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Global solvability for certain classes of underdetermined systems of vector fields

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0 Introduction

This work can be regarded as a natural continuation of [BCM], where a study of the regularity of the global solutions to certain systems of vector fields defined over compact manifolds was presented. The general fact (already pointed out in the late fifties by F. Treves; see [T1], Theorem 52.2) that a hypoelliptic linear partial differential operator has a locally solvable transpose, leads us to consider the question of the global solvability for the transpose of the systems considered in [BCM].

Our study is inserted in the context of *involutive structures* as described in [T3]. Let M denote a compact, connected, orientable, real analytic manifold of dimension n. To a real analytic, complex, closed 1-form ω over M we associate, in a natural way, an involutive structure over $M \times S^1$ (S^1 is the unit circle); furthermore, to such a structure we attach a complex of differential operators $\{\mathbb{L}^j\}_{0 \leqq j \leqq n-1}$ defined by the vector fields orthogonal to $dx - \omega$ (x denotes the angular coordinate in S^1).

One of the results in [BCM] gives, when the characteristic set of the involutive structure is free from singularities, a necessary and sufficient condition for the global hypoellipticity of $\mathbb{L}^0$. The transpose of $\mathbb{L}^0$ is identified to $\mathbb{L}^{n-1}$ and the standard functional analysis argument applies: the global hypoellipticity of $\mathbb{L}^0$ implies the global solvability of $\mathbb{L}^{n-1}$. This fact is the starting point of our study and, under the above mentioned hypothesis on the characteristic set, a necessary and sufficient condition for the global solvability of $\mathbb{L}^{n-1}$ can then be derived (Theorem 1.9). As one would expect in this situation, the presence of the imaginary part of ω brings to the picture the natural generalization of the so-called Condition (P) of Nirenberg–Treves whereas the real part of ω contributes with the diophantine approximation aspects of the problem. It is

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worth recalling that the latter is just relevant because we are dealing with a global question: the corresponding local solvability can be described simply in terms of Condition (P) ([Co-Ho]).

There are two cases for which Theorem 1.9 is a consequence of results obtained previously: when $n = 1$ it follows from results in [H] and when ω is exact it is contained in the main theorem in [Ca-Ho].

Finally we point out that our arguments can be carried out under the hypothesis of the real part of ω being only $\mathscr{C}^\infty$. Furthermore, when the imaginary part of ω vanishes identically, even M can be assumed only $\mathscr{C}^\infty$.

1 Preliminaries and statement of the main theorem

In this work M denotes a compact, orientable, connected, real-analytic manifold of dimension $n \geqq 1$ and S^1 is the unit circle.

Our basic datum is a complex, real-analytic, closed 1-form ω defined on M. To ω we associate the line subbundle $T' \subset \mathbb{C} \otimes T^*(M \times S^1)$ spanned by the 1-form $\vartheta = dx - \omega$, where x denotes the angular variable in S^1. Its orthogonal $\mathscr{L} = (T')^\perp \subset \mathbb{C} \otimes T(M \times S^1)$ is then a vector subbundle of $\mathbb{C} \otimes T(M \times S^1)$ of fiber dimension n that can locally be described as follows: if $(V, t_1, \ldots, t_n)$ is a coordinate system on M such that $d\lambda = \omega$ in V for some $\lambda \in C^\omega(V)$, the pairwise commuting vector fields

$$L_j = \frac{\partial}{\partial t_j} + \frac{\partial \lambda}{\partial t_j} \frac{\partial}{\partial x}, \quad j = 1, \ldots, n \tag{1.1}$$

span $\mathscr{L}$ over $V \times S^1$. Thus $\mathscr{L}$ defines a *locally integrable structure of codimension one over* $M \times S^1$, see [T3].

To the structure $\mathscr{L}$ it is possible to associate, in a natural way, a complex of differential operators. The intrinsic construction of such a complex is given in [T3]; here we briefly recall the definition.

Let $\Lambda^{p,0}$ $(0 \leqq p \leqq n)$ be the subbundle of $\Lambda^p(\mathbb{C} \otimes T^*(M \times S^1))$ characterized by the following property: if $(V, t_1, \ldots, t_n)$ is a coordinate system on M then $\Lambda^{p,0}|_{V \times S^1}$ is spanned by dt_J, $|J| = p$; we are using the notation $J = (j_1, \ldots, j_p)$, $1 \leqq j_1 < j_2 < \cdots < j_p \leqq n$, $dt_J = dt_{j_1} \wedge \cdots \wedge dt_{j_p}$. Due to the isomorphism $C^\infty(M \times S^1; \Lambda^{p,0}) \simeq C^\infty(S^1; \Lambda^p C^\infty(M))$ the exterior derivative d_t in M may be thought of as an operator

$$d_t : C^\infty(M \times S^1; \Lambda^{p,0}) \to C^\infty(M \times S^1; \Lambda^{p+1,0}).$$

Similarly the operator ∂_x may be considered as an endomorphism

$$\partial_x : C^\infty(M \times S^1; \Lambda^{p,0}) \to C^\infty(M \times S^1; \Lambda^{p,0}).$$

For $p \in \{0, 1, \ldots, n\}$ we define

$$\mathbb{L} = \mathbb{L}^p : C^\infty(M \times S^1; \Lambda^{p,0}) \to C^\infty(M \times S^1; \Lambda^{p+1,0}) \tag{1.2}$$

by the expression

$$\mathbb{L}u = (d_t + \omega \wedge \partial_x)u, \quad u \in C^\infty(M \times S^1; \Lambda^{p,0}).$$

Since $\mathbb{L} \circ \mathbb{L} = 0$ (which is in fact equivalent to $d\omega = 0$), (1.2) indeed defines a complex of differential operators.

A similar construction can be carried out if we allow the coefficients of the forms to be distributions: we obtain a new complex by replacing C^∞ by $\mathscr{D}'$ in (1.2). Notice that, if $(V, t_1, \ldots, t_n)$ is a coordinate system on M and if $u \in \mathscr{D}'(M \times S^1; \Lambda^{p,0})$ then we can represent $u|_{V\times S^1} = \sum_{|J|=p} u_J(t,x)\, dt_J$ and

$$(\mathbb{L}u)|_{V\times S^1} = \sum_{j=1}^{n} \sum_{|J|=p} L_j u_J(t,x)\, dt_j \wedge dt_J . \tag{1.3}$$

Our main goal is to find conditions for the global solvability of $\mathbb{L}^{n-1}u = f$ where $f \in C^\infty(M \times S^1; \Lambda^{n,0})$ is given and $u \in \mathscr{D}'(M \times S^1; \Lambda^{n-1,0})$ is sought. There exist natural compatibility conditions on f for the solution u to exist, namely

Lemma 1.1 *If $f \in C^\infty(M\times S^1; \Lambda^{n,0})$ and if there exists $u \in \mathscr{D}'(M\times S^1; \Lambda^{n-1,0})$ with $\mathbb{L}^{n-1}u = f$ then*

$$\int_{M\times S^1} h(t,x) f(t,x) \wedge dx = 0 \tag{1.4}$$

for every $h \in C^\infty(M \times S^1)$ satisfying $\mathbb{L}^0 h = 0$.

Proof. First notice that if $v \in \mathscr{D}'(M \times S^1; \Lambda^{p,0})$ then $\mathbb{L}^p v \wedge \vartheta = d(v \wedge \vartheta)$, where d is the exterior derivative in $M \times S^1$; moreover, when $p = n-1$ we also have $\mathbb{L}^{n-1}v \wedge dx = \mathbb{L}^{n-1}v \wedge \vartheta$. Hence, if u, f and h are as above we obtain

$$\int_{M\times S^1} hf \wedge dx = \int_{M\times S^1} h\mathbb{L}^{n-1}u \wedge dx = \int_{M\times S^1} hd(u \wedge \vartheta)$$

$$= -\int_{M\times S^1} dh \wedge u \wedge \vartheta = -\int_{M\times S^1} \mathbb{L}^0 h \wedge u \wedge \vartheta = 0 . \qquad \square$$

In view of this lemma it is natural to pose the following

Definition 1.2 *Let $\mathbb{E}$ be the set*

$$\mathbb{E} = \left\{ f \in C^\infty(M \times S^1; \Lambda^{n,0}) : \int_{M\times S^1} hf \wedge dx = 0 \right.$$

$$\left. \text{for all } h \in C^\infty(M \times S^1) \text{ such that } \mathbb{L}^0 h = 0 \right\}.$$

We say that $\mathbb{L}^{n-1}$ is **globally solvable** *(GS) if for every $f \in \mathbb{E}$ there exists $u \in \mathscr{D}'(M \times S^1; \Lambda^{n-1,0})$ such that $\mathbb{L}^{n-1}u = f$.*

As will be seen, there is an intimate connection between the global solvability of $\mathbb{L}^{n-1}$ and the global hypoellipticity of $\mathbb{L}^0$; the latter property was the subject of [BCM] and we recall its definition.

Definition 1.3 *We say that* $\mathbb{L}^0$ *is* **globally hypoelliptic** (GH) *if the conditions* $u \in \mathscr{D}'(M \times S^1)$ *and* $\mathbb{L}^0 u \in C^\infty(M \times S^1; \Lambda^{1,0})$ *imply* $u \in C^\infty(M \times S^1)$.

In order to state our theorem we now recall a few definitions and results.

Definition 1.4 (See [BCM]) *For a closed, real* $a \in \Lambda^1 C^\infty(M)$, *we define*:

(i) a *is* **integral** *if* $\frac{1}{2\pi}\int_\sigma a \in \mathbb{Z}$ *for any* 1-*cycle* σ *in* M.

(ii) a *is* **rational** *if there exists* $q \in \mathbb{N}$ *such that* qa *is an integral* 1-*form.*

(iii) a *is* **Liouville** *if* a *is not rational and there exist a sequence of closed, integral* 1-*forms* $\{a_j\}$ *and a sequence of integers* $q_j \geqq 2$ *such that* $\{q_j^j(a - \frac{1}{q_j}a_j)\}$ *is bounded in* $\Lambda^1 C^\infty(M)$.

As in [BCM] we use the notations $\omega = a + ib$ and $\Sigma = \{t \in M : b(t) = 0\}$.

Proposition 1.5 (See [BCM], Proposition 3.1) *There exist an open set* $\mathscr{U}$ *with* $\Sigma \subset \mathscr{U} \subset M$ *and a function* $\varphi \in C^\omega(\mathscr{U})$ *such that* $d\varphi = b$ *in* $\mathscr{U}$, $\varphi = 0$ *on* Σ. *Moreover* φ *is uniquely determined as a germ of an analytic function on* Σ.

In this work we always assume that the following condition is satisfied:

$$\textit{Each component of } \Sigma \textit{ is an embedded analytic submanifold of } M\,. \quad (1.5)$$

We will denote by $\mathscr{A}$ the set of all connected components Σ' of Σ such that either $\varphi = 0$ or $\varphi > 0$ or $\varphi < 0$ in $\mathscr{U}'\backslash\Sigma'$ for some open subset $\Sigma' \subset \mathscr{U}' \subset \mathscr{U}$. Notice that $\mathscr{A} = \{M\}$ when $b \equiv 0$.

Proposition 1.6 (See [BCM], Corollary 5.4) *Assume that* (1.5) *holds. The operator* $\mathbb{L}^0$ *is globally hypoelliptic if and only if each* $\Sigma' \in \mathscr{A}$ *has dimension* $\geqq 1$ *and the pull-back,* $i^*_{\Sigma'}(a)$, *of the* 1-*form* a *to* Σ' *is neither rational nor Liouville.*

Next we recall condition $(\psi)_{n-1}$ from [T2] (see also [Ca-Ho]). Assume that b is exact, that is, there exists $\varphi_\bullet \in C^\infty(M)$ such that $d\varphi_\bullet = b$. Set, for an arbitrary real number r, $M_r^- = \{t \in M : \varphi_\bullet(t) < r\}$, $M_r^+ = \{t \in M : \varphi_\bullet(t) > r\}$, and consider the natural homomorphisms

$$i_{n-1}^\pm : H_{n-1}(M_r^\pm) \to H_{n-1}(M)$$

induced by the inclusion maps $M_r^\pm \subset M$, where $H_{n-1}(M_r^\pm)$ and $H_{n-1}(M)$ stand, respectively, for the $(n-1)^{\text{th}}$ de Rham homology space of $M_r^\pm$ and M over $\mathbb{C}$.

Definition 1.7 (See [T2], [Ca-Ho]) *We say that property* $(\psi)_{n-1}$ *holds* (*for* $\mathbb{L}^{n-1}$) *if* $i_{n-1}^\pm$ *are injective for every real number* r.

Proposition 1.8 (See [Ca-Ho]) *If* ω *is exact then* $\mathbb{L}^{n-1}$ *is globally solvable if and only if* $(\psi)_{n-1}$ *holds.*

We are now ready to state the main result of the present work.

Theorem 1.9 *If* (1.5) *holds then the property*

(I) $\mathbb{L}^{n-1}$ *is globally solvable,*
is equivalent to
(II) *One of the following conditions is satisfied:*

$(\mathrm{II})_1$ $\mathbb{L}^0$ *is globally hypoelliptic;*

$(\mathrm{II})_2$ *b is exact,* $(\psi)_{n-1}$ *holds, a is rational, and if* $q \in \mathbb{N}$ *is the smallest integer such that qa is integral then* $q = \min\{l \in \mathbb{N} : l i^*_{\Sigma'}(a) \text{ is integral}\}$ *for every* $\Sigma' \in \mathscr{A}$.

We now present an example in order to clarify the meaning of condition $(\mathrm{II})_2$: Let $M = \mathbb{R}^2/2\pi\mathbb{Z}^2$ be the two-dimensional torus, where the coordinates are written as $t = (t_1, t_2)$. Take $a = \tau dt_1 + (1/3)dt_2$, where τ is a rational number, and $b = \cos t_1 dt_1$. It is easily seen that condition $(\psi)_1$ is satisfied. Moreover $\Sigma = \{t : t_1 = \pm\pi/2\}$ and both of its components belong to $\mathscr{A}$. If $\tau = 4/3$ then $q = 3$ and $(\mathrm{II})_2$ is satisfied whereas if $\tau = 1/2$ then $q = 6$ and $(\mathrm{II})_2$ is not satisfied.

2 Study of Ker $\mathbb{L}^0$

The purpose of this section is to describe Ker $\mathbb{L}^0$, where $\mathbb{L}^0 = d_t + (a+ib)\wedge\partial_x$ is the operator defined in Sect. 1, acting on $C^\infty(M \times S^1)$.

In the statements below we will make use of the universal covering of M, $\Pi : \widehat{M} \to M$.

Lemma 2.1 *Let* $\alpha, \beta \in \Lambda^1 C^\infty(M)$ *be real and closed and consider the operator* $\mathbb{D} : C^\infty(M) \to \Lambda^1 C^\infty(M)$ *defined by*

$$\mathbb{D} = d_t + i(\alpha + i\beta)\wedge\ .$$

Then:

(i) *if either* β *is non-exact or* α *is non-integral then* Ker $\mathbb{D} = \{0\}$;

(ii) *if* β *is exact and* α *is integral then* Ker $\mathbb{D}$ *is spanned by the function* $e^{-i\psi}\cdot e^{\varphi^0}$ *where* $\psi \in C^\infty(\widehat{M})$ *is such that* $d\psi = \Pi^*\alpha$ *and* $\varphi^0 \in C^\infty(M)$ *is such that* $d\varphi^0 = \beta$.

Proof. Let $u \in C^\infty(M)$ be such that $\mathbb{D}u = 0$. Take $\chi \in C^\infty(\widehat{M})$ such that $d\chi = \Pi^*(\alpha + i\beta)$. Then we have $d_t(\Pi^*u) + id\chi(\Pi^*u) = 0$ and, consequently,

$$d_t(e^{i\chi}\Pi^*u) = 0 \quad \text{in } \widehat{M}\ .$$

Hence $\Pi^*u = ce^{-i\chi}$, for some constant $c \in \mathbb{C}$.

For all $A, B \in \widehat{M}$ with $\Pi(A) = \Pi(B) = p$ we have $u(p) = ce^{-i\chi(A)} = ce^{-i\chi(B)}$. If $c \neq 0$ then, writing $\chi = \psi_\bullet + i\widetilde{\varphi}$, we must have

$$e^{i[\psi_\bullet(B) - \psi_\bullet(A)] - [\widetilde{\varphi}(B) - \widetilde{\varphi}(A)]} = 1\ .$$

This implies $\widetilde{\varphi}(A) = \widetilde{\varphi}(B)$ and $e^{i[\psi_\bullet(B)-\psi_\bullet(A)]} = 1$ for all A, B as above; these facts imply, respectively, that β is exact and α is integral. Thus part (i) is proved.

Furthermore, when β is exact and α is integral, it is easy to see that the function $e^{-i\psi} \cdot e^{\varphi^0}$ indeed belongs to Ker $\mathbb{D}$. The proof is complete. □

We can now describe Ker $\mathbb{L}^0$.

Lemma 2.2 (i) *If either b is non-exact or a is not rational then* Ker $\mathbb{L}^0 = \mathbb{C}$;

(ii) *if b is exact and a is rational then* Ker $\mathbb{L}^0$ *consists of the functions $h \in C^\infty(M \times S^1)$ whose x-Fourier series are of the form $h(t,x) = \sum_{j=-\infty}^{+\infty} h_j(t)$ e^{ijx} where*

$$h_j = \begin{cases} 0, & \text{if } j \neq qN,\ \forall N \in \mathbb{Z} \\ C_{qN} e^{-iN\widetilde{\psi}} \cdot e^{qN\varphi_\bullet}, & \text{if } j = qN,\ N \in \mathbb{Z} \end{cases}$$

where $q = \min\{l \in \mathbb{N} : la \text{ is integral}\}$, $\widetilde{\psi} \in C^\infty(\widehat{M})$ with $d\widetilde{\psi} = \Pi^*(qa)$, $\varphi_\bullet \in C^\infty(M)$ with $d\varphi_\bullet = b$, and $C_{qN} \in \mathbb{C}$.

Proof. Let $h \in C^\infty(M \times S^1)$. By using the x-Fourier series we see that $\mathbb{L}^0 h = 0$ if and only if $\mathbb{D}_j h_j(t) = 0$, for all $j \in \mathbb{Z}$, where $\mathbb{D}_j = d_t + ij(a + ib)\wedge$.

If b is non-exact (respectively, a is not rational) then jb is non-exact (respectively, ja is non-integral) for all $j \in \mathbb{Z}\backslash\{0\}$. Now Lemma 2.1 implies $h_j(t) = 0$, for all $j \in \mathbb{Z}\backslash\{0\}$.

If $j = 0$ then $\mathbb{D}_0 = d_t$ and so $h_0(t) =$ constant, since M is connected.

Assume now that b is exact and a is rational. Then jb is exact for all $j \in \mathbb{Z}$. Also ja is integral if and only if $j = qN$, with $N \in \mathbb{Z}$. Now Lemma 2.1 implies that $h_j(t) = 0$, for all $j \notin q\mathbb{Z}$. Lemma 2.1 also implies that, for $j = qN$, $N \in \mathbb{Z}$, $h_{qN}(t) = C_{qN} e^{-iN\widetilde{\psi}(t)} \cdot e^{qN\varphi_\bullet(t)}$, where $d\widetilde{\psi} = \Pi^*(qa)$. Thus if $\mathbb{L}^0 h = 0$ then h has the form stated in lemma.

It is easy to see that functions of this form indeed belong to Ker $\mathbb{L}^0$. This concludes the proof. □

3 Necessity

Assume that neither $(\mathrm{II})_1$ nor $(\mathrm{II})_2$ hold and reason in the following way. Suppose first that b is not exact; then Proposition 1.6 implies that (3.3) below holds. If otherwise b is exact then the conjunction of Proposition 1.6 with the negation of $(\mathrm{II})_2$ implies that we are in one of the remaining four situations below:

(3.1) $b \equiv 0$ and a is Liouville;

(3.2) b is exact, a is rational, and $(\psi)_{n-1}$ does not hold;

(3.3) b is non-exact and there exists $\Sigma' \in \mathscr{A}$ such that $i^*_{\Sigma'}(a)$ is either rational or Liouville;

(3.4) b is exact, $b \not\equiv 0$, a is not rational, and there exists $\Sigma' \in \mathscr{A}$ such that $i^*_{\Sigma'}(a)$ is either rational or Liouville;

(3.5) b is exact, $b \not\equiv 0$, a is rational, $(\psi)_{n-1}$ holds, and there exist $l \in \mathbb{N}$ and $\Sigma' \in \mathscr{A}$ such that $li^*_{\Sigma'}(a)$ is integral but la is not integral.

We will show that each of these conditions implies $\mathbb{L}^{n-1}$ not globally solvable by violating the inequality appearing in the next lemma. The latter is a variation of the classical result Lemma 6.1.2 of Hörmander [H1], and its proof will be omitted.

Lemma 3.1 *If $\mathbb{L}^{n-1}$ is globally solvable then there exist $C > 0$ and $m \in \mathbb{Z}_+$ such that*

$$\left| \int_{M\times S^1} g(t,x)f(t,x) \wedge dx \right| \leqq C\|f\|_m \|\mathbb{L}^0 g\|_m ,$$

for all $g \in C^\infty(M \times S^1)$ and all $f \in \mathbb{E}$ (here $\| \ \|_m$ denotes some norm which defines the C^m topology).

In order to violate this inequality we will construct sequences $\{f_j\}_{j\in\mathbb{N}} \subset \mathbb{E}$, $\{g_j\}_{j\in\mathbb{N}} \subset C^\infty(M \times S^1)$ such that, for arbitrary $m \in \mathbb{Z}_+$,

$$\|f_j\|_m \|\mathbb{L}^0 g_j\|_m \to 0 \quad \text{as } j \to \infty$$

and

$$\int_{M\times S^1} g_j f_j \wedge dx \to \text{a nonzero number} .$$

In cases (3.3)–(3.5) where there is $\Sigma' \in \mathscr{A}$ we will reason under the assumption that $\varphi > 0$ in $U'\backslash\Sigma'$, the case $\varphi < 0$ being similar.

(3.1) $\Rightarrow\sim$ (I). By assumption there exist a sequence of closed, integral 1-forms $\{a_j\}$ and a sequence of integers $q_j \geqq 2$ such that $\{q_j^j(a - \frac{1}{q_j}a_j)\}$ is bounded in $\Lambda^1 C^\infty(M)$. Let $\psi_j \in C^\infty(\widehat{M})$ be such that $d\psi_j = \Pi^* a_j$ and let also $\Omega \in \Lambda^n C^\infty(M)$ be nowhere vanishing. Recalling that $e^{i\psi_j} \in C^\infty(M)$ (cf. Lemma 2.3 in [BCM]) we set, for $j = 1,2,\ldots$,

$$f_j(t,x) = e^{-i(q_j x - \psi_j(t))}\Omega(t), \qquad g_j(t,x) = e^{i(q_j x - \psi_j(t))} .$$

For each j we have $f_j \in E$ since $\operatorname{Ker} \mathbb{L}^0 = \mathbb{C}$ and

$$\int_{M\times S^1} f_j \wedge dx = \left(\int_M e^{i\psi_j}\Omega \right) \left(\int e^{iq_j x} dx \right) = 0 .$$

Also

$$\int_{M\times S^1} g_j f_j \wedge dx = \int_{M\times S^1} \Omega \wedge dx = 2\pi \int_M \Omega \neq 0 . \tag{3.6}$$

On the other hand

$$\mathbb{L}^0 g_j = iq_j \left(a - \frac{1}{q_j} a_j \right) g_j$$

and a simple computation shows that, for each $m \in \mathbb{Z}_+$, there exists $C_m > 0$ such that, for all $j \in \mathbb{N}$,

$$\|f_j\|_m \leqq C_m q_j^m$$

and

$$\|\mathbb{L}^0 g_j\|_m \leqq C_m q_j^{m+1-j} .$$

Thus

$$\|f_j\|_m \|\mathbb{L}^0 g_j\|_m \to 0, \quad \text{as } j \to \infty . \tag{3.7}$$

Now the conjunction of (3.6) and (3.7) shows that the inequality in Lemma 3.1 is violated.

(3.2) $\Rightarrow\sim$ (I). Here $\operatorname{Ker}\mathbb{L}$ is spanned by the functions

$$e^{-iN(\tilde{\psi}+iq\varphi_\bullet)} \cdot e^{iqNx}, \quad N \in \mathbb{Z},$$

where q is the smallest natural number such that qa is integral, $d\tilde{\psi} = \Pi^*(qa)$, and $d\varphi_\bullet = b$.

Since $(\psi)_{n-1}$ does not hold we may assume (changing $x \mapsto -x$ if necessary) that, for some r, i_r^- is not injective. Thus there exist a closed 1-form $v \in \Lambda^1 C_c^\infty(M_r^-)$ such that v is exact in M but not compactly exact in M_r^-. This allows us to choose $\chi \in C^\infty(M)$ such that $d\chi = v$ and also, due to de Rham's theorem, a form $\mu \in \Lambda^{n-1} C^\infty(M_r^-)$ such that $d\mu = 0$ in M_r^- and $\int_M \mu \wedge v \neq 0$. Next take $r_0 < r$ such that $\operatorname{supp} v \subset M_{r_0}^-$ and for a fixed $\varepsilon > 0$ with $r_0 + 2\varepsilon < r$ we also select $\zeta \in C_c^\infty(M_{r_0+2\varepsilon}^-)$ with $\zeta = 1$ in $M_{r_0+\varepsilon}^-$. We have

$$0 \neq \int_M \mu \wedge v = \int_M \zeta\mu \wedge d\chi = -\int_M d(\zeta\mu) \wedge \chi = -\int \tilde{\mu} \wedge \chi ,$$

where we have set $\tilde{\mu} = d(\zeta\mu)$. We note that $\varphi_\bullet(t) \geqq r_0 + \varepsilon$ on $\operatorname{supp} \tilde{\mu}$.

Set, for $N = 1, 2, \ldots,$

$$f_N(t,x) = e^{-iqN(x-i\varphi_\bullet(t))+iN\widetilde{\psi}(t)} \widetilde{\mu}(t), \qquad g_N(t,x) = e^{iqN(x-i\varphi_\bullet(t))-iN\widetilde{\psi}(t)} \chi(t) .$$

Since $\widetilde{\mu}$ is exact it follows that each f_N belongs to $\mathbb{E}$. Furthermore,

$$\mathbb{L}^0 g_N = e^{iqN(x-i\varphi_\bullet(t))-iN\widetilde{\psi}(t)} \cdot v(t)$$

$$\|\mathbb{L}^0 g_N\|_m \leqq C_m N^m e^{r_0 N_q}$$

$$\|f_N\|_m \leqq C_m N^m e^{-N(r_0+\varepsilon)q}$$

$$\int_{M\times S^1} g_N f_N \wedge dx = 2\pi \int_M \widetilde{\mu} \wedge \chi \neq 0 .$$

Thus the inequality in Lemma 3.1 is also violated in this case.

We now proceed to prove that each of the conditions (3.3), (3.4) and (3.5) imply the negation of (I). First, however, we pause to state a result that will be necessary in the argument.

Lemma 3.2 *Let Σ' be a component of $\Sigma, \Pi' : \hat{\Sigma}' \to \Sigma'$ a universal covering and let $U \supseteq \Sigma'$ be a tubular neighborhood of Σ' in M with an analytic retraction map $\rho : U \to \Sigma'$. Let finally $a \in \Lambda^1 C^\infty(M)$ be real and closed and take $\eta \in C^\infty(U)$ so that $d\eta = a - \rho^*(i_{\Sigma'}^*(a))$.*

(A) *If $i_{\Sigma'}^*(a)$ is integral then there is $\psi^0 \in C^\infty(\hat{\Sigma}')$ such that $e^{i\psi^0} \in C^\infty(\Sigma')$ and $(d_t - ia)[(e^{i\psi^0}) \circ \rho e^{i\eta}] = 0$ in U.*

(B) *If* $i^*_{\Sigma'}(a)$ *is Liouville then there are* $\{\psi_j^0\} \subseteq C^\infty(\hat{\Sigma}')$ *and* $\{q_j\} \subseteq \mathbb{Z}_+$, $2 \leqq q_j \to \infty$ *such that* $\{(e^{i\psi_j^0}) \circ \rho\}$ *and* $\{q_j^j(\mathrm{Re}\,\mathbb{L}^0)[e^{-iq_j(x-\eta)}(e^{i\psi_j^0} \circ \rho)]\}$ *are bounded sequences in* $C^\infty(U)$ *and* $\Lambda^1 C^\infty(U \times S^1)$, *respectively.*

Proof. (A) If $\psi^0 \in C^\infty(\hat{\Sigma}')$ is such that $d_{\hat{\Sigma}'}(\psi^0) = \Pi'^* i^*_{\Sigma'}(a)$ then $e^{i\psi^0} \in C^\infty(\Sigma')$ [cf. Lemma 2.3 in [BCM]). The remaining conclusions in this part follows from direct computation.

(B) We take sequences $\{q_j\} \subseteq \mathbb{Z}_+$, $2 \leqq q_j \to \infty$, and $\{a_j\} \subseteq \Lambda^1 C^\infty(\Sigma')$, each a_j integral, such that $\{q_j^j(i^*_{\Sigma'}(a) - \frac{a_j}{q_j})\}$ is bounded in $\Lambda^1 C^\infty(\Sigma')$. We also take $\psi_j^0 \in C^\infty(\hat{\Sigma}')$ such that $d_{\hat{\Sigma}'}\psi_j^0 = \Pi'^*(a_j)$. Since $\{a_j\}$ is bounded in $\Lambda^1 C^\infty(\Sigma')$ we obtain, by the chain rule, that $\{e^{i\psi_j^0} \circ \rho\}$ is bounded in $C^\infty(U)$. The chain rule also implies that $\{q_j^j(\rho^* i^*_{\Sigma'}(a) - \frac{\rho^*(a_j)}{q_j})\}$ is bounded in $\Lambda^1 C^\infty(U)$. Finally we have

$$\begin{aligned}
&(\mathrm{Re}\,\mathbb{L}^0)[e^{-iq_j(x-\eta)}(e^{i\psi_j^0} \circ \rho)] \\
&= \left\{d_t + \left[\rho^*(i^*_{\Sigma'}(a)) - \frac{\rho^*(a_j)}{q_j} + d\eta + \frac{\rho^*(a_j)}{q_j}\right] \wedge \partial_x\right\} [e^{-iq_j(x-\eta)}(e^{i\psi_j} \circ \rho)] \\
&= -iq_j \left[\rho^*(i^*_{\Sigma'}(a)) - \frac{\rho^*(a_j)}{q_j}\right] [e^{-iq_j(x-\eta)}(e^{i\psi_j} \circ \rho)] \\
&\quad + \left\{d_t + \left[d\eta + \frac{\rho^*(a_j)}{q_j}\right] \wedge \partial_x\right\} [e^{-iq_j(x-\eta)}(e^{i\psi_j} \circ \rho)] \\
&= -iq_j \left[\rho^*(i^*_{\Sigma'}(a)) - \frac{\rho^*(a_j)}{q_j}\right] [e^{-iq_j(x-\eta)}(e^{i\psi_j} \circ \rho)]
\end{aligned}$$

where in the last equality we have used the fact that

$$d_t[(e^{i\psi_j}) \circ \rho] = \rho^*[d_{\Sigma'}(e^{i\psi_j})] = \rho^*[ia_j e^{i\psi_j}] = i\rho^*(a_j)(e^{i\psi_j} \circ \rho)\,. \qquad \square$$

We now prove the implications (3.3) $\Rightarrow\sim$ (I) and (3.4) $\Rightarrow\sim$ (I).

Case (i). Assume that there exists $\Sigma' \in \mathscr{A}$ such that $i^*_{\Sigma'}(a)$ is rational.

Let $q = \min\{l \in \mathbb{N} : l i^*_{\Sigma'}(a) \text{ is integral}\}$. Lemma 3.2(A) implies the existence of $\psi^0 \in C^\infty(\hat{\Sigma}')$ such that $d\psi^0 = \Pi'^*(q i^*_{\Sigma'}(a))$, $e^{i\psi^0} \in C^\infty(\Sigma')$, and, for $j = 1,2,\ldots$,

$$(d_t - ijqa)[(e^{ij\psi^0} \circ \rho)e^{ij\eta}] = 0 \quad \text{in } U\,. \tag{3.8}$$

Take $\chi \in C_c^\infty(U)$ with $\chi = 1$ in a neighborhood of Σ' and $\varphi(t) \geqq \varepsilon > 0$ on $\operatorname{supp}(d\chi)$, for some $\varepsilon > 0$. Finally we select $\theta \in C_c^\infty(U)$ with $\varphi(t) \leqq \varepsilon/2$ on $\operatorname{supp}\theta$ and

$$\int_M \theta(t)\chi(t)\Omega(t) \neq 0\,. \tag{3.9}$$

For $j = 1,2,\ldots$, we define

$$f_j(t,x) = e^{ijq(x-i\varphi(t))}(e^{-ij\psi^0} \circ \rho)e^{-ij\eta(t)}\theta(t)\Omega(t)$$

and

$$g_j(t,x) = e^{-ijq(x-i\varphi(t))}(e^{ij\psi^0} \circ \rho)e^{ij\eta(t)}\chi(t) .$$

We have $\{f_j\} \subset \mathbb{E}$ since $\operatorname{Ker} \mathbb{L}^0 = \mathbb{C}$ and

$$\int_{M\times S^1} f_j \wedge dx = \left(\int_M e^{jq\varphi}(e^{-ij\psi^0} \circ \rho)e^{-ij\eta}\theta\Omega\right)\left(\int_{S^1} e^{ijqx}\,dx\right) = 0 .$$

Also, by (3.9),

$$\int_{M\times S^1} g_j f_j \wedge dx = C \neq 0 \tag{3.10}$$

with C independent of j. On the other hand, by using (3.8), we obtain

$$\mathbb{L}^0 g_j = (d\chi)e^{-ijq(x-i\varphi(t))}(e^{ij\psi^0} \circ \rho)e^{ij\eta(t)} .$$

We have

$$\|f_j\|_m \leqq C_m j^m e^{\varepsilon j/2}, \qquad \|\mathbb{L}^0 g_j\|_m \leqq C_m j^m e^{-\varepsilon j}$$

and, consequently,

$$\|f_j\|_m \, \|\mathbb{L}^0 g_j\|_m \to 0 \quad \text{as } j \to \infty . \tag{3.11}$$

The conjunction of (3.10) and (3.11) show that the inequality in Lemma 3.1 is also violated in this case.

Case (ii). Assume that there exists $\Sigma' \in \mathscr{A}$ such that $i^*_{\Sigma'}(a)$ is Liouville. We take χ as in Case (i) and select $t_0 \in \Sigma'$ and a chart (V,γ) with $\gamma : V \to B \doteq B(0,1)$, $\gamma(t_0) = 0$ and $V \subset \{\chi = 1\}$. Since $\varphi(t_0) = 0$ we have $\varphi(t) \leqq C|\gamma(t)|$, for all $t \in V$ and some constant $C > 0$. Next, choose $\theta \in C_c^\infty(B)$ with $\int \theta \neq 0$ and set, for $j = 1,2,\ldots,$

$$\theta_j(t) = \begin{cases} \theta(q_j\gamma(t)), & t \in V \\ 0, & t \in M\setminus V , \end{cases}$$

where $\{q_j\}$ is as in Lemma 3.2(B). We have $\varphi(t) \leqq C/q_j$ on $\operatorname{supp}\theta_j$.

According to Lemma 3.2(B) we set, for $j = 1,2\ldots,$

$$f_j(t,x) = q_j^n\theta_j(t)e^{q_j\varphi(t)}e^{iq_j(x-\eta(t))}(e^{-i\psi_j^0} \circ \rho)\Omega(t) ,$$

$$g_j(t,x) = \chi(t)e^{-q_j\varphi(t)}e^{-iq_j(x-\eta(t))}(e^{i\psi_j^0} \circ \rho) .$$

As in Case (i) we have $\{f_j\} \subset \mathbb{E}$. We also have

$$\mathbb{L}^0 g_j = (d\chi)e^{-q_j\varphi}e^{-iq_j(x-\eta)}(e^{i\psi_j^0} \circ \rho) + \chi(t)\cdot \operatorname{Re}\mathbb{L}^0[e^{-iq_j(x-\eta)}(e^{i\psi_j^0} \circ \rho)]e^{-q_j\varphi}$$

and thus, since Lemma 3.2(B) implies the estimates

$$\|(d\chi)e^{-q_j\varphi}e^{-iq_j(x-\eta)}(e^{i\psi_j^0} \circ \rho)\|_m \leqq C_m q_j^m e^{-\varepsilon q_j}$$

and

$$\|\chi(t)\mathrm{Re}\,\mathbb{L}^0[e^{-iq_j(x-\eta)}(e^{i\psi_j^0}\circ\rho)]e^{-q_j\varphi}\|_m \leqq C_m q_j^{-j+m},$$

we obtain

$$\|\mathbb{L}^0 g_j\|_m \leqq C_m q_j^m(e^{-\varepsilon q_j}+q_j^{-j}).$$

We also have

$$\|f_j\|_m \leqq C_m q_j^n q_j^m e^{q_j\frac{C}{q_j}} \leqq \tilde{C}_m q_j^{n+m}$$

which then give

$$\|f_j\|_m\|\mathbb{L}^0 g_j\|_m \leqq C_m\tilde{C}_m q_j^{2m+n}(e^{-\varepsilon q_j}+q_j^{-j}) \overset{j\to\infty}{\longrightarrow} 0. \tag{3.12}$$

On the other hand, writing $(\gamma^{-1})^*\Omega = h\,ds_1\wedge\cdots\wedge ds_n$,

$$\int_{M\times S^1} g_j f_j\wedge dx = q_j^n \int_{M\times S^1}\theta_j(t)\chi(t)\Omega(t)\wedge dx = 2\pi q_j^n\int_V \theta_j(t)\Omega(t)$$

$$= 2\pi q_j^n\int_B(\gamma^{-1})^*(\theta_j\Omega) = 2\pi q_j^n\int_B \theta(q_j s)h(s)\,ds_1\wedge\cdots\wedge ds_n$$

$$= 2\pi q_j^n q_j^{-n}\int\theta(s')h\left(\frac{s'}{q_j}\right)ds_1'\wedge\cdots\wedge ds_n'$$

and the last expression converges, as $j\to\infty$, to the non-zero value $2\pi h(0)\int\theta$.
Thus the inequality in Lemma 3.1 is violated.

Finally we show that (3.5) $\Rightarrow\sim$ (I).
Take q, l, Σ' as in the assumption and $\eta\in C^\infty(U)$ such that $d\eta = la - \rho^*(li_{\Sigma'}^*(a))$. Next we apply Lemma 3.2(A), with a replaced by la, and get the existence of $\psi^0\in C^\infty(\hat{\Sigma}')$ such that $e^{i\psi^0}\in C^\infty(\Sigma')$ and

$$(d_t - ila)[(e^{i\psi^0}\circ\rho)e^{i\eta}] = 0 \quad \text{in } U.$$

With the notation $q_j = l+qj$, $q_j' = q_j/l$ we note that, for $j=1,2,\ldots,$

$$(d_t - iq_j a)[(e^{iq_j'\psi^0}\circ\rho)e^{iq_j'\eta}] = 0 \quad \text{in } U.$$

Finally we select $\chi\in C_c^\infty(U)$ with $\chi = 1$ in a neighborhood of Σ' and $\varphi(t)\geqq\varepsilon>0$ on $\operatorname{supp}(d\chi)$, and also $\theta\in C_c^\infty(U)$ such that $\varphi(t)\leqq\varepsilon/2$ on $\operatorname{supp}\theta$ and such that (3.9) holds. Set, for $j=1,2,\ldots,$

$$f_j(t,x) = e^{iq_j(x-i\varphi(t))}(e^{-iq_j'\psi^0}\circ\rho)e^{-iq_j'\eta}\theta(t)\Omega(t),$$

$$g_j(t,x) = e^{-iq_j(x-i\varphi(t))}(e^{iq_j'\psi^0}\circ\rho)e^{iq_j'\eta}\chi(t).$$

We have $\{f_j\}\subset\mathbb{E}$ because $\operatorname{Ker}\mathbb{L}^0$ is spanned by the functions

$$h_N = e^{-iN(\widetilde{\psi}+iq\varphi_\bullet)}e^{iqNx},\quad N\in\mathbb{Z},$$

and, for all $j\in\mathbb{N}$ and $N\in\mathbb{Z}$, we have

$$\int_{S^1} e^{i(q_j+qN)x}\,dx = 0.$$

Now,

$$\mathbb{L}^0 g_j = (d\chi)e^{-iq_j(x-i\varphi(t))}(e^{iq_j'\psi^0} \circ \rho)e^{iq_j'\eta}$$

and hence we obtain

$$\|\mathbb{L}^0 g_j\|_m \leqq C_m j^m e^{-\varepsilon q j} .$$

On the other hand

$$\|f_j\|_m \leqq C_m j^m e^{\varepsilon q j/2}$$

and so

$$\|f_j\|_m \|\mathbb{L}^0 g_j\|_m \leqq C_m^2 j^{2m} e^{-\varepsilon q j/2} \to 0, \quad \text{as } j \to \infty .$$

Finally, since

$$\int_{M\times S^1} f_j g_j \wedge dx = 2\pi \int_M \theta(t)\chi(t)\Omega(t) \neq 0 ,$$

the inequality in Lemma 3.1 is also violated in this final case.

4 Sufficiency

For the proof of the sufficiency it will be convenient for us to split the action of $\mathbb{L}^p$ into certain subspaces, as follows. Let $A \subset \mathbb{Z}$. Define

$$\mathscr{D}_A'(M \times S^1) = \left\{ u \in \mathscr{D}'(M \times S^1) : u(t,x) = \sum_{j\in A} u_j(t)e^{ijx} \right\} .$$

The space $\mathscr{D}_A'(M\times S^1; \Lambda^{p,0})$ is defined in an analogous fashion. More generally, if $F \subset \mathscr{D}'(M \times S^1; \Lambda^{p,0})$ then we set $F_A = F \cap \mathscr{D}_A'(M \times S^1; \Lambda^{p,0})$. Notice that if $\mathbb{L}^p F \subset G$ then $\mathbb{L}^p F_A \subset G_A$; we use the notation $\mathbb{L}_A^p$ for the operator $\mathbb{L}^p$ acting from F_A into G_A. Also, any decomposition $\mathbb{Z} = A \cup B$ with $A \cap B = \emptyset$ induces direct sum decompositions $F = F_A \oplus F_B$ and $\mathbb{L}^p F = \mathbb{L}_A^p F_A \oplus \mathbb{L}_B^p F_B$. Furthermore, when F is a Hilbert space these decompositions are orthogonal direct sums.

We may talk about global solvability and global hypoellipticity of $\mathbb{L}^p$ relative to the subspaces F_A; more precisely,

Definition 4.1 *Let $A \subset \mathbb{Z}$. We say that $\mathbb{L}_A^{n-1}$ is* **globally solvable** *(GS) if, for every $f \in \mathbb{E}_A$, there exists $u \in \mathscr{D}_A'(M \times S^1; \Lambda^{n-1,0})$ such that $\mathbb{L}_A^{n-1} u = f$. We say that $\mathbb{L}_A^0$ is* **globally hypoelliptic** *(GH) if the conditions $u \in \mathscr{D}_A'(M \times S^1)$ and $\mathbb{L}_A^0 u \in C_A^\infty(M \times S^1; \Lambda^{1,0})$ imply $u \in C_A^\infty(M \times S^1)$.*

When $A = \mathbb{Z}$ we of course recover the previous notions of (GS) and (GH); it is also clear that $\mathbb{L}^{n-1}$ is (GS) if and only if $\mathbb{L}_A^{n-1}$ and $\mathbb{L}_B^{n-1}$ are (GS).

The next result is a variation of Theorem 26.1.7 in [H2]; the proof will be omitted.

Proposition 4.2 *Let $B \subset \mathbb{Z}$ and assume that $\mathbb{L}_B^0$ is (GH) and that* $\operatorname{Ker} \mathbb{L}_B^0 = \{0\}$. *Then*:

(i) *there exist $C > 0$ and $k \in \mathbb{Z}_+$ such that*

$$\|u\|_{(1)} \leqq C\|\mathbb{L}_B^0 u\|_{(k)}, \quad \text{for all } u \in C_B^\infty(M \times S^1)\,,$$

where $\| \; \|_{(1)}$ and $\| \; \|_{(k)}$ are Sobolev norms on $M \times S^1$;
(ii) $\mathbb{L}_B^{n-1}$ *is (GS).*

We now move on to the proof of the sufficiency.

$(\mathrm{II})_1 \Rightarrow (\mathrm{I})$. In this case $\operatorname{Ker} \mathbb{L}^0 = \mathbb{C}$ by Proposition 1.6 and Lemma 2.2(i). We choose $A = \{0\}$ and $B = \mathbb{Z}\backslash\{0\}$. Notice that $\mathscr{D}_A'(M \times S^1) \cong \mathscr{D}'(M)$, $\mathbb{L}_A^{n-1} = d_t$, $\operatorname{Ker} \mathbb{L}_A^0 = \mathbb{C}$, and $\mathbb{E}_A \cong \{f \in \Lambda^n C^\infty(M) : \int_M f = 0\}$. Thus each $f \in \mathbb{E}_A$ is an exact n-form on M, that is, there exists $u \in \Lambda^{n-1}C^\infty(M)$ such that $d_t u = f$. Therefore $\mathbb{L}_A^{n-1}$ is (GS).

Since $\mathbb{L}^0$ is (GH) the same is true of $\mathbb{L}_B^0$. Also $\operatorname{Ker} \mathbb{L}_B^0 = \{0\}$ and then Proposition 4.2 applies to yield the fact that $\mathbb{L}_B^{n-1}$ is (GS). Thus $\mathbb{L}^{n-1}$ is (GS).

$(\mathrm{II})_2 \Rightarrow (\mathrm{I})$. Here we choose $A = q\mathbb{Z}$ and $B = \mathbb{Z}\backslash q\mathbb{Z}$. We have $\operatorname{Ker} \mathbb{L}^0 \subset C_A^\infty(M \times S^1)$ and so $\operatorname{Ker} \mathbb{L}_B^0 = \{0\}$.

In order to show that $\mathbb{L}_A^{n-1}$ is (GS) we will make use of the following isomorphism of $\mathscr{D}_A'(M \times S^1; \Lambda^{p,0})$:

$$T\left(\sum_{N\in\mathbb{Z}} u_{qN}(t)e^{iqNx}\right) = \sum_{N\in\mathbb{Z}} u_{qN}(t)e^{-iN\widetilde{\psi}(t)}e^{iqNx}\,,$$

where $\widetilde{\psi} \in C^\infty(\widehat{M})$ is such that $d\widetilde{\psi} = \Pi^*(qa)$. Notice that T is also an isomorphism of $C_A^\infty(M \times S^1; \Lambda^{p,0})$ and a direct computation gives

$$T^{-1}\mathbb{L}_A^{n-1}T = d_t + ib \wedge \partial_x\,.$$

Let $f \in \mathbb{E}_A$ be given. Since b is exact and $(\psi)_{n-1}$ holds, it is easily seen that Proposition 1.8 applies to yield the existence of $v \in \mathscr{D}'(M \times S^1; \Lambda^{n-1,0})$ with $(d_t + ib \wedge \partial_x)v = T^{-1}f$. Since $\mathbb{L}^{n-1}$ acts in an invariant way the component v_A of v satisfies $\mathbb{L}_A^{n-1}(Tv_A) = f$. Hence $\mathbb{L}_A^{n-1}$ is (GS).

In particular the proof is complete when $B = \emptyset$ (which is equivalent to the property that a is integral). From now on we assume $q \geqq 2$ and proceed to show that $\mathbb{L}_B^0$ is (GH) which, according to Proposition 4.2, will finish the proof.

Let then $u \in \mathscr{D}_B'(M \times S^1)$ be such that $\mathbb{L}_B^0 u \in C_B^\infty(M \times S^1; \Lambda^{1,0})$. We will show that $(\{p\} \times S^1) \cap \operatorname{singsupp}(u) = \emptyset$ for all $p \in M$; we consider three separate cases.

Case (i). $p \in M\backslash\Sigma$. Here it suffices to recall that $\mathbb{L}^0$ is elliptic in the set $(M\backslash\Sigma) \times S^1$.

Case (ii). $p \in \Sigma\backslash(\bigcup_{\Sigma' \in \mathscr{A}} \Sigma')$. In this case φ is an open map at some point in the same connected component of p in Σ. Corollary 4.8 in [BCM] yields then the desired conclusion.

Case (iii). $p \in \Sigma'$, with $\Sigma' \in \mathscr{A}$. For this case we need to recall some facts from [BCM], p. 271ff. We take the pullback $i^*_{\Sigma'}(a)$ and associate to it the vector $\vec{v} \doteq I([i^*_{\Sigma'}(a)]) \in \frac{1}{q}\mathbb{Z}^\nu$ (cf. Proposition 2.2 in [BCM]). It is easy to see that there exists $\delta > 0$ such that $|\vec{v} - \vec{r}/s| \geqq \delta/|s|$ for all $\vec{r} \in \mathbb{Z}^\nu$ and all $s \in B$. Thus $\vec{v}$ is non-Liouville with respect to denominators belonging to B. Consider then the operator $\mathbb{L}^{0'} : \mathscr{D}'(\Sigma' \times S^1) \to \mathscr{D}'(\Sigma' \times S^1; \Lambda^{1,0})$ defined by $\mathbb{L}^{0'} = d' + i^*_{\Sigma'}(a) \wedge \partial_x$, where d' denotes the exterior derivative on Σ'. A minor modification of the proof of Theorem 2.4 in [BCM] implies that $\mathbb{L}^{0'}_B$ is (GH), hence $u(p, \cdot) \in C^\infty_B(S^1)$. Finally, Theorem 4.1 in [BCM] implies the desired result, namely $(\{p\} \times S^1) \cap \operatorname{singsupp}(u) = \emptyset$.

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