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A Liouville property for spherical averages in the plane

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Abstract. It is shown that any continuous bounded function f on $\mathbb{R}^2$ such that

$$f(x) = \frac{1}{(2\pi)} \int_0^{2\pi} f(x + r(x)e^{it})\, dt \ ,$$

$x \in \mathbb{R}^2$, is constant provided r is a strictly positive real function on $\mathbb{R}^2$ satisfying

$$\limsup_{|x|\to\infty} (r(x) - |x|) < +\infty \ .$$

The proof is based on a minimum principle exploiting that $\lim_{|x|\to\infty} \ln |x| = \infty$ and on a study of (σ, r)-stable sets, i.e., sets A such that the circle of radius $r(x)$ centered at x is contained in A whenever $x \in A$. The latter reveals that there is no disjoint pair of non-empty closed (σ, r)-stable subsets in $\mathbb{R}^2$ unless $\limsup_{|x|\to\infty} r(x)/|x| \geq 3$ (taking spheres this holds for any $\mathbb{R}^d$, $d \geq 2$). A counterexample is given where $\limsup_{|x|\to\infty} r(x)/|x| = 4$.

1 Introduction

The main result of this paper is the following:

Theorem 1.1. *Let r be a strictly positive real function on $\mathbb{R}^2$ such that*

$$\limsup_{|x|\to\infty} (r(x) - |x|) < +\infty$$

and let f be a continuous bounded function on $\mathbb{R}^2$ such that, for every $x \in \mathbb{R}^2$,

$$f(x) = \frac{1}{2\pi} \int_0^{2\pi} f(x + r(x)e^{it})\, dt. \tag{1.1}$$

Then f is constant.

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In [Fe] the same conclusion is obtained under the considerably stronger assumption that the function r is bounded. To discuss the background for the problem which is solved by Theorem 1.1 let us first recall some general notation which has been used in the survey paper [Ha1] and in [Ha2].

Let λ denote Lebesgue measure and let σ be the $(d-1)$-dimensional Hausdorff measure on $\mathbb{R}^d$, $d \geq 1$. For every $x \in \mathbb{R}^d$ and $r > 0$ let

$$B(x,r) = \left\{y \in \mathbb{R}^d : |y-x| < r\right\}, \quad S(x,r) = \{y \in \mathbb{R}^d : |y-x| = r\},$$

and define

$$\lambda_{x,r} = \big(\lambda(B(x,r))\big)^{-1} 1_{B(x,r)}\lambda, \quad \sigma_{x,r} = (\sigma(S(x,r)))^{-1} 1_{S(x,r)}\sigma,$$

i.e., for functions f on $B(x,r)$ (on $S(x,r)$ resp.) $\lambda_{x,r}(f)$ is the volume mean of f on $B(x,r)$ (the spherical mean of f on $S(x,r)$ resp.). It is well known that harmonic functions on a domain U, i.e., functions $h \in \mathcal{C}^2(U)$ satisfying $\Delta h = 0$, can be characterized by mean value properties, for example: A locally bounded measurable function f on U is harmonic if and only if $\lambda_{x,r}(f) = f(x)$ $(\sigma_{x,r}(f) = f(x)$ resp.) for every $x \in U$ and every $r > 0$ such that $\overline{B(x,r)} \subset U$.

The problem to what extent harmonicity of f is already a consequence of knowing that for every $x \in U$ there exists *one* radius $r(x) > 0$ such that $\lambda_{x,r(x)}(f) = f(x)$ $(\sigma_{x,r(x)}(f) = f(x)$ resp.) has a long history (see e.g. [NV] and [Ha1]). We say that a real function f on U is (λ, r)-*median* ((σ, r)-*median* resp.) if r is a strictly positive real function on U such that $B(x, r(x)) \subset U$ $(\overline{B(x, r(x))} \subset U$ resp.) and

$$\lambda_{x,r(x)}(f) = f(x) \qquad (\sigma_{x,r(x)}(f) = f(x) \text{ resp.})$$

for every $x \in U$ (where we implicitly assume that f has the necessary measurability and integrability properties).

Since bounded harmonic functions on $\mathbb{R}^d$ are constant, Theorem 1.1 can be restated as follows: Every continuous bounded (σ, r)-median function on $\mathbb{R}^2$ is harmonic provided $r \leq |\cdot| + M$ at infinity. The growth condition $r \leq |\cdot| + M$ seems to be natural, since it is a consequence of $B(r, r(x)) \subset U$ if $U^c \neq \emptyset$ (see also the counterexample given by Remark 5.1 and Proposition 6.1 where $r \leq 4(|\cdot| + 1)$).

What is known for median functions on domains $U \neq \emptyset$ in $\mathbb{R}^d$, $d \geq 1$? For simplicity let us restrict our attention to continuous bounded functions f on U (and assume that $r \leq |\cdot| + M$ if $U = \mathbb{R}^d$). Then f is always harmonic (no restriction on U or the dimension d) if f is (λ, r)-median (see [Hu] for real intervals and [HN1], [HN3] for the other cases). Suppose now that f is only (σ, r)-median (only, since every (λ, r)-median function is (σ, r')-median for some function $r' \leq r$). Does this imply that f is harmonic? For $d \geq 3$ this is an open problem (for any given U). For $d = 1$ the answer is negative (whatever

U might be). For $d = 2$ the answer is still negative if U is the unit disk ([HN2]), whereas Theorem 1.1 yields a positive answer for $U = \mathbb{R}^2$!

Our proof of Theorem 1.1 will be based on a minimum principle for (σ, r)-supermedian functions on $\mathbb{R}^2$ (Proposition 2.1). We recall that a l.s.c. lower bounded function f on $\mathbb{R}^d$ is called (σ, r)-supermedian ((λ, r)-supermedian resp.) if, for every $x \in \mathbb{R}^d$, $\sigma_{x,r(x)}(f) \le f(x)$ ($\lambda_{x,r(x)}(f) \le f(x)$ resp.). An immediate consequence of the minimum principle is the following (see Corollary 3.1):

Corollary 1.2. *Let r be a stricly positive real function on $\mathbb{R}^2$ such that*

$$\limsup_{|x|\to\infty} (r(x) - |x|) < +\infty$$

and let f be a l.s.c. lower bounded (λ, r)-supermedian function on $\mathbb{R}^2$. Then f is constant.

Let us note that such a result (assuming that even $\sup_{x\in\mathbb{R}^2}(r(x) - |x|) < \infty$) has been proved in [HN3] and [Ha2] using an entirely different and rather involved technique.

To deduce Theorem 1.1 from our minimum principle we shall show that assuming

$$\limsup_{|x|\to\infty} \frac{r(x)}{|x|} < 3$$

it is impossible to have two disjoint non-empty closed sets A_0, A_1 in $\mathbb{R}^2$ which are (σ, r)-*stable*, i.e., satisfy $S(x, r(x)) \subset A_j$ whenever $x \in A_j$, $j \in \{0, 1\}$ (Proposition 4.1). For sets which are invariant under rotations $\limsup_{|x|\to\infty} r(x)/|x| = 4$ is the borderline for the existence of (σ, r)-stable pairs (A_0, A_1) (Proposition 6.1).

2 A minimum principle for (σ, r)-supermedian functions

The essential step for the proof of our main result is the following minimum principle in the plane (where $B_r := B(0, r)$):

Proposition 2.1. *Let $M > 0$ and let f be a l.s.c. lower bounded function on $\mathbb{R}^2$ such that, for every $x \in \overline{B}_M^c$, there exists $0 < r(x) \le |x| + M$ with $\sigma_{x,r(x)}(f) \le f(x)$. Then there exists a point $x_0 \in \overline{B}_M$ such that $f \ge f(x_0)$.*

We observe that this result is rather optimal: Defining $f(x) := (M - |x|)^+$, $x \in \mathbb{R}^2$, and $r(x) := |x| + M$ for $|x| \le M$, $r(x) := |x| - M$ for $|x| > M$, the function f is (σ, r)-supermedian and $f(x_0) > \inf f(\mathbb{R}^2)$ for all $x_0 \in B_M$!

For the present let us fix $\varepsilon > 0$ and construct a continuous real function $\varphi \geq 0$ on $\mathbb{R}^2$ such that $\sigma_{x,r(x)}(\varphi) \leq \varphi(x)$ for every $x \in B^c_{M+2\varepsilon}$ and $\lim_{|x|\to\infty} \varphi(x) = \infty$. To that end we define

$$\varphi_z(y) := (\ln|y-z| - \ln M)^+ \qquad (y, z \in \mathbb{R}^2)$$

and fix a continuous real function ψ on $\mathbb{R}^2$ such that $0 \leq \psi \leq 1$ on $\mathbb{R}^2$, $\psi = 0$ on $B_{M+\varepsilon}$ and $\psi = 1$ on $B^c_{M+2\varepsilon}$. Then obviously

$$\psi(x) - \sigma_{x,r(x)}(\psi) = \sigma_{x,r(x)}(1-\psi) \geq \sigma_{x,r(x)}(B_{M+\varepsilon}) \quad \text{for all } x \in B^c_{M+2\varepsilon}.$$

So the following lemma shows that the function

$$\varphi := \varphi_0 + \frac{4\pi M}{\varepsilon}\,\psi$$

has the desired properties.

Lemma 2.2. *For every $x \in B^c_{M+2\varepsilon}$,*

$$\varphi_0(x) - \sigma_{x,r(x)}(\varphi_0) \geq -\frac{4\pi M}{\varepsilon}\sigma_{x,r(x)}(B_{M+\varepsilon}).$$

Proof. Fix $x \in B^c_{M+2\varepsilon}$. Since φ_0 is harmonic on $\overline{B}^c_M$, we have $\varphi_0(x) - \sigma_{x,r(x)}(\varphi_0) = 0$ if $|x| - r(x) > M$. So let us suppose that $|x| - r(x) \leq M$.. Knowing that $|x| - r(x) \geq -M$ by assumption we conclude that $S(x, r(x))$ intersects the closed disk $\overline{B}_M$ and therefore

$$\sigma_{x,r(x)}(B_{M+\varepsilon}) \geq 2 \cdot \frac{\varepsilon}{2\pi r(x)}. \tag{2.1}$$

In order to get an estimate for $\varphi_0(x) - \sigma_{x,r(x)}(\varphi_0)$ let us consider the point

$$z := -2M\frac{x}{|x|}.$$

Since $|x| - r(x) \geq -M$, the circle $S(x, r(x))$ is contained in the closed halfplane

$$H := \{y \in \mathbb{R}^2 : |y-z| \geq |y|\}.$$

This implies that

$$\sigma_{x,r(x)}(\varphi_z) = \varphi_z(x) \quad \text{and} \quad \varphi_z - \varphi_0 \geq 0 \quad \text{on } S(x, r(x)).$$

Consequently,

$$\begin{aligned} \varphi_0(x) - \sigma_{x,r(x)}(\varphi_0) &= \varphi_0(x) - \varphi_z(x) + \sigma_{x,r(x)}(\varphi_z - \varphi_0) \\ &\geq \varphi_0(x) - \varphi_z(x) \end{aligned} \tag{2.2}$$

where

$$\varphi_z(x) - \varphi_0(x) = \ln \frac{|x - z|}{|x|} = \ln(1 + \frac{2M}{|x|}) \le \frac{2M}{|x|}. \tag{2.3}$$

Since $|x| \ge M$, we know that $2|x| \ge |x| + M \ge r(x)$. Thus, by (2.2) and (2.3),

$$\varphi_0(x) - \sigma_{x,r(x)}(\varphi_0) \ge -\frac{4M}{r(x)}. \tag{2.4}$$

The proof is finished combining (2.1) and (2.4). □

Proof of Proposition 2.1. Fix $\delta > 0$, and define

$$g := f + \delta\varphi - \inf f(B_{M+2\varepsilon}).$$

By Lemma 2.2,

$$\sigma_{x,r(x)}(g) \le g(x) \qquad \text{for all } x \in B^c_{M+2\varepsilon}.$$

Since $\lim_{|x|\to\infty} \varphi(x) = \infty$ and f is lower bounded, there exists $R > 0$ such that $g > 0$ on B^c_R. Consider now the set $\mathcal{F}$ of all l.s.c. lower bounded functions u on $X := \overline{B}_{R+M}$ such that

$$\sigma_{x,r(x)}(u) \le u \qquad \text{for all } M + 2\varepsilon \le |x| \le R$$

(observe that $S(x, r(x)) \subset X$ whenever $|x| \le R$). Then $\mathcal{F}$ is a convex cone containing $g|_X$ and all affinely linear functions on X. Let us recall that the *Choquet boundary* $\mathrm{Ch}_{\mathcal{F}}X$ of X with respect to $\mathcal{F}$ is the set of all points $x \in X$ such that the Dirac measure at x is the only Radon measure $\mu \ge 0$ on X satisfying $\mu(u) \le u(x)$ for every $u \in \mathcal{F}$ (see e.g. [BH], p. 21). By our definition of $\mathcal{F}$ it is obvious that $\mathrm{Ch}_{\mathcal{F}}X$ does not contain points $x \in X$ such that $M + 2\varepsilon \le |x| \le R$. Having $g(x) \ge 0$ if $|x| < M + 2\varepsilon$ or $|x| > R$ and therefore $g \ge 0$ on $\mathrm{Ch}_{\mathcal{F}}X$, Bauer's minimum principle (cf. [Ba1,Ba2] or [BH]) yields that $g \ge 0$ on X whence $g \ge 0$ on $\mathbb{R}^2$. Since $\delta > 0$ is arbitrary, we conclude that

$$f \ge \inf\{f(x) : x \in B_{M+2\varepsilon}\} \quad \text{on } \mathbb{R}^2.$$

This inequality holds for every $\varepsilon > 0$. Thus the lower semi-continuity of f implies that

$$f \ge \inf\{f(x) : x \in \overline{B}_M\} \quad \text{on } \mathbb{R}^2$$

and that $\inf\{f(x) : x \in \overline{B}_M\} = f(x_0)$ for some point $x_0 \in \overline{B}_M$. □

3 Application to (λ, r)-supermedian functions

If f is a l.s.c. lower bounded (λ, r)-supermedian function, then f is obviously (σ, r')-supermedian for some strictly positive function $r' \le r$. Therefore Proposition 2.1 has the following consequence (cf. [HN3, Ha2]):

Corollary 3.1. *Let r be a strictly positive real function on $\mathbb{R}^2$ such that* $\limsup_{|x|\to\infty}(r(x) - |x|) < +\infty$ *and let f be a l.s.c. lower bounded (λ, r)-supermedian function on $\mathbb{R}^2$. Then f is constant.*

Indeed, by the assumption on r there exists $M > 0$ such that $r(x) \le |x| + M$ for all $x \in B_M^c$. Therefore, by Proposition 2.1, there exists a point $x_0 \in \mathbb{R}^2$ such that $f \ge f(x_0)$ on $\mathbb{R}^2$. Obviously, the set

$$A := \{x \in \mathbb{R}^2 : f(x) = f(x_0)\}$$

is closed and $x_0 \in A$. Moreover, for every $x \in A$, the inequality $\lambda_{x,r(x)}(f) \le f(x)$ and the lower semi-continuity of f imply that $B(x, r(x)) \subset A$. Thus A is open as well, and we conclude that $A = \mathbb{R}^2$, i.e., that $f = f(x_0)$ on $\mathbb{R}^2$.

4 (σ, r)-stable sets

The proof of Corollary 3.1 breaks down if we only know that the function f is (σ, r)-supermedian, since then the closed set $A = \{f = f(x_0)\}$ is only (σ, r)-*stable*, i.e., we have $S(x, r(x)) \subset A$ for all $x \in A$, and this of course does not imply that A is open. Obviously, the conclusion of Corollary 3.1 itself does not hold if f is only (σ, r)-supermedian, since every continuous function $f \ge 0$ with compact support is (σ, r)-supermedian for a suitable (even bounded) function r. If, however, f is a continuous bounded (σ, r)-*median* function the situation is different. Applying Proposition 2.1 to f and $-f$ we obtain points $x_0, x_1 \in \mathbb{R}^2$ such that $f(x_0) \le f \le f(x_1)$ and then we have two non-empty closed (σ, r)-stable subsets $\{f = f(x_0)\}$, $\{f = f(x_1)\}$. We shall see that this is impossible unless $f(x_0) = f(x_1)$ or r grows too fast at infinity.

Let us say that (A_0, A_1) is a (σ, r)-*stable pair*, if A_0, A_1 are non-empty closed (σ, r)-stable sets in $\mathbb{R}^d$ such that $A_0 \cap A_1 = \emptyset$.

Proposition 4.1. *Let $d \ge 2$ and suppose that there exists a (σ, r)-stable pair of subsets in $\mathbb{R}^d$. Then, for every line L in $\mathbb{R}^d$,*

$$\limsup_{x\in L, |x|\to\infty} \frac{r(x)}{|x|} \ge 3. \tag{4.1}$$

We observe that of course $\limsup_{x\in L,|x|\to\infty} r(x)/|x| \le 1$ if $r \le |\cdot| + M$. Before passing to the proof of Proposition 4.1 let us note the following consequence:

Corollary 4.2. *Let f be a continuous (σ, r)-median function on $\mathbb{R}^d$, $d \geq 2$, and suppose that there exist points $x_0, x_1 \in \mathbb{R}^d$ such that*

$$f(x_0) \leq f \leq f(x_1) \quad \text{on } \mathbb{R}^d. \tag{4.2}$$

Then f is constant provided $\limsup_{x \in L, |x| \to \infty} r(x)/|x| < 3$ for some line L in $\mathbb{R}^d$.

Proof. If f is not constant, then the sets

$$A_j := \{x \in \mathbb{R}^2 : f(x) = f(x_j)\} \qquad (j \in \{0, 1\})$$

form a (σ, r)-stable pair and therefore (4.1) holds for every line L in $\mathbb{R}^d$. □

Proof of Proposition 4.1. Let (A_0, A_1) be a (σ, r)-stable pair in $\mathbb{R}^d$. First let us fix points $a_0 \in A_0$, $a_1 \in A_1$, and denote by L the line containing a_0 and a_1 (our proof of (4.1) for this line will imply that every line in $\mathbb{R}^d$ intersects A_0 and A_1). Let e be the unit vector $(a_1 - a_0)/|a_1 - a_0|$. Using the bijective mapping $t \mapsto a_0 + te$ from $\mathbb{R}$ on L we obtain an order $\leq$ on L. Clearly, the sets $A_j \cap L$, $j \in \{0, 1\}$, are closed and non-empty. Since A_0, A_1 are (σ, r)-stable, we know that

$$x \pm r(x)e_L \in A_j \cap L \qquad \text{whenever } x \in A_j \cap L,\ j \in \{0, 1\}. \tag{4.3}$$

In particular, the subsets $A_j \cap L$ of L, $j \in \{0, 1\}$, are neither bounded from below nor bounded from above (with respect to $\leq$). Let us define

$$A_{j+2k} := A_j \qquad (k \in \mathbb{N},\ j \in \{0, 1\}).$$

We claim that there exist points $x_n^{\pm} \in A_n$ such that, for every $n \in \mathbb{N}$,

$$x_n^- < x_{n-1}^- \leq x_{n-1}^+ < x_n^+$$

and the line segment

$$I_{n-1} := \{x \in L : x_{n-1}^- - r(x_{n-1}^-)e \leq x \leq x_{n-1}^+ + r(x_{n-1}^+)e\}$$

does not intersect the spheres $S(x, r(x))$ for $x \in A_n \cap (L \setminus I_{n-1})$. Indeed, we may take $x_0^{\pm} = a_0$. Any sphere $S(x, r(x))$, $x \in L \setminus I_0$, intersecting I_0 contains points of $S(a_0, r(a_0))$ which by (4.3) is impossible if $x \in A_1 \cap (L \setminus I_0)$, since $A_0 \cap A_1 = \emptyset$. Suppose now that $n \in \mathbb{N}$ and that $x_0^{\pm}, \ldots, x_{n-1}^{\pm}$ are already constructed. Define

$$x_n^+ := \min\{x \in A_n \cap L : x \geq x_{n-1}^+ + r(x_{n-1}^+)e\},$$
$$x_n^- := \max\{x \in A_n \cap L : x \leq x_{n-1}^- - r(x_{n-1}^-)e\}.$$

Then the open line segment from x_n^- to x_n^+ does not intersect the spheres $S(x_n^-, r(x_n^-))$, $S(x_n^+, r(x_n^+))$ and therefore

$$r(x_n^{\pm}) \geq |x_n^+ - x_n^-| = |x_n^+ - a_0| + |a_0 - x_n^-|. \tag{4.4}$$

Any sphere $S(x, r(x))$, $x \in A_{n+1} \cap (L \setminus I_n)$, intersecting I_n would contain points of the subset $S(x_n^-, r(x_n^-)) \cup S(x_n^+, r(x_n^+))$ of A_n which is impossible. So $S(x, r(x)) \cap I_n = \emptyset$ for every $x \in A_{n+1} \cap (L \setminus I_n)$.

If $\sup_n |x_n^+| < \infty$, then $\lim_{n\to\infty} x_n^+ \in A_0 \cap A_1$. Impossible! Therefore $\sup |x_n^+| = \infty$ and, similarly, $\sup |x_n^-| = \infty$, i.e.,

$$\lim_{n\to\infty} |x_n^\pm| = \infty. \tag{4.5}$$

Of course (4.4) implies that

$$r(x_n^+) \geq 2\,|x_n^+ - a_0| \quad \text{or} \quad r(x_n^-) \geq 2\,|x_n^- - a_0| \tag{4.6}$$

whence $\limsup_{|x|\to\infty} r(x)/|x| \geq 2$ which would be sufficient for the proof of Theorem 1.1. To prove (4.1) we finally define

$$y_{n-1}^+ := \max\{y \in A_{n-1} \cap L : y \leq x_n^- + r(x_n^-)e\} \qquad (n \in \mathbb{N}).$$

Obviously, $y_{n-1}^+ \geq x_{n-1}^+$, since $x_{n-1}^+ < x_n^+ \leq x_n^- + r(x_n^-)$ by (4.4) so that

$$\lim_{n\to\infty} |y_{n-1}^+| = \infty.$$

For a moment let us fix $n \in \mathbb{N}$. Since $0 \leq y_{n-1}^+ - a_0 \leq r(x_n^-)e - (a_0 - x_n^-)$, we know that

$$|y_{n-1}^+ - a_0| \leq r(x_n^-) - |a_0 - x_n^-|. \tag{4.7}$$

Since the open line segment $\{y \in L : y_{n-1}^+ < y < x_n^- + r(x_n^-)\}$ and the sphere $S(x_n^-, r(x_n^-))$ are contained in the complement of A_{n-1}, we conclude that

$$r(y_{n-1}^+) \geq |y_{n-1}^+ - (x_n^- - r(x_n^-))|. \tag{4.8}$$

We claim that

$$r(x_n^-) > 3|x_n^- - a_0| \quad \text{or} \quad r(y_{n-1}^+) \geq 3|y_{n-1}^+ - a_0|. \tag{4.9}$$

Indeed, suppose that $r(x_n^-) \leq 3|x_n^- - a_0|$. Then

$$|x_n^- - a_0| + r(x_n^-) \geq 2(r(x_n^-) - |x_n^- - a_0|).$$

Using (4.8), (4.7), and the inequality $x_n^- \leq a_0 \leq y_{n-1}^+$ we obtain that

$$r(y_{n-1}^+) \geq |y_{n-1}^+ - a_0| + |a_0 - x_n^-| + r(x_n^-) \geq 3|y_{n-1}^+ - a_0|.$$

Thus (4.9) holds and we conclude that

$$\limsup_{x\in L, |x|\to\infty} \frac{r(x)}{|x|} \geq 3.$$

Finally let $\tilde{L}$ be any line in $\mathbb{R}^2$. Since $\lim_{n\to\infty} |x_n^\pm| = \infty$, there exists $k \in \mathbb{N}$ such that $B(a_0, |a_0 - x_n^\pm|) \cap \tilde{L} \neq \emptyset$ for every $n \geq 2k$. Fix $j \in \{0, 1\}$ and let $n = 2k + j$. By (4.6), $B(a_0, |a_0 - x_n^+|) \subset B(x_n^+, r(x_n^+))$ or $B(a_0, |a_0 - x_n^-|) \subset B(x_n^-, r(x_n^-))$ and therefore $S(x_n^+, r(x_n^+)) \cap \tilde{L} \neq \emptyset$ or $S(x_n^-, r(x_n^-)) \cap \tilde{L} \neq \emptyset$ whence $A_j \cap \tilde{L} \neq \emptyset$. This finishes the proof. □

5 Proof of Theorem 1.1

As already indicated our main result now follows immediately: Suppose that we have $\limsup_{|x|\to\infty}(r(x) - |x|) < +\infty$ and that f is a continuous bounded (σ, r)-median function on $\mathbb{R}^2$. Then there exists $M > 0$ such that $r(x) \le |x|+M$ whenever $|x| \ge M$. Since the functions f and $-f$ are (σ, r)-supermedian, we conclude from Proposition 2.1 that there exist points $x_0, x_1 \in \mathbb{R}^2$ such that $f \ge f(x_0)$ and $-f \ge -f(x_1)$, i.e., that $f(x_0) \le f \le f(x_1)$. Thus f is constant by Corollary 4.2.

Remark 5.1. If we only require that $r : \mathbb{R}^d \to]0, \infty[$ satisfies

$$\limsup_{|x|\to\infty} \frac{r(x)}{|x|} \le 4,$$

then it is possible that there is a continuous bounded (σ, r)-median function which is not constant (but invariant under rotations).

Indeed, Proposition 6.1 will show that there exists $r : \mathbb{R}^d \to]0, \infty[$ and a (σ, r)-stable pair (A_0, A_1) which is invariant under rotations such that $\limsup_{|x|\to\infty} r(x)/|x| = 4$. Let f denote the continuous bounded function on $\mathbb{R}^d$ which is harmonic on $\mathbb{R}^d \setminus (A_0 \cup A_1)$ and equal to j on A_j, $j \in \{0, 1\}$. Let $\tilde{r} : \mathbb{R}^d \to]0, \infty[$ such that $\tilde{r} = r$ on $A_0 \cup A_1$ and $\tilde{r} \le \min(r, \operatorname{dist}(\cdot, A_0 \cup A_1))$ on $\mathbb{R}^d \setminus (A_0 \cup A_1)$. Then the (non-constant) function f is obviously $(\sigma, \tilde{r})$-median and invariant under rotations.

6 Rotationally invariant (σ, r)-stable sets

The following result will complete our considerations:

Proposition 6.1. *Suppose that $d \ge 2$.*

1. *There exists $r : \mathbb{R}^d \to]0, \infty[$ and a (σ, r)-stable pair which is invariant under rotations such that*

$$\limsup_{|x|\to\infty} \frac{r(x)}{|x|} = 4.$$

2. *Conversely, if $r : \mathbb{R}^d \to]0, \infty[$ such that there exists a (σ, r)-stable pair which is invariant under rotations, then*

$$\limsup_{|x|\to\infty} \frac{r(x)}{|x|} \ge 4. \tag{6.1}$$

Proof. 1. Taking $\beta_0 = 0$ we define recursively

$$\alpha_n := \beta_{n-1} + 1, \quad \beta_n := 3\alpha_n.$$

Denote

$$V_n := \{x \in \mathbb{R}^d : \alpha_n \le |x| \le \beta_n\} \qquad (n \in \mathbb{N})$$

and let

$$A_0 := \bigcup_{k=1}^{\infty} V_{2k}, \quad A_1 := \bigcup_{k=1}^{\infty} V_{2k-1}.$$

Finally, we define $r : \mathbb{R}^d \to \,]0, \infty[$ by

$$r(x) := \begin{cases} 2\alpha_n, & |x| = \alpha_n, n \in \mathbb{N}, \\ 4(\beta_n + 1), & |x| = \beta_n, n \in \mathbb{N}, \\ \operatorname{dist}(x, \partial(A_0 \cup A_1)), & x \notin \partial(A_0 \cup A_1). \end{cases}$$

It is immediately seen that

$$\limsup_{|x|\to\infty} \frac{r(x)}{|x|} = 4$$

and $S(x, r(x)) \subset A_j$ for all $x \in A_j \setminus \partial A_j$, $j \in \{0, 1\}$. For every $x \in \mathbb{R}^d$ and for every $r > 0$,

$$\begin{aligned} &\min\{|z| : z \in S(x, r(x))\} = |r - |x||, \\ &\max\{|z| : z \in S(x, r(x))\} = r + |x|. \end{aligned} \tag{6.2}$$

For every $n \in \mathbb{N}$,

$$[r(\alpha_n) - \alpha_n, r(\alpha_n) + \alpha_n] = [\alpha_n, 3\alpha_n] = [\alpha_n, \beta_n]$$

and $\alpha_{n+2} = \beta_{n+1} + 1 = 3\alpha_{n+1} + 1 = 3\beta_n + 4$, $\beta_{n+2} = 9\beta_n + 12$ whence

$$[r(\beta_n) - \beta_n, r(\beta_n) + \beta_n] = [3\beta_n + 4, 5\beta_n + 4] \subset [\alpha_{n+2}, \beta_{n+2}].$$

This shows that $S(x, r(x)) \subset A_j$ for every $x \in \partial A_j$, $j \in \{0, 1\}$. Thus (A_0, A_1) is a (σ, r)-stable pair.

2. We shall prove (6.1) by contradiction. Let us suppose that $r : \mathbb{R}^d \to \,]0, \infty[$ and (A_0, A_1) is a (σ, r)-stable pair such that, for some real $K > 0$,

$$r(x) \le 4|x| \qquad \text{whenever } |x| \ge K. \tag{6.3}$$

Let us identify $\mathbb{R}$ with the line $\mathbb{R} \times \{0\}^{d-1}$ in $\mathbb{R}^d$ and introduce

$$A_{j+2k} := A_j \qquad (k \in \mathbb{N}, j \in \{0, 1\}).$$

Choosing an arbitrary $\alpha_0 \in A_0 \cap [K, \infty[$ we define recursively

$$\alpha_n = \inf(A_n \cap \big[\alpha_{n-1}, \infty\big[), \quad \beta_n = \sup(A_n \cap [\alpha_n, \alpha_{n+1}]) \qquad (n \in \mathbb{N}).$$

Then, for every $n \in \mathbb{N}$,

$$\alpha_n, \beta_n \in A_n, \quad \alpha_n \le \beta_n < \alpha_{n+1} \le \beta_{n+1} \tag{6.4}$$

and

$$A_0 \cap [\alpha_1, \infty[\subset \bigcup_{k=1}^{\infty} [\alpha_{2k}, \beta_{2k}], \quad A_1 \cap [\alpha_1, \infty[\subset \bigcup_{k=0}^{\infty} [\alpha_{2k+1}, \beta_{2k+1}]. \tag{6.5}$$

By (6.2) we conclude that, for every $n \in \mathbb{N}$, there exist (unique) numbers k_n and m_n such that $k_n \ge n$, $m_n > n$, the differences $k_n - n$ and $m_n - n$ are even, and

$$[|r(\alpha_n) - \alpha_n|, r(\alpha_n) + \alpha_n] \subset [\alpha_{k_n}, \beta_{k_n}],$$

$$[|r(\beta_n) - \beta_n|, r(\beta_n) + \beta_n] \subset [\alpha_{m_n}, \beta_{m_n}].$$

In particular, $|r(\alpha_n) - \alpha_n| \ge \alpha_{k_n} \ge \alpha_n$, hence $|r(\alpha_n) - \alpha_n| = r(\alpha_n) - \alpha_n$. Similarly, $|r(\beta_n) - \beta_n| = r(\beta_n) - \beta_n$. Using (6.3) we obtain that

$$\alpha_{k_n} \le 3\alpha_n, \quad \alpha_{k_n} + 2\alpha_n \le \beta_{k_n}, \tag{6.6}$$

$$\alpha_{m_n} \le 3\beta_n, \quad \alpha_{m_n} + 2\beta_n \le \beta_{m_n}. \tag{6.7}$$

Having $\alpha_n \le \alpha_{k_n}$ and $5\alpha_{k_n} \le 3\alpha_{k_n} + 6\alpha_n$ this implies that

$$3\alpha_n \le \beta_{k_n}, \qquad 5\alpha_{k_n} \le 3\beta_{k_n}. \tag{6.8}$$

Similarly,

$$3\beta_n \le \beta_{m_n}, \qquad 5\alpha_{m_n} \le 3\beta_{m_n}. \tag{6.9}$$

Let J denote the set consisting of all k_n and m_n, $n \in \mathbb{N}$, and let us remove all points x with $\alpha_i \le |x| \le \beta_i$ for some $i \in \mathbb{N} \setminus J$ from A_0 and A_1. Then the reduced pair $(\tilde{A}_0, \tilde{A}_1)$ which we obtain is of course still (σ, r)-stable and invariant under rotations. It leads to sets $\tilde{A}_n$ and intervals $[\tilde{\alpha}_n, \tilde{\beta}_n]$ in $\tilde{A}_n$ such that $5\tilde{\alpha}_n \le 3\tilde{\beta}_n$ for every $n \in \mathbb{N}$ (each interval $[\tilde{\alpha}_n, \tilde{\beta}_n]$) contains an interval $[\alpha_{k_i}, \beta_{k_i}]$ or $[\alpha_{m_i}, \beta_{m_i}]$, $i \in \mathbb{N}$). In other words, we may assume from the very beginning that

$$5\alpha_n \le 3\beta_n \quad \text{for all } n \in \mathbb{N}. \tag{6.10}$$

Then by (6.6), (6.4), and (6.10),

$$\alpha_{k_n} \le 3\alpha_n \le 3 \cdot \left(\frac{3}{5}\right)^3 \alpha_{n+3} < \alpha_{n+3},$$

$$\alpha_{m_n} \le 3\beta_n \le 3 \cdot \left(\frac{3}{5}\right)^3 \beta_{n+3} < \beta_{n+3} < \alpha_{m+4}$$

whence for all $n \in \mathbb{N}$

$$k_n \in \{n, n+2\}, \qquad m_n = n+2.$$

If $n \geq 2$, then $m_{n-1} = n + 1$ and therefore

$$\beta_n < \alpha_{m_{n-1}} \leq 3\beta_{n-1} < 3\alpha_n \leq \beta_{k_n}.$$

Thus in fact

$$k_n = n + 2 \qquad \text{for all } n \geq 2.$$

Using (6.6), (6.7), and (6.10) we conclude that, for every $n \geq 2$,

$$\frac{\alpha_{n+3}}{\alpha_{n+2}} > \frac{\beta_{n+2}}{\alpha_{n+2}} \geq 1 + 2 \cdot \frac{\beta_n}{\alpha_n} : \frac{\alpha_{n+2}}{\alpha_n} \geq 1 + 2 \cdot \frac{5}{3} : 3 > 2$$

and therefore

$$\alpha_6 = \alpha_{k_4} \leq 3\alpha_4 \leq \frac{3}{2^2}\alpha_6 < \alpha_6.$$

This contradiction finishes the proof. □

Remark 6.2. Note that $k_n = n$ in the example given for the first part of Proposition 6.1. In fact, a closer analysis would reveal that for every reduced (σ, r)-stable pair which is invariant under rotations we even have $\limsup_{|x|\to\infty} r(x)/|x| \geq 5$ unless $k_n = n$ for almost every $n \in \mathbb{N}$.

Final remark. Suppose that U is a proper subset of $\mathbb{R}^d$, $d \geq 2$, and that $0 < r < \text{dist}(\cdot, U^c)$. Then there is no (σ, r)-stable pair in U. To see this it suffices to proceed similarly as in the proof of Proposition 4.1 using an extension of a polygonal arc intersecting A_0 and A_1. Consequently, if g, h are harmonic functions on U and f is a (σ, r)-median continuous function on U such that $g \leq f \leq h$ and $g(x) = f(x)$, $f(y) = h(y)$ for some points $x, y \in U$, then $f = g = h$.

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