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Criteria of motivic equivalence for quadratic forms and central simple algebras

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Abstract. We give a short, elementary, and characteristic independent proof of the criterion for motivic isomorphism of two projective quadrics discovered by A. Vishik [24]. We also give a criterion for motivic isomorphism of two Severi-Brauer varieties.

0. Introduction

We consider non-degenerate quadratic forms over fields of characteristic not 2. Let ϕ_1 and ϕ_2 be two quadratic forms over a field F. We call them *splitting-equivalent* in the sense of A. Vishik (*s-equivalent* for short) and write $\phi_1 \overset{\mathrm{s}}{\sim} \phi_2$, if they are of the same dimension and for any field extension E/F the Witt indexes of the E-forms $(\phi_1)_E$ and $(\phi_2)_E$ coincide (the condition $\dim\phi_1 = \dim\phi_2$ is really needed only in the case of two completely split forms and is superfluous in other cases). Following [5], we call ϕ_1 and ϕ_2 *motivic-equivalent* (*m-equivalent* for short) and write $\phi_1 \overset{\mathrm{m}}{\sim} \phi_2$, if the motives of the projective quadrics X_{ϕ_1} and X_{ϕ_2}, given by ϕ_1 and ϕ_2, are isomorphic (note that X_{ϕ_1} and X_{ϕ_2} are isomorphic as algebraic varieties iff the quadratic forms ϕ_1 and ϕ_2 are *similar*, i.e. $\phi_1 \simeq a\cdot\phi_2$ for some non-zero $a \in F$).

Our category of motives is simply the classical category of correspondences (see Section 1), or, more precisely, the category $\mathcal{CV}^0$ of Chow-correspondences of degree 0. The motive of an algebraic variety is then simply the variety itself considered as an object of this category.

Here is the criterion of motivic equivalence for quadratic forms we are meaning (compare [24, Statement 1.4.1]):

Criterion 0.1 (A. Vishik). *Two quadratic forms are m-equivalent if and only if they are s-equivalent.*

The proof is given in the end of Sect. 5.

It may be remarked that there is another standard classical motivic category – the category of Grothendieck Chow-motives (see, for instance, [3, Example

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16.1.12]). This category contains $\mathcal{CV}^0$ as a full subcategory, is slightly bigger then $\mathcal{CV}^0$, and is obtained from $\mathcal{CV}^0$ by applying a very formal procedure (the procedure of pseudo-abelian completion). Of course one may replace $\mathcal{CV}^0$ by this category in the definition of m-equivalence.

In the original proof of the criterion, given in [24], the characteristic of the base field is assumed to be 0 and the category of correspondences $\mathcal{CV}^0$ is replaced by the triangulated category of motivic complexes of Voevodsky [25, §3.1] (denoted as DM_-^{eff} in the literature) using the theorem that in the characteristic 0 case the category $\mathcal{CV}^0$ is a full subcategory of the DM_-^{eff} ([25, Theorem 3.2.6 and Corollary 4.2.6]). This replacement is needed because the proof makes use of the motives of the standard simplicial schemes (associated to the varieties of totally isotropic subspaces of the quadratic forms) which do not live in $\mathcal{CV}^0$ (they also do not live in the Grothendieck category of Chow-motives and even not in Voevodsky's triangulated category of geometrical motives $DM_{gm}^{eff} \subset DM_-^{eff}$).

In our proof we stay all the time in the category of Chow-correspondences and, what is probably important to mention, we do not pay for this by making the proof more complicated or less conceptual. In some sense our proof is almost a word by word translation of Vishik's proof to a more elementary language. Only "almost", because there are some further simplifications, e.g., we work only with quadrics and do not work with other varieties of isotropic subspaces of ϕ_1 and ϕ_2. And only "in some sense" because we do not really do equivalent things.

Some more remarks about Criterion 0.1 should be made. The s-equivalence is a very natural equivalence relation for quadratic forms. However it was not satisfactory (if at all) considered previously (a systematic investigation is started recently in [5]).

Since m-equivalent forms should have the same dimension and Witt index and remain m-equivalent over any extension of scalars (because an extension of scalars is a functor on motives), m-equivalence easily implies s-equivalence (see Corollary 2.5). Thus, the essential part of the criterion is the inverse implication.

In the case where the dimension of the quadratic forms is odd, this part was recently depreciated by O. Izhboldin. Using the framework of quadratic forms exclusively, he showed that two s-equivalent odd-dimensional quadratic forms are similar (and thus their quadrics are isomorphic already as algebraic varieties !). By this reason, we removed the case of odd-dimensional forms from our consideration, being in fact very sad about this, because the proof for the odd dimensions is much more elegant than for even ones. In the case where dimension (of the quadratic form and therefore of the quadric) is even, one has to "struggle" with the algebraic cycles on the quadric having the middle codimension. The difficulties do not seem to be really important, but they require rather long additional computations (made in Sect. 6). In fact all considerations which are specific for the even-dimensional case are easily recognized in the text; what would be left without them is the proof for the odd case.

We refer to [5, §4] and [6] for examples of s-equivalent non-similar quadratic forms of even dimension (in [6], such examples are constructed for every even dimension starting from 8 and except 12; as shown in [5, Proposition 3.1], s-equivalent forms of any dimension up to 7 are similar). Note that since any two non-similar quadratic forms determine non-isomorphic (as algebraic varieties) projective quadrics, every example of this kind gives two non-isomorphic quadrics with isomorphic motives. In the final section, we show that this situation also occurs to the Severi-Brauer varieties. Namely, Criterion 7.1. states that the motives of two Severi-Brauer varieties are isomorphic if and only if the varieties are isomorphic or opposite (the latter condition means that one of the varieties is given by the algebra opposite to the algebra giving the other variety; note that the varieties are not isomorphic in this case, iff the algebras are of an exponent bigger than 2). Let us discuss some starting points of Criterion 7.1.

It is a straight-forward idea to try to extend Vishik's criterion to a wider class of projective homogeneous varieties. However it is already not always straight-forward, how to define the s-type equivalence for a given type of such varieties (and it is also not straight-forward how to prove, because the proof in the case of quadrics uses many quadrics specific things). In the case of Severi-Brauer varieties, it seems at least to be clear how to start: let us say that two finite-dimensional central simple F-algebras A_1 and A_2 are s-equivalent (and write $A_1 \stackrel{s}{\sim} A_2$), if the algebras have the same dimension over F (see Remark 7.2) and for any field extension E/F the E-algebras $(A_1)_E$ and $(A_2)_E$ have the same Schur index. Substituting for E the function fields of the Severi-Brauer varieties X_{A_1} and X_{A_2} of A_1 and A_2, and using an old theorem of Amitsur, one translates the condition $A_1 \stackrel{s}{\sim} A_2$ for two algebras of the same dimension as follows (Lemma 7.13): A_1 and A_2 generate the same subgroup in the Brauer group of F. It follows that in the case where $A_1 \stackrel{s}{\sim} A_2$, the direct product $X_{A_1} \times X_{A_2}$ is a projective space bundle over X_{A_1} as well as over X_{A_2}. This produces a motivic isomorphism of $X_{A_1} \times \mathbb{P}^{i-1}$ and $X_{A_2} \times \mathbb{P}^{i-1}$, where i is the Schur index of A_1 and A_2, and $\mathbb{P}^{i-1}$ is the $(i-1)$-dimensional projective space. This is already a rather strong relation between the motives of X_{A_1} and X_{A_2} (for example, one concludes immediately that $H^*(X_{A_1}) \simeq H^*(X_{A_2})$ for any *geometric cohomology theory* H^* (see [11, §2] for the definition) such that the group $H^*(X_{A_1})$ is finitely generated). However the condition $A_1 \stackrel{s}{\sim} A_2$ turns out to be non-sufficient for m-equivalence of A_1 and A_2 (that is for motivic isomorphism of X_{A_1} and X_{A_2}): as Criterion 7.1. states, $A_1 \stackrel{m}{\sim} A_2$ if and only if $[A_1] = \pm[A_2]$ in the Brauer group of F.

Terminology and notation concerning algebraic varieties, cycles, and correspondences are introduced in Sect. 1. We only emphasize here the following agreement. Let X be an F-variety and E/F a field extension. We say that a cycle $\alpha \in \mathrm{CH}^*(X_E)$ is *defined over* F and write α/F if α is in the image of the restriction homomorphism $\mathrm{res}_{E/F}\colon \mathrm{CH}^*(X) \to \mathrm{CH}^*(X_E)$. A correspondence is said to be defined over F if it is defined over F as a cycle.

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I have an impression, that everything in the "Integral Motives of Quadrics", except Sect. 6, can be rewritten in pure Chow-motivic terms, using Rost's Nilpotence Theorem instead of Lemma 3.10. I just find my language as more convenient to think in. I never bothered by the $\mathrm{char}(k) = 0$ *assumption: it is completely clear, that in all other characteristics (but 2) everything is the same.*

(A. Vishik. A letter to the author.)

1. Category of correspondences

The definition of the category of degree 0 Chow-correspondences $\mathcal{CV}^0(F)$ is classical. We recall it briefly in order to fix necessary notation. However we do not have a reference for the category $\mathcal{CV}(F)$ (which we call the *category of Chow-correspondences* and which might be called the *category of Chow-correspondences with twists*) also introduced in this section. The construction of $\mathcal{CV}(F)$ is a very natural and simple variation of known ones.

We write $\mathcal{V}(F)$ for the category of smooth complete not necessarily connected F-varieties (we also include $\emptyset$ in $\mathcal{V}$). For any $X \in \mathcal{V}(F)$ we write $\mathrm{CH}^*(X)$ for the Chow ring of algebraic cycles on X modulo rational equivalence, graded by codimension of cycles (sometimes we also write $\mathrm{CH}_*(X)$ for the gradation by the dimension of cycles). Abusing terminology, we sometimes refer to the elements of $\mathrm{CH}^*(X)$ as to *cycles* (and not as to *classes of cycles*).

A *correspondence* from X_1 to X_2, where $X_1, X_2 \in \mathcal{V}(F)$, is by definition a cycle in $\mathrm{CH}^*(X_1 \times X_2)$ (a correspondence "on X" is a correspondence from X to itself). For two correspondences $c_{12} \in \mathrm{Hom}(X_1, X_2)$ and $c_{23} \in \mathrm{Hom}(X_2, X_3)$, the composition $c_{23} \circ c_{12}$ is defined by the classical formula (compare to [3, Definition 16.1.1]) $c_{23} \circ c_{12} := (pr_{13})_*\big((c_{12} \times X_3) \cdot (X_1 \times c_{23})\big)$, where $\cdot$ stays for the multiplication of cycles in $\mathrm{CH}^*(X_1 \times X_2 \times X_3)$ and $(pr_{13})_*$ is the push-forward with respect to the projection $pr_{13}\colon X_1 \times X_2 \times X_3 \to X_1 \times X_3$.

In our work here, we only shall apply this formula to the case, where the correspondences c_{12} and c_{23} are decomposed (and homogeneous). In this case the composition can be computed as follows:

Lemma 1.1. *Let* $X_1, X_2, X_3 \in \mathcal{V}(F)$ *and* $c_1 \in \mathrm{CH}^*(X_1)$, $c_2, c_2' \in \mathrm{CH}^*(X_2)$, $c_3 \in \mathrm{CH}^*(X_3)$. *Suppose that the variety* X_2 *is connected (or, more generally, equidimensional) and that the cycles* c_2 *and* c_2' *are homogeneous. Then*

$$(c_2' \times c_3) \circ (c_1 \times c_2) = (pr_{13})_*(c_1 \times (c_2 \cdot c_2') \times c_3)$$
$$= \begin{cases} \deg(c_2 \cdot c_2') \cdot (c_1 \times c_3) & \textit{if} \;\; \mathrm{codim}(c_2) + \mathrm{codim}(c_2') = \dim X_2, \\ 0 & \textit{otherwise}, \end{cases}$$

where $\deg(-)$ *stays for the degree of a 0-cycle (see [3, Definition 1.4]).* □

Let $X, Y \in \mathcal{V}(F)$ and suppose that Y is connected (or, more generally, equidimensional). A correspondence from X to Y is said to be of degree p if it is given by a (homogeneous) cycle from $\mathrm{CH}^{\dim Y + p}(X \times Y)$. This definition is extended to the case of arbitrary $Y \in \mathcal{V}(F)$ by taking the direct sum of the groups of the degree p correspondences from X to the connected components of Y.

Since degrees of correspondences are added while the correspondences are composed ([3, Example 16.11]), the composition of degree 0 correspondences has degree 0 as well. This legitimates the following definition.

The additive category $\mathcal{CV}^0(F)$ (called the category of degree 0 correspondences) has the same objects as $\mathcal{V}(F)$, while $\mathrm{Hom}(X, Y)$ is defined to be the group of degree 0 correspondences from X to Y. We refer to [3, §16.1] for checking that $\mathcal{CV}^0(F)$ is really a category and only note here that id_X is given by the diagonal class on $X \times X$.

Although the category $\mathcal{CV}^0(F)$ is already satisfactory for the definition of the motivic equivalence, we will sometimes need a bigger category $\mathcal{CV}(F)$, e.g. to formulate Decomposition 1.2.

The objects of $\mathcal{CV}(F)$ are formal finite direct sums of pairs (X, i), where $X \in \mathcal{V}(F)$ and $i \in \mathbb{Z}$. One writes $X(i)$ for (X, i). For $X(i), Y(j) \in \mathcal{CV}(F)$, the group $\mathrm{Hom}(X(i), Y(j))$ is defined as the group of degree $j - i$ correspondences from X to Y. Note that the evident functor $\mathcal{CV}^0(F) \to \mathcal{CV}(F)$, $X \mapsto X := X(0)$ is a full imbedding.

Here is an example:

Decomposition 1.2 ([19, Proposition 2]). *Let* ψ *be a quadratic form over* F, $\phi := \mathbb{H} \perp \psi$, *and* X, Y *the projective quadrics given by* ϕ, ψ. *In* $\mathcal{CV}(F)$ *there is an isomorphism*

$$X \simeq \mathbf{pt} \oplus Y(1) \oplus \mathbf{pt}(n) \;,$$

where $\mathbf{pt} := \mathrm{Spec}(F)$ *and* $n := \dim X$.

Remark 1.3. Decomposition 1.2 is a particular case of the motivic decompositions of the isotropic flag varieties (established in [11]) and is produced by a certain *relative cellular structure* (see [11, Definition 6.1]) on X.

2. Trivial implications and reduction to anisotropic case

In this section, we prove that two m-equivalent forms are s-equivalent and that the inverse implication should be proved only for anisotropic forms.

We fix the following notation: ϕ_1 and ϕ_2 are quadratic forms over F; X_1 and X_2 are the corresponding projective quadrics.

Lemma 2.1. *If* $\phi_1 \overset{\mathrm{m}}{\sim} \phi_2$, *then* $\dim \phi_1 = \dim \phi_2$.

Proof. Let $X \neq \emptyset$ be any variety from $\mathcal{V}(F)$. The formula

$$\mathrm{Hom}(\mathbf{pt}(i), X) = \begin{cases} \mathrm{CH}^0(X) \neq 0 & \text{for } i = \dim X, \\ \mathrm{CH}^{\dim X - i}(X) = 0 & \text{for } i > \dim X \end{cases}$$

shows that the dimension of X is determined by the isomorphism class of its motive. □

Lemma 2.2. *Suppose that* $\phi_1 = \mathbb{H} \perp \psi_1$ *and* $\phi_2 = \mathbb{H} \perp \psi_2$ *for some quadratic forms* ψ_1 *and* ψ_2. *Then* $\phi_1 \overset{\mathrm{m}}{\sim} \phi_2$ *if and only* $\psi_1 \overset{\mathrm{m}}{\sim} \psi_2$.

Proof. We write Y_1 and Y_2 for the projective quadrics determined by ψ_1 and ψ_2. Any of two conditions $\phi_1 \overset{\mathrm{m}}{\sim} \phi_2$ and $\psi_1 \overset{\mathrm{m}}{\sim} \psi_2$ implies that $\dim X_1 = \dim X_2$ (by Lemma 2.1), so we may assume that $\dim X_1 = \dim X_2 = n$. By Decomposition 1.2 we have motivic isomorphisms

$$X_1 = \mathbf{pt} \oplus Y_1(1) \oplus \mathbf{pt}(n) \quad \text{and} \quad X_2 = \mathbf{pt} \oplus Y_2(1) \oplus \mathbf{pt}(n) \ ,$$

where $\dim Y_1 = \dim Y_2 = n - 2$. Therefore

$$\mathrm{Hom}(X_\rho, X_\delta) = \mathrm{End}(\mathbf{pt}) \times \mathrm{Hom}(Y_\rho, Y_\delta) \times \mathrm{End}(\mathbf{pt})$$

for any $\rho, \delta \in \{1, 2\}$ (note that there are no homomorphisms between $\mathbf{pt}$, Y_ρ, and $\mathbf{pt}(n)$ by the simple dimension count reason). Consequently, $X_1 \simeq X_2$ in $\mathcal{CV}(F)$ if and only if $Y_1 \simeq Y_2$. □

Lemma 2.3. *If* $\phi_1 \overset{\mathrm{m}}{\sim} \phi_2$, *then either* ϕ_1 *and* ϕ_2 *are both anisotropic or they are both isotropic.*

Proof. We write n for the dimension of X_1 and X_2. Let E/F be a field extension such that the forms $(\phi_1)_E$ and $(\phi_2)_E$ are isotropic. Consider the restriction functor $\mathcal{CV}(F) \to \mathcal{CV}(E)$. For $\rho = 1, 2$, we have $\mathrm{Hom}(\mathbf{pt}_E, (X_\rho)_E) = \mathrm{End}(\mathbf{pt}_E) = \mathbb{Z}$ and the homomorphism

$$\mathrm{CH}_0(X_\rho) = \mathrm{Hom}(\mathbf{pt}, X_\rho) \xrightarrow{\mathrm{res}_{E/F}} \mathrm{Hom}(\mathbf{pt}_E, (X_\rho)_E) = \mathbb{Z}$$

is given by taking the degrees of 0-cycles. By the Springer theorem [15, Theorem 2.3 of Chapter Seven] the cokernel of this homomorphism is 0 if and only if the form ϕ_ρ is isotropic (in the anisotropic case, the cokernel equals $\mathbb{Z}/2$). Therefore the forms ϕ_1 and ϕ_2 can be isotropic only simultaneously. □

Remark 2.4. There is a complete description of the group $\mathrm{CH}_0(X_\rho)$ (see [23] or [7], where the kernel of the degree homomorphism $\mathrm{CH}_0(X_\rho) \to \mathbb{Z}$ (whose image is detected by Springer's theorem) is shown to be zero). However this information is superfluous in the proof of Lemma 2.3.

Corollary 2.5. *If $\phi_1 \overset{\mathrm{m}}{\sim} \phi_2$, then $\phi_1 \overset{\mathrm{s}}{\sim} \phi_2$.*

Proof. First of all, we have $\dim \phi_1 = \dim \phi_2$ by Lemma 2.1. Lemmas 2.2 and 2.3 together imply that the Witt indexes of the forms ϕ_1 and ϕ_2 coincide. Finally, since m-equivalent forms remain m-equivalent over any extension E/F (as seen by applying the restriction functor $\mathcal{CV}^0(F) \to \mathcal{CV}^0(E)$), we get $\phi_1 \overset{\mathrm{s}}{\sim} \phi_2$. □

For the proof of the inverse implication we need the following simple

Lemma 2.6. *Let ϕ_1 and ϕ_2 be of even dimension. If $\phi_1 \overset{\mathrm{s}}{\sim} \phi_2$, then* $\mathrm{disc}\phi_1 = \mathrm{disc}\phi_2$.

Proof. We prove this by induction on $n := \dim \phi_1 = \dim \phi_2$. If $n = 2$, then the form $(\phi_1)_{F(\sqrt{\mathrm{disc}\phi_1})}$ is isotropic; therefore $(\phi_2)_{F(\sqrt{\mathrm{disc}\phi_1})}$ is isotropic as well what implies that $\mathrm{disc}\phi_1 = \mathrm{disc}\phi_2$.

Let $n > 2$. If ϕ_1 and ϕ_2 are isotropic, we may cancel one hyperbolic plane contained in each of them and apply the induction hypothesis. If the forms are anisotropic, we may pass to the function field of X_1 (note that F is algebraically closed in $F(X_1)$). □

Remark 2.7. It deserves to be mentioned that the even Clifford algebras of s-equivalent forms are isomorphic. The proof of this statement is easily reduced to the case where the forms have an even dimension and trivial discriminant. Then the statement is a particular case of the following observation (also generalizing Lemma 2.6), due to O. Izhboldin: if the classes in the Witt ring of two s-equivalent quadratic forms belong to the n-th Knebusch ideal J_n (see [13, §6]), then they are congruent modulo J_{n+1}.

3. Nilpotence theorem

The following theorem is due to M. Rost. It was announced in [18]; a proof was given in [19, Proposition 9]. A new proof (only for the characteristic 0 case) is given in [24, Lemma 3.10]; this proof produces a better nilpotence exponent.

Theorem 3.1 (Nilpotence theorem). *Let X be a projective quadric over F, $c \in \mathrm{End}(X)$ a correspondence on X, and E/F a field extension. If $c_E = 0 \in \mathrm{End}(X_E)$, then c is nilpotent.*

Corollary 3.2. *Let X_1 and X_2 be projective quadrics over F,*

$$c_{12} \in \mathrm{Hom}(X_1, X_2)\ , \quad c_{21} \in \mathrm{Hom}(X_2, X_1)$$

some correspondences, and E/F a field extension. If $(c_{12})_E$ and $(c_{21})_E$ are mutually inverse isomorphisms (of $(X_1)_E$ and $(X_2)_E$ in $\mathcal{CV}^0(E)$), then c_{12} and c_{21} themselves are isomorphisms (but, may be, not mutually inverse ones).

Proof. Since the correspondence $\varepsilon_1 := c_{21} \circ c_{12} - id_1$ vanishes over E, it is nilpotent (by Theorem 3.1). Therefore $c_{21} \circ c_{12} = id_1 + \varepsilon_1$ is an isomorphism (the inverse is given by the finite sum $id_1 - \varepsilon_1 + \varepsilon_1^2 - \ldots$). By the symmetry, $c_{12} \circ c_{21}$ is an isomorphism as well. □

Remark 3.3. In fact, an enhanced version of Corollary 3.2 holds for two projective quadrics X_1 and X_2 (compare with [19, Corollary 11]): if a correspondence from X_1 to X_2 becomes to be an isomorphism after an extension of scalars, then it is an isomorphism already over the base field (so, one does not really need to assume that the inverse is defined over the base field). This enhancement is superfluous for us here (it is replaced by Lemma 6.3).

4. Correspondences on split quadrics

Let $\mathfrak{X}$ be a split projective quadric of an even dimension n. By saying "split" we mean that the $(n+2)$-dimensional quadratic form, determining $\mathfrak{X}$, is isomorphic to the direct sum of $(n+2)/2$ hyperbolic planes. First of all we need a description of the Chow ring $\mathrm{CH}^*(\mathfrak{X})$. The description given in Lemma 4.1 is classical and is reproduced in many references. Essentially it is contained already in [4]. In more appropriate terms it is obtained in [22]. We formulate it here in order to be self-contained and to have an occasion to introduce our notation for the generators.

We denote by $h \in \mathrm{CH}^1(\mathfrak{X})$ the class of a hyperplane section of $\mathfrak{X}$ (that is, the pull-back of the hyperplane class with respect to the imbedding of $\mathfrak{X}$ into the projective space). For any $i \geq 0$, the i-th power h^i of h taken in the ring $\mathrm{CH}^*(\mathfrak{X})$ gives us an element of $\mathrm{CH}^i(\mathfrak{X})$ (which is, of course, 0 if $i > n$ and $1 = [\mathfrak{X}]$ if $i = 0$).

For every $i = 0, \ldots, n/2$, a totally isotropic $(i+1)$-dimensional subspace of the quadratic form determining $\mathfrak{X}$ gives rise to a closed i-dimensional subvariety of $\mathfrak{X}$ (which is a linear subspace of the projective space containing $\mathfrak{X}$). Its class in $\mathrm{CH}^{n-i}(\mathfrak{X})$ do not depend on the choice of the subspace iff $i \neq n/2$. For $i = n/2$, there are precisely two different classes; we write l and l' for them. Sometimes we also write $l^{(1)}$ for l' and $l^{(0)}$ for l.

Lemma 4.1 ([22, Theorem 13.3]). *The additive group $\mathrm{CH}^*(\mathfrak{X})$ is torsion free. For $i < n/2$ the group $\mathrm{CH}^i(\mathfrak{X})$ is generated by h^i; for $i > n/2$ by $h^i/2$; and* *for $i = n/2$ by two independent generators l and l'. The multiplicative structure of the commutative ring $\mathrm{CH}^*(\mathfrak{X})$ is described by the formulas $h^{n/2} = l + l'$, $h \cdot l = h \cdot l' = h^{n/2+1}/2$, and*

$$l \cdot l' = \begin{cases} 0 & \text{if } n/2 \text{ is even,} \\ h^n/2 & \text{if } n/2 \text{ is odd;} \end{cases} \qquad l \cdot l = l' \cdot l' = \begin{cases} h^n/2 & \text{if } n/2 \text{ is even,} \\ 0 & \text{if } n/2 \text{ is odd} \end{cases}$$

(we do not care about minimality of this list of relations). For every $i > n/2$ the element $h^i/2$ coincides with the class of any $(n-i)$-dimensional linear subspace lying on $\mathfrak{X}$ (in particular, $h^n/2$ is the class of a rational point).

The above calculation can be also obtained using Decomposition 1.2. Anyway, it can be easily generalized to produce a description of the Chow group $\mathrm{CH}^*(\mathfrak{X} \times T)$ in terms of $\mathrm{CH}^*(T)$ for an arbitrary F-variety T. Taking $T = \mathfrak{X}$ and $T = \mathfrak{X} \times \mathfrak{X}$, one gets

Lemma 4.2. *The homomorphisms*

$$\begin{aligned} \mathrm{CH}^*(\mathfrak{X}) \otimes \mathrm{CH}^*(\mathfrak{X}) &\to \mathrm{CH}^*(\mathfrak{X} \times \mathfrak{X}), \quad \alpha \otimes \beta \mapsto \alpha \times \beta \ \text{ and} \\ \mathrm{CH}^*(\mathfrak{X}) \otimes \mathrm{CH}^*(\mathfrak{X}) \otimes \mathrm{CH}^*(\mathfrak{X}) &\to \mathrm{CH}^*(\mathfrak{X} \times \mathfrak{X} \times \mathfrak{X}), \quad \alpha \otimes \beta \otimes \gamma \\ &\mapsto \alpha \times \beta \times \gamma \end{aligned}$$

are isomorphism of rings. In particular, $\mathrm{CH}^n(\mathfrak{X} \times \mathfrak{X})$ is a free abelian group on $l^{(u)} \times l^{(v)}$ and $(h^i \times h^{n-i})/2$ with $u, v \in \{0, 1\}$ and $i \in \{0, 1, \ldots, n\} \setminus \{n/2\}$.

Thus we almost have described the ring $\mathrm{End}(\mathfrak{X})$ of correspondences on $\mathfrak{X}$. It remains only to describe the multiplicative structure. This is done by the following formulas which are easily verified with use of Lemmas 1.1 and 4.1. In what follows we write N for the set of indexes $\{0, 1, \ldots, n\} \setminus \{n/2\}$ appeared in Lemma 4.2.

Lemma 4.3. *(1) For any $0 \le i, j \le n$ one has*

$$((h^j \times h^{n-j})/2) \circ ((h^i \times h^{n-i})/2) = \begin{cases} (h^i \times h^{n-i})/2 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

(2) For any $u, v \in \{0, 1\}$ and any $i \in N$, the correspondences $l^{(u)} \times l^{(v)}$ and $(h^i \times h^{n-i})/2$ are orthogonal.

(3) Finally, for any $u, v, v', w \in \{0, 1\}$, one has

$$\left(l^{(v')} \times l^{(w)}\right) \circ \left(l^{(u)} \times l^{(v)}\right) = \begin{cases} l^{(u)} \times l^{(w)} & \text{if } v = v' \text{ and } n/2 \text{ is even or} \\ & \text{if } v \ne v' \text{ and } n/2 \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

□

Corollary 4.4. *The diagonal class in* $\mathrm{CH}^n(\mathfrak{X} \times \mathfrak{X})$ *equals*

$$\sum_{i \in N} (h^i \times h^{n-i})/2 + \begin{cases} l \times l + l' \times l' & \text{if } n/2 \text{ is even,} \\ l \times l' + l' \times l & \text{if } n/2 \text{ is odd.} \end{cases}$$

Proof. It follows from Lemma 4.3, that multiplication of any generator of the additive group $\mathrm{End}(\mathfrak{X}) = \mathrm{CH}^n(\mathfrak{X} \times \mathfrak{X})$ by the sum written down does not change the generator. Therefore the sum is the unit of the ring $\mathrm{End}(\mathfrak{X})$ which is known to be the class of the diagonal. □

Now let $\mathfrak{X}_1$ and $\mathfrak{X}_2$ be two split projective quadrics of some even dimension n (of course they are isomorphic, we just prefer to have different notation for them). We are going to introduce notations for certain elements of the groups $\mathrm{Hom}(\mathfrak{X}_\rho, \mathfrak{X}_\delta) = \mathrm{CH}^n(\mathfrak{X}_\rho \times \mathfrak{X}_\delta)$ for $\rho, \delta \in \{1, 2\}$.

For the introduced above elements h, l, and l' of the group $\mathrm{CH}^*(\mathfrak{X}_\rho)$ we use from now on the notation h_ρ, l_ρ, and l'_ρ. For any subset $I \subset N$ and any matrix

$$A = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \in M_2(\mathbb{Z}) \ ,$$

we define a cycle $c_{\rho\delta}(I, A) \in \mathrm{CH}^n(\mathfrak{X}_\delta \times \mathfrak{X}_\rho)$ as follows

$$c_{\rho\delta}(I, A) := \sum_{i \in I} \left(h_\rho^i \times h_\delta^{n-i}\right)/2 + \begin{cases} \displaystyle\sum_{u,v \in \{0,1\}} a_{uv} \cdot \left(l_\rho^{(v)} \times l_\delta^{(u)}\right) & \text{if } n/2 \text{ is even,} \\ \displaystyle\sum_{u,v \in \{0,1\}} a_{uv} \cdot \left(l_\rho^{(v)} \times l_\delta^{(1-u)}\right) & \text{if } n/2 \text{ is odd.} \end{cases}$$

The change of definition (in the case of odd $n/2$ with respect to the case of even $n/2$) is justified by the following lemma, which is a simple consequence of Lemma 4.3 and Corollary 4.4:

Lemma 4.5. *For any* $I, J \subset N$*, any* $A, B \in M_2(\mathbb{Z})$*, and any* $\rho, \delta, \sigma \in \{1, 2\}$*, one has*

$$c_{\delta\sigma}(J, B) \circ c_{\rho\delta}(I, A) = c_{\rho\sigma}(J \cap I, B \cdot A) \ .$$

Besides, the diagonal class on $\mathfrak{X}_\rho \times \mathfrak{X}_\rho$ *equals* $c_{\rho\rho}(N, 1)$. □

Definition 4.6. For any $A \in M_2(\mathbb{Z})$ we write $A^{\diagdown}$ for the transposition of A (i.e. the turn over of A with respect to its main diagonal) and we write $A^{\diagup}$ for its turn over with respect to the second diagonal.

The following assertions are evident:

Lemma 4.7. *For any $I \subset N$, any $A \in M_2(\mathbb{Z})$, and any $\rho, \delta \in \{1, 2\}$, one has*

$$c_{\rho\delta}(I, A)^t = \begin{cases} c_{\delta\rho}\left(n - I, A^{\diagdown}\right) & \text{if } n/2 \text{ is even,} \\ c_{\delta\rho}\left(n - I, A^{\diagup}\right) & \text{if } n/2 \text{ is odd,} \end{cases}$$

where $c_{\rho\delta}(I, A)^t \in \mathrm{CH}^n(\mathfrak{X}_\delta \times \mathfrak{X}_\rho)$ denotes the transposition of the correspondence $c_{\rho\delta}(I, A)$. □

Lemma 4.8. *Two correspondences $\alpha \in \mathrm{Hom}(\mathfrak{X}_1, \mathfrak{X}_2)$ and $\beta \in \mathrm{Hom}(\mathfrak{X}_2, \mathfrak{X}_1)$ are mutually inverse isomorphisms if and only if $\alpha = c_{12}(I, B) - c_{12}(N \setminus I, 0)$ and $\beta = c_{21}(I, B^{-1}) - c_{21}(N \setminus I, 0)$ for some invertible matrix $B \in M_2(\mathbb{Z})$ and some $I \subset N$.* □

5. Proof of the criterion for quadratic forms

In this section, we finish the proof of Criterion 0.1 (using the computation of Sect. 6).

Let X_1 and X_2 be two (not necessary split) projective quadrics of even dimension n, given by quadratic forms ϕ_1 and ϕ_2 (of dimension $n+2$). We assume that the criterion is already proved for quadrics of dimension less than n (this will be really needed only in the proof of Propositions 5.7 and 5.11).

We fix a field extension $\mathcal{F}/F$ such that the quadrics $(X_1)_\mathcal{F}$ and $(X_2)_\mathcal{F}$ are split (for instance, $\mathcal{F}$ can be an algebraic or a separable closure of F) and write $\mathfrak{X}_1$ for $(X_1)_\mathcal{F}$ and $\mathfrak{X}_2$ for $(X_2)_\mathcal{F}$. We use the notation for certain elements of $\mathrm{CH}^n(\mathfrak{X}_\rho \times \mathfrak{X}_\delta)$ ($\rho, \delta \in \{1, 2\}$) introduced in the previous section. We recall that $N := \{0, 1, \ldots, n\} \setminus \{n/2\}$.

Let X be an F-variety and E/F an arbitrary field extension. As agreed in Introduction, we say that a cycle $\alpha \in \mathrm{CH}^*(X_E)$ is defined over F and write α/F, if α is in the image of the restriction homomorphism $\mathrm{CH}^*(X) \to \mathrm{CH}^*(X_E)$. A correspondence is said to be defined over F if it is defined over F as a cycle.

The central assertion of this section (and, in fact, of the whole quadratic form part of the article) is

Theorem 5.1. *If $\phi_1 \overset{s}{\sim} \phi_2$, then $c_{12}(N, B)/F$ and $c_{21}(N, B^{-1})/F$ for some invertible matrix B.*

Definition 5.2. A subset $I \subset N$ is said to be *admissible*, if $c_{12}(I, A_1)/F$ for some $A_1 \in M_2(\mathbb{Z})$ *and* $c_{21}(I, A_2)/F$ for some $A_2 \in M_2(\mathbb{Z})$. A subset $I \subset N$ is said to be *weakly admissible*, if $c_{11}(I, A_1)/F$ *and* $c_{22}(I, A_2)/F$ for some $A_1, A_2 \in M_2(\mathbb{Z})$.

The following lemma is a consequence of the computation of the diagonal class (Corollary 4.4, reformulated in Lemma 4.5):

Lemma 5.3. *The complement $N \setminus I$ of a weakly admissible set I is weakly admissible as well.* $\square$

Definition 5.4. A subset $I \subset N$ is said to be *symmetric*, if $I = n - I$, where $n - I := \{n - i \mid i \in I\}$. For any $I \subset N$ the set $I \cup (n - I)$ is the smallest symmetric set containing I; it can be called the *symmetrization* of I.

Remark 5.5. The formula of Lemma 4.7 shows that the definition of admissibility can be shortened in the case of symmetric I: it suffices only to require that $c_{12}(I, A)/F$ for some A.

Proposition 5.6. *(1) An admissible set is weakly admissible.*
(2) The symmetrization of an admissible set is admissible.
(3) A union of admissible sets is admissible.

Proof. (1): This follows from the formulas $c_{21}(I, B) \circ c_{12}(I, A) = c_{11}(I, B \cdot A)$ and $c_{12}(I, A) \circ c_{21}(I, B) = c_{22}(I, A \cdot B)$ given in Lemma 4.5.

(3): Let I and J be admissible sets. We have to show that the set of cycles $c_{12}(I \cup J, *)$ contains a cycle defined over F and the set of cycles $c_{21}(I \cup J, *)$ contains a cycle defined over F. The statement on $c_{12}(I \cup J, *)$ is served by the evident formula $c_{12}(I \cup J, *) \supset c_{12}(I, *) + c_{12}(J, *) - c_{12}(I \cap J, *)$ together with the formula $c_{12}(I \cap J, *) \supset c_{12}(J, *) \circ c_{11}(I, *)$ easily deduced from Lemma 4.5. The statement on $c_{21}(I \cup J, *)$ is proved analogously.

(2): If a set $I \subset N$ is admissible, we have $c_{12}(I, A_1)/F$ and $c_{21}(I, A_2)/F$ for some $A_1, A_2 \in M_2(\mathbb{Z})$. Applying Lemma 4.7, we get $c_{21}(n - I, A_1')/F$ and $c_{12}(n - I, A_2')/F$ for $'$ being either $\diagdown$ or $\diagup$. $\square$

Proposition 5.7. *Suppose that $\phi_1 \overset{s}{\sim} \phi_2$. Let $r \in N$ be an index smaller than $n/2$. If r is the smallest index of some weakly admissible set, then r is contained in an admissible set.*

Corollary 5.8. *If $\phi_1 \overset{s}{\sim} \phi_2$, then the set N is admissible.*

Proof. Note that $\emptyset$ is a symmetric admissible set. Let I_0 be a symmetric admissible set. It suffices to show that if $I_0 \neq N$ then I_0 is contained in a strictly bigger symmetric admissible set I_1.

By Item 1 of Proposition 5.6, the set I_0 is weakly admissible. By Lemma 5.3, it follows that the set $I := N \setminus I_0$ is weakly admissible as well. Set $r := \min I$. Since I is symmetric (because I_0 was symmetric), the condition $r < n/2$ is satisfied and Proposition 5.7 provides us with an admissible set J containing r. By Item 3 of Proposition 5.6, the union $I_0 \cup J$ is an admissible set; we take as I_1 its symmetrization. The set I_1 is admissible (Item 2 of Proposition 5.6), symmetric, and contains I_0 properly ($I_0 \neq I_1$ because $r \in I_1 \setminus I_0$). $\square$

Proof of Proposition 5.7. Multiplying the generic point morphism

$$X_1 \leftarrow \operatorname{Spec} F(X_1)$$

by $X_1 \times X_2$ (on the left), we get a flat morphism

$$X_1 \times X_2 \times X_1 \leftarrow (X_1 \times X_2)_{F(X_1)} .$$

Taking the pull-back with respect to it, we obtain an epimorphism

$$\mathrm{CH}^n(X_1 \times X_2 \times X_1) \twoheadrightarrow \mathrm{CH}^n\big((X_1 \times X_2)_{F(X_1)}\big) .$$

The corresponding epimorphism over $\mathcal{F}$

$$f\colon \mathrm{CH}^n(\mathfrak{X}_1 \times \mathfrak{X}_2 \times \mathfrak{X}_1) \to \mathrm{CH}^n\big((\mathfrak{X}_1 \times \mathfrak{X}_2)_{\mathcal{F}(\mathfrak{X}_1)}\big)$$

can be easily computed in terms of generators of the Chow group $\mathrm{CH}^n(\mathfrak{X}_1 \times \mathfrak{X}_2 \times \mathfrak{X}_1)$ given by Lemmas 4.1 and 4.2, because for any homogeneous cycles $\alpha, \gamma \in \mathrm{CH}^*(\mathfrak{X}_1)$ and $\beta \in \mathrm{CH}^*(\mathfrak{X}_2)$

$$f(\alpha \times \beta \times \gamma) = \begin{cases} 0 & \text{if } \operatorname{codim}\gamma > 0, \\ (\alpha \times \beta)_{\mathcal{F}(\mathfrak{X}_1)} & \text{if } \gamma = [\mathfrak{X}_1]. \end{cases}$$

Since the quadratic forms $(\phi_1)_{F(X_1)}$ and $(\phi_2)_{F(X_1)}$ are isotropic, Lemma 2.2 together with the induction hypothesis, formulated in the beginning of this section, imply that $(\phi_1)_{F(X_1)} \overset{\mathrm{m}}{\sim} (\phi_2)_{F(X_1)}$. By Lemma 4.8, it follows that the cycle $\big(c_{12}(I, A) - c_{12}(N \setminus I, 0)\big)_{\mathcal{F}(\mathfrak{X}_1)} \in \mathrm{CH}^n\big((\mathfrak{X}_1 \times \mathfrak{X}_2)_{\mathcal{F}(\mathfrak{X}_1)}\big)$ (for some matrix A and some $I \subset N$) is defined over $F(X_1)$. Since $2c_{12}(N \setminus I, 0)/F$, the cycle $c_{12}(N, A)_{\mathcal{F}(\mathfrak{X}_1)}$ is defined over $F(X_1)$. Therefore, the set of preimages of $c_{12}(N, A)_{\mathcal{F}(\mathfrak{X}_1)}$ with respect to f contains a defined over F cycle as well. Any cycle in this set of preimages has the form

$$(\dagger) \qquad c_{12}(N, A) \times 1 + \sum \alpha \times \beta \times \gamma \ ,$$

where the sum is taken over some homogeneous α, β, γ with positive $\operatorname{codim}_{\mathfrak{X}_1}\gamma$. In what follows we assume that (†) is a cycle defined over F.

Let I be a weakly admissible set such that $r = \min I$. We have $c_{11}(I, A')/F$ for some $A' \in M_2(\mathbb{Z})$. Considering the cycle (†) as a correspondence from $\mathfrak{X}_1$ to $\mathfrak{X}_2 \times \mathfrak{X}_1$, we may take the composition $(\dagger) \circ c_{11}(I, A')$. The result is a defined over F cycle on $\mathfrak{X}_1 \times \mathfrak{X}_2 \times \mathfrak{X}_1$ which is equal to

$$(\dagger\dagger) \qquad c_{12}(I, B) \times 1 + \sum \alpha \times \beta \times \gamma \ ,$$

where $B := A \cdot A'$ and the sum is taken over some (other) homogeneous α, β, γ such that $\operatorname{codim}\gamma > 0$ and $\operatorname{codim}\alpha \geq r$. Let us take the pull-back of the cycle (††) with respect to the morphism $\mathfrak{X}_1 \times \mathfrak{X}_2 \times \mathfrak{X}_1 \leftarrow \mathfrak{X}_1 \times \mathfrak{X}_2$ induced by the diagonal of $\mathfrak{X}_1$. The result is a defined over F cycle on $\mathfrak{X}_1 \times \mathfrak{X}_2$ which is equal to

$$(\dagger\dagger\dagger) \qquad c_{12}(I, B) + \sum(\alpha \cdot \gamma) \times \beta \ ,$$

where $\operatorname{codim}(\alpha \cdot \gamma) > r$. Since $(\alpha \cdot \gamma) \times \beta$ is a multiple of $(h^i \times h^{n-i})/2$ (if $i := \operatorname{codim}(\alpha \cdot \gamma) \neq n/2$) or a linear combination with integer coefficients of $l^{(u)} \times l^{(v)}$ (if $i = n/2$), and since the cycles $h^i \times h^{n-i}$ are defined over F, one sees that

$$\sum(\alpha \cdot \gamma) \times \beta \equiv c_{12}(I', B') \quad (\text{mod cycles defined over } F)$$

for some $I' \subset N$ with $\min I' > r$. Therefore $(\dagger\dagger\dagger) \equiv c_{12}(J', B + B')$ with $J' := (I \cup I') \setminus (I \cap I')$. It follows that $J' \ni r$ and $c_{12}(J', B + B')/F$.

By the symmetry argument (or, other speaking, repeating the procedure with X_1 and X_2 interchanged), we may find a set $J'' \ni r$ and a matrix B'' such that $c_{21}(J'', B'')/F$. Then $J := J' \cap J''$ is a required admissible set, because of the inclusion

$$c_{12}(J, *) \supset c_{12}(J', *) \circ c_{21}(J'', *) \circ c_{12}(J', *)$$

and because of the similar inclusion for $c_{21}(J, *)$. □

Definition 5.9. We say that the set N is 0-admissible, if $c_{12}\ (N, 0)/F$ and $c_{21}(N, 0)/F$. We say that the set N is 1-admissible, if $c_{12}(N, B)/F$ and c_{21} $(N, B^{-1})/F$ for some invertible matrix B.

Proposition 5.10. *Suppose that* $\operatorname{disc}\phi_1 = \operatorname{disc}\phi_2$. *If the set N is admissible, then it is 0-admissible or 1-admissible.*

Proof. We apply Propositions 6.1 and 6.4 (of the next section) to the following data:

$$C_{\rho\delta} := \{1 \times A \mid A \in M_2(\mathbb{Z}) \text{ such that } c_{\rho\delta}(N, A)/F\}$$
$$\cup\{0 \times A \mid A \in M_2(\mathbb{Z}) \text{ such that } c_{\rho\delta}(\emptyset, A)/F\} \ .$$

We claim that in the case where the discriminants $\operatorname{disc}\phi_1 = \operatorname{disc}\phi_2$ are trivial the conditions of Proposition 6.1 are satisfied, while in the case of non-trivial discriminants the conditions of Proposition 6.4 hold.

(i): $C_{\rho\delta} + C_{\rho\delta} \subset C_{\rho\delta}$ since

$$c_{\rho\delta}(\emptyset, A) + c_{\rho\delta}(\emptyset, B) = c_{\rho\delta}(\emptyset, A + B) \ ,$$
$$c_{\rho\delta}(\emptyset, A) + c_{\rho\delta}(N, B) = c_{\rho\delta}(N, A + B) \ , \text{and}$$
$$c_{\rho\delta}(N, A) + c_{\rho\delta}(N, B) = c_{\rho\delta}(\emptyset, A + B) + \sum_{i \in N} h^i \times h^{n-i},$$

where the sum $\sum_{i\in N} h^i \times h^{n-i}$ is defined over F; $-C_{\rho\delta} \subset C_{\rho\delta}$ since

$$c_{\rho\delta}(I, A) + c_{\rho\delta}(I, -A) = \sum_{i\in I} h^i \times h^{n-i}$$

for any $I \subset N$ (we use this only for $I = \emptyset, N$).

(ii): $C_{\delta\sigma} \cdot C_{\rho\delta} \subset C_{\rho\sigma}$ by the composition formula of Lemma 4.5.

(iii): $1 \times 1 \in C_{\rho\rho}$ by Lemma 4.4.

(iv): Lemma 4.7.

(v): Clearly, we may assume that $\mathcal{F}/F$ is a Galois extension of degree 2^r for some $r \geq 0$ (one may take as $\mathcal{F}/F$ a finite tower of quadratic extensions). For any $\tau \in \mathrm{Gal}(\mathcal{F}/F)$, one has (see [22, Lemma 13.5])

$$\tau(l_\rho^{(u)}) = \begin{cases} l_\rho^{(u)} & \text{if } \tau(d_\rho) = d_\rho, \\ l_\rho^{(1-u)} & \text{if } \tau(d_\rho) = -d_\rho, \end{cases}$$

where d_ρ is a square root of discϕ_ρ in $\mathcal{F}$.

If discϕ_1 = discϕ_2 = 1, it follows that the Galois group $\mathrm{Gal}(\mathcal{F}/F)$ acts trivially on every $l_\rho^{(u)} \times l_\delta^{(v)}$. Thus, taking the transfer, we see that

$$2^r \cdot \left(l_\rho^{(u)} \times l_\delta^{(v)}\right)/F\ .$$

This gives the assumption (v) of Proposition 6.1.

If the discriminants are non-trivial, then the transfer argument shows that

$$2^{r-1} \cdot \left(l_\rho \times l_\delta + l'_\rho \times l'_\delta\right)/F \quad \text{and} \quad 2^{r-1} \cdot \left(l_\rho \times l'_\delta + l_\rho \times l'_\delta\right)/F$$

(this is one part of the assumption (v) of Proposition 6.4). On the other hand, if a cycle $c_{\rho\delta}\left(I, \begin{matrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{matrix}\right)$ is defined over F, then it is stable under the action of $\mathrm{Gal}(\mathcal{F}/F)$. In particular, it is stable under the action of an element $\tau \in \mathrm{Gal}(\mathcal{F}/F)$ such that $\tau(d_1) = -d_1$, where d_1 is a square root of discϕ_1 in $\mathcal{F}$ (such τ exists because $d_1 \notin F$). This implies the desired conditions $a_{00} = a_{11}$ and $a_{01} = a_{10}$ and finishes checking of (iv) for Proposition 6.4.

(vi): $c_{\rho\delta}\left(\emptyset, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right) = h^{n/2} \times h^{n/2}$ and $(h^{n/2} \times h^{n/2})/F$.

We have checked that the conditions (i)–(vi) are satisfied. If the set N is admissible, then the projection $C_{12} \to \mathbb{Z}/2$ is non-zero by the very definition of the admissibility (Definition 5.2). Thus Propositions 6.1 and 6.4 serve the assertion under proof. □

Proposition 5.11. *Suppose that $\phi_1 \overset{s}{\sim} \phi_2$. If the set N is 0-admissible, then it is also 1-admissible.*

Proof. Let us repeat the proof of Proposition 5.7, making some changes or more precise.

First of all, the matrix A in (†) can be chosen the way that it is invertible and $c_{21}(N, A^{-1})_{\mathcal{F}(\mathfrak{X}_1)}/F(X_1)$. Further, let us note that since N is 0-admissible, the cycle

$$c_{11}(\emptyset, 1) = c_{11}(N, 1) - c_{21}(N, 0) \circ c_{12}(N, 0)$$

is defined over F. Then the composition (†) $\circ\, c_{11}(\emptyset, 1)$ is a defined over F cycle on $\mathfrak{X}_1 \times \mathfrak{X}_2 \times \mathfrak{X}_1$ which is equal to

$$(\dagger\dagger) \qquad c_{12}(\emptyset, A) \times 1 + \sum \alpha \times \beta \times \gamma \ ,$$

where $\operatorname{codim}\gamma > 0$ and $\operatorname{codim}\alpha = n/2$. The pull-back of (††) with respect to the morphism $\mathfrak{X}_1 \times \mathfrak{X}_2 \times \mathfrak{X}_1 \leftarrow \mathfrak{X}_1 \times \mathfrak{X}_2$ induced by the diagonal of $\mathfrak{X}_1$ is a defined over F cycle on $\mathfrak{X}_1 \times \mathfrak{X}_2$ which is equal to

$$(\dagger\dagger\dagger) \qquad c_{12}(\emptyset, A) + \sum(\alpha \cdot \gamma) \times \beta \ ,$$

where $\operatorname{codim}(\alpha \cdot \gamma) > n/2$. Composing with $c_{11}(\emptyset, 1)$ once again, we get

$$(\dagger\dagger\dagger) \circ c_{11}(\emptyset, 1) = c_{12}(\emptyset, A) \ .$$

Consequently the cycle $c_{12}(N, A) = c_{12}(N, 0) + c_{12}(\emptyset, A)$ is defined over F. Similarly, $c_{21}(N, A^{-1})/F$. Thus N is 1-admissible. □

Proof of Theorem 5.1. We assume that $\phi_1 \stackrel{s}{\sim} \phi_2$. By Lemma 2.6 we have $\operatorname{disc}\phi_1 = \operatorname{disc}\phi_2$ in this case. The set N is admissible by Corollary 5.8. Therefore, by Proposition 5.10, it is 0-admissible or 1-admissible. Finally, Proposition 5.11 says that N should be 1-admissible what is already the desired assertion. □

Proof of Criterion 0.1. If $\phi_1 \stackrel{m}{\sim} \phi_2$, then $\phi_1 \stackrel{s}{\sim} \phi_2$ by Corolary 2.5.

Now assume that $\phi_1 \stackrel{s}{\sim} \phi_2$. By Theorem 5.1 the correspondences $c_{12}(N, B)$ and $c_{21}(N, B^{-1})$ (for some invertible matrix B) are defined over F. By Lemma 4.8, they are mutually inverse isomorphisms of $\mathfrak{X}_1$ and $\mathfrak{X}_2$. Therefore, by Corollary 3.2, there is a motivic isomorphism between X_1 and X_2, i.e. $\phi_1 \stackrel{m}{\sim} \phi_2$. □

6. Some matrix computations

In this section, we work with the ring $(\mathbb{Z}/2) \times M_2(\mathbb{Z})$. Our aim here is Propositions 6.1 and 6.4, which are used in the proof of Proposition 5.10.

Let A be a matrix in $M_2(\mathbb{Z})$. Recall that we agreed upon to write $A^{\diagdown}$ for the usual transposition of A and $A^{\diagup}$ for the transposition with respect to the second diagonal. Besides, we shall write $A^{|}$ for the interchanging of the columns of A and A^{-} for the interchanging of the rows. For an element x of the ring

$(\mathbb{Z}/2) \times M_2(\mathbb{Z})$, we write $x^{\diagdown}$, $x^{\diagup}$, $x^{|}$, and $x^{—}$ for the element with the matrix component changed in the given way and with the unchanged $(\mathbb{Z}/2)$-component.

The unit and the zero of $M_2(\mathbb{Z})$ are denoted simply by 1 and 0. We write $a \times A$ for the element of $(\mathbb{Z}/2) \times M_2(\mathbb{Z})$ having the $(\mathbb{Z}/2)$-component a and the matrix component A.

Proposition 6.1. *Let C_{11}, C_{12}, C_{21}, and C_{22} be some subsets of the ring $(\mathbb{Z}/2) \times M_2(\mathbb{Z})$. Assume that for any $\rho, \delta, \sigma \in \{1, 2\}$, one has*

(i) *the set $C_{\rho\delta}$ is an additive subgroup of the ring $(\mathbb{Z}/2) \times M_2(\mathbb{Z})$;*
(ii) $C_{\delta\sigma} \cdot C_{\rho\delta} \subset C_{\rho\sigma}$;
(iii) $C_{\rho\rho} \ni 1 \times 1$ *(thus, C_{11} and C_{22} are subrings with unit);*
(iv) $C^{\diagup}_{\rho\delta} \subset C_{\delta\rho}$ *or* $C^{\diagdown}_{\rho\delta} \subset C_{\delta\rho}$;
(v) $0 \times (2^r \cdot M_2(\mathbb{Z})) \subset C_{\rho\delta}$ *for some positive integer r;*
(vi) $0 \times \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in C_{\rho\delta}$.

If the projection $C_{12} \to \mathbb{Z}/2$ is non-zero (that is, if $1 \times A \in C_{12}$ for some matrix $A \in M_2(\mathbb{Z})$), then at least one of the following two conditions holds:

– *there exists an invertible matrix B such that $1 \times B \in C_{12}$; and $1 \times B^{-1} \in C_{21}$;*
– *C_{12} and C_{21} contain the element 1×0.*

We prove Proposition 6.1 after two following preliminary lemmas:

Lemma 6.2. *We assume* (i)–(iv) *and* (vi) *(the assumption* (v) *is not needed here). For any $\rho, \delta \in \{1, 2\}$, the following inclusions hold:*

$$C^{\diagup}_{\rho\delta} \subset C_{\delta\rho}, \qquad C^{\diagdown}_{\rho\delta} \subset C_{\delta\rho},$$
$$C^{|}_{\rho\delta} \subset C_{\rho\delta}, \qquad C^{—}_{\rho\delta} \subset C_{\rho\delta}\ .$$

Proof. Let us fix some arbitrary ρ and δ. Since 1×1 and $0 \times \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ are in $C_{\rho\rho}$ (by (iii) and (vi)), the difference

$$0 \times \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - 1 \times 1 = 1 \times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 1 \times 1^{|}$$

is also there. Since $(a \times A) \cdot (1 \times 1^{|}) = a \times A^{|}$ for any $a \times A \in \mathbb{Z}/2 \times M_2(\mathbb{Z})$ and $C_{\rho\delta} \cdot C_{\rho\rho} \subset C_{\rho\delta}$ (this inclusion holds by (ii)), it follows that $C^{|}_{\rho\delta} \subset C_{\rho\delta}$.

The inclusion $C^{—}_{\rho\delta} \subset C_{\rho\delta}$ is proved in the similar way by using the formula

$$(1 \times 1^{—}) \cdot (a \times A) = a \times A^{—}$$

and the inclusion $1 \times 1^{—} \in C_{\delta\delta}$ (to avoid misunderstanding let us note that $1^{—} = 1^{|}$).

One of the two remaining inclusions holds by the assumption (iv). Using the equality $A^{\diagup} = ((A^{|})^{\diagdown})^{|}$ or the equality $A^{\diagdown} = ((A^{|})^{\diagup})^{|}$ (which is true for any matrix A) as well as the already proved inclusion for $|$, we obtain the last inclusion remained. □

Lemma 6.3. *Let us assume* (i)–(iv) *(the assumptions* (v) *and* (vi) *are not needed here). If* $1 \times B \in C_{12}$ *for an invertible matrix* B*, then* $1 \times B^{-1} \in C_{21}$.

Proof. (Compare with [19, Proof of Corollary 10].) By (iv) we have $1 \times B' \in C_{21}$ for either $B' := B^{\diagdown}$ or $B' := B^{\diagup}$. Thus $x := 1 \times B' \cdot B \in C_{11}$ and it suffices to show that $x^{-1} \in C_{11}$ as well.

Let $f(t)$ be the characteristic polynomial of the matrix $B' \cdot B$. We have $f(t) \in \mathbb{Z}[t]$ and the free coefficient of $f(t)$ is $\det(B' \cdot B) = 1$. The element x is a zero of the polynomial $(t-1) \cdot f(t) \in \mathbb{Z}[t]$. We have $(t-1) \cdot f(t) = t^3 + at^2 + bt - 1$ with $a, b \in \mathbb{Z}$. Therefore $x^{-1} = x^2 + ax + b \in C_{11}$. □

Proof of Proposition 6.1. (Compare with [24, case D in the proof of Lemma 3.24].) Let A be a matrix such that $1 \times A \in C_{12}$. We write A mod 2 for the image of A in $M_2(\mathbb{Z}/2)$. There are five possibilities:

0. A mod $2 = 0$;
1. A mod 2 has precisely one non-zero entry;
2. A mod 2 has precisely two non-zero entries;
3. A mod 2 has precisely three non-zero entries;
4. A mod 2 has no zero entries.

We consider them one by one.

4. Since $0 \times \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in C_{12}$ by (vi), we may reduce the case 4 to the case 0.

3. The same argument reduces the case 3 to the case 1.

0. For any positive integer r, we have $1 \times (A \cdot (A^{\diagdown} \cdot A)^r) \in C_{12}$ by Lemma 6.2 and by the assumption (ii). All entries of the matrix $A \cdot (A^{\diagdown} \cdot A)^r$ are divisible by 2^r (more precisely, they are divisible by 2^{2r+1}). Taking r as in (v), we prove that $1 \times 0 \in C_{12}$. Then we also have $1 \times 0 = 1 \times 0^{\diagdown} \in C_{21}$.

1. By Lemma 6.2, we may assume that A mod $2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Since $(A^{\diagdown})^{|}$ mod $2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, the product $(A^{\diagdown})^{|} \cdot A$ is zero modulo 2. Therefore $(A(A^{\diagdown})^{|}A)$ is zero modulo 2 as well. Since $1 \times (A \cdot (A^{\diagdown})^{|} \cdot A) \in C_{12}$, we come to the case 0.

2. Suppose that the two non-zero entries of A mod 2 are in the same column. Then $A^{\diagdown} \cdot A$ mod $2 = 0$ and we come to the case 0.

In the case where the two non-zero entries of A lie in the same row, we have $A \cdot A^{\diagdown}$mod $2 = 0$.

Thus, we may assume that the two non-zero entries of A mod 2 are neither in the same column, nor in the same row. Then either $A \bmod 2 = 1$ or $A \bmod 2 = 1^{\intercal}$. Replacing A by $A^{\intercal}$ in the second case, we come to the situation with $A \bmod 2 = 1$ anyway.

In the rest of the proof we assume that $A \bmod 2 = 1$. Denote the entries of A as follows: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

The sum of the integers $a + d - b - c$ and $a + b - c - d$ is $2(a - c)$ and $a - c$ is odd. Therefore either $a + d - b - c$ or $a + b - c - d$ is not divisible by 4. We consider these two cases separately.

1) First assume that the integer $a + d - b - c$ is not divisible by 4. Since $0 \times \begin{pmatrix} 2^r & 0 \\ 0 & 0 \end{pmatrix} \in C_{12}$ (by (v)), we may assume that $a + d - b - c = 2$.

The determinant of the matrix $B := A + x \cdot \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, where $x := (1 - \det A)/2 \in \mathbb{Z}$, equals $\det A + (a + d - b - c) \cdot x = \det A + 2x = 1$, and $1 \times B \in C_{12}$. Therefore we are done by Lemma 6.3.

2) Now assume that $a + b - c - d$ is not divisible by 4. As in the previous case, we may then assume that $a + b - c - d = 2$.

The determinant of the matrix $B := A + x \cdot \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$, where $x := (\det A - 1)/2 \in \mathbb{Z}$, equals $\det A - (a + b - c - d) \cdot x = \det A - 2x = 1$. Therefore it suffices to show that $0 \times \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \in C_{12}$.

We have

$$(a+c-b-d) \cdot 0 \times \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = 1 \times A + 1 \times A^{\intercal} - (b+d) \left(0 \times \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right) \in C_{12} .$$

Since the sum $(a + b - c - d) + (a + c - b - d) = 2(a - d)$ is divisible by 4 (because a and d are odd), the integer $a + c - b - d$ is not divisible by 4. Taking (v) in account, we see that $0 \times \begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix} \in C_{12}$. Therefore

$$0 \times \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = 0 \times \begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix} - 0 \times \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in C_{12} ,$$

and the proposition is proven. □

Proposition 6.4. *Let us modify the assumption* (v) *of Proposition 6.1 as follows:*

(v) $0 \times 2^r \in C_{\rho\delta}$ *for some positive integer* r *and the image of the projection* $C_{\rho\delta} \to M_2(\mathbb{Z})$ *is contained in the additive subgroup of* $M_2(\mathbb{Z})$ *generated by* 1 *and* $1^{\intercal}$.

Assume (i)–(vi). *If the projection* $C_{12} \to \mathbb{Z}/2$ *is non-zero (that is, if* $1 \times A \in C_{12}$ *for some matrix* $A \in M_2(\mathbb{Z})$*), then at least one of the following two conditions holds:*

- C_{12} *and* C_{21} *contain the element* 1×1*;*
- C_{12} *and* C_{21} *contain the element* 1×0.

Proof. (Compare with [24, case B in the proof of Lemma 3.24].) Let A be a matrix such that $1 \times A \in C_{12}$. By (v), $A = a + b^{|}$ for some $a, b \in \mathbb{Z}$. We have

$$1 \times (a - b) = 1 \times A - b\left(0 \times \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right) \in C_{12}\ ,$$

what means, we may assume b to be zero.

Suppose that a is odd. Multiplying by 2, we have $0 \times (2a) \in C_{12}$. Since (by (v)) $0 \times 2^r \in C_{12}$ for some r, it follows that $0 \times 2 \in C_{12}$. Therefore

$$1 \times 1 = 1 \times a - ((a-1)/2)(0 \times 2) \in C_{12}$$

and the proof is finished in the case of odd a.

Now suppose that a is even. Since $1 \times a^2 = (1 \times a^{\diagdown}) \cdot (1 \times a) \in C_{11}$ and $0 \times 2 = 2 \cdot (1 \times 1) \in C_{11}$, it follows that $1 \times 0 \in C_{11}$. Consequently $1 \times 0 = (1 \times a) \cdot (1 \times 0) \in C_{12}$ and $1 \times 0 = 1 \times 0^{\diagdown} \in C_{21}$. $\square$

7. Criterion for central simple algebras

In this section, F is an arbitrary field (any characteristic, even 2, is allowed). We say that two central simple F-algebras are *opposite*, if one of them is isomorphic to the opposite algebra of the other (see e.g. [14, §1 of Chapter I] for the definition of the opposite algebra).

Criterion 7.1. *Two central simple algebras are m-equivalent if and only if they are isomorphic or opposite.*

We start with terminology and notation of this section: for a central simple F-algebra A, X_A is its Severi-Brauer variety and $\mathfrak{X}_A := (X_A)_{\mathcal{F}}$, where $\mathcal{F}$ is a fixed separable closure of F ($\mathcal{F}$ may be almost always replaced by an arbitrary field extension of F, splitting all the algebras under consideration).

We recall the definition of the Severi-Brauer variety X_A (compare to [11, Definition 10.3]): for any commutative F-algebra R, the set of R-points $X_A(R)$ is defined as the set of the right ideals I of the R-algebra $A_R := A \otimes_F R$ such that I, as an R-module, is a direct summand of A_R of the constant rank $\deg A$ (the *degree* $\deg A$ of A is defined as $\deg A := \sqrt{\dim_F A}$).

We say that two central simple F-algebras A_1 and A_2 are *m-equivalent* (and write $A_1 \overset{\mathrm{m}}{\sim} A_2$), if X_{A_1} and X_{A_2} are isomorphic as objects of $\mathcal{CV}^0(F)$ (the category

$\mathcal{CV}^0(F)$ is defined in Sect. 1). Note that the varieties X_{A_1} and X_{A_2} are isomorphic iff the algebras A_1 and A_2 are so.

We say that A_1 and A_2 are *s-equivalent* (and write $A_1 \overset{\mathrm{s}}{\sim} A_2$), if $\deg A_1 = \deg A_2$ and $\operatorname{ind}(A_1)_E = \operatorname{ind}(A_2)_E$ for any field extension E/F (where $\operatorname{ind} A := \deg D$ with D a division algebra Brauer-equivalent to A).

Remark 7.2. The notion of the Schur index $\operatorname{ind} A$ of a central simple algebra A is not really analogous to the notion of Witt index of a quadratic form. The right notion would be $\operatorname{ind}' A := \deg A/\operatorname{ind} A$, because presicely $\operatorname{ind}' A$, not $\operatorname{ind} A$, measures how much A is split. Defining the s-equivalence with the help of ind′, one may avoid the condition on the degrees: $A_1 \overset{\mathrm{s}}{\sim} A_2$ iff $\operatorname{ind}'(A_1)_E = \operatorname{ind}'(A_2)_E$ for any E/F (compare with the definition of s-equivalent quadratic forms).

Let A be a central simple algebra and i an integer; we write A^i for the (determined up to an isomorphism) central simple F-algebra such that $\deg A^i = \deg A$ and $[A^i] = i \cdot [A] \in \operatorname{Br}(F)$. In particular, A^{-1} is the opposite algebra of A. Note that $A^{\otimes i}$ is isomorphic to a matrix algebra over A^i, namely (comparing the degrees) $A^{\otimes i} \simeq M_{(\deg A)^{i-1}}(A^i)$.

For an arbitrary Severi-Brauer variety X_A/F, we write $h \in \operatorname{CH}^1(\mathfrak{X}_A)$ for the class of a hyperplane (note that X_A is isomorphic to the projective space of dimension $\deg A - 1$ iff A is split).

One of the implications of Criterion 7.1. is given by the following

Proposition 7.3. *For any central simple algebras A, one has $A \overset{\mathrm{m}}{\sim} A^{-1}$.*

Proof. We are going to use the following simple assertion (for its last part see e.g. [12, Lemma I.3]):

Claim 7.4. *For any two central simple F-algebras A_1 and A_2, the tensor product of the ideals gives rise to a closed imbedding $X_{A_1} \times X_{A_2} \hookrightarrow X_{A_1 \otimes A_2}$. If A_1 and A_2 are split, this is the Segre imbedding of the product of the projective spaces into the projective space. In particular, the pull-back of $h \in \operatorname{CH}^1(\mathfrak{X}_{A_1 \otimes A_2})$ equals $1 \times h + h \times 1 \in \operatorname{CH}^1(\mathfrak{X}_{A_1} \times \mathfrak{X}_{A_2})$.* □

Claim 7.5. *The cycle $1 \times h + h \times 1 \in \operatorname{CH}^1(\mathfrak{X}_A \times \mathfrak{X}_{A^{-1}})$ is defined over F.*

Proof. (Also see Remark 7.17.) Since the tensor product $A \otimes A^{-1}$ is a split algebra, the cycle $h \in \operatorname{CH}^1(\mathfrak{X}_{A \otimes A^{-1}})$ is defined over F. To finish, we apply Claim 7.4.. □

Claim 7.6. *Assume that we are given a positive integer n and a subring S of the integral polynomial ring in two variables $\mathbb{Z}[x, y]$ such that S contains $x + y$ as well as $\binom{n+1}{i} \cdot x^i$ for all $i = 0, \ldots, n$. Then S contains the element $\sum_{i=0}^{n} (-1)^i \cdot (x^i y^{n-i})$.*

Proof. The assertion is served by the formula

$$\sum_{i=0}^{n}(-1)^i x^i y^{n-i} = \sum_{i=0}^{n}(-1)^i \binom{n+1}{i} x^i (x+y)^{n-i} .$$

Since this formula is homogeneous, it suffices to prove it after the substitution $x = 1$. One has

$$\begin{aligned}\sum_{i=0}^{n}(-1)^i y^{n-i} &= \frac{(-1)^n + y^{n+1}}{1+y} \\ &= \frac{(-1)^n + (-1 + (1+y))^{n+1}}{1+y} \\ &= \sum_{i=0}^{n}(-1)^i \binom{n+1}{i} (1+y)^{n-i} .\end{aligned}$$

□

Claim 7.7. *Set* $n := \dim X_A$ *(*$= \deg A - 1$*). The cycle* $\binom{n+1}{i} \cdot h^i \in \mathrm{CH}^i(\mathfrak{X}_A)$ *is defined over* F *for any* $i = 0, 1, \ldots, n$.

Proof. We explain two different proofs. The first one is based on the observation that the binomial coefficient $\binom{n+1}{i}$ is divisible by the integer

$$(n+1)/(i, n+1)$$

for any i, where $(i, n+1)$ stays for the greatest common divisor of i and $n+1$. A transfer argument shows (see [8, Lemma 3]), that the cycle $\big((n+1)/(i, n+1)\big) \cdot h^i$ is defined over F.

The other way of proving uses Chern classes. The total Chern class (see [10, Definition 2.11])

$$c_t\colon K_0(\mathfrak{X}_A) \to \left(\sum_{i=0}^{\infty} \mathrm{CH}^i(\mathfrak{X}_A) \cdot t^i\right)^{\times} ,$$

where $K_0(\mathfrak{X}_A)$ is the Grothendieck group and t is a formal variable, is defined over F and is a group homomorphism. For the class $[\mathcal{T}] \in K_0(\mathfrak{X}_A)$ of the tautological vector bundle $\mathcal{T}$ on the projective space $\mathfrak{X}_A$, one has $c_t(\mathcal{T}) = 1 - h \cdot t$ (by the very definition of c_t). Since the element $(\deg A) \cdot [\mathcal{T}] = (n+1) \cdot [\mathcal{T}]$ is defined over F (it coincides with the image under the restriction $K_0(X_A) \to K_0(\mathfrak{X}_A)$ of the class in $K_0(X_A)$ of the tautological vector bundle on X_A, see [17, Sect. 8.4]) and $c_t((n+1)[\mathcal{T}]) = c_t([\mathcal{T}])^{n+1} = \sum_{i=0}^{n+1}(-1)^i \binom{n+1}{i} \cdot h^i \cdot t^i$, we are done. □

Claim 7.8. *Set* $n := \dim X_A$ *(*$= \deg A - 1$*). The cycle* $\sum_{i=0}^{n}(-1)^i(h^i \times h^{n-i}) \in \mathrm{CH}^n(\mathfrak{X}_A \times \mathfrak{X}_{A^{-1}})$ *is defined over* F.

Proof. Consider the homomorphism of rings

$$f : \mathbb{Z}[x, y] \to \mathrm{CH}^*(\mathfrak{X}_A \times \mathfrak{X}_{A^{-1}}) \ , \quad x \mapsto h \times 1 \ , \quad y \mapsto 1 \times h \ .$$

Let S be the set of those elements of $\mathbb{Z}[x, y]$ whose image under f is defined over F. Then S is a subring, contains $x + y$ (Claim 7.5.), and contains $\binom{n+1}{i} \cdot x^i$ for all i (Claim 7.7.). Therefore, according to Claim 7.6., S also contains the sum $\sum_{i=0}^{n}(-1)^i \cdot (x^i y^{n-i})$. The image under f of this sum is the desired cycle $\sum_{i=0}^{n}(-1)^i(h^i \times h^{n-i})$. □

Since the composition of the correspondence $\sum_{i=0}^{n}(-1)^i \cdot (h^i \times h^{n-i})$ (from Claim 7.8.) with its transposition (in any order) gives $\sum_{i=0}^{n} h^i \times h^{n-i}$, which is the class of the diagonal, this correspondence determines an isomorphism between $\mathfrak{X}_A$ and $\mathfrak{X}_{A^{-1}}$.

The nilpotence theorem (i.e., the theorem analogous to Theorem 3.1) for the Severi-Brauer varieties is proved in [9, Proposition 2.2.3] (it can be also proved using the method of proving of Theorem 3.1). Thus we have the analog of Corollary 3.2 for the Severi-Brauer varieties, what finishes the proof of the proposition. □

Remark 7.9. It is remarkable to look at the isomorphisms between the Chow groups $\mathrm{CH}^i(\mathfrak{X}_A) = \mathbb{Z}$ (with the canonical generator h^i) and $\mathrm{CH}^i(\mathfrak{X}_{A^{-1}}) = \mathbb{Z}$ (with the canonical generator h^i), induced by the constructed motivic isomorphism of X_A and $X_{A^{-1}}$: they are "identical" for even i and they are "mulitplication by -1" for odd i (compare with Proposition 7.10).

The inverse implication of Criterion 7.1. is contained in the following

Proposition 7.10. *If central simple* F*-algebras* A_1 *and* A_2 *are m-equivalent, then* $A_1 \simeq A_2$ *or* $A_1 \simeq A_2^{-1}$*. If there exists a motivic isomorphism* $X_{A_1} \to X_{A_2}$ *inducing "identical" isomorphism* $\mathrm{CH}^*(\mathfrak{X}_{A_1}) \to \mathrm{CH}^*(\mathfrak{X}_{A_2})$*, then* $A_1 \simeq A_2$.

We give the proof after several simple lemmas, which are, to our mind, of independent interest. The following one is parallel to Corollary 2.5.

Lemma 7.11. *If* $A_1 \overset{\mathrm{m}}{\sim} A_2$*, then* $A_1 \overset{\mathrm{s}}{\sim} A_2$.

Proof. We assume that $A_1 \overset{\mathrm{m}}{\sim} A_2$. Since $\dim X_{A_\rho} = \deg A_\rho - 1$ (for $\rho = 1, 2$), it follows that $\deg A_1 = \deg A_2$ (the proof is the same as for Lemma 2.1).

Since the homomorphism $\mathrm{res}_{\mathcal{F}/F} : \mathrm{CH}_0(X_{A_\rho}) \to \mathrm{CH}_0(\mathfrak{X}_{A_\rho}) = \mathbb{Z}$ is given by taking the degrees of 0-cycles, the cokernel of $\mathrm{res}_{\mathcal{F}/F}$ has the order $\mathrm{ind} A_\rho$ (to see it at once, note that $\operatorname{ind} A_\rho$ is equal to the minimal degree of a field extension E/F such that $[A_\rho]_E = 0 \in \operatorname{Br}(E)$ and that the variety $(X_{A_\rho})_E$ has a rational point over an extension E/F iff $[A_\rho]_E = 0$). Therefore, $\operatorname{ind} A_1 = \operatorname{ind} A_2$ (compare to the proof of Lemma 2.3).

Finally, since m-equivalent algebras remain m-equivalent over any extension of scalars, we conclude that $A_1 \overset{s}{\sim} A_2$. □

Remark 7.12 (Compare to Remark 2.4). There is a complete description of the group $\mathrm{CH}_0(X_A)$ (see [16], where the kernel of the degree homomorphism $\mathrm{CH}_0(X_A) \to \mathbb{Z}$ is shown to be zero). However this information is superfluous in the proof of Lemma 7.11.

Lemma 7.13. *Assume that* $\deg A_1 = \deg A_2$. *The relation* $A_1 \overset{s}{\sim} A_2$ *holds if and only if the classes of* A_1 *and* A_2 *generate the same subgroup in the Brauer group of* F.

Proof. If $A_1 \overset{s}{\sim} A_2$, then, in particular, A_2 splits over the function field of X_{A_1}. The index reduction formula for Severi-Brauer varieties [21, Theorem 1.3] (or actually a simpler and earlier result of Amitsur [2, Theorem 9.3] describing the kernel of $\operatorname{Br}(F) \to \operatorname{Br}(F(X_{A_1}))$) shows that $[A_2]$ is in the subgroup of $\operatorname{Br}(F)$ generated by $[A_1]$. The rest of the proof is evident. □

Lemma 7.14. *Assume that* $A_1 \overset{m}{\sim} A_2$. *Among the* 2^{n+1} *cycles*

$$\sum_{i=0}^{n} \pm(h^i \times h^{n-i}) \in \mathrm{CH}^n(\mathfrak{X}_{A_1} \times \mathfrak{X}_{A_2}) ,$$

there is a cycle defined over F. *Moreover, if there exists a motivic isomorphism of* X_{A_1} *and* X_{A_2} *inducing "identical" isomorphism* $\mathrm{CH}^*(\mathfrak{X}_{A_1}) \to \mathrm{CH}^*(\mathfrak{X}_{A_2})$, *then the cycle* $\sum_{i=0}^{n} h^i \times h^{n-i}$ *is defined over* F.

Proof. Any motivic isomorphism between $\mathfrak{X}_{A_1}$ and $\mathfrak{X}_{A_2}$ is given by a correspondence of the kind $\sum_{i=0}^{n} \pm(h^i \times h^{n-i})$ (compare with Lemma 4.8). Since there is a motivic isomorphism between X_{A_1} and X_{A_2} (over F), we are done with the first assertion.

For the second assertion, it suffices to note, that if a motivic isomorphism of $\mathfrak{X}_{A_1}$ onto $\mathfrak{X}_{A_2}$ induces "identical" isomorphism on the Chow group, then it is equal to the correspondence $\sum_{i=0}^{n} h^i \times h^{n-i}$. □

Lemma 7.15. *For an arbitrary central simple* F*-algebra* D *of degree* $n+1$, *and for any integer* $s > 0$, *the cycle* $\sum_{i=0}^{n} h^i \times h^{n-i} \in \mathrm{CH}^n(\mathfrak{X}_{M_s(D)} \times \mathfrak{X}_D)$ *is defined over* F.

Proof. For any commutative F-algebra R, the standard imbedding of $M_{s-1}(D_R)$ into $M_s(D_R)$ (by adding 0 entries in the s-th rows and in the s-th column) gives a map of the sets of ideals $X_{M_{s-1}(D)}(R) \to X_{M_s(D)}(R)$ (by taking the generated right ideal). This determines a closed imbedding $X_{M_{s-1}(D)} \hookrightarrow X_{M_s(D)}$. Since the codimension of $X_{M_{s-1}(D)}$ in $X_{M_s(D)}$ is $n+1 > n$, the pull-back $\mathrm{CH}^n(X_{M_s(D)} \times X_D) \to \mathrm{CH}^n\big((X_{M_s(D)} \setminus X_{M_{s-1}(D)}) \times X_D\big)$ to the open subvariety is an isomorphism. In its turn, the difference $X_{M_s(D)} \setminus X_{M_{s-1}(D)}$ is a vector bundle over X_D (see [11, Theorem 10.9]). Therefore, the pull-back with respect to the morphism $(X_{M_s(D)} \setminus X_{M_{s-1}(D)}) \times X_D \leftarrow X_D \times X_D$, given by the zero-section of the vector bundle, is an isomorphism $\mathrm{CH}^n\big((X_{M_s(D)} \setminus X_{M_{s-1}(D)}) \times X_D\big) \to \mathrm{CH}^n(X_D \times X_D)$.

Let us now extend the scalars up to $\mathcal{F}$. Since the image in $\mathrm{CH}^n(\mathfrak{X}_D \times \mathfrak{X}_D)$ of the sum $\sum_{i=0}^n h^i \times h^{n-i} \in \mathrm{CH}^n(\mathfrak{X}_{M_s(D)} \times \mathfrak{X}_D)$ is given by the same sum (and therefore coincides with the diagonal class), it is defined over F. □

Remark 7.16. In [9, Corollary 1.3.2] a motivic decomposition

$$X_{M_s(D)} \simeq \bigoplus_{i=0}^{s-1} X_D(i \cdot (n+1))$$

(in the category $\mathcal{CV}(F)$ constructed in Sect. 1) is established (which is a particular case of the motivic decompositions of the isotropic flag varieties obtained in [11]). Lemma 7.15 gives a way to refine this result by the additional information that the induced over $\mathcal{F}$ isomorphisms of the Chow groups are “identical” (i.e., “multiplication by -1” does not occur).

Proof of Proposition 7.10. (See also Remark 7.17.) We assume that $A_1 \overset{\mathrm{m}}{\sim} A_2$. Lemmas 7.11 and 7.13 imply that $A_2 \simeq A_1^{-r}$ for some integer r. Since $[A_1]$ (as well as all other elements of $\mathrm{Br}(F)$) is of finite order, we may assume that r is positive. To prove the first assertion of the proposition, it suffices to show that $\exp A_1$ (this is the order of $[A_1]$ in $\mathrm{Br}(F)$) divides $r+1$ or $r-1$; for the second assertion we have to show that $\exp A_1$ divides $r+1$ under the additional assumption of the second assertion.

We set $A := A_1$ and $B := A^{-1}$. We have $B^r \overset{\mathrm{m}}{\sim} A$. Taking the composition of a defined over F correspondence $\mathfrak{X}_{B^r} \to \mathfrak{X}_A$ from Lemma 7.14 and the correspondence $\mathfrak{X}_A \to \mathfrak{X}_B$ from Claim 7.5., we see that at least one of two cycles $1 \times h \pm h \times 1 \in \mathrm{CH}^1(\mathfrak{X}_{B^r} \times \mathfrak{X}_B)$ is defined over F. Let $\varepsilon \in \{0, 1\}$ be such that $\big(1 \times h + (-1)^\varepsilon (h \times 1)\big)/F$.

We have $B^{\otimes r} \simeq M_{(\deg B)^{r-1}}(B^r)$. Considering the composition of the defined over F correspondence $\mathfrak{X}_{B^{\otimes r}} \to \mathfrak{X}_{B^r}$ of Lemma 7.15 with the correspondence $1 \times h + (-1)^\varepsilon (h \times 1) \in \mathrm{CH}^1(\mathfrak{X}_{B^r} \times \mathfrak{X}_B)$, we conclude that the cycle $1 \times h + (-1)^\varepsilon (h \times 1) \in \mathrm{CH}^1(\mathfrak{X}_{B^{\otimes r}} \times \mathfrak{X}_B)$ is defined over F.

Taking the pull-back of the cycle $1 \times h + (-1)^{\varepsilon}(h \times 1) \in \mathrm{CH}^1(\mathfrak{X}_{B^{\otimes r}} \times \mathfrak{X}_B)$ with respect to the composition of morphisms

$$\mathfrak{X}_B \longrightarrow \mathfrak{X}_B^{\times(r+1)} = \mathfrak{X}_B^{\times r} \times \mathfrak{X}_B \xrightarrow{\text{segre}\times id} \mathfrak{X}_{B^{\otimes r}} \times \mathfrak{X}_B \ ,$$

where the first one is the $(r+1)$-diagonal while segre: $\mathfrak{X}_B^{\times r} \to \mathfrak{X}_{B^{\otimes r}}$ is the r-fold Segre imbedding (see Claim 7.4.), we get a defined over F cycle $(r + (-1)^{\varepsilon}) \cdot h \in \mathrm{CH}^1(\mathfrak{X}_B)$. Therefore the order of $\mathrm{Coker}\big(\mathrm{CH}^1(X_B) \to \mathrm{CH}^1(\mathfrak{X}_B)\big)$ divides $r + (-1)^{\varepsilon}$. On the other hand, by the classical computation of the Piccard group for the Severi-Brauer varieties ([1, Sect. 2]), the order of this cokernel is equal to $\exp B$. This proves the first assertion of the proposition.

For the second assertion it suffices to note that if a motivic isomorphism between X_{B^r} and X_B gives the "identity" on the Chow groups over $\mathcal{F}$, then over $\mathcal{F}$ this isomorphism is given by the correspondence $\sum_{i=0}^{n} h^i \times h^{n-i}$. Thus this correspondence is defined over F, and one may choose $\varepsilon = 1$. □

Remark 7.17. The following shorter proof of Proposition 7.10 is suggested by A. S. Merkurjev. For any central simple F-algebras A_1 and A_2, the exact sequence

$$\mathrm{CH}^1(X_{A_1} \times X_{A_2}) \xrightarrow{\mathrm{res}_{\mathcal{F}/F}} \mathrm{CH}^1(\mathfrak{X}_{A_1} \times \mathfrak{X}_{A_2}) \xrightarrow[1\times h \mapsto [A_2]]{h\times 1 \mapsto [A_1]} \mathrm{Br}(F)$$

(compare to [20, Lemme 6.3 (i)] and [1, Sect. 2.1]) proves that the cycle

$$1 \times h + (-1)^{\varepsilon}(h \times 1) \in \mathrm{CH}^1(\mathfrak{X}_{A_1} \times \mathfrak{X}_{A_2})$$

is defined over F if and only if $[A_1] = (-1)^{1-\varepsilon}[A_2]$ (this also gives another proof for Claim 7.5.). In particular, the cycle $1 \times h - h \times 1 \in \mathrm{CH}^1(\mathfrak{X}_{A_1} \times \mathfrak{X}_{A_1})$ is defined over F. Having a motivic isomorphism of X_{A_1} and X_{A_2} we therefore may conclude (using Lemma 7.14) that at least one of the two cycles $1 \times h \pm h \times 1 \in \mathrm{CH}^1(\mathfrak{X}_{A_1} \times \mathfrak{X}_{A_2})$ is defined over F as well (if the motivic isomorphism induces "identity" on the Chow groups, then the cycle with the minus sign is defined over F); thus $[A_1] = \pm[A_2]$ (in the latter case $[A_1] = [A_2]$).

Proof of Criterion 7.1. One of the implications of the criterion is Proposition 7.3; the inverse implication is contained in Proposition 7.10. □

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