(y)\leqslant R} \frac{1}{d(x,y)^Q} \frac{d(x,y)^Q}{t^{Q/2}}$$
$$\times\ e^{-a\ d(x,y)^2/(2t)} e^{-a\ d(x,y)^2/(2t)} u_0(y)dy$$
$$\leqslant C\delta/2 + \frac{C}{(d(x)-R)^Q} \int_{d(y)\leqslant R} e^{-a\ d(x,y)^2/(2t)} u_0(y)dy$$
$$\leqslant C\delta/2 + C\frac{R^Q}{(d(x)-R)^Q} ||u_0||_{L^\infty}$$
$$\leqslant C\delta,$$

when $d(x)$ is sufficiently large. This proves (b). q.e.d.

Lemma 2.2. *(a). Given $\alpha \in (0,1]$, suppose $u_0 \in L^\infty$, $u_0 \geqslant 0$ and $u_0(x) \leqslant C/d(x)^\alpha$ when $d(x)$ is large, then*

$$h_a(x,t) \leqslant C(1+||u_0||_{L^\infty})/d(x)^{\frac{Q}{Q+1}\alpha}, \tag{2.3}$$

when $d(x)$ is large.

(b). Suppose $0 \leqslant u_0(x) \leqslant C/(1+d(x)^Q)$ and $u_0 \in L^1(\mathbf{R}^n)$, then

$$h_a(x,t) \leqslant C(1+||u_0||_{L^1})/(1+d(x)^Q),$$

for all $t > 0$ and $x \in \mathbf{R}^n$.

Proof. First we prove (a). The proof starts as that of part (b) of the previous lemma. Without loss of generality we assume that $0 < u_0(x) \leqslant C/(1+d(x)^\alpha)$ for a constant C. Given $R > 0$ and when $d(x) > R$ we have

$$\begin{aligned} h_a(x,t) &= \int_{d(y)>R} \frac{1}{t^{Q/2}} e^{-a\ d(x,y)^2/t}\ u_0(y)dy \\ &\quad + \int_{d(y)\leqslant R} \frac{1}{t^{Q/2}} e^{-a\ d(x,y)^2/t}\ u_0(y)dy \\ &\leqslant \frac{C}{1+R^\alpha} \int_{d(y)>R} \frac{1}{t^{Q/2}} e^{-a\ d(x,y)^2/t} dy \\ &\quad + \int_{d(y)\leqslant R} \frac{1}{d(x,y)^Q} \frac{d(x,y)^Q}{t^{Q/2}} \\ &\quad \times e^{-a\ d(x,y)^2/(2t)} e^{-a\ d(x,y)^2/(2t)} u_0(y)dy \\ &\leqslant \frac{C}{1+R^\alpha} + \frac{C}{(d(x)-R)^Q} \int_{d(y)\leqslant R} e^{-a\ d(x,y)^2/(2t)} u_0(y)dy \\ &\leqslant \frac{C}{1+R^\alpha} + C\frac{R^Q}{(d(x)-R)^Q} ||u_0||_{L^\infty}. \end{aligned}$$

In the above we have used the fact that $\int_{\mathbf{R}^n} \frac{1}{t^{Q/2}} e^{-a\ d(x,y)^2/t} dy \leqslant C$.

For each x, we pick R so that $d(x) = R(1+d(x)^{1/(Q+1)})$. Then

$$h_a(x,t) \leqslant \frac{C}{1+[d(x)/(1+d(x)^{1/(Q+1)})]^\alpha} + \frac{C||u_0||_{L^\infty}}{d(x)^{Q/(Q+1)}}.$$

When $d(x) \geqslant 1$, the last inequality implies

$$h_a(x,t) \leqslant \frac{C(1+||u_0||_{L^\infty})}{d(x)^{\alpha Q/(Q+1)}},$$

when $\alpha \in (0,1]$.

Next we prove (b). Following the proof of part (a) of the lemma (replace α by Q) we find, for $d(x) > R$,

$$
\begin{aligned}
h_a(x,t) &= \int_{d(y)>R} \frac{1}{t^{Q/2}} e^{-a\ d(x,y)^2/t}\ u_0(y)dy \\
&\quad + \int_{d(y)\leqslant R} \frac{1}{t^{Q/2}} e^{-a\ d(x,y)^2/t}\ u_0(y)dy \\
&\leqslant \frac{C}{1+R^Q} + \frac{C}{(d(x)-R)^Q} \int_{d(y)\leqslant R} e^{-a\ d(x,y)^2/(2t)} u_0(y)dy \\
&\leqslant \frac{C}{1+R^Q} + \frac{C}{(d(x)-R)^Q} ||u_0||_{L^1(\mathbf{R}^n)}.
\end{aligned}
$$

Part (b) is thus proved by taking R so that $d(x) = 2R$. q.e.d.

3 Proof of Theorem A

This section is divided into three parts. In the first part we list a number of notations and symbols, which include an integral operator and an appropriate function space. In the next part we will prove Lemma 3.1 which states that the integral operator has a fixed point in the function space. Theorem A will be proved in the end of the section.

First we recall and define a number of notations. Given a positive $u_0 \in L^\infty(\mathbf{R}^n)$, write

$$h(x,t) = \int_{\mathbf{R}^n} G(x,t;y,0)u_0(y)dy. \tag{3.1}$$

Here G is the fundamental solution of the operator H_0 in (1.1). By (1.3), there are positive constants C and b such that

$$G(x,t;y,s) \leqslant \frac{C}{(t-s)^{Q/2}} \exp(-b\frac{d(x,y)^2}{t-s}) = CG_b(x,t;y,s),$$

for all $t > s$ and $x, y \in \mathbf{R}^n$.

For $u \in L^\infty(\mathbf{R}^n \times [0,\infty))$, we define T to be the integral operator:

$$Tu\ (x,t) = h(x,t) + \int_0^t \int_{\mathbf{R}^n} G(x,t;y,s)u^p(y,s)dyds. \tag{3.2}$$

For any constants $a > 0$, $d > 1$ and $M > 1$, the space S_d is defined by

$$S_d = \{u(x,t) \in C(\mathbf{R}^n \times [0,d]) \mid 0 \leqslant u(x,t) \leqslant Mh_a(x,t)\}$$

where the function h_a is given by (2.1).

From this moment, we fix the number a to be a positive number strictly less than b, which is the constant in the Gaussian upper bound for G. This choice of a is crucial when we prove Lemma 3.1 below. Since $a < b$ we have

$$G(x,t;y,s) \leqslant CG_b(x,t;y,s) \leqslant CG_a(x,t;y,s),$$

$$h(x,t) \leqslant Ch_b(x,t) \leqslant Ch_a(x,t).$$

Next we present a Lemma which will lead to a proof of Theorem A. The idea is to show that the operator T has a fixed point in S_d.

Lemma 3.1. *Given $p > 1 + \frac{2}{Q}$, for any $d > 1$, there exist constants $C > 1$, $M > 1$ and $b_0 > 0$ independent of d such that the integral operator (3.2) has a fixed point in S_d, provided that $u_0 \in C^2(\mathbf{R}^n) \cap L^1(\mathbf{R}^n)$ and $0 \leqslant u_0 \leqslant b_0/(1+d(x)^Q)$.*

Proof. Step 1. We want to use the Schauder fixed point theorem. To this end we need to check the following conditions.

(i). S_d is nonempty, closed, bounded and convex.
(ii). $TS_d \subset S_d$.
(iii). TS_d is a compact subset of S_d in L^∞ norm.
(iv). T is continuous.

Step 2. Condition (i) is obviously true. So let's verify (ii), which requires us to show that $0 \leqslant Tu \leqslant Mh_a$ when $0 \leqslant u \leqslant Mh_a$.

Since $p > 1 + \frac{2}{Q}$ we can write $p = p_1 + p_2$ such that $p_1 > 1$ and $p_2 > \frac{2}{Q}$. For any $u \in S_d$, $u \leqslant Mh_a$; hence Lemma 2.1 (a) implies, since $||u_0||_{L^\infty} \leqslant b_0$,

$$u^{p_1}(y,s) \leqslant CM^{p_1}||u_0||^{p_1-1}_{L^\infty} h_a(y,s) \leqslant CM^{p_1} b_0^{p_1-1} h_a(y,s).$$

Lemma 2.2 (b) implies

$$u^{p_2}(y,s) \leqslant M^{p_2} h_a^{p_2}(y,s) \leqslant M^{p_2} [\frac{C}{1+d(y)^Q}]^{p_2} \leqslant CM^{p_2} \frac{1}{1+d(y)^{p_2 Q}}, \tag{3.3}$$

for all $s > 0$. For convenience we write

$$V(y) = \frac{1}{1+d(y)^{p_2 Q}}.$$

Therefore

$$u^{p_2}(y,s) \leqslant CM^{p_2} V(y).$$

Recalling the definition of h_a in (2.1) and using the previous inequalities, we have

$$
\begin{aligned}
u^p(y,s) &= u^{p_1}(y,s)u^{p_2}(y,s) \\
&\leqslant CM^p b_0^{p_1-1} V(y) \int_{\mathbf{R}^n} G_a(y,s;z,0)u_0(z)dz.
\end{aligned} \tag{3.4}
$$

Substituting (3.4) into (3.2) and using Fubini's theorem we obtain

$$
\begin{aligned}
Tu(x,t) \leqslant h(x,t) + CM^p b_0^{p_1-1} \int_{\mathbf{R}^n} \int_0^t \int_{\mathbf{R}^n} \\
\times G(x,t;y,s)|V(y)|G_a(y,s;z,0)dyds\, u_0(z)dz,
\end{aligned} \tag{3.5}
$$

Remembering that

$$
G(x,t;y,s) \leqslant \frac{C}{(t-s)^{Q/2}} \exp(-b\, d(x,y)^2/(t-s)) = CG_b(x,t;y,s),
$$

we have

$$
\begin{aligned}
&\int_0^t \int_{\mathbf{R}^n} G(x,t;y,s)|V(y)|G_a(y,s;z,0)dyds \\
&\qquad \leqslant C \int_0^t \int_{\mathbf{R}^n} G_b(x,t;y,s)|V(y)|G_a(y,s;z,0)dyds
\end{aligned}
$$

At this stage we quote the following inequality (3.6) which was first proved in [Zhang3] for the Euclidean case. We will prove (3.6) as part of Lemma 6.1 in Sect. 6. Given $b > a$,

$$
\begin{aligned}
&\int_0^t \int_{\mathbf{R}^n} G_b(x,t;y,s)|V(y)|G_a(y,s;z,0)dyds \\
&\leqslant CC_{a,b} N_{c,\infty}(V) G_a(x,t;z,0),
\end{aligned} \tag{3.6}
$$

for all $t > 0$ and some positive c and $C_{a,b}$. Here the expression $N_{c,\infty}(V)$, defined in (1.7), denotes a number related to any given function V. By Proposition 6.1 in Sect. 5, we know that $N_{c,\infty}(V)$ is a finite number since $V(y) = 1/(1 + d(y)^{p_2 Q})$ and $p_2 Q > 2$ by our choice of $p_2 > \frac{2}{Q}$.

Combining (3.6) with (3.5), we reach

$$
Tu(x,t) \leqslant h(x,t) + CM^p b_0^{p_1-1} C_{a,b} N_{c,\infty}(V) \int_{\mathbf{R}^n} G(x,t;z,0)u_0(z)dz,
$$

which yields

$$
Tu(x,t) \leqslant (C + CM^p b_0^{p_1-1} C_{a,b} N_{c,\infty}(V)) h_a(x,t) \tag{3.7}
$$

By taking $M > 2C$ and b_0 suitably small we find that

$$
0 \leqslant Tu(x,t) \leqslant Mh_a(x,t). \tag{3.8}
$$

Thus condition (ii) is satisfied.

Step 3. Now we need to check condition (iii). We note that the local regularity theory for solutions of uniformly parabolic equations can be transplanted to the operator H_0 in (1.1). For brevity we refer the reader to the papers [KS1] or [Sa] for details. By our choice, functions u in S_d are uniformly bounded and therefore, Tu is equicontinuous and in fact Hölder continuous. This is because Tu actually satisfies, in the weak sense, $H_0(Tu) = -u^p$ in $\mathbf{R}^n \times (0, d)$ and $Tu(x, 0) = u_0(x)$ and $u_0 \in C^2(\mathbf{R}^n)$. Taking into account that

$$0 \leqslant \lim_{d(x)\to\infty} Tu(x,t) \leqslant C \lim_{d(x)\to\infty} h_a(x,t) = 0$$

uniformly (by Lemma 2.1 (b)), we know, from (1.2), that

$$\lim_{|x|\to\infty} Tu(x,t) = 0$$

Hence TS_d is a relatively compact subset of S_d. This is an easy modification of the classical Ascoli-Arzela theorem (see [Zhao]). Hence we have verified (iii).

Step 4. Finally we need to check condition (iv).

Given u_1 and u_2 in S_d, we have, by (3.2),

(3.9)

$$(Tu_1 - Tu_2)(x,t) = \int_0^t \int_{\mathbf{R}^n} G(x,t;y,s))\, [u_1^p(y,s) - u_2^p(y,s)] dyds.$$

Next we notice that

$$|u_1^p(y,s) - u_2^p(y,s)| \leqslant p \max\{u_1^{p-1}(y,s), u_2^{p-1}(y,s)\} |u_1(y,s) - u_2(y,s)|.$$

Using (3.3) on u_1^{p-1} and u_2^{p-1} we have

$$|u_1^p(y,s) - u_2^p(y,s)| \leqslant \frac{CM^{p-1}}{(1+d(y)^{p_2 Q})^{(p-1)/p_2}} |u_1(y,s) - u_2(y,s)|.$$

i.e.

$$|u_1^p(y,s) - u_2^p(y,s)| \leqslant CM^{p-1}[V(y)]^{(p-1)/p_2} |u_1(y,s) - u_2(y,s)|.$$

Substituting the last inequality to (3.9) we obtain

$$\begin{aligned} ||Tu_1 - Tu_2||_{L^\infty} &\leqslant ||u_1 - u_2||_{L^\infty} \int_0^t \int_{\mathbf{R}^n} G(x,t;y,s)[V(y)]^{(p-1)/p_2} dyds \\ &\leqslant C||u_1 - u_2||_{L^\infty} N_{c,\infty}(V^{(p-1)/p_2}). \end{aligned}$$

Here c is a suitable constant. Since $p = p_1 + p_2$ and $p_1 > 1$ we know that $p-1 > p_2$. Taking into account that $V(x) \leqslant 1$ we have $N_{c,\infty}(V^{(p-1)/p_2}) \leqslant$

$N_{c,\infty}(V)$. The later is a finite constant due to Proposition 6.1. This proves the continuity of T and the lemma. q.e.d.

Now we are ready to give the

Proof of Theorem A.
For any $d > 1$, let u_d be a fixed point of T in the space S_d as given in Lemma 3.1. Define

$$U_d(x,t) = \begin{cases} u_d(x,t), & t \leqslant d; \\ u_d(x,d), & t > d. \end{cases}$$

Then from the proof of Lemma 3.1 ((3.8) e.g.), we know that $\{U_d\}$ is uniformly bounded and equicontinuous. Hence there is a subsequence $\{U_{d_m} | m = 1, 2, ..\}$ which converges uniformly to a function u in any compact region of $\mathbf{R}^n \times [0,\infty)$. For any fixed $(x,t) \in \mathbf{R}^n \times [0,\infty)$ and m sufficiently large, we know that

$$U_{d_m}(x,t) = h(x,t) + \int_0^t \int_{\mathbf{R}^n} G(x,t;y,s) U^p_{d_m}(y,s) dyds.$$

This is because U_{d_m} is a fixed point of T in S_{d_m}. Now by the dominated convergence theorem, u satisfies

$$u(x,t) = h(x,t) + \int_0^t \int_{\mathbf{R}^n} G(x,t;y,s) u^p(y,s) dyds,$$

for all $(x,t) \in \mathbf{R}^n \times [0,\infty)$. Moreover, by (3.8) We know

$$0 < u(x,t) \leqslant M h_a(x,t)$$

for all $(x,t) \in \mathbf{R}^n \times [0,\infty)$. Clearly u is a global positive solution of (1.1). This finishes the proof. q.e.d.

4 Proof of Theorem B

In this section we shall prove Theorem B.

Proof of Theorem B. The proof closely follows the lines of that of Theorem A except that we will choose a new initial function. From this moment, while keeping other symbols unchanged, we assume that the initial function u_0 satisfies

$$(4.1) \qquad u_0 \in C^2(\mathbf{R}^n), \quad 0 \leqslant u_0 \leqslant b_0 \quad \text{and} \quad u_0(x) \leqslant b_0/(1 + d(x)).$$

Just like the proof of Theorem A, we start by proving the counter part of Lemma 3.1, which is stated as the subsequent claim.

Claim. Given $p > 3 + \frac{2}{Q}$, for and $d > 1$, there exist $C > 1$, $M > 1$ and $b_0 > 0$ independent of d such that the integral operator (3.2) has a fixed point in S_d, provided that the initial value u_0 satisfies (4.1).

The proof of the claim is very similar to the proof of Lemma 3.1. We need to verify that

(i). S_d is nonempty, closed, bounded and convex;

(ii). $TS_d \subset S_d$;

(iii). TS_d is a compact subset of S_d in L^∞ norm;

(iv). T is continuous.

Comparing to the proof of Lemma 3.1, the only significant difference occurs in the proof of (ii), which, for simplicity, will be the only one given in detail below.

We need to show $TS_d \subset S_d$. Since $p > 3 + \frac{2}{Q}$ we can write $p = p_1 + p_2$ such that $p_1 > 1$ and $p_2 > 2 + \frac{2}{Q}$. For any $u \in S_d$, $u \leqslant M h_a$; hence Lemma 2.1 (a) implies

$$u^{p_1}(y,s) \leqslant CM^{p_1} ||u_0||_{L^\infty}^{p_1-1} h_a(y,s) \leqslant CM^{p_1} b_0^{p_1-1} h_a(y,s).$$

Lemma 2.2 (a) with $\alpha = 1$ implies

$$\begin{aligned} u^{p_2}(y,s) &\leqslant M^{p_2} h_a^{p_2}(y,s) \leqslant M^{p_2} [\frac{C}{1+d(y)^{Q/(Q+1)}}]^{p_2} \\ &\leqslant CM^{p_2} \frac{1}{1+d(y)^{p_2 Q/(Q+1)}}, \end{aligned} \tag{4.2}$$

for all $s > 0$. For convenience we write

$$W(y) = \frac{1}{1+d(y)^{p_2 Q/(Q+1)}}.$$

Therefore

$$u^{p_2}(y,s) \leqslant CM^{p_2} W(y)$$

for all $s > 0$. Recalling the definition of h_a in (2.1) and using the last two inequalities, we have

$$\begin{aligned} u^p(y,s) &= u^{p_1}(y,s) u^{p_2}(y,s) \\ &\leqslant CM^p b_0^{p_1-1} W(y) \int_{\mathbf{R}^n} G_a(y,s;z,0) u_0(z) dz. \end{aligned} \tag{4.3}$$

Substituting (4.3) into (3.2) and using Fubini's theorem we obtain

$$\begin{aligned} Tu(x,t) \leqslant h(x,t) &+ CM^p b_0^{p_1-1} \int_{\mathbf{R}^n} \int_0^t \int_{\mathbf{R}^n} \\ &\times G(x,t;y,s) |W(y)| G_a(y,s;z,0) dyds\, u_0(z) dz, \end{aligned} \tag{4.4}$$

Remembering that

$$G(x,t;y,s) \leqslant \frac{C}{(t-s)^{Q/2}} \exp(-b\, d(x,y)^2/(t-s)) = CG_b(x,t;y,s),$$

we have

$$\begin{aligned}\int_0^t \int_{\mathbf{R}^n} G(x,t;y,s)|W(y)|G_a(y,s;z,0)dyds \\ \leqslant C \int_0^t \int_{\mathbf{R}^n} G_b(x,t;y,s)|W(y)|G_a(y,s;z,0)dyds \\ \leqslant CC_{a,b}N_{c,\infty}(W)G_a(x,t;z,0).\end{aligned}$$

To reach the last inequality, we again quote (3.6) which will be proved as Lemma 6.1 in Sect. 6. By Proposition 6.1 in Sect. 5, we know that $N_{c,\infty}(W)$ is a finite number since $W(y) = 1/(1+d(y)^{p_2 Q/(Q+1)})$ and $p_2 Q/(Q+1) > 2$ by our choice of $p_2 > 2 + \frac{2}{Q}$.

Substituting the last inequality into (4.4), we obtain

$$Tu(x,t) \leqslant h(x,t) + CM^p b_0^{p_1-1} C_{a,b} N_{c,\infty}(W) \int_{\mathbf{R}^n} G(x,t;z,0)u_0(z)dz,$$

which yields

$$Tu(x,t) \leqslant (C + CM^p b_0^{p_1-1} C_{a,b} N_{c,\infty}(W))h_a(x,t) \tag{4.5}$$

By taking $M > 2C$ and b_0 suitably small we find that

$$0 \leqslant Tu(x,t) \leqslant Mh_a(x,t). \tag{4.6}$$

This proves the claim. The rest of the proof for Theorem B is identical to that of Theorem A and is hence omitted. q.e.d.

5 Proof of Theorem C

Proof of part (a). In this part, H_0 in (1.1) is just an uniformly parabolic operator in $\mathbf{R}^n \times (0,\infty)$ and the distance $d(x,y)$ is just the Euclidean one.

Suppose u is a global positive solution of (1.1), by Definition 1.1, we know that u solves the integral equation

(5.1)

$$u(x,t) = \int_{\mathbf{R}^n} G(x,t;y,0)u_0(y)dy + \int_0^t \int_{\mathbf{R}^n} G(x,t;y,s)u^p(y,s)dyds,$$

for all $(x,t) \in \mathbf{R}^n \times [0,\infty)$. By the lower bound in (1.3) and using the notation in (1.4), we can find positive constants c and C such that

$$G(x,t;y,s) \geqslant CG_c(x,t;y,s) \tag{5.2}$$

for all $x, y \in \mathbf{R}^n$ and $t > s$. Merging (5.1) with (5.2), we obtain

$$u(x,t) \geqslant C \int_{\mathbf{R}^n} G_c(x,t;y,0) u_0(y) dy$$

$$+C \int_0^t \int_{\mathbf{R}^n} G_c(x,t;y,s) u^p(y,s) dy ds, \tag{5.3}$$

for all $(x,t) \in \mathbf{R}^n \times [0,\infty)$.

Given $t > 0$, choosing $T > t$, multiplying $G_c(x,T;0,t)$ on both sides of (5.3) and integrating with respect ot x, we obtain

$$\int_{\mathbf{R}^n} G_c(x,T;0,t) u(x,t) dx$$

$$\geqslant C \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} G_c(x,T;0,t)\, G_c(x,t;y,0) dx\, u_0(y) dy +$$

$$+C \int_0^t \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} G_c(x,T;0,t)\, G_c(x,t;y,s) dx\, u^p(y,s) dy ds. \tag{5.4}$$

Since we are dealing with Euclidean distance, we have $G_c(x,T;0,t) = G_c(0,T;x,t)$. Therefore by the reproducing property of the heat kernel, we reach

$$\int_{\mathbf{R}^n} G_c(x,T;0,t)\, G_c(x,t;y,0) dx = C G_c(0,T;y,0) = C G_c(y,T;0,0),$$

$$\int_{\mathbf{R}^n} G_c(x,T;0,t)\, G_c(x,t;y,s) dx = C G_c(0,T;y,s) = C G_c(y,T;0,s).$$

Substituting the last two equalities into (5.4), we see that

$$\int_{\mathbf{R}^n} G_c(x,T;0,t) u(x,t) dx$$

$$\geqslant C \int_{\mathbf{R}^n} G_c(y,T;0,0) u_0(y) dy \tag{5.5}$$

$$+ C \int_0^t \int_{\mathbf{R}^n} G_c(y,T;0,s) u^p(y,s) dy ds.$$

Using Hölder's inequality and the fact that $\int_{\mathbf{R}^n} G_c(y,T;0,s) dy = C$, we obtain

$$\int_{\mathbf{R}^n} G_c(y,T;0,s) u(y,s) dy$$

$$= \int_{\mathbf{R}^n} G_c^{1/q}(y,T;0,s) G_c^{1/p}(y,T;0,s) u(y,s) dy$$

$$\leqslant C [\int_{\mathbf{R}^n} G_c(y,T;0,s) u^p(y,s) dy]^{1/p},$$

where $1/p+1/q=1$. Inequality (5.5) then implies

$$\begin{aligned}&\int_{\mathbf{R}^n} G_c(x,T;0,t)u(x,t)dx\\ &\qquad\geqslant C\int_{\mathbf{R}^n} G_c(y,T;0,0)u_0(y)dy\\ &\qquad+C\int_0^t[\int_{\mathbf{R}^n} G_c(y,T;0,s)u(y,s)dy]^p ds.\end{aligned}\tag{5.6}$$

Without loss of generality we can assume that $u_0(x)\geqslant\epsilon$ when $d(x)\leqslant\delta$ where ϵ and δ are two small positive numbers. This assumption together with (1.4) enable us to find a constant $C>0$ so that

$$\begin{aligned}&\int_{\mathbf{R}^n} G_c(y,T;0,0)u_0(y)dy\\ &\qquad\geqslant\int_{d(y)\leqslant\delta}\frac{1}{T^{n/2}}e^{-cd^2(y)/T}\epsilon dy\geqslant C/T^{n/2},\qquad T>1.\end{aligned}\tag{5.7}$$

Going back to (5.6) and writing $J(t)\equiv\int_{\mathbf{R}^n} G_c(x,T;0,t)u(x,t)dx$, we have

$$J(t)\geqslant C/T^{n/2}+C\int_0^t J^p(s)ds,\quad T>t, T>1.\tag{5.8}$$

Using the notation $g(t)\equiv\int_0^t J^p(s)ds$, we obtain, from (5.8),

$$g'(t)/(1/T^{n/2}+g(t))^p\geqslant C.\tag{5.9}$$

Integrating (5.9) from 0 to T and noticing $g(0)=0$, we have

$$-\frac{1}{(1/T^{n/2}+g(t))^{p-1}}\Big|_0^T\geqslant(p-1)CT$$

and therefore

$$T^{n(p-1)/2}\geqslant(p-1)CT,\tag{5.10}$$

for all $T>1$. This is possible only when $p\geqslant 1+\frac{2}{n}$. Part (a) is thus proved.

Proof of part (b). In this part, H_0 in (1.1) is the heat equation in $\mathbf{H}^m\times(0,\infty)$.

Suppose u is a global positive solution of (1.1), by Definition 1.1, we know that u solves the integral equation

(5.11)

$$u(x,t)=\int_{\mathbf{R}^n} G(x,t;y,0)u_0(y)dy+\int_0^t\int_{\mathbf{R}^n} G(x,t;y,s)u^p(y,s)dyds,$$

for all $(x,t)\in\mathbf{R}^n\times[0,\infty)$. Here $n=2m+1$.

Given $t > 0$, choosing $T > t$, multiplying $G(x, T; 0, t)$ on both sides of (5.11) and integrating with respect ot x, we obtain

(5.12)
$$\begin{aligned}&\int_{\mathbf{R}^n} G(x,T;0,t)u(x,t)dx \\ &\qquad \geqslant C\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} G(x,T;0,t)\ G(x,t;y,0)dx\ u_0(y)dy + \\ &\qquad + C\int_0^t\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} G(x,T;0,t)\ G(x,t;y,s)dx\ u^p(y,s)dyds.\end{aligned}$$

Even though H_0 is an operator with variable coefficients, the fundamental solution G still enjoys the symmetry

$$G(x,T;y,t) = G(y,T;x,t)$$

for all $x, y \in \mathbf{H}^m$ and $T > t$ (see [G]). Therefore by the reproducing property of the heat kernel, we reach

$$\int_{\mathbf{R}^n} G(x,T;0,t)\ G(x,t;y,0)dx = CG(0,T;y,0) = CG(y,T;0,0),$$

$$\int_{\mathbf{R}^n} G(x,T;0,t)\ G(x,t;y,s)dx = CG(0,T;y,s) = CG(y,T;0,s).$$

Substituting the last two equalities into (5.4), we see that

$$\begin{aligned}&\int_{\mathbf{R}^n} G(x,T;0,t)u(x,t)dx \\ &\qquad \geqslant C\int_{\mathbf{R}^n} G(y,T;0,0)u_0(y)dy \\ &\qquad + C\int_0^t\int_{\mathbf{R}^n} G(y,T;0,s)u^p(y,s)dyds.\end{aligned} \tag{5.13}$$

Using Hölder's inequality and the fact that $\int_{\mathbf{R}^n} G(y,T;0,s)dy = 1$, we obtain

$$\begin{aligned}&\int_{\mathbf{R}^n} G(y,T;0,s)u(y,s)dy \\ &\qquad = \int_{\mathbf{R}^n} G^{1/q}(y,T;0,s)G^{1/p}(y,T;0,s)u(y,s)dy \\ &\qquad \leqslant C[\int_{\mathbf{R}^n} G(y,T;0,s)u^p(y,s)dy]^{1/p},\end{aligned}$$

where $1/p + 1/q = 1$. Inequality (5.13) then implies

$$\begin{aligned} &\int_{\mathbf{R}^n} G(x,T;0,t)u(x,t)dx \\ (5.14) \qquad &\geqslant C \int_{\mathbf{R}^n} G(y,T;0,0)u_0(y)dy \\ &+ C \int_0^t [\int_{\mathbf{R}^n} G(y,T;0,s)u(y,s)dy]^p ds. \end{aligned}$$

The rest of the proof is similar to part (a).

Without loss of generality we assume that u_0 is strictly positive in a neighborhood of 0. Using the lower bound in (1.3) for G, we can then find a constant $C > 0$ so that

$$(5.15) \qquad \int_{\mathbf{R}^n} G(y,T;0,0)u_0(y)dy \geqslant C/T^{Q/2}, \qquad T > 1.$$

Going back to (5.14) and writing $J(t) \equiv \int_{\mathbf{R}^n} G(x,T;0,t)u(x,t)dx$, we have

$$(5.16) \qquad J(t) \geqslant C/T^{Q/2} + C\int_0^t J^p(s)ds, \quad T > t, T > 1.$$

As in part (a), (5.16) implies

$$T^{Q(p-1)/2} \geqslant (p-1)CT,$$

for all $T > 1$. This is possible only when $p \geqslant 1 + \frac{2}{Q}$. q.e.d.

Remark 5.1. In the general case we do not know whether the relations

$$\int_{\mathbf{R}^n} G_c(x,t;z,\tau)G_c(z,\tau;y,s)dz \geqslant CG_c(x,t;y,s)$$

or $G(x,t;y,s) = G(y,t;x,s)$ still hold. This is the difficulty in adopting the proof of this section to treat (1.1) under the basic assumptions in Sect. 1.

6 Two inequalities

In this section we mainly present Proposition 6.1 and Lemma 6.1. The later contains two new inequalities involving heat kernels, including (3.6) which we used in the proof of the Theorems A and B. The proof of the inequalities in the Euclidean case was first given in [Zhang3].

Proposition 6.1. *Suppose* $V = V(x,t)$ *satifies* $0 \leqslant V(x,t) \leqslant \frac{C}{1+d(x)^\beta}$ *for* $\beta > 2$*, then*

$$N_{c,\infty}(V) < \infty \qquad \textit{for all} \quad c > 0.$$

Proof. Observing that

$$\int_{-\infty}^t G_c(x,t;y,s)ds \leqslant C/d(x,y)^{Q-2},$$
$$\int_s^\infty G_c(x,t;y,s)dt \leqslant C/d(x,y)^{Q-2},$$

we know

$$\begin{aligned}
N_{c,\infty}(V) &\leqslant \sup_{x\in\mathbf{R}^n} \int_{\mathbf{R}^n} \frac{C}{1+d(y)^\beta}\frac{C}{d(x,y)^{Q-2}}dy \\
&= \sup_{x\in\mathbf{R}^n} \int_{d(x,y)\geqslant d(y)} \frac{C}{1+d(y)^\beta}\frac{C}{d(x,y)^{Q-2}}dy + \\
&\qquad \sup_{x\in\mathbf{R}^n} \int_{d(x,y)\leqslant d(y)} \frac{C}{1+d(y)^\beta}\frac{C}{d(x,y)^{Q-2}}dy \\
&\leqslant \sup_{x\in\mathbf{R}^n} \int_{d(x,y)\leqslant d(y)} \frac{C}{1+d(y)^\beta}\frac{C}{d(y)^{Q-2}}dy + \\
&\qquad \sup_{x\in\mathbf{R}^n} \int_{d(x,y)\leqslant d(y)} \frac{C}{1+d(x,y)^\beta}\frac{C}{d(x,y)^{Q-2}}dy \\
&\leqslant \int_{\mathbf{R}^n} \frac{C}{1+d(y)^\beta}\frac{C}{d(y)^{Q-2}}dy \\
&\quad + \sup_{x\in\mathbf{R}^n} \int_{\mathbf{R}^n} \frac{C}{1+d(x,y)^\beta}\frac{C}{d(x,y)^{Q-2}}dy \\
&\leqslant C\int_0^\infty \frac{r}{1+r^\beta}dr.
\end{aligned}$$

Since $\beta > 2$, we know $N_{c,\infty}(V) < \infty$.q.e.d.

Lemma 6.1. *Suppose* $0 < a < b$*, there exist positive constants* $C_{a,b}$ *and* c *depending only on* a *and* b *such that*

$$\begin{aligned}
(i).\ & \int_s^t \int_{\mathbf{R}^n} G_a(x,t;z,\tau)\,|V(z,\tau)|\,G_b(z,\tau;y,s)dzd\tau \\
& \leqslant C_{a,b}\, N_{c,\infty}(V) G_a(x,t;y,s); \\
(ii).\ & \int_s^t \int_{\mathbf{R}^n} G_b(x,t;z,\tau)\,|V(z,\tau)|\,G_a(z,\tau;y,s)dzd\tau \\
& \leqslant C_{a,b}\, N_{c,\infty}(V) G_a(x,t;y,s).
\end{aligned}$$

Remark 6.1. The condition $a < b$ is indispensable for Lemma 6.1.

Proof of the Lemma. We will only give a proof of (i) since (ii) can be handled similarly. For simplicity we write

$$J(x,t;y,s) = \int_s^t \int_{\mathbf{R}^n} G_a(x,t;z,\tau)\,|V(z,\tau)|\,G_b(z,\tau;y,s)dzd\tau. \tag{6.1}$$

Clearly we can assume that $s = 0$ and hence we only need to show that

$$J(x,t;y,0) \leqslant CN_{c,\infty}(V)G_a(x,t;y,0). \tag{6.2}$$

Let $\rho \in (0,1)$ to be chosen later, we have

$$\begin{aligned} J(x,t;y,0) &= \int_0^{\rho t} \int_{\mathbf{R}^n} G_a\,|V|\,G_b dzd\tau + \int_{\rho t}^t \int_{\mathbf{R}^n} ...dzd\tau \\ &\equiv J_1 + J_2. \end{aligned} \tag{6.3}$$

We will estimate J_1 first. To this end let us recall the inequality

$$\frac{d(x,z)^2}{t-\tau} + \frac{d(z,y)^2}{\tau - s} \geqslant \frac{d(x,y)^2}{t-s}, \quad 0 < s < \tau < t. \tag{6.4}$$

By (6.3) we know that

$$\begin{aligned} J_1 &= \int_0^{\rho t} \int_{\mathbf{R}^n} \frac{\exp(-a\frac{d(x,z)^2}{t-\tau})}{(t-\tau)^{Q/2}}\,|V(z,\tau)|\,\frac{\exp(-b\frac{d(z,y)^2}{\tau})}{(\tau)^{Q/2}} dzd\tau \\ &= \int_0^{\rho t} \int_{\mathbf{R}^n} \frac{\exp(-a[\frac{d(x,z)^2}{t-\tau} + \frac{d(z,y)^2}{\tau}])}{(t-\tau)^{Q/2}}\,|V(z,\tau)| \\ &\quad \times \frac{\exp(-(b-a)\frac{d(z,y)^2}{\tau})}{(\tau)^{Q/2}} dzd\tau \end{aligned} \tag{6.5}$$

By (6.4),

$$\exp(-a[\frac{d(x,z)^2}{t-\tau} + \frac{d(z,y)^2}{\tau}]) \leqslant \exp(-a\frac{d(x,y)^2}{t}).$$

Moreover $t - \tau \geqslant (1-\rho)t$ for $\tau \in (0, \rho t)$. Therefore (3.5) implies

$$J_1 \leqslant \frac{\exp(-a\frac{d(x,y)^2}{t})}{((1-\rho)t)^{Q/2}} \int_0^{\rho t} \int_{\mathbf{R}^n} |V(z,\tau)|\,\frac{\exp(-(b-a)\frac{d(z,y)^2}{\tau})}{(\tau)^{Q/2}} dzd\tau,$$

which means

$$J_1 \leqslant (1-\rho)^{-Q/2} N_{b-a,\infty}(V) G_a(x,t;y,0). \tag{6.6}$$

Next we estimate J_2. From (6.3) we have

(6.7)

$$
\begin{aligned}
J_2 &= \int_{\rho t}^t \int_{\mathbf{R}^n} \frac{\exp(-a\frac{d(x,z)^2}{t-\tau})}{(t-\tau)^{Q/2}}\, |V(z,\tau)|\, \frac{\exp(-b\frac{d(z,y)^2}{\tau})}{(\tau)^{Q/2}} dz d\tau \\
&= \int_{\rho t}^t \int_{d(z,y)\geqslant d(x,y)(a/b)^{1/2}} ...dzd\tau + \int_{\rho t}^t \int_{d(z,y)\leqslant d(x,y)(a/b)^{1/2}} ...dzd\tau \\
&\equiv J_{21} + J_{22}.
\end{aligned}
$$

When $d(z,y) \geqslant d(x,y)(a/b)^{1/2}$ and $\tau \in (\rho t, t)$, then

$$
\frac{\exp(-b\frac{d(z,y)^2}{\tau})}{(\tau)^{Q/2}} \leqslant \frac{\exp(-a\frac{d(x,y)^2}{t})}{(\rho t)^{Q/2}}.
$$

Therefore

$$
\begin{aligned}
J_{21} \leqslant \frac{\exp(-a\frac{d(x,y)^2}{t})}{(\rho t)^{Q/2}} \int_{\rho t}^t \int_{d(z,y)\geqslant d(x,y)(a/b)^{1/2}} \\
\times \frac{\exp(-a\frac{d(x,z)^2}{t-\tau})}{(t-\tau)^{Q/2}}\, |V(z,\tau)| dz d\tau,
\end{aligned}
$$

which gives

$$
J_{21} \leqslant (\rho)^{-Q/2} N_{a,\infty}(V) G_a(x,t;y,0). \tag{6.8}
$$

Finally we will estimate J_{22}. From (6.7), we have

(6.9)

$$
J_{22} \leqslant (\rho t)^{-Q/2} \int_{\rho t}^t \int_{d(z,y)\leqslant d(x,y)(a/b)^{1/2}} \frac{\exp(-a\frac{d(x,z)^2}{t-\tau})}{(t-\tau)^{Q/2}}\, |V(z,\tau)| dz d\tau.
$$

If $d(z,y) \leqslant d(x,y)(a/b)^{1/2}$, then

$$
d(x,z) \geqslant d(x,y) - d(z,y) \geqslant d(x,y)\ (1-(a/b)^{1/2}).
$$

Hence

$$
\begin{aligned}
\exp(-a\frac{d(x,z)^2}{t-\tau}) &= \exp(-a\frac{d(x,z)^2}{2(t-\tau)}) \exp(-a\frac{d(x,z)^2}{2(t-\tau)}) \\
&\leqslant \exp(-a\frac{d(x,z)^2}{2(t-\tau)}) \exp(-a\frac{d(x,y)^2}{2(t-\tau)}(1-(a/b)^{1/2})^2) \\
&\leqslant \exp(-a\frac{d(x,z)^2}{2(t-\tau)}) \exp(-a\frac{d(x,y)^2}{2(1-\rho)t}(1-(a/b)^{1/2})^2).
\end{aligned}
$$

Here we have used the fact that $0 < t - \tau \leqslant (1-\rho)t$. Now taking ρ so that

$$\frac{(1-(a/b)^{1/2})^2}{2(1-\rho)} = 1, \tag{6.10}$$

we obtain,

$$\exp(-a\frac{d(x,z)^2}{t-\tau}) \leqslant \exp(-a\frac{d(x,z)^2}{2(t-\tau)}) \exp(-a\frac{d(x,y)^2}{t}). \tag{6.11}$$

Substituting (6.11) to (6.9) we have

$$J_{22} \leqslant \frac{\exp(-a\frac{d(x,y)^2}{t})}{(\rho t)^{Q/2}} \int_{\rho t}^{t} \int_{d(z,y)\leqslant d(x,y)(a/b)^{1/2}} \frac{\exp(-a\frac{d(x,z)^2}{2(t-\tau)})}{(t-\tau)^{Q/2}} |V(z,\tau)| dz d\tau,$$

which yields

$$J_{22} \leqslant (\rho)^{-Q/2} N_{a/2,\infty}(V) G_a(x,t;y,0). \tag{6.12}$$

Combining (6.8) and (6.12) we have

$$J_2 \leqslant 2(\rho)^{-Q/2} N_{a/2,\infty}(V) G_a(x,t;y,0). \tag{6.13}$$

Inequalities (6.6) and (6.13) infer (6.2) with $c = \min\{b-a, a/2\}$ and the lemma. q.e.d.

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References

[A] D.G. Aronson, Non-negative solutions of linear parabolic equations, Annali della Scuola Norm. Sup. Pisa **22** (1968), 607–694.

[FeS] C. Fefferman and A. Sanchez-Calle, Fundamental solutions for second order subelliptic operators, Ann. of Math. **124** (1986), 247–272.

[Fu] H. Fujita, On the blowup of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$, J. Fac. Sci. Univ. Tokyo, Sect I **13** (1966), 109–124.

[G] B. Gaveau, Principle de moindre action, propagation de la chaleur et éstimées sous élliptic sur certain groupes nilpotents, Acta Math. **139** (1977), 95–153.

[KS1] S. Kusuoka and D. Stroock, Applications of Malliavin Calculus, III, J. Fac. Sci. Univ. Tokyo, Sect. 1 A, Math **34** (1987), 391–442.

[KS2] S. Kusuoka and D. Stroock, Long time estimates for the heat kernel associated with a uniformly subelliptic symmetric second order equation, Ann. of Math **127** (1988), 165–189.

[Le] H. Levine, The role of critical exponents in blowup theorems, SIAM Review **32** (1990), 269–288.

[LN] Tzong-Yow Lee and Wei-Ming Ni, Global existence, large time behavior and life span of solutions of a semilinear parabolic Cauchy problem, Tranactions AMS **333** (1992), 365–378.

[LSU] O.A. Ladyzhenskaya and V. A. Solonnikov and N. N. Uralceva, Linear and quasilinear equations of parabolic type, Providence, AMS

[Me] P. Meier, On the critical exponent for reaction-diffusion equations, Arch. Rat. Mech. Anal. **109** (1990), 63–71.

[Mo] J. Moser, A Harnack inequality for parabolic differential equations, Comm. Pure and Appl. Math. **17** (1964), 101–134.

[NSW] A. Nagel, E. M. Stein and S. Wainger Balls and metrics defined by vector fields I, Acta Mathematica **155** (1985), 103–147.

[Sa] L. Saloff-Coste, A note on Poincare, Sobolev and Harnack inequality, IMRN, Duke Math. J. **2** (1992), 27–38.

[U] Y. Uda, The critical exponent for a weakly coupled system of the generalized Fujita type reaction-diffusion equations, ZAMP **46** (1995), 366–383.

[W] X. Wang, On the Cauchy problem for reaction-diffusion equations, Trans. AMS. **337** (1993), 545–590.

[Zhao] Z. Zhao, On the existence of positive solutions of nonlinear elliptic equations- A probabilistic potential theory approach, Duke Math J. **69** (1993), 247–258.

[Zhao2] Z. Zhao, Subcriticality, positivity, and guageability of the Schrödinger operator, Bull. AMS **23** (1990), 513–517.

[Zhang1] Qi Zhang, On a parabolic equation with a singular lower order term, Transactions of AMS **348** (1996), 2811–2844.

[Zhang2] Qi Zhang, On a parabolic equation with a singular lower order term, Part II, Indiana U. Math. J., to appear

[Zhang3] Qi Zhang, Global existence and local continuity of solutions for semilinear parabolic equations, Comm. PDE **22** (1997), 1529–1557.