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Operator–valued Fourier multiplier theorems and maximal L_p-regularity

Lutz Weis

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Abstract. We prove a Mihlin–type multiplier theorem for operator–valued multiplier functions on UMD–spaces. The essential assumption is R–boundedness of the multiplier function. As an application we give a characterization of maximal L_p–regularity for the generator of an analytic semigroup T_t in terms of the R–boundedness of the resolvent of A or the semigroup T_t.

1. Introduction

Let X and Y be Banach spaces, $B(X, Y)$ be the space of bounded linear operators from X to Y, and $\mathcal{S}(X)$ be the space of rapidly decreasing functions from $\mathbb{R}$ to X. For $f \in L_1(\mathbb{R}, X)$ we write $\hat{f}(t) = \int e^{-its} f(s)ds$ for the Fourier transform of f and $\check{f}$ for the inverse Fourier transform of f.

We say that a function $M : \mathbb{R} \setminus \{0\} \to B(X, Y)$ is a **Fourier multiplier** on $L_p(\mathbb{R}, X)$ if the expressions

$$Kf = (M(\cdot)[\hat{f}(\cdot)])\check{} \quad \text{where } f \in \mathcal{S}(X) \tag{1}$$

are well defined and K extends to a bounded operator $K : L_p(\mathbb{R}, X) \to L_p(\mathbb{R}, Y)$.

It is a well known result of L. Schwartz (see e. g. [BL], Sect. 6.1) that, in the case that X and Y are both Hilbert spaces, the Mihlin multiplier theorem extends to operator–valued multiplier functions: if $M : \mathbb{R} \setminus \{0\} \to B(X, Y)$ satisfies, for some constant C,

$$\|M(t)\| \leq C, \quad \|tM'(t)\| \leq C \quad \text{for each } t \in \mathbb{R} \setminus \{0\} \tag{2}$$

then M is a Fourier multiplier on $L_p(\mathbb{R}, X)$ with $1 < p < \infty$, in the sense of (1). Pisier observed that the converse is true: if $X = Y$ and all M satisfying (2) are Fourier multipliers on $L_2(\mathbb{R}, X)$, then X is isomorphic to a Hilbert space. Therefore, additional hypotheses are needed to obtain multiplier theorems in more general Banach spaces.

L. Weis
Mathematisches Institut I, Universität Karlsruhe, 76128 Karlsruhe, Germany
(e-mail: Lutz.Weis@math.uni-karlsruhe.de)

In [Bou 2] (see also [Zi]) Bourgain has shown that for $M(t) = m(t)I$, with a scalar–valued function m, the Mihlin multiplier theorem holds provided X is an UMD–space. UMD spaces may be defined by the fact that the Hilbert transform

$$Hf(t) = PV - \int \frac{1}{t-s} f(s)ds, \quad f \in \mathcal{S}(X)$$

extends to a bounded operator on $L_p(\mathbb{R}, X)$ for $1 < p < \infty$, i. e. $m(t) = \text{sign}(t)$ is a Fourier multiplier on $L_p(\mathbb{R}, X)$. All closed subspaces and quotient spaces of a $L_q(\Omega, \mu)$–space with $1 < q < \infty$ are examples of UMD–spaces (see e. g. [Bu]).

In Sect. 3 of this paper we extend Bourgain's result to operator–valued functions $M(t) \in B(X, Y)$ for UMD–spaces X and Y: if M satisfies in place of (2) the stronger assumption that the sets

(3) $$\{M(t) : t \in \mathbb{R} \setminus \{0\}\} \text{ and } \{tM'(t) : t \in \mathbb{R} \setminus \{0\}\}$$

are R–bounded, then M is a Fourier multiplier on $L_p(\mathbb{R}, X)$ for all $1 < p < \infty$.

A set $\tau \subset B(X, Y)$ is called R–bounded if there is a constant C such that for all $T_1, \dots, T_n \in \tau,\ x_1, \dots, x_n \in X,\ n \in \mathbb{N}$

(4) $$\int_0^1 \Big\| \sum_{j=0}^n r_j(u) T_j(x_j) \Big\|_Y du \le C \int_0^1 \Big\| \sum_{j=0}^n r_j(u) x_j \Big\|_X du$$

where (r_j) is a sequence of independent symmetric $\{-1, 1\}$–valued random variables on $[0, 1]$, e. g. the Rademacher functions $r_j(t) = \text{sign}(\sin(2^j \pi t))$. This concept was already used in [Bou2] and [BG] in connection with multiplier theorems and more recently a detailed study was given in [CPSW]. If $X = Y$ is a $L_q(\Omega, \mu)$ space, then (4) is equivalent to

(5) $$\Big\| \Big(\sum_{j=1}^n |T_j x_j|^2 \Big)^{1/2} \Big\|_{L_q} \le C \Big\| \Big(\sum_{j=1}^n |x_j|^2 \Big)^{1/2} \Big\|_{L_q}$$

and so the connection with square function estimates and Paley–Littlewood decompositions in harmonic analysis becomes clear. Note that in a Hilbert space every norm–bounded set τ is R–bounded; therefore, our result can be viewed as an extention of the theorem of Schwartz from the Hilbert space to the Banach space setting. We also show that R–boundedness conditions for the multiplier function are necessary for the multiplier theorem to hold.

The multiplier theorem provides a useful characterization of maximal L_p–regularity. Let A be the generator of a bounded analytic semigroup T_t on X. Consider the Cauchy problem

(6) $$y'(t) = Ay(t) + f(t), \qquad t \ge 0, \qquad y(0) = 0$$

for a given $f \in L_p(\mathbb{R}_+, X)$. We say that A has maximal L_p–regularity if the unique solution y of (6) satisfies

$$\|y'\|_{L_p(\mathbb{R}_+,X)} + \|Ay\|_{L_p(\mathbb{R}_+,X)} \leq C\|f\|_{L_p(\mathbb{R}_+,X)}. \tag{7}$$

It is well known that every bounded analytic semigroup on a Hilbert space has maximal L_p–regularity. Indeed, if we differentiate the solution formula

$$y(t) = \int_0^t T_{t-s}(f(s))ds$$

we see that maximal L_p–regularity is equivalent to the boundedness of the "convolution operator"

$$Kf(t) = \int_0^t AT_{t-s}(f(s))ds \tag{8}$$

on $L_p(\mathbb{R}_+, X)$ for $1 < p < \infty$. Since $\|AT_t\| \sim \frac{1}{t}$ as $t \to 0$ this operator looks like a singular integral operator. By the analyticity of T_t,

$$M(t) = (AT_\cdot)\widehat{\ }(t) = AR(it, A) = itR(it, A) - I \tag{9}$$

satisfies $\|M(t)\| \leq C$, $\|t'M(t)\| \leq C$, and so we may apply the Schwartz multiplier theorem for Hilbert spaces.

On the other hand, it was shown quite recently by Kalton and Lancien [KL], that on every separable Banach lattice, which is not isomorphic to a Hilbert space, there are analytic semigroups without maximal L_p–regularity. While this characterizes Banach spaces in which all analytic semigroups have maximal L_p–regularity, we can apply our multiplier theorem to (8) and (9) to obtain a characterization of maximal L_p-regularity for individual generators A of analytic semigroups on an UMD-space X.

We will show that A has maximal L_p–regularity if and only if the sets $\{tR(it, A) : t \in \mathbb{R} \setminus \{0\}\}$ or $\{T_z : |\arg(z)| \leq \varphi\}$ for some small $\varphi > 0$ are R–bounded. (see Sect. 4 for details; an independent proof using a different method was given by N. Kalton.)

A large number of sufficient conditions for maximal L_p–regularity are known: e. g., if T_t is a contraction semigroup on $L_q(\Omega)$ for all $1 \leq q \leq \infty$ ([La]), T_t has Gaussian bounds ([HP], [CD]), or A has bounded imaginary powers with $\|(-A)^{it}\| \leq Ce^{d|t|}$, $d < \frac{\pi}{2}$, and X is an UMD–space ([DV]). In Sect. 5 and in some further papers ([We 1], [We 2]) we will show how these conditions can be obtained and, in some cases, also improved by using our characterization.

In Sect. 2 we give some general properties of R–bounded sets which will be useful for our main results.

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2. *R*–boundedness

The notion of R–bounded collections of operators is basic for our results. In this section we collect some examples of R–bounded sets and some methods to obtain new R–bounded sets from known ones.

2.1. Definition. *$\tau \subset B(X,Y)$ is called R–**bounded**, if there is a constant $C < \infty$ such that for all $T_1, \dots, T_n \in \tau$, $x_1, \dots, x_n \in X$ and $n \in \mathbb{N}$*

$$\int_0^1 \left\| \sum_{j=1}^n r_j(u) T_j(x_j) \right\| du \le C \int_0^1 \left\| \sum_{j=1}^n r_j(u) x_j \right\| du \,, \tag{1}$$

where (r_j) is a sequence of independent, symmetric $\{1,-1\}$–valued random variables on $[0,1]$, e. g. the Rademacher functions. The smallest constant C, for which (1) holds is denoted by $R(\tau)$.

2.2. Remark. This notion appears implicitly in [Bou2] and was named Riesz–property in [BG]. [CPSW] contains a detailed study of property (1) and "R" was reinterpreted as 'Randomized bounded'. Note that for a Banach function space X, which is concave for some $q < \infty$, (see [LT], Theorem 1.d.6), (1) is equivalent to

$$\left\| \left(\sum_{j=1}^n |T_j(x_j)|^2 \right)^{1/2} \right\|_X \le C \left\| \left(\sum_{j=1}^n |x_j|^2 \right)^{1/2} \right\|_X . \tag{2}$$

In this form and for $X = L_q(\Omega, \mu)$ the condition is well known in harmonic analysis in connection with square function estimates.

If X is a Hilbert space, e. g. $X = L_2(\Omega, \mu)$, then Fubini's theorem applied to (2) shows that **every** bounded set in $B(X)$ is R–bounded.

2.3. Remark. Here are some technical, but very useful points:

a) In the definition we allow that some of the T_j's are identical. It is shown in [CPSW], Lemma 3.3, that we can add the restriction $T_i \ne T_j$ for $i \ne j$ in 2.1. and obtain an equivalent definition.
b) By Kahane's inequality condition (1) is equivalent to

$$\left(\int_0^1 \left\| \sum_{j=1}^n r_j(u) T_j(x_j) \right\|^p du \right)^{1/p} \le C_p \left(\int_0^1 \left\| \sum_{j=1}^n r_j(u) x_j \right\|^p du \right)^{1/p} \tag{3}$$

for all $1 \le p < \infty$.
c) If τ is R–bounded, then the closure of τ in the strong operator topology is also R–bounded.

d) By Khintchine's inequality, we have for all $x_i \in X$, $i = 1 \dots n$, that

$$\begin{aligned} \sup_{1\le i\le n} \|x_i\| &\le \sup\{(\sum_{i=1}^{n} |y^*(x_i)|^2)^{1/2} : y^* \in Y^*, \|y^*\| \le 1\} \\ &\le C' \int_0^1 \|\sum_{i=1}^{n} r_i(u) x_i\| du \end{aligned} \tag{4}$$

2.4. Lemma. *Let G be an index set and assume that*

$$T(\mu) = \sum_{n=1}^{\infty} T_n(\mu), \qquad \mu \in G$$

converges in the strong operator topology of $B(X,Y)$ for all $\mu \in G$. Then

$$R(\{T(\mu) : \mu \in G\}) \le \sum_{n=1}^{\infty} R(\{T_n(\mu) : \mu \in G\}).$$

Proof. Put $S_N(\mu) = \sum_{n=1}^{N} T_n(\mu)$. For all $x_1, \dots, x_m \in X$ and $\mu_1, \dots, \mu_m \in G$ we have

$$\begin{aligned} &\int_0^1 \|\sum_{j=1}^{m} r_j(u) S_N(\mu_j)(x_j)\| du \\ &\le \sum_{n=1}^{N} \int_0^1 \|\sum_{j=1}^{m} r_j(u) T_n(\mu_j)(x_j)\| du \\ &\le \sum_{n=1}^{N} R(\{T_n(\mu) : \mu \in G\}) \int_0^1 \|\sum_{j=1}^{m} r_j(u) x_j\| du \ . \end{aligned} \tag{5}$$

Since $T(\mu) = \lim_{N\to\infty} S_N(\mu)$ in the strong operator topology we appeal to 2.3.c). □

The following proposition holds more generally for functions of bounded variation (see [SW]).

2.5. Proposition. *Let $J \subset \mathbb{R}$ be an interval and $t \in J \to M(t) \in B(X,Y)$ have an integrable derivative. Then $\{M(t) : t \in J\}$ is R–bounded.*

Proof. For a singleton $\tau = \{T\}$ we have $R(\tau) = \|T\|$. If $J = [a, b), a < b \leq \infty$, and $\sigma = \{t_0, t_1, \ldots, t_n\}$ is a partition of J, put

$$M_\sigma(t) = M(a) + \sum_{j=1}^{n} \chi_{[t_{j-1}, b)}(t) \int_{t_{j-1}}^{t_j} M'(s)ds.$$

By our first observation, Remark 2.3 a) and Lemma 2.4 we have

$$\begin{aligned} R(\{M_\sigma(t) : t \in J\}) &\leq \|M(a)\| + \sum_{j=1}^{n} \left\| \int_{t_{j-1}}^{t_j} M'(s)ds \right\| \\ &\leq \|M(a)\| + \int_a^b \|M'(s)\|ds. \end{aligned}$$

Choosing a sequence σ_n of partitions with $M_{\sigma_n}(t) \to M(t)$ for $t \in J$, the claim follows from 2.3. c). □

In a similar way we can show:

2.6. Proposition. *Let $\lambda \in G \to M(\lambda) \in B(X, Y)$ be analytic on the open set G and $K \subset G$ be compact. Then $\{M(\lambda) : \lambda \in K\}$ is R–bounded.*

Proof. For $\mu \in K$ consider the power series expansion

$$M(\lambda) = \sum_{j=0}^{\infty} (\lambda - \mu)^j \frac{1}{j!} M^{(j)}(\mu), \quad |\lambda - \mu| < r(\mu)$$

with radius of convergence $r(\mu)$. Then by Lemma 2.4 and $R(\{T\}) = \|T\|$

$$R(\{M(\lambda) : |\lambda - \mu| < \frac{r(\mu)}{2}\}) \leq \sum_{j=0}^{\infty} \left(\frac{r(\mu)}{2}\right)^j \frac{1}{j!} \|M^{(j)}(\mu)\| < \infty.$$

Since K is covered by finitely many balls $\{\lambda : |\lambda - \mu_i| < \frac{r(\mu_i)}{2}\}$, the claim follows. □

We quote an extremely useful result from [CPSW], Lemma 3.2.

2.7. Lemma. *Let $\tau \subset B(X)$ be a R–bounded collection with R–bound M. Then the closure of the absolute convex hull of τ in the strong operator topology is also R–bounded. For the real absolute convex hull the R–bound is again M; for the complex convex hull the R–bound is at most $2M$.*

As a first consequence of this convexity result we present

2.8. Proposition. *Let $G \subset \mathbb{C}$ be a simply connected Jordan region such that $\mathbb{C} \setminus G$ has interior points. Let $\lambda \in \overline{G} \to N(t) \in B(X, Y)$ be a bounded, strongly measurable function, analytic in G. Assume that $\{N(\lambda) : \lambda \in \partial G\}$ is R–bounded.*

a) Then $\{N(\lambda) : \lambda \in \overline{G}\}$ is R–bounded.
b) For all $n \in \mathbb{N}$, $\{d(\lambda)^n N^{(n)}(\lambda) : \lambda \in G\}$ is R–bounded, where $d(\lambda) = \inf\{|\lambda - \mu| : \mu \in \partial G\}$.

Proof. a) By the Riemann mapping theorem there is a conformal mapping g from $D = \{\lambda : |\lambda| < 1\}$ onto G which extends to a bijection of the boundary, i. e. $g(\partial D) = \partial G$ (see e. g. [Ru] 14.19, 14.20). So we may restrict ourselves to the case $G = D$. If $p_r(\theta)$ denotes the Poisson kernel

$$p_r(\theta) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2}$$

for D, then for all $\lambda = re^{i\theta} \in D$

$$N(\lambda) = \frac{1}{2\pi} \int_{-\pi}^{\pi} p_r(\theta - t) N(e^{it}) dt$$

and the claim follows from 2.7 since

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} p_r(\theta) d\theta = 1 \quad \text{for all } r \in (0, 1).$$

b) If $\lambda \in G$ and $\Gamma(\lambda) = \{\mu : |\mu - \lambda| = d(\lambda)\}$, then by Cauchy's formula

$$N^{(n)}(\lambda) = n! \frac{1}{2\pi i} \int_{\Gamma(\lambda)} \frac{1}{(\mu - \lambda)^{n+1}} N(\mu) d\mu$$

with $\frac{n!}{2\pi} \int_{\Gamma(\lambda)} |\mu - \lambda|^{-n-1} d\mu = n! d(\lambda)^{-n}$. Hence we may appeal to 2.7 again. □

The following special case is of particular interest to us.

2.9. Example. Put $\Sigma(\theta) = \{\lambda \in \mathbb{C} : |\arg(\lambda)| < \theta\}$. For a bounded analytic function $\lambda \in \Sigma(\theta) \to M(\lambda) \in B(X, Y)$ we get from 2.7:

a) If $\{M(\lambda) : |\arg\lambda| = \theta_1\}$, $\theta_1 < \theta$, is R–bounded, then so is $\{M(\lambda) : \lambda \in \Sigma(\theta_1)\}$.
b) If $\{M(\lambda) : \lambda \in \Sigma(\theta)\}$ is R–bounded, then so is $\{\lambda M'(\lambda) : \lambda \in \Sigma(\theta_1)\}$ for $\theta_1 < \theta$.

(Of course a) follows directly from 2.7 and Poisson's formula for the halfplane applied to $\lambda \in \mathbb{C}_+ \to M(\lambda^\alpha)$, $\alpha = \frac{2\theta_1}{\pi}$, and b) follows from 2.7 and the formula

$$\lambda M'(\lambda) = \frac{1}{2\pi i} \int\limits_{\Gamma(\lambda)} \frac{\lambda}{(\mu - \lambda)^2} M(\mu) d\mu$$

with $\Gamma(\lambda) = \{\mu : |\mu - \lambda| = |\lambda| \cdot \sin(\theta - \theta_1)\}$.) □

Next we consider the Laplace transform. Let $t \in \mathbb{R}_+ \to N(t) \in B(X, Y)$ be bounded and put

(6) $$\hat{N}(\lambda) := \int\limits_0^\infty e^{-\lambda t} N(t) dt, \quad \operatorname{Re} \lambda > 0.$$

Then, if $\mathcal{N} = \{N(t) : t \in \mathbb{R}_+\}$ is R–bounded, so is $\hat{\mathcal{N}}_\varphi = \{\lambda \hat{N}(\lambda) : \lambda \in \Sigma(\varphi)\}$, with $\varphi < \frac{\pi}{2}$. Indeed, by 2.7

$$R(\hat{\mathcal{N}}_\varphi) \leq 2C R(\mathcal{N}) \text{ with } C = \int\limits_0^\infty r e^{-r \cos\varphi t} dt = \frac{1}{\cos\varphi}.$$

If $N(t)$ extends analytically to a sector $\Sigma(\theta)$ we can show:

2.10. Theorem. *Let $\lambda \in \Sigma(\theta) \to N(\lambda) \in B(X, Y)$ be bounded and analytic and $\lim_{\lambda \to 0} N(\lambda)$ exists for the weak operator topology. If we define $\hat{N}(\lambda)$ by (6), then $\hat{N}(\cdot)$ extends to $\Sigma(\theta + \frac{\pi}{2})$ and the following conditions are equivalent:*

a) $\{N(\lambda) : \lambda \in \Sigma(\theta_1)\}$ *is R–bounded for all $\theta_1 < \theta$*
b) $\{\lambda \hat{N}(\lambda) : \lambda \in \Sigma(\frac{\pi}{2} + \theta_1)\}$ *is R–bounded for all $\theta_1 < \theta$.*

Proof. We fix θ_1, θ_2 and ψ with $\theta_1 < \psi < \theta_2 < \theta$. By Cauchy's formula we can represent $\lambda \hat{N}(\lambda)$ for $\operatorname{Re} \lambda > 0$ as a contour integral

(7) $$\lambda \hat{N}(\lambda) = \int\limits_\gamma \lambda e^{-\lambda\mu} N(\mu) d\mu, \quad \gamma = \{t e^{-i\psi} : t \in \mathbb{R}_+\}.$$

By analytic continuation this holds for all $\lambda = s e^{i(\frac{\pi}{2} + \varphi)}$ with $0 < \varphi < \theta_1$ and $s > 0$, since for $\mu \in \gamma$

$$\begin{aligned} \operatorname{Re} \mu\lambda &= st \operatorname{Re}(i \cos(\varphi - \psi) + \sin(\psi - \varphi)) \\ &= st \sin(\psi - \varphi) \geq st \sin(\psi - \theta_1) \end{aligned}$$

and $\int\limits_0^\infty |\lambda e^{-\lambda\mu}| d\mu \le (\sin(\psi - \theta_1))^{-1}$. If $-\theta_1 < \varphi < 0$ we can use $\gamma = \{te^{i\psi t} : t \in \mathbb{R}_+\}$ to get a similar representation for $\lambda\hat{N}(\lambda)$. Hence $\{\lambda\hat{N}(\lambda) : \lambda \in \Sigma(\frac{\pi}{2} + \theta_1)\}$ is R–bounded if $\{N(\lambda) : \lambda \in \Sigma(\theta_2)\}$ is R–bounded (by 2.7).

Conversely, we assume that $\{\lambda\hat{N}(\lambda) : \lambda \in \Sigma(\frac{\pi}{2} + \theta_2)\}$ is R–bounded for some $\theta_2 < \theta$. Then $\|\lambda\hat{N}(\lambda)\|$ is uniformly bounded on $\Sigma(\frac{\pi}{2} + \theta_2)$. Fix $\theta_1 < \theta_2$ and a $\psi > \frac{\pi}{2}$ with $\psi - \frac{\pi}{2} < \theta_2 - \theta_1$. By the inversion formula for the Laplace transform (see e. g. [Wi] Sect. II.7; the necessary deformation of the path of integration is possible since $N(\lambda)x \to N_0 x$ for $\lambda \to 0$ implies that $\lambda\hat{N}(\lambda)x \to N_0 x$ for $\lambda \to \infty$ by [Wi] Sect. V.1) we have

$$N(t) = \frac{1}{2\pi i} \int\limits_{\gamma_t} e^{t\lambda} \hat{N}(\lambda) d\lambda$$

$$(8) \qquad = \frac{1}{2\pi i} \int\limits_{\gamma_t} e^{\lambda} [\hat{N}(\lambda/t)\lambda/t] \frac{d\lambda}{\lambda},$$

where $\gamma_t = \{se^{\pm i\psi} : |s| \ge \frac{1}{t}\} \cup \{\frac{1}{t}e^{is} : |s| < \psi\}$. By analytic continuation, we get for $\mu = te^{i\varphi}$

$$N(\mu) = \frac{1}{2\pi i} \int\limits_{\gamma_t} e^{\lambda} [\hat{N}(\lambda\mu^{-1})\lambda\mu^{-1}] \frac{d\lambda}{\lambda}$$

as long as $\arg(\lambda\mu^{-1}) = \pm\psi - \varphi \in (-\frac{\pi}{2} - \theta_2, \frac{\pi}{2} + \theta_2)$ and this is the case if $\psi - \frac{\pi}{2} < \theta_2 - \theta_1$ and $|\varphi| < \theta_1$. Since $\int\limits_{\gamma_t} |\frac{1}{\lambda} e^{\lambda}| d\lambda < \infty$, we can use 2.7 to obtain the R–boundedness of $\{N(\lambda) : \lambda \in \Sigma(\theta_1)\}$. □

We will need the following extension result:

2.11. Proposition. *For $1 \le p < \infty$ and $T \in B(X, Y)$ denote by $\tilde{T} \in B(L_p(\mathbb{R}, X), L_p(\mathbb{R}, Y))$ the operator $(\tilde{T} f)(t) = T(f(t))$ for $f \in L_p(\mathbb{R}, X)$ and $t \in \mathbb{R}$. Then, if $\tau \subset B(X, Y)$ is R–bounded the collection $\tilde{\tau} = \{\tilde{T} : T \in \tau\} \subset B(L_p(\mathbb{R}, X), L_p(\mathbb{R}, Y))$ is also R–bounded.*

Proof. For $f_1, \ldots, f_n \in L_p(\mathbb{R}, X)$ and $\tilde{T}_1 \ldots \tilde{T}_n \in \tilde{\tau}$ we get by 2.3.b) and Fubini's theorem

$$\int\limits_0^1 \|\sum r_j(u)\tilde{T}_j(f_j)\|^p_{L_p(\mathbb{R},Y)} du = \int\limits_{\mathbb{R}} \int\limits_0^1 \|\sum r_j(u) T_j(f_j(t))\|^p_X du\, dt$$

$$\le \int\limits_{\mathbb{R}} \int\limits_0^1 \|\sum r_j(u) f_j(t)\|^p_Y du\, dt = \int\limits_0^1 \|\sum r_j(u) f_j\|^p_{L_p(\mathbb{R},X)} du \qquad \square$$

Besides Hilbert spaces there is another interesting situation, where bounded sets of operators are automatically R–bounded.

2.12. Example. If Y is a Banach space of type 2 and X is a Banach space of cotype 2, then every bounded set $\tau \subset B(X, Y)$ is R–bounded.

This follows directly from the definitions (cf. [LT]): Y has **type 2**, if there is a constant C, such that for all $y_1 \dots y_n \in Y$ we have

$$\int \|\Sigma r_j(u) y_j\| du \le C \left(\Sigma \|y_j\|^2 \right)^{1/2}$$

and X has **cotype 2**, if for all $x_1, \dots x_n \in X$

$$\int \|\Sigma r_j(u) x_j\| du \ge \frac{1}{c} \left(\sum_j \|x_j\|^2 \right)^{1/2}.$$

3. Multiplier Theorems

To obtain our multiplier theorems we will combine R-boundedness of the multiplier function with the following vector-valued Paley-Littlewood decomposition from [Bou2] and [Zi]. In this statement we will need the so called 'partial sum operators' S_j that we define by

$$(S_j f)\hat{} = \chi_j \hat{f} .$$

Here χ_j are the characteristic functions of $\{t \in \mathbb{R} : 2^j \le |t| \le 2^{j+1}\}$, $j \in \mathbb{Z}$.

By $(r_j)_{j \in \mathbb{Z}}$ we denote a sequence of independent symmetric $\{1, -1\}$–valued random variables on $[0, 1]$.

3.1. Theorem. *Let X be an UMD–space and $1 < p < \infty$. Then there is a constant C_p such that*

$$\frac{1}{C_p} \|f\|_{L_p(\mathbb{R}, X)} \le \int_0^1 \|\sum_{j \in \mathbb{Z}} r_j(u) S_j f\| du \le C_p \|f\|_{L_p(\mathbb{R}, X)}$$

Proof. Proposition 4 of [Zi] gives for all $1 \le q \le \infty$

$$(1) \qquad (\int_0^1 \|\sum_{j \in \mathbb{Z}} r_j(u) S_j f\|^q du)^{1/q} \le C_q \|f\| .$$

To obtain the converse inequality, we choose a $g \in L_{p'}(\mathbb{R}, X^*)$, $\|g\| \le 1$, with $\|f\| = < g, f >$.

Since $< S_j g, S_k f >=< (S_j g)\widehat{\ }, (S_k f)\widehat{\ } >= 0$ for $j \neq k$ we get

$$\|f\| = < g, f > = \int_0^1 < \sum r_j(u) S_j g, \sum r_j(u) S_j f > du$$
$$\leq C_p \int_0^1 \| \sum r_j(u) S_j f \| du$$

where we applied (1) to g and X^* with $q = \infty$. For the discrete case, see also [Bou2], Theorem 3. □

As a first step towards a Mihlin–type multiplier theorem, we extend the Marcinkiewicz type multiplier theorem from [Bou2].

3.2. Theorem. *Let X and Y be UMD–spaces. Consider a function* $M : \mathbb{R}\backslash\{0\} \to B(X, Y)$ *of the form*

$$M(t) = \sum_{j\in\mathbb{Z}} \chi_j(t) m(t) M_j \ ,$$

where the scalar function m *satisfies* $\sup_j$ *Var* $(\chi_j m) < \infty$ *and the set* $\{M_j, \ j \in \mathbb{Z}\} \subset B(X, Y)$ *is R–bounded. Then*

$$\mathcal{K} f = [M(\cdot)\hat{f}(\cdot)]^\vee, \quad f \in \mathcal{S}(X),$$

extends to a bounded operator $\mathcal{K} : L_p(\mathbb{R}, X) \to L_p(\mathbb{R}, Y)$ *for all* $1 < p < \infty$.

3.3. Remark. By 3.1, the S_j define an unconditional Schauder decomposition of $L_p(\mathbb{R}, X)$. Therefore this result is related to Theorem 3.4 of [CPSW]. Higher dimensional and discrete versions of Theorem 3.2. and 3.4. will be contained in [SW].

Proof. We estimate $\mathcal{K} f$ using the Paley–Littlewood decomposition 3.1. and the extension result 2.11.:

$$\begin{aligned}
\|\mathcal{K} f\|_{L_p(\mathbb{R},X)} &\leq C_p \int_0^1 \| \sum_j r_j(u) [\chi_j m M_j(\hat{f})]^\vee \| du \\
&= C_p \int_0^1 \| \sum_j r_j(u) \tilde{M}_j [\chi_j m \hat{f}]^\vee \| du \\
&\leq C_p D \int_0^1 \| \sum_j r_j(u) [\chi_j m \hat{f}]^\vee \| du \\
&\leq C_p^2 \cdot D \sup \mathrm{Var}(\chi_j m) \, \|f\|_{L_p(X)}.
\end{aligned}$$

In the last step we used Proposition 5 of [Zi]. There it is assumed in addition that $\chi_j m$ has a distributional derivative in L_1, but this is inessential in the one–dimensional case, since there is always a sequence φ_k on supp χ_j with this additional property and $\varphi_k(s) \to (\chi_j m)(s)$, Var $(\varphi_k) \leq$ Var $(\chi_j m)$ for all k. □

By $\mathcal{D}_0(X)$ we denote the C^∞ functions $f : \mathbb{R} \to X$ with compact support in $\mathbb{R} \setminus \{0\}$. Note that $\mathcal{D}_0(X)$ is dense in $L_p(\mathbb{R}, X)$ for $1 < p < \infty$.

3.4. Theorem. *Let X and Y be UMD–spaces. Let $t \in \mathbb{R} \setminus \{0\} \to M(t) \in B(X, Y)$ be a differentiable function such that the sets*

$$\{M(t) : t \in \mathbb{R} \setminus \{0\}\}, \quad \{tM'(t); t \in \mathbb{R} \setminus \{0\}\}$$

are R–bounded.

Then $\mathcal{K} f = [M(\cdot)\hat{f}(\cdot)]^\vee$, $f \in \mathcal{D}_0(X)$, extends to a bounded operator $\mathcal{K} : L_p(\mathbb{R}, X) \to L_p(\mathbb{R}, Y)$ for $1 < p < \infty$.

3.5. Remark. a) The following formally weaker conditions are sufficient for the conclusion of Theorem 3.4.:

i) $\|M(t)\| \leq C$ for all $t \in \mathbb{R} \setminus \{0\}$ and $R(\{M(\pm 2^n) : n \in \mathbb{Z}\}) \leq C$

ii) $\|tM'(t)\| \leq C$ for all $t \in \mathbb{R} \setminus \{0\}$ and $R(\{a2^n M'(a2^n) : n \in \mathbb{Z}\}) \leq C$ for all $1 \leq |a| \leq 2$.

The following proof also shows that i) and ii) already imply the R-boundedness of $\{M(t) : t \in \mathbb{R} \setminus \{0\}\}$.

b) As we noted in 2.2 and 2.12, the R–boundedness of $\{M(t)\}$ and $\{tM'(t)\}$ follows from the norm boundedness of these sets, if X is a Hilbert space, or if Y has type 2 and X has cotype 2.

Proof. If $f \in \mathcal{D}_0(X)$, $\hat{f}(0) = 0$ then $\mathcal{K} f$ belongs to $L_p(\mathbb{R}, X)$ for $1 < p < \infty$. To simplify our notation we may assume that $M(t) = 0$ for $t < 0$.
In order to apply Theorem 3.2, we approximate M by a sequence M_k such that each M_k is a convex combination of multipliers of the special form considered in 3.2: For $j \in \mathbb{Z}$, $k \in \mathbb{N}$ we put

$$\chi_{k,j,l}(t) = \chi_{(2^j + l2^{j-k}, 2^{j+1}]}(t), \quad l = 1 \ldots 2^k .$$

Then $M_k : (0, \infty) \to B(X, Y)$ is defined by

$$M_k(t) = \sum_{j \in \mathbb{Z}} \{\chi_j(t) M(2^j) + \sum_{l=1}^{2^k} \chi_{k,j,l}(t) 2^{j-k} M'(2^j + (l-1)2^{j-k}) .$$

Since $\|M'(t)\| \leq C/t$ we have for all $t \in (0, \infty)$

$$\|M_k(t)\| \leq 2C, \quad \lim_{k \to \infty} M_k(t) = M(t) . \tag{2}$$

Indeed, e. g. for $t \in [2^j, 2^{j+1})$ we have for $k \to \infty$

$$M_k(t) \to M(2^j) + \int_{2^j}^{t} M'(s)ds = M(t).$$

Now we decompose the functions M_k as $M_k = M_{k,0} + \frac{1}{2^k} \sum_{l=1}^{2^k} M_{k,l}$, where

$$M_{k,0}(t) = \sum_{j\in\mathbb{Z}} \chi_j(t) M(2^j)$$
$$M_{k,l}(t) = \sum_{j\in\mathbb{Z}} \chi_{k,j,l}(t) 2^j M'(2^j + (l-1)2^{j-k}).$$

Each $M_{k,l}$ is a function of the form considered in 3.2. with $m = \sum_{r\in\mathbb{Z}} \chi_{k,r,l}$ and $M_j = 2^j M'(2^j a),\ a = 1 + (l-1)2^{-k} \in [1, 2]$. Our assumptions allow to apply 3.2. and we obtain

$$\|(M_k \hat{f})^\vee\| \le D\|f\|, \quad k \in \mathbb{N}.$$

This uniform estimate together with (2) gives the claim of the theorem for $k \to \infty$.

To justify Remark 3.5., note that we only used i) and ii) of 3.5. to show that the sets $\{M_{k,l}(t) : t \in \mathbb{R} \setminus \{0\}\}$ are uniformly R–bounded. Hence $\{M_k(t) : t \in \mathbb{R} \setminus \{0\}\}$ is uniformly R–bounded with respect to k and the R–boundedness of $\{M(t), t \neq 0\}$ follows from (2) and 2.3.c). □

There is a partial converse to Theorem 3.4 showing that a R-boundedness condition is necessary for the theorem to hold.

3.6. Proposition. *Let X and Y be Banach spaces and let $t \in \mathbb{R} \setminus \{0\} \to M(t) \in B(X, Y)$ satisfy*

(3) $$\|M(t)\| \le C, \quad \|tM'(t)\| \le C \quad \textit{for all } t \in \mathbb{R}.$$

Assume that $\mathcal{K}f = [M(\cdot)\hat{f}(\cdot)]^\vee$, $f \in \mathcal{D}_0(X)$, defines a bounded operator $\mathcal{K} : L_p(\mathbb{R}, X) \to L_p(\mathbb{R}, Y)$, for some $1 < p < \infty$. Then

$$\{a2^n M(a2^n) : n \in \mathbb{Z}, a \neq 0\}$$

is R-bounded.

Proof. For all $x_n \in X$ and $1 < p < \infty$ we have the inequality (cf [Pi])

$$\frac{1}{D^p}\int_0^1 \|\sum_n r_n(u)x_n\|^p du \le \int_{-\pi}^{\pi} \|\sum_{n=0}^{\infty} \mathrm{e}^{\mathrm{i}2^n t} x_n\|^p dt \le D^p \int_0^1 \|\sum_n r_n(u)x_n\|^p du,$$

where r_n denote the Rademacher functions. Hence our claim is equivalent to the inequality

$$\int_{-\pi}^{\pi} \|\sum_{n=0}^{\infty} \mathrm{e}^{\mathrm{i}2^n t} M(a2^n)x_n\|^p dt \le C^p \int_{-\pi}^{\pi} \|\sum_{n=0}^{\infty} \mathrm{e}^{\mathrm{i}2^n t} x_n\|^p dt \tag{4}$$

for $a \neq 0$. First we put $a = 1$. For $f(t) = \sum_n \mathrm{e}^{\mathrm{i}2^n t}\mathrm{e}^{-|t|}x_n$ and $f_M(t) = \sum_n \mathrm{e}^{\mathrm{i}2^n t}\mathrm{e}^{-|t|}M(2^n)x_n$ we have

$$\begin{aligned}\Big(\int_{-\pi}^{\pi} \|\sum \mathrm{e}^{\mathrm{i}2^n t} M(2^n)x_n\|^p dt\Big)^{1/p} &\le \|(f_M - \mathcal{K}f)\chi_{[-\pi,\pi]}\|_{L_p} + \|\mathcal{K}f\|_{L_p(X)} \\ &\le 2\pi\|f_M - \mathcal{K}f\|_{L_\infty(X)} + \|\mathcal{K}\|\,\|f\|_{L_p(X)}.\end{aligned} \tag{5}$$

Note that

$$\begin{aligned}\|f\|^p_{L_p(\mathbb{R},X)} &\le \sum_{k=0}^{\infty} \mathrm{e}^{-\pi kp} \int_{-\pi}^{\pi} \|\sum_n \mathrm{e}^{\mathrm{i}2^n t}\mathrm{e}^{-|t|}x_n\|^p dt \\ &\le C\int_{-\pi}^{\pi} \|\sum_n \mathrm{e}^{\mathrm{i}2^n t} x_n\|^p dt\,.\end{aligned} \tag{6}$$

To obtain (4) for $a = 1$ from (5) and (6) it remains to show that

$$\|f_n - \mathcal{K}f\|_{L_\infty(X)} \le C\Big(\int_{-\pi}^{\pi} \|\sum_{n=0}^{\infty} \mathrm{e}^{\mathrm{i}2^n t} x_n\|^p dt\Big)^{1/p}. \tag{7}$$

To see (7) we use the Fourier transform. For $g(t) = \mathrm{e}^{-|t|}$ we have $\hat{g}(t) = \dfrac{2}{1+t^2}$. Furthermore,

$$(f_M)\widehat{\ }(s) = \sum_{n=0}^{\infty} \hat{g}(s-2^n)M(2^n)x_n\,,$$

$$(\mathcal{K}f)\widehat{\ }(s) = \sum_{n=0}^{\infty} \hat{g}(s-2^n)M(s)x_n\,.$$

Then, with $E_n = [2^n - n^2, 2^n + n^2],\ F_n = \mathbb{R} \setminus E_n$, we get

$$\begin{aligned}
&\int_{-\infty}^{\infty} \| \sum_{n\geq 5} \hat{g}(s-2^n)[M(s)x_n - M(2^n)x_n]\| ds \\
&\leq \sum_{n\geq 5} [\int_{E_n} \|\hat{g}\|_\infty \|M(s) - M(2^n)\| ds + \int_{F_n} |\hat{g}(s-2^n) 2\|M\|_\infty ds] \sup_m \|x_m\| \\
&\leq 2 \sum_{n\geq 5} [\sup_{s\in E_n} \|M'(s)(s-2^n)\| \mu(E_n) + \|M\|_\infty \int_{|s|\geq n^2} \hat{g}(s) ds] \sup_m \|x_m\| \\
&\leq 2 \sum_{n\geq 5} [\frac{2n^2 \cdot n^2}{2^n - n^2} \|sM'(s)\|_\infty + C_0 \frac{1}{n^2} \|M\|_\infty] \sup_m \|x_m\| \\
&\leq C[\|M\|_\infty + \|sM(s)\|_\infty] \sup_m \|x_m\|.
\end{aligned}$$

Hence

$$\|f_M - \mathcal{K}f\|_{L_\infty} \leq \|\hat{f}_n - (\mathcal{K}f)\hat{}\ \|_{L_1} \leq C_1[\|M\|_\infty + \|sM(s)\|_\infty] \sup_m \|x_m\|.$$

Furthermore,

$$\begin{aligned}
\sup_n \|x_n\| &\leq \sup\left\{ (\sum_{n=0}^{\infty} |x^*(x_n)|^2)^{1/2} : \|x^*\| \leq 1,\ x^* \in X^* \right\} \\
&\leq \left(\int_{-\pi}^{\pi} \| \sum_{n=0}^{\infty} \mathrm{e}^{\mathrm{i}2^n t} x_n \|^p dt \right)^{1/p},
\end{aligned}$$

since the lacunary sequence $(\mathrm{e}^{\mathrm{i}2^n t})$ in $L_p[-\pi, \pi]$ is equivalent to the unit vector basis of l_2. The last two estimates prove (7) and therefore (4) for $a = 1$.

For general a, in particular $a = 2^{-k}$, consider $f_a(t) = \frac{1}{a} f(\frac{1}{a})$ and $\mathcal{K}_a$ defined by

$$(\mathcal{K}_a g)\hat{}\,(s) = M_a(s)\hat{g}(s)\,,$$

where $M_a(s) = M(as)$. Then $\mathcal{K}_a(f_a) = (\mathcal{K}f)_a$ and

$$\|\mathcal{K}_a\| = \|\mathcal{K}\|, \quad \|M_a\|_\infty = \|M\|_\infty, \quad \|sM_a'(s)\|_\infty = \|sM'(s)\|_\infty.$$

□

In the next corollary we use the notation $S(\theta) = \{r\mathrm{e}^{\mathrm{i}\varphi} : r \in \mathbb{R} \setminus \{0\}, |\varphi| < \theta\}$ for a double sector with $0 < \theta < \dfrac{\pi}{2}$.

3.7. Corollary. *Let X and Y be UMD–spaces. Assume that $\lambda \in S(\theta) \to M(\lambda) \in B(X,Y)$ is a bounded analytic function. Then the following statements are equivalent:*
a) The operators $K_\varphi = [M(\mathrm{e}^{\mathrm{i}\varphi}\cdot)\hat{f}(\cdot)]^\vee$, $f \in \mathcal{D}_0$ extend to bounded operators from $L_p(\mathbb{R}, X)$ to $L_p(\mathbb{R}, Y)$ with

$$\sup\{\|K_\varphi\|_p : |\varphi| < \theta_1\} < \infty$$

for all $\theta_1 < \theta$ and $1 < p < \infty$.
b) For all $\theta_1 < \theta$, there is a constant C such that

$$R(\{M(ae^{\mathrm{i}\varphi}2^n) : n \in \mathbb{Z}\}) \le C \text{ for all } 1 \le |a| \le 2 \text{ and } |\varphi| < \theta_1 \,. \tag{8}$$

c) $\{M(\lambda) : \lambda \in S(\theta_1)\}$ is R–bounded for all $\theta_1 < \theta$.

Proof. $a) \Longrightarrow b)$ Since $M(\lambda)$ is bounded on $S(\theta)$, $\lambda M'(\lambda)$ is bounded on $S(\theta_1)$ (cf the proof of Example 2.9). Hence $a) \Longrightarrow b)$ follows from Proposition 3.6.

$b) \Longrightarrow c)$ For $\theta_1 < \theta$ the following Lemma 3.8 shows that 8) implies condition ii) of Remark 3.5.

$c) \Longrightarrow d)$ Example 2.9.b) shows that the assumption of Theorem 3. 4. is fulfilled.

3.8. Lemma. *Assume that the analytic function $\lambda \in S(\theta_1) \to M(\lambda) \in B(X,Y)$ satisfies (8). Then for $\theta_2 < \theta_1$ the set $\{M(\lambda) : \lambda \in S(\theta_2)\}$ and $\{\lambda M'(\lambda) : \lambda \in S(\theta_2)\}$ are R–bounded.*

Proof. We show that (8) implies

$$R\{a\mathrm{e}^{\mathrm{i}\varphi}2^j M'(a\mathrm{e}^{\mathrm{i}\varphi}2^j) : j \in \mathbb{Z}\} \le D \text{ for all } 1 \le |a| \le 2 \text{ and } |\varphi| \le \theta_2. \tag{9}$$

Consider first the case $\varphi = 0$ and $a \in [1,2]$. By Cauchy's formula we have for $s \in \mathbb{R} \setminus \{0\}$

$$sM'(s) = \frac{1}{2\pi i}\int_{\Gamma_s} \frac{s}{(s-z)^2} M(z)dz = \frac{1}{2\pi \sin\theta_2}\int_0^{2\pi} \mathrm{e}^{-i\psi} M(z_s(\psi))d\psi,$$

where $\Gamma_s = \{z : |z - s| = (\sin\theta_2)s\}$ and $z_s(\psi) = s + s(\sin\theta_2)e^{i\psi}$. For $x_j \in X$ we obtain then

$$\int_0^1 \|\sum_j r_j(t)a2^j M'(a2^j)x_j\|dt$$
$$= \frac{1}{2\pi\sin\theta_2}\int_0^1 \|\sum_j r_j(t)\int_0^{2\pi} e^{-i\psi}M(a2^j(1+\sin\theta_2 e^{i\psi}))x_j\,d\psi\|dt$$
$$\le \frac{1}{2\pi\sin\theta_2}\int_0^{2\pi}\int_0^1 \|\sum_j r_j(t)M(\tilde{a}e^{i\tilde{\psi}}2^j)x_j\|dt\,d\psi$$
$$\le \frac{C}{\sin\theta_2}\int_0^1 \|\sum r_j(t)x_j\|dt$$

by (8), where $\tilde{a} \in [0, 2a]$ and $\tilde{\psi} \in (-\theta_2, \theta_2)$ are determined by $a(1+(\sin\theta_2)e^{i\psi}) = \tilde{a}e^{i\tilde{\psi}}$. Since $\tilde{a}$ is of the form $\tilde{a} = 2^{k_0}a_1$ for some $k_0 \in \mathbb{Z}$ and $a_1 \in [1, 2]$, (9) follows for $\varphi = 0$ and $a \in [1, 2]$. This argument can be adopted to $-a$ and all $|\varphi| < \theta_2$.

The R–boundedness of $\{M(\lambda), \lambda M'(\lambda) : \lambda \in S(\theta_2)\}$ follows now from Remark 3.5. □

4. Maximal L_p–regularity

Let A be a generator of a bounded analytic semigroup T_t on a Banach space X. It is well known that the Cauchy problem

(1) $$y'(t) = Ay(t) + f(t), \quad t \ge 0, \quad y(0) = 0$$

has an unique mild solution $y \in L_{p,loc}(\mathbb{R}, X_+)$ for every $f \in L_p(\mathbb{R}_+, X)$.

4.1. Definition. *We say that (1) has maximal L_p–regularity, $1 < p < \infty$, on $[0, T), 0 < T \le \infty$, if for every $f \in L_p([0, T), X)$ the solution is almost everywhere differentiable, has values in $D(A)$ and there is a constant $C < \infty$ with*

(2) $$\|y'\|_{L_p([0,T),X)} + \|Ay\|_{L_p([0,T),X)} \le C\|f\|_{L_p([0,T),X)}.$$

This definition is slightly weaker than the usual one, which also requires $y \in L_p([0, T), X)$. But for $T = \infty$ this additional condition implies already that $s(A) = \sup\{\mathrm{Re}\lambda : \lambda \in \sigma(A)\} < 0$. Since we want to include the case $0 \in \sigma(A)$ in our analysis, we use (2). It is well known (cf [Do]) that the two definitions are equivalent for semigroups with $s(A) < 0$. We state now our characterization of maximal L_p–regularity:

4.2. Theorem. *Let X be an UMD–space and T_t a bounded analytic semigroup with generator A. Then the following conditions are equivalent:*
1) A has maximal L_p–regularity.
2) There is a constant $C < \infty$ such that

$$R(\{a2^n R(ia2^n, A) : n \in \mathbb{Z}\}) \leq C \textit{ for all } 1 \leq |a| \leq 2\,,$$

3) There is a $\theta > 0$ such that the set

$$\{\lambda R(\lambda, A) : \lambda \in \Sigma(\tfrac{\pi}{2} + \theta)\}$$

is R–bounded.
4) There is a $\theta > 0$ such that the set

$$\{T_z : z \in \Sigma(\theta)\}$$

is R–bounded.
5) There is a $\theta > 0$ and a constant C such that for all $a \in [1, 2]$, $|\varphi| \leq \theta$

$$R(\{T_{a2^n e^{i\varphi}} : n \in \mathbb{Z}\}) \leq C.$$

4.3. Remark. For complemented subspaces X of $L_q(\Omega, \mu)$, $1 < q < \infty$, (or more generally, for complemented subspaces X of a q concave, $q < \infty$, Banach function space with UMD) this result was shown by the author in the Fall of 98, using a variant of the operator sum method. The first proof for general Banach spaces with UMD is due to N. Kalton, who used the Haar System in $L_p(X)$. The following proof was found independently.

Proof. Since A is analytic we may assume that $\|\lambda R(\lambda, A)\| \leq C$ for $\lambda \in \Sigma(\frac{\pi}{2} + \theta_0)$ for some $0 < \theta_0 < \frac{\pi}{2}$. The unique solution of (1) and its derivative are given by

$$y(t) = \int_0^t T_{t-s}(f(s))ds, \quad y'(t) = \int_0^t AT_{t-s}(f(s))ds + f(t).$$

Hence maximal L_p–regularity is equivalent to the boundedness of

$$\mathcal{K}_0 f(t) = \int_0^t AT_{t-s}(f(s))ds$$

on $L_p(\mathbb{R}_+, X)$. By a standard argument this is also equivalent to the boundedness of

$$\mathcal{K} f(t) = \int_{-\infty}^{\infty} AT(t-s)(f(s))ds \tag{3}$$

on $L_p(\mathbb{R}, X)$, where $AT(t) = AT_t$ for $t > 0$ and $AT(t) = 0$ for $t \leq 0$. By taking the Fourier transform of (3) we obtain for $f \in \mathcal{S}(D(A))$

$$(4) \qquad (AT(\cdot) * f(\cdot))\widehat{\ }(t) = AR(it, A)(\hat{f}(t)) = itR(it, A)(\hat{f}(t)) - \hat{f}(t).$$

1) $\Longrightarrow$ 2) If A has maximal L_p–regularity, then by (3) and (4) the function $M(t) = itR(it, A)$ satisfies the assumptions of Theorem 3.6.

2) $\Longrightarrow$ 1) By (3) and (4) it is enough to show that $t \to tR(it, A)$ is a multiplier on $L_p(\mathbb{R}, X)$. To this end, we will show that for some $\theta < \theta_0$ the function $\lambda \in S(\theta) \to \lambda R(i\lambda, A)$ satisfies the assumption of Corollary 3.7. We use the power series expansion:

$$(5) \qquad \lambda R(\lambda, A) = \sum_{m=0}^{\infty} \lambda(it - \lambda)^m R(it, A)^{m+1} .$$

For $\lambda = ia\mathrm{e}^{\mathrm{i}\varphi}2^n$, $t = a2^n$ with $a \in [1, 2]$ we get

$$a\mathrm{e}^{\mathrm{i}(\varphi+\frac{\pi}{2})}2^n R(a\mathrm{e}^{\mathrm{i}(\varphi+\frac{\pi}{2})}2^n, A) = \mathrm{e}^{\mathrm{i}\varphi} \sum_{m=0}^{\infty}(1 - \mathrm{e}^{\mathrm{i}\varphi})^m [ia2^n R(ia2^n, A)]^{m+1} .$$

If $C = R(\{ia2^n R(ia2^n, A) : n \in \mathbb{Z}\})$ we choose $\theta < \theta_0$ so small that $|1 - \mathrm{e}^{\mathrm{i}\varphi}| < \frac{1}{2C}$ for $|\varphi| < \theta$. For an arbitrary $\varphi \in [-\theta, \theta]$ we have by (5) and Lemma 2.4.

$$\begin{aligned} &R(\{a\mathrm{e}^{\mathrm{i}(\varphi+\frac{\pi}{2})}2^n R(a\mathrm{e}^{i(\varphi+\frac{\pi}{2})}2^n, A) : n \in \mathbb{Z}\}) \\ &\qquad \leq \sum_{m=0}^{\infty} \left(\frac{1}{2C}\right)^m R(\{[ia2^n R(ia2^n, A)]^{m+1} : n \in \mathbb{Z}\}) \\ &\qquad \leq \sum_{m=0}^{\infty} \left(\frac{1}{2C}\right)^m R(\{ia2^n R(ia2^n, A) : n \in \mathbb{Z}\})^{m+1} \\ &\qquad \leq \sum_{m=0}^{\infty} \frac{C^{m+1}}{(2C)^m} = 2C. \end{aligned}$$

In the same way one may argue for $a \in [-2, -1]$.

Now apply Corollary 3.7 to see that the convolution operator (3) is bounded on $L_p(\mathbb{R}, X)$.

2) $\Longleftrightarrow$ 3) 2) implies that for some $\theta > 0$, there is a $C < \infty$ such that for all $a \in [1, 2]$, $|\varphi| \leq 2\theta$

$$R(\{a2^n\mathrm{e}^{\mathrm{i}(\frac{\pi}{2}+\varphi)} R(a2^n\mathrm{e}^{\mathrm{i}(\frac{\pi}{2}+\varphi)}, A) : n \in \mathbb{Z}\}) \leq C .$$

This was shown in 2) $\Longrightarrow$ 1). Now Lemma 3.8 implies that the set $\{t\mathrm{e}^{\mathrm{i}(\frac{\pi}{2}+\theta)} R(t\mathrm{e}^{\pm i(\frac{\pi}{2}+\theta)}, A) : t \in \mathbb{R}_+\}$ is R–bounded. An appeal to 2.9 a) gives 3). The converse is clear.

3) $\Longleftrightarrow$ 4) Since $R(\lambda, A) = \int_0^\infty e^{-\lambda t} T_t dt$ we may apply 2.10.

4) $\Longleftrightarrow$ 5) By Lemma 3.8 we obtain from 5) that $R(\{T_{te^{i\pm\varphi}} : t \in \mathbb{R}_+\})) \leq C$ for some $\varphi < \theta$. Then by 2.9. a) $\{T_z : z \in \Sigma(\varphi)\}$ is R-bounded. The converse is clear. □

In the following statement we collect some further characterizations of maximal L_p–regularity, which will be useful in future work:

4.4. Corollary. *Let X be a UMD–space and A the generator of a bounded analytic semigroup on X. Then each of the following conditions is equivalent to the maximal L_p–regularity of A:*

i) *For some $n \in \mathbb{N}$ the set*

$$\{\lambda^n R(\lambda, A)^n : \lambda \in i\mathbb{R}\} \text{ is } R-\text{bounded}\ .$$

ii) *The sets $\{T_t, t > 0\}$ and $\{tAT_t : t > 0\}$ are R–bounded.*

iii) *For some $\theta > 0$ the set*

$$\{\frac{1}{t}\int_0^t T_{e^{i\varphi}s} ds : t > 0, |\varphi| < \theta\} \text{ is } R-\text{bounded}\ .$$

Proof. i) 4.4.i) follows from 4.2.3) since

$$R(\{[\lambda R(\lambda, A)]^n : \lambda \in i\mathbb{R}\}) \leq R(\{\lambda R(\lambda, A) : \lambda + i\mathbb{R}\})^n\ .$$

Conversely, we have that

$$R(it, A)^{n-1} = (n-1)i \int_t^\infty R(is, A)^n ds, \quad \text{or}$$

$$(it)^{n-1} R(it, A)^{n-1} = \int_0^\infty h(s)(is)^n R(is, A)^n ds$$

where $h(s) = (n-1)t^{n-1}s^{-n} \cdot \chi_{[t,\infty)}(s)$ with $\int_0^\infty h(s)ds = 1$ for $n \geq 2$. Hence the R–boundedness of $\{\lambda^{n-1} R(\lambda, A)^{n-1} : \lambda \in i\mathbb{R}\}$ follows from the R–boundedness of $\{\lambda^n R(\lambda, A)^n : \lambda \in i\mathbb{R}\}$ by 2.7., and we can iterate this step.

ii) Since $A^n T_t = \dfrac{d^n}{dt^n} T_t$, Condition 4.4i) follows from 4.2.4) by Example 2.9.b).

Conversely, put $C = R(\{T_t, tAT_t : t > 0\})$ and choose an $\varepsilon > 0$ such that $|e^{\pm i\varepsilon} - 1| < \dfrac{1}{2eC}$. Now we use the power series expansion for $t > 0$

$$T_z = \sum_{n=0}^\infty \frac{1}{n!} T_t^{(n)} (z-t)^n = \sum_{n=0}^\infty \frac{1}{n!} A^n T_t (z-t)^n, \quad t > 0\ .$$

For $z = \mathrm{e}^{\mathrm{i}\varphi}t$, $|\varphi| < \varepsilon$ we obtain

$$T_{\mathrm{e}^{\mathrm{i}\varphi}t} = \sum_{n=0}^{\infty} \frac{1}{n!}[t^n A^n T_t](\mathrm{e}^{\mathrm{i}\varphi} - 1)^n \ .$$

Since $t^n A^n T_t = n^n \left(\frac{t}{n} A T_{t/n}\right)^n$ we have

$$R\Big(\{t^n A^n T_t\}\Big) \le n^n C^n \ .$$

Hence by 2.4. and $n^n \le n!\mathrm{e}^n$

$$R\{T_{\mathrm{e}^{\mathrm{i}\varphi}t} : t > 0, |\varphi| < \varepsilon\} \le C + \sum_{n=1}^{\infty} \frac{1}{n!} n^n C^n (\frac{1}{2eC})^n$$
$$\le C + \sum_{n=0}^{\infty} 2^{-n} < \infty \ .$$

So we may apply 4.2.5).
iii) 4.4.iii) follows from 4.2.4) by the convexity property 2.7. For the converse, note that for $M(t) = t^{-1} \int_0^t T_{\mathrm{e}^{\mathrm{i}\varphi}s} ds$ we have

$$T_{\mathrm{e}^{\mathrm{i}\varphi}t} = M(t) + tM'(t).$$

Since $\{tM'(t) : t > 0\}$ is R–bounded by 2.9.b) the R–boundedness of $\{T_{\mathrm{e}^{\mathrm{i}\varphi}t} : t > 0\}$ follows. □

If $0 \in \varrho(A)$ one can relax the conditions of Theorem 4.2.

4.5. Corollary. *Let X be a UMD–space and A the generator of a bounded analytic semigroup with $0 \in \varrho(A)$. Then A has maximal L_p–regularity if and only if there is a constant C and $\theta > 0$ such that one of the following conditions is fulfilled:*

$$(6) \qquad \begin{aligned} &R(\{a2^n R(ia2^n, A) : n \in \mathbb{N}\}) \le C \quad \textit{for all } 1 \le |a| \le 2 \\ &R(\{T_a 2^{-n} \mathrm{e}^{\mathrm{i}\varphi} : n \in \mathbb{N}\}) \le C \textit{ for all } a \in [1,2], |\varphi| \le \theta \ . \end{aligned}$$

Proof. To get maximal L_p–regularity from the first condition, combine 4.2. and 2.6.
For the second condition, observe that for $s(A) < \varepsilon < 0$

$$\int_1^{\infty} \|\frac{d}{dt} T_{\mathrm{e}^{\mathrm{i}\varphi}t}\| dt = \int_1^{\infty} \|A T_{\mathrm{e}^{\mathrm{i}\varphi}t}\| dt \le C \int_1^{\infty} t^{-1} \mathrm{e}^{-\varepsilon t} dt < \infty$$

and we can combine 4.2 and 2.5. □

4.6. Remark. a) Condition (6) characterizes maximal L_p–regularity on finite intervals $[0, T]$ for all bounded analytic semigroups on an UMD–space.

b) Notice that we did not use the UMD–assumption when we showed that (6) or condition 4.2.2) are necessary for maximal L_p–regularity.

4.7. Remark. Theorem 4.2 can be used to prove the following general perturbation theorem for maximal L_p–regularity: Let A be the generator of an analytic semigroup on an UMD–space X with maximal L_p–regularity. Assume that B is a closed operator on X which is relatively bounded with respect to A, i.e.

$$D(B) \supset D(A), \qquad \|Bx\| \leq a\|Ax\| + b\|x\| \quad \forall x \in D(A).$$

Then there exists a constant a_0 which only dependent on A such that for all $a < a_0$ $A + B$ is the generator of an analytic semigroup with the maximal L_p-regularity on finite intervals $[0, T]$, i.e. the usual perturbation theorem for analytic semigroups also preserves maximal L_p-regularity. For variants of this statement and details see [We2].

4.8. Remark. Let X be a complemented subspace of a Banach function space E on a measure space (Ω, μ). If E is q–concave for some $q < \infty$ (see [LT], Theorem 1.d.6) we mentioned already that R–boundedness can be expressed by the following square–function estimate:

$$\left\| \left(\sum_i |T_i x_i|^2 \right)^{1/2} \right\|_E \leq C \left\| \left(\sum_i |x_i|^2 \right)^{1/2} \right\|_E$$

(Of course $(\sum |x_i|^2)^{1/2}$ is in general not in X anymore, but this expression makes sense in E.) . In [We1] we give further square function estimates that characterize maximal L_p–regularity. For example, if E has the UMD–property and A generates a bounded analytic semigroup T_t on X, then the following conditions are equivalent to maximal L_p–regularity.

a) $$\left\|\left(\int_{-\infty}^{\infty} |tR(it, A) f(t)|^2 dt \right)^{1/2} \right\|_E \leq C \left\|\left(\int_{-\infty}^{\infty} |f(t)|^2 dt \right)^{1/2} \right\|_E,$$

b) There is a $\theta > 0$, such that for all $|\varphi| \leq \theta$

$$\|(\int_{-\infty}^{\infty} |T_{te^{i\varphi}}(f(t))|^2 dt)^{1/2}\|_E \leq C \|(\int_{-\infty}^{\infty} |f(t)|^2 dt)^{1/2}\|_E.$$

(One shows that these expressions make sense at least for finite step functions $f : (a, b) \to D(A)$ and then uses density arguments.)

4.9. Remark. a) If $X = L_2(\Omega, \mu)$, then by Fubini's theorem condition a) or b) of 4.8 is satisfied if $\|tR(it, A)\| \leq C$ for $t \in \mathbb{R}$ or $\|T_z\| \leq C$ for $z \in \Sigma(\theta)$. Hence we obtain the well known result (see [Do] for references) that all bounded analytic semigroups on a Hilbert space have maximal L_p–regularity.

b) Conditions a) and b) of 4.8 show immediately that maximal L_p–regularity is inherited by domination. For example, if T_t satisfies Gaussian estimates, i. e. $|T_{te^{i\varphi}} f| \leq bG_{at}|f|$ for some constants a, b, where G_t is the Gaussian semigroup, then T_z satisfies condition b) of 4.8, since G_t satisfies condition b). (see [We1] for details). More general Poisson estimates as in [HP], [CD] can be considered, too.

c) Condition 4.4.iii) can be used to improve a result of Lamberton ([La]): If T_t is a positive, analytic contraction semigroup on $L_q(\Omega, \mu)$ for one q with $1 < q < \infty$ (not for all $1 < q < \infty$), then A has already maximal L_p–regularity. Indeed, it is enough that T_t satisfies the following maximal ergodic estimate (see [We1] for details)

$$\left\| \sup_{t \geq 0} \left| \frac{1}{t} \int_0^t T_s f\, ds \right| \right\|_{L_q} \leq C \|f\|_{L_q}.$$

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