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Very ampleness for Theta on the compactified Jacobian

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1 Introduction

If X is a smooth, complete, connected curve over an algebraically closed field, then the Jacobian J_0, parametrizing invertible sheaves on X of Euler characteristic 0, is projective and admits a canonical ample divisor Θ, the Theta divisor. If g denotes the genus of X, then Θ is the scheme-theoretic image of the Abel-Jacobi morphism $X^{g-1} \to J_0$, given by

$$(p_1, \ldots, p_{g-1}) \to \mathscr{O}_X(p_1 + \ldots + p_{g-1}) .$$

It follows from [14, Sect. 17, p. 163] that 3Θ is very ample.

In the singular case, D'Souza has constructed a natural compactification $\overline{J}_0$ for the Jacobian J_0 of a complete, integral curve over an algebraically closed field [5]. The scheme $\overline{J}_0$ parametrizes torsion-free, rank 1 sheaves of Euler characteristic 0 on X. A natural question in this context is whether there is a canonical Cartier divisor on $\overline{J}_0$ extending the notion of the classical Theta divisor.

The above question was partially and independently answered in [6] and [19]. In these two works the same canonical line bundle $\mathscr{L}$ on $\overline{J}_0$ and the same global section θ of $\mathscr{L}$ are defined. For smooth curves, the zero scheme of θ is the classical Theta divisor Θ. In [19] Soucaris shows that the zero scheme of the restriction of θ to the maximum reduced subscheme of $\overline{J}_0$ is a Cartier divisor. Both [6] and [19] show that $\mathscr{L}$ is ample. It remains to determine whether the zero scheme of θ on $\overline{J}_0$ is a Cartier divisor in general, and what is the minimum n such that $\mathscr{L}^{\otimes n}$ is very ample.

In this article our main concern is with the latter question. We will show that $\mathscr{L}^{\otimes n}$ is very ample for n at least equal to a specified lower bound (Theorem 7). If X has at most ordinary nodes or cusps as singularities, then our lower bound is 3. Our main tool is to use theta sections θ_E associated to vector bundles E on X.

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The theta sections were used by Faltings [9] to construct the moduli of semistable vector bundles on a smooth, complete curve without using Geometric Invariant Theory (see also [18]). In a forthcoming work [7], [8] we will apply such method to construct the compactified Jacobian for families of reduced curves.

The importance of Theorem 7 is that we obtain a canonical projective embedding of $\overline{J}_0$ in $\mathbf{P}(H^0(\overline{J}_0, \mathscr{L}^{\otimes n}))$, for n minimum such that $\mathscr{L}^{\otimes n}$ is very ample. By studying the structure of the homogeneous coordinate ring of $\overline{J}_0$ in $\mathbf{P}(H^0(\overline{J}_0, \mathscr{L}^{\otimes n}))$, maybe in a way analogous to Mumford's in [15] and [16], we might be able to understand better the algebraic structure of $\overline{J}_0$.

Notation. We will often deal with parameter spaces, that is, spaces whose points are classes representing certain objects. In such context, we will employ the usual bracket notation $[F]$ for the point representing the object F. If E is a vector bundle on a scheme Y, we denote by $\mathbf{P}_Y(E)$ the corresponding projective bundle over Y. By a point we mean a closed point.

2 The compactified Jacobian

Let X be a complete, integral curve over an algebraically closed field k. Denote by g the arithmetic genus of X, and by ω the dualizing sheaf on X. A coherent sheaf I on X is *torsion-free* if I_x is a torsion-free $\mathscr{O}_x$-module for every $x \in X$. A coherent sheaf I on X is *rank 1* if I is generically invertible. By [4, p. 96], the sheaf ω is torsion-free, rank 1. Fix an ample line bundle $\mathscr{O}_X(1)$ on X. For every coherent sheaf F on X, let $\mathrm{Quot}_X^{p(t)}(F)$ denote Grothendieck's Quot-scheme [10], parametrizing quotients of F with Hilbert polynomial $p(t)$ with respect to $\mathscr{O}_X(1)$. We will drop the superscript $p(t)$ whenever it is not important.

For every integer d, let $\overline{\mathbf{J}}_d$ denote the *compactified Jacobian functor*. For each k-scheme S, the set $\overline{\mathbf{J}}_d(S)$ consists of equivalence classes of S-flat coherent sheaves $\mathscr{I}$ on $X \times S$ such that $\mathscr{I}(s)$ is torsion-free, rank 1 of Euler characteristic d for every $s \in S$. (Two sheaves $\mathscr{I}_1$ and $\mathscr{I}_2$ are called equivalent if there is an invertible sheaf N on S such that $\mathscr{I}_1 \cong \mathscr{I}_2 \otimes N$). D'Souza [5] and Altman and Kleiman [3], [4] have shown that $\overline{\mathbf{J}}_d$ is represented by a (projective) scheme $\overline{J}_d$, the *compactilied Jacobian*. Here we present yet another proof of the representability of $\overline{\mathbf{J}}_d$ by a scheme, a proof more suitable for treating the question of very ampleness in Sect. 4.

For every torsion-free, rank 1 sheaf I on X, let

$$e(I) := \max_{x \in X} \dim_k I(x) .$$

Since I is generically invertible, then $1 \leq e(I) < \infty$.

Proposition 1 *Let I be a torsion-free, rank 1 sheaf on X of Euler characteristic d. Then, for every integer $r \geq \max(e(\underline{Hom}_X(I, \omega)), 2)$, there is a vector bundle E on X of rank r and degree $-rd - 1$ such that:*
(i) $h^0(X, I \otimes E) = 0$ and $h^1(X, I \otimes E) = 1$;
(ii) the unique (modulo k^*) non-zero homomorphism $I \to E^* \otimes \omega$ is an embedding with torsion-free cokernel.

Proof. Let $m \gg 0$ be an integer such that $H^0(X, I(-m)) = 0$ and $\underline{Hom}_X(I, \omega)(m)$ is generated by global sections. Since $r \geq \max(e(\underline{Hom}_X(I, \omega)), 2)$ and k is infinite, then there is a surjection $p : \mathcal{O}_X^{\oplus r} \twoheadrightarrow \underline{Hom}_X(I, \omega)(m)$. Applying $\underline{Hom}_X(\cdot, \omega)$ to p, we obtain an embedding $I(-m) \hookrightarrow \omega^{\oplus r}$, whose cokernel is torsion-free since $\underline{Ext}^1_X(F, \omega) = 0$ for every torsion-free sheaf F on X [4, p. 96]. Twisting by $\mathcal{O}_X(m)$ and letting $E := \mathcal{O}_X(-m)^{\oplus r}$, we get that $H^0(X, I \otimes E) = 0$ and there is a short exact sequence on X of the form

$$0 \to I \xrightarrow{\mu} E^* \otimes \omega \xrightarrow{q} C \to 0 ,$$

where C is torsion-free.

Let $h := h^1(X, I \otimes E)$. If $h = 1$, then the proposition is proved. We will show by descending induction on h that we can choose E as in the above paragraph with $h = 1$. Suppose $h > 1$. Let $\lambda : I \to E^* \otimes \omega$ be a homomorphism that is not a multiple of μ. Since I is simple by [4, Lemma 5.4, p. 83], then the composition $p := q \circ \lambda$ is not zero. Since C is torsion-free, then there is a regular point $x \in X$ such that $p(x) \neq 0$. Let $\sigma : C(x) \to I(x)$ be a splitting for $\rho(x)$. Let

$$F := (\ker(E^* \to E^*(x) \xrightarrow{q(x)} C(x) \xrightarrow{\sigma} I(x)))^* .$$

(We implicitly chose a trivialization of ω at x. Any other choice of trivialization yields the same subsheaf F.) Then F is a vector bundle of $\deg F = \deg E + 1$ and rank r. By definition of F, we have that μ factors through an embedding $\mu' : I \to F^* \otimes \omega$, but λ does not. Thus $h^1(X, I \otimes F) < h^1(X, I \otimes E)$. Since $\deg F = \deg E + 1$, then $H^0(X, I \otimes F) = 0$. It is clear that the cokernel of μ' is torsion-free. The induction proof is complete. □

Corollary 2 *The functor* $\overline{\mathbf{J}}_d$ *is representable by a scheme.*

Proof. First note that properties (i) and (ii) in the statement of Proposition 1 are open on I. More precisely, given a vector bundle E on X of rank r and degree $-rd - 1$, the subfunctor $\mathbf{U}_E \subseteq \overline{\mathbf{J}}_d$, parametrizing sheaves I satisfying properties (i) and (ii) in the statement of Proposition 1, is open. By Proposition 1, the subfunctors $\mathbf{U}_E$ cover $\overline{\mathbf{J}}_d$. Thus to show that $\overline{\mathbf{J}}_d$ is representable we need only show that each $\mathbf{U}_E$ is representable.

Fix a vector bundle E on X of rank r and degree $-rd - 1$. Let

$$V \subseteq \mathrm{Quot}_X(E^* \otimes \omega)$$

be the open subscheme parametrizing those quotients $q : E^* \otimes \omega \to G$ such that both G and $\ker(q)$ are torsion-free, $\ker(q)$ has rank 1,

$$h^0(X, \ker(q) \otimes E) = 0 \quad \text{and} \quad h^1(X, \ker(q) \otimes E) = 1 . \tag{2.1}$$

There is a morphism of functors $V \to \mathbf{U}_E$ sending a quotient $[q] \in V$ to its kernel, $[\ker(q)] \in \mathbf{U}_E$. It follows from (2.1) that the latter morphism is an isomorphism. The proof is complete. □

We will say that a vector bundle E on X of rank r and degree $-rd - 1$ satisfying properties (i) and (ii) in the statement of Proposition 1 *represents* I. We remark that the property of representing I is open.

3 The Theta divisor

Assume from now on that $g \geq 0$. Let $\mathscr{I}$ be a universal relatively torsion-free, rank 1 sheaf on $X \times \overline{J}_0$ over $\overline{J}_0$. Denote by $p : X \times \overline{J}_0 \to \overline{J}_0$ the projection map. Define

$$\mathscr{L} := (\det Rp_*(\mathscr{I}))^{-1} ,$$

where $\det Rp_*$ denotes the determinant of cohomology associated with the projection p. (For a brief description of Rp_* see [6] or [19]; for a more in-depth development of the theory of determinants, see [12].) Since the sheaf $\mathscr{I}$ has relative Euler characteristic 0 over $\overline{J}_0$, then $\mathscr{L}$ is independent on the choice of a universal sheaf $\mathscr{I}$, and there is a canonical global section θ of $\mathscr{L}$ whose zero scheme Θ parametrizes torsion-free, rank 1 sheaves I of Euler characteristic 0 on X such that

$$h^0(X, I) = h^1(X, I) \neq 0 .$$

Equivalently, by Serre's duality, Θ consists of the torsion-free, rank 1 sheaves of Euler characteristic 0 that can be embedded into the dualizing sheaf ω. In other words, Θ is (set-theoretically) the image of the $(g-1)$-th component of the Abel-Jacobi map:

$$\mathscr{A}^{g-1} : \mathrm{Quot}_X^{g-1}(\omega) \to \overline{J}_0 ,$$

where $\mathscr{A}^{g-1}$ sends a quotient $[q] \in \mathrm{Quot}_X^{g-1}(\omega)$ to its kernel, $[\ker(q)] \in \overline{J}_0$ (cf. [4, p. 87].) We say that $\mathscr{L}$ is the *Theta line bundle*, and Θ is the *Theta divisor* (even though it is not known whether Θ is actually a Cartier divisor in general).

If X is smooth, then

$$\mathrm{Quot}_X^{g-1}(\omega) \cong \mathrm{Hilb}_X^{g-1} = \mathrm{Symm}^{g-1}(X) ,$$

where $\mathrm{Hilb}_X^{g-1} := \mathrm{Quot}_X^{g-1}(\mathscr{O}_X)$ is the Hilbert scheme, parametrizing $(g-1)$-uples of points in X, and $\mathrm{Symm}^{g-1}(X)$ is the symmetric product of $(g-1)$ copies of X. Hence Θ corresponds to the classical Theta divisor (cf. Sect. 1).

Assume that X is locally planar, that is, that the embedding dimension of each point of X is at most 2. Equivalently, assume that X can be embedded into a quasi-projective smooth surface [2]. Then $\mathrm{Quot}_X^{g-1}(\omega)$ and $\overline{J}_0$ are integral, local complete intersections of dimensions $g-1$ and g, respectively. (Since locally planar curves are Gorenstein, then $\mathrm{Quot}_X^{g-1}(\omega) \cong \mathrm{Hilb}_X^{g-1}$, and thus our statement follows from [1, Cor. 7 and Thm. 9].) In this case, Θ is an irreducible, local complete intersection, effective Cartier divisor on $\overline{J}_0$. Moreover, it is clear that $\mathscr{A}^{g-1}$ is an isomorphism over the open subscheme of Θ parametrizing torsion-free, rank 1 sheaves I with $h^1(X, I) = 1$. From [4, Prop. 3.5.ii, p. 76], this open subscheme is dense. Since Θ is Cohen-Macaulay and irreducible, and $\mathrm{Quot}_X^{g-1}(\omega)$ is integral, then Θ is also integral. We observe that the assumption that X is locally planar is essential in the above argument. If X is not locally planar, then $\overline{J}_0$ is not irreducible (cf. [11] or [17, Thm. A]), and may have dimension greater than g (cf. [1, Ex. 13, p. 10]).

We observe that the above notions and arguments can be extended to families of integral, complete curves without difficulty [6]. Moreover, the formation of θ and $\mathscr{L}$ commutes with base change, since so does the determinant of cohomology. From this observation it follows that Poincaré's formula holds for locally planar curves. Namely, we claim that, if X is locally planar, then the self-intersection Θ^g is equal to $g!$. In fact, the claim is known for smooth curves [13, Sect. 2]. Since every locally planar curve is part of a family whose general member is a smooth curve, then we may apply the principle of conservation of intersection number to prove our claim.

As we have already remarked, it is not known whether Θ is always a Cartier divisor. Nevertheless, Soucaris showed that the zero scheme of the restriction of the canonical section θ to the maximum reduced subscheme of $\overline{J}_0$ is a Cartier divisor [19, Thm. 8, p. 236].

4 Very ampleness

Recall the notations of Sect. 3. If E is a vector bundle on X with $\deg E = 0$, then $\mathscr{I} \otimes E$ has relative Euler characteristic 0 over $\overline{J}_0$. Therefore, the invertible sheaf

$$\mathscr{L}_E = (\det Rp_*(\mathscr{I} \otimes E))^{-1}$$

on $\overline{J}_0$ has a canonical global section θ_E, whose zero scheme Θ_E parametrizes torsion-free, rank 1 sheaves I of Euler characteristic 0 on X such that

$$h^0(X, I \otimes E) = h^1(X, I \otimes E) \neq 0 \ .$$

As before, $\mathscr{L}_E$ and θ_E are independent on the choice of a universal sheaf $\mathscr{I}$.

Lemma 3 *Let E and F be vector bundles on X of same rank and degree 0. If* $\det E \cong \det F$, *then* $\mathscr{L}_E \cong \mathscr{L}_F$.

Proof. By Seshadri in [18, Lemma 2.5, p. 165]. □

By Lemma 3, if E is a vector bundle on X of rank n and $\det E \cong \mathscr{O}_X$, then $\mathscr{L}_E \cong \mathscr{L}^{\otimes n}$. Thus we may consider θ_E as a global section of $\mathscr{L}^{\otimes n}$ under the latter isomorphism. In this case we say that θ_E is a *theta section of degree n*. We now have a convenient way to produce sections of powers of $\mathscr{L}$.

For every integer d, let J^d be the Jacobian of X, parametrizing invertible sheaves of degree d on X. Recall that J^d is connected, quasi-projective and smooth.

Lemma 4 *Let $n \geq 2$. For each $i = 1, \ldots, n$, let d_i be an integer and $U_i \subseteq J^{d_i}$ be a non-empty, open subset. Let L be an invertible sheaf of degree $d_1 + \ldots + d_n$. Then there are points $[L_i] \in U_i$ for every $i = 1, \ldots, n$ such that*

$$L \cong \bigotimes_{i=1}^{n} L_i \ .$$

Proof. Consider the morphism $\phi : U_1 \times \ldots \times U_{n-1} \to J^{d_n}$, given by

$$([M_1], \ldots, [M_{n-1}]) \mapsto [L \otimes M_1^{-1} \otimes \ldots \otimes M_{n-1}^{-1}] .$$

It is clear that the image V of ϕ is open in J^{d_n}. Since J^{d_n} is irreducible, then $V \cap U_n \neq \emptyset$. Thus there is a point $[L_i] \in U_i$ for each $i = 1, \ldots, n$ such that

$$L_n \cong L \otimes L_1^{-1} \otimes \ldots L_{n-1}^{-1} .$$

The proof is complete. □

Theorem 5 *The sheaf $\mathscr{L}^{\otimes n}$ is generated by global sections if $n \geq 2$.*

Proof. Fix $n \geq 2$. Let I be a torsion-free, rank 1 sheaf on X of Euler characteristic 0. We will show that there is a vector bundle E on X of rank n and $\det E \cong \mathscr{O}_X$ such that

(5.1) $$h^0(X, I \otimes E) = h^1(X, I \otimes E) = 0 .$$

In this case, the section θ_E generates $\mathscr{L}^{\otimes n}$ at $[I]$, thereby proving the theorem.

By the proof of [19, Prop. 7, p. 235], there is an invertible sheaf L on X of degree 0 such that

$$h^0(X, I \otimes L) = h^1(X, I \otimes L) = 0 .$$

By semicontinuity, there is an open, dense subset $U \subseteq J^0$, containing $[L]$, such that if $[M] \in U$, then

$$h^0(X, I \otimes M) = h^1(X, I \otimes M) = 0 .$$

From Lemma 4, with $U_i := U$ for every $i = 1, \ldots, n$, there are invertible sheaves $M_1, \ldots M_n$ of degree 0 on X such that

$$h^0(X, I \otimes M_i) = h^1(X, I \otimes M_i) = 0$$

for every $i = 1, \ldots, n$, and

$$M_1 \otimes \ldots \otimes M_n \cong \mathscr{O}_X .$$

If we now let $E := M_1 \oplus \ldots \oplus M_n$, then E satisfies (5.1) and $\det E \cong \mathscr{O}_X$. The proof is complete. □

Soucaris had used [19, Prop. 7, p. 235] to show that the pullback of $\mathscr{L}^{\otimes 2}$ to the normalization of $\overline{J}_0$ is generated by global sections [19, Prop. 9, p. 236].

If S is a k-scheme and $\mathscr{F}$ is a vector bundle on $X \times S$ of relative degree d over S, then we denote by $\pi_{\mathscr{F}} : S \to J^d$ the determinant morphism, mapping $s \in S$ to $[\det \mathscr{F}(s)] \in J^d$.

Lemma 6 *Let $F_1, \ldots, F_n$ be vector bundles on X of same rank r and same degree d. Then there are a connected, smooth k-scheme S and a vector bundle $\mathscr{F}$ on $X \times S$ such that $\pi_{\mathscr{F}}$ is smooth, and $F_i \cong \mathscr{F}(s_i)$ for some $s_i \in S$, for each $i = 1, \ldots, n$.*

Proof. Let $m \gg 0$ be such that $F_i(m)$ is generated by global sections for every $i = 1, \dots, n$. Since k is infinite, then there is an exact sequence of the form

$$0 \to \mathscr{O}_X(-m)^{\oplus r-1} \to F_i \to (\det F_i)((r-1)m) \to 0 \tag{6.1}$$

for each $i = 1, \dots, n$. Let $\mathscr{P}$ be a universal sheaf on $X \times J^d$. Let $p : X \times J^d \to J^d$ denote the projection map, and let $\mathscr{V} := R^1 p_*(\mathscr{P}^{-1}(-rm))^{\oplus r-1}$. Choose $m \gg 0$ such that $\mathscr{V}$ is locally free, and let $T := \mathbf{P}_{J^d}(\mathscr{V}^*)$. Since $\mathscr{V}$ is locally free, then T is smooth over J^d. Since J^d is connected, smooth and quasi-projective, then so is T. The scheme T parametrizes $\mathscr{O}_X$-module extensions of $L((r-1)m)$ by $\mathscr{O}_X(-m)^{\oplus r-1}$ for invertible sheaves L on X of degree d. Thus there is $s_i \in T$ corresponding to (6.1) for each $i = 1, \dots, n$. Since T is quasi-projective, then there is an affine open subscheme $S \subseteq T$ containing $s_1, \dots, s_n$. Since S is affine, then

$$\mathscr{V}(S) = \mathrm{Ext}^1_{X \times S}(\mathscr{P}|_{X \times S}((r-1)m), \mathscr{O}_{X \times S}(-m)^{\oplus r-1}).$$

Let $q : \mathscr{V}_T^* \to \mathscr{Q}$ be the universal quotient on T over J^d. Then q induces an extension of the form

$$0 \to \mathscr{O}_{X \times S}(-m)^{\oplus r-1} \otimes \mathscr{Q} \to \mathscr{F} \to \mathscr{P}|_{X \times S}((r-1)m) \to 0$$

on $X \times S$ that specializes to (6.1) over s_i, for each $i = 1, \dots, n$. By construction, $\pi_{\mathscr{F}}$ is equal to the restriction to S of the structure morphism $T \to J^d$. Thus $\pi_{\mathscr{F}}$ is smooth. The proof is complete. $\square$

Let $e_X := \max_I e(I)$, where the maximum runs over all torsion-free, rank 1 sheaves on X. If S is a k-scheme, we say that an S-flat coherent sheaf $\mathscr{C}$ on $X \times S$ is *relatively torsion-free* if $\mathscr{C}(s)$ is torsion-free for every $s \in S$.

Theorem 7 *The sheaf $\mathscr{L}^{\otimes n}$ is very ample for every $n \geq \max(e_X, 2) + 1$.*

Proof. Fix $n \geq \max(e_X, 2) + 1$. By Theorem 5, the sheaf $\mathscr{L}^{\otimes n}$ is generated by global sections. We need only show that $\mathscr{L}^{\otimes n}$ separates points and tangent vectors on $\overline{J}_0$. The former is Step 1, while the latter is Step 2 below.

Step 1. Let I_1 and I_2 be non-isomorphic torsion-free, rank 1 sheaves on X of Euler characteristic 0. Then there is a vector bundle E on X of rank n and $\det E \cong \mathscr{O}_X$ such that $\theta_E([I_1]) \neq 0$, but $\theta_E([I_2]) = 0$.

Proof of Step 1. By Proposition 1, since $n \geq \max(e_X, 2) + 1$, there is a vector bundle F_i on X of rank $n-1$ and degree -1 representing I_i for each $i = 1, 2$. From Lemma 6, since the property of representing a torsion-free, rank 1 sheaf is open, we may assume that there are a non-empty, connected, smooth k-scheme S, and a vector bundle $\mathscr{F}$ on $X \times S$ of rank $n-1$ and relative degree -1 over S such that the determinant morphism $\pi_{\mathscr{F}}$ is smooth, and $\mathscr{F}(s)$ represents both I_1 and I_2 for every $s \in S$.

By replacing S with an open, dense subscheme if necessary, we may assume that for each $i = 1, 2$ there is an exact sequence

$$0 \to I_i \otimes \mathscr{O}_S \xrightarrow{\lambda_i} \mathscr{F}^* \otimes \omega \xrightarrow{q_i} \mathscr{C}_i \to 0$$

on $X \times S$, where $\mathscr{C}_i$ is a relatively torsion-free sheaf over S. If the composition $\rho := q_2 \circ \lambda_1$ were zero over a certain $s \in S$, then $\lambda_1(s)$ would factor through I_2, and since $\chi(I_1) = \chi(I_2)$ we would have that $I_1 \cong I_2$. Thus $\rho : I_1 \otimes \mathscr{O}_S \to \mathscr{C}_2$ is an embedding with S-flat cokernel. Since $\mathscr{C}_2$ is relatively torsion-free, by replacing S with an open, dense subscheme if necessary, we may assume that there is a regular point $x \in X$ such that $\rho(x) : I_1(x) \otimes \mathscr{O}_S \to \mathscr{C}_2(x)$ is an embedding with free cokernel. Let $\sigma : \mathscr{C}_2(x) \to I_1(x) \otimes \mathscr{O}_S$ be a splitting for $\rho(x)$. Let

$$\mathscr{G} := (\ker(\mathscr{F}^* \to \mathscr{F}^*(x) \xrightarrow{q_2(x)} \mathscr{C}_2(x) \xrightarrow{\sigma} I_1(x) \otimes \mathscr{O}_S))^* \ .$$

(As in the proof of Proposition 1, we implicitly chose a trivialization of ω at x.) Then $\mathscr{G}$ is a vector bundle on $X \times S$ of rank $n-1$ and relative degree 0 over S. Moreover, $\det \mathscr{G}(s) \cong \det \mathscr{F}(s) \otimes \mathscr{O}_X(x)$ for every $s \in S$. Thus the determinant morphism $\pi_{\mathscr{G}}$ is also smooth. In addition, λ_2 factors through $\mathscr{G}^* \otimes \omega$, but $\lambda_1(s)$ does not factor through $\mathscr{G}^*(s) \otimes \omega(s)$ for any $s \in S$. Thus

$$h^0(X, I_1 \otimes \mathscr{G}(s)) = 0 \ , \quad \text{but } h^0(X, I_2 \otimes \mathscr{G}(s)) \neq 0$$

for every $s \in S$.

By the proof of [19, Prop. 7, p. 235] (see the proof of Theorem 5), there is an open dense subset $U \subseteq J^0$ such that

$$h^0(X, I_1 \otimes L) = h^0(X, I_2 \otimes L) = 0$$

for every $[L] \in U$. By Lemma 4 applied to $U_1 := \pi_{\mathscr{G}}(S)$ and $U_2 := U$, there are $s \in S$ and $[L] \in U$ such that

$$(\det \mathscr{G}(s)) \otimes L \cong \mathscr{O}_X \ .$$

It is clear that $E := \mathscr{G}(s) \oplus L$ satisfies $\det E \cong \mathscr{O}_X$ and

$$h^0(X, I_1 \otimes E) = 0, \quad \text{but } h^0(X, I_2 \otimes E) \neq 0 \ .$$

Thus $\theta_E([I_1]) \neq 0$, but $\theta_E([I_2]) = 0$. The proof of Step 1 is complete. □

Step 2. Let I be a torsion-free, rank 1 sheaf on X with $\chi(I) = 0$. Let $v \in T_{\bar{J}_0,[I]}$ be a non-zero tangent vector on $\bar{J}_0$ at $[I]$. Then there is a vector bundle E on X of rank n and $\det E \cong \mathscr{O}_X$ such that Θ_E contains $[I]$ but not v.

Proof of Step 2. As in Step 1, we may assume that there are a non-empty, connected, smooth k-scheme S, and a vector bundle $\mathscr{F}$ on $X \times S$ of rank $n-1$ and relative degree -1 over S, such that the determinant morphism $\pi_{\mathscr{F}}$ is smooth, and $\mathscr{F}(s)$ represents I for every $s \in S$.

By replacing S with an open, dense subscheme if necessary, we may assume that there is an exact sequence

$$0 \to I \otimes \mathscr{O}_S \xrightarrow{\lambda} \mathscr{F}^*\omega \xrightarrow{q} \mathscr{C} \to 0$$

on $X \times S$, where $\mathscr{C}$ is a relatively torsion-free sheaf over S. By the proof of Corollary 2, we have natural identifications

$$T_{\bar{J}_0,[I]} = T_{Q(s),[q(s)]} = Hom_X(I, \mathscr{C}(s)) \tag{7.1}$$

for every $s \in S$, where $Q(s) := \mathrm{Quot}_X(\mathscr{F}^*(s) \otimes \omega)$. So there is a homomorphism $\nu : I \otimes \mathscr{O}_S \to \mathscr{C}$ such that $\nu(s) = v$ under the identification (7.1) for every $s \in S$. Since $v \neq 0$, then ν is an embedding with S-flat cokernel. Since $\mathscr{C}$ is relatively torsion-free, by replacing S with an open, dense subscheme if necessary, there is a regular point $x \in X$ such that $\nu(x)$ is an embedding with free cokernel. Let $\sigma : \mathscr{C}(x) \to I(x) \otimes \mathscr{O}_S$ be a splitting for $\nu(x)$. Let

$$\mathscr{G} := (\ker(\mathscr{F}^* \to \mathscr{F}^*(x) \stackrel{q(x)}{\to} \mathscr{C}(x) \stackrel{\sigma}{\to} I(x) \otimes \mathscr{O}_S))^* .$$

(As in the proof of Step 1, we implicitly chose a trivialization of ω at x.) Then $\mathscr{G}$ is a vector bundle on $X \times S$ of rank $n-1$ and relative degree 0 over S. Moreover, $\det \mathscr{G}(s) \cong \det \mathscr{F}(s) \otimes \mathscr{O}_X(x)$ for every $s \in S$. Thus the determinant morphism $\pi_{\mathscr{G}}$ is smooth. In addition, λ factors through $\mathscr{G}^* \otimes \omega$. Thus $[I] \in \Theta_{\mathscr{G}(s)}$ for every $s \in S$. On the other hand, since $\nu(x)$ is an embedding, then v does not belong to $\Theta_{\mathscr{G}(s)}$ for any $s \in S$.

The reader is invited to repeat the argument in the last paragraph of the proof of Step 1 to finish the proof of Step 2. The proof of Theorem 7 is complete. □

Remark 8 Let $x \in X$. Let $\overline{\mathscr{O}}_x$ denote the normalization of $\mathscr{O}_x$. Let δ_x denote the length of $\overline{\mathscr{O}}_x/\mathscr{O}_x$. If I is a torsion-free, rank 1 module over $\mathscr{O}_x$, then it is easy to show that I is isomorphic to a submodule of $\overline{\mathscr{O}}_x$ containing $\mathscr{O}_x$. Thus

$$\dim_k I(x) \leq \delta_x + 1 . \tag{8.1}$$

If the conductor, $\mathscr{C}_x := (\mathscr{O}_x : \overline{\mathscr{O}}_x)$, is the maximal ideal m_x of $\mathscr{O}_x$, then equality in (8.1) is achieved for $I = \overline{\mathscr{O}}_x$ only; otherwise the inequality (8.1) is always strict. Let

$$\delta_X := \max_{x \in X} \delta_x .$$

Since X is generically non-singular, then $\delta_X < \infty$. It follows from (8.1) that $e_X \leq \delta_X + 1$.

Theorem 7 states that $\mathscr{L}^{\otimes 3}$ is very ample if $e_X \leq 2$. This is the case for X non-singular, or with at most ordinary nodes or cusps as singularities, as $\delta_X \leq 1$. It is clear that if $e_X \leq 2$ then X is locally planar. If $\delta_X = 2$, then $e_X \leq 2$ if and only if X is locally planar. If $\delta_X = 3$, then $e_X \leq 2$ if and only if X is locally planar and $m_x^2 \neq \mathscr{C}_x$ for every $x \in X$. Note that the planar curve $X \subseteq \mathbf{P}_k^2$, given as the zero scheme of $u^3 w - v^4$, has $e_X = 3$.

Question 9 It follows from the proof of the theorem in [14, Sect. 17, p. 163] that, if X is smooth, then 3Θ is very ample, and the sections θ_E associated to completely decomposable vector bundles E (that is: vector bundles E of the form $L_1 \oplus L_2 \oplus L_3$, where L_i is an invertible sheaf of degree 0 for $i = 1, 2, 3$, and $L_1 \otimes L_2 \otimes L_3 \cong \mathscr{O}_X$), are enough to embed J_0 into a projective space. We might ask: for which integral curves X are such sections enough to embed $\bar{J}_0$ into a projective space? The proof of Theorem 7 shows that the sections θ_E associated to vector bundles E of the form $F \oplus L$, where F is a vector bundle of rank $\max(e_X, 2)$ and degree 0, the sheaf L is invertible of degree 0 and $(\det F) \otimes L \cong \mathscr{O}_X$, are enough to embed $\overline{J}_0$ into a projective space.

Example 10 Let X be a complete, integral curve of arithmetic genus $g = 1$. As a subset, Θ is the locus of torsion-free, rank 1 sheaves I with Euler characteristic 0 such that $h^0(X, I) > 0$. Since $\chi(\mathscr{O}_X) = 0$, then any non-zero section $\mathscr{O}_X \to I$ must be an isomorphism. Since Θ is integral by Sect. 3, then $\Theta = [\mathscr{O}_X]$, as Cartier divisors of $\overline{J}_0$.

By [3, Ex. 8.9.iii, p. 109], the first component of the Abel-Jacobi map,

$$\begin{aligned} \mathscr{A}^1 : \quad & X \to \overline{J}_{-1} \\ & x \mapsto [m_x] , \end{aligned}$$

where m_x denotes the maximal ideal sheaf of x, is an isomomorphism. Fix a regular point $x \in X$. Then we have an isomorphism $\phi_x : \overline{J}_{-1} \to \overline{J}_0$, by sending $[I] \in \overline{J}_{-1}$ to $[I(x)] \in \overline{J}_0$. Under the composition $\psi := \phi_x \circ \mathscr{A}^1$, the Cartier divisor Θ corresponds to the Cartier divisor $[x]$ in X.

Let $n \geq 3$ be an integer. The complete linear system associated to $\mathscr{O}_X(nx)$ gives rise to an embedding $X \hookrightarrow \mathbf{P}^{n-1}$. If $H \subseteq \mathbf{P}^{n-1}$ is a hyperplane intersecting X at regular points $y_1, \ldots, y_n$, then $[y_1] + \ldots + [y_n]$ is a Cartier divisor on X whose associated invertible sheaf is $\mathscr{O}_X(nx)$. Under ψ, the divisor $[y_1] + \ldots + [y_n]$ corresponds to Θ_E, where

$$E = (\mathscr{O}_X(y_1) \oplus \ldots \oplus \mathscr{O}_X(y_n)) \otimes \mathscr{O}_X(-x) .$$

It follows now from Bertini's theorem that the theta sections of degree n associated to completely decomposable vector bundles generate $H^0(\overline{J}_0, \mathscr{L}^{\otimes n})$ for every $n \geq 0$. (In case $n \leq 2$ it is easy to check the latter statement directly.) Thus, for the case of curves of arithmetic genus 1, Question 9 is answered in the affirmative.

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