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Curved flats and isothermic surfaces

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1. Introduction

These notes grew out of a series of discussions on a recent paper by J. Cieśliński, P. Goldstein and A. Sym [6]: these authors give a characterization of isothermic surfaces as “soliton surfaces” by introducing a spectral parameter. In trying to understand the geometric meaning of this spectral parameter, we observed some analogies with the theory of conformally flat hypersurfaces in a four-dimensional space form [10]. Inspired by this analogies we found a relation between the concept of Darboux transformations of isothermic surfaces [8] and the recently developed concept of curved flats in symmetric spaces [9] — at this point, we would like to thank Prof. Alexander Bobenko for his interest in our work and for helpful discussions on the Darboux transform[1]. Considering the “limiting geometry” where the spectral parameter $\lambda \to 0$, we obtain pairings of isothermic surfaces by Christoffel transformations (duality) [5] (cf. also [2]). This was a key tool in the recently discovered theory of “discrete isothermic surfaces” [3].

2. Curved Flats

A curved flat is the natural generalization of a developable surface in Euclidean space: it is a submanifold $M \subset G/K$ of a (pseudo-Riemannian) symmetric space for which the curvature operator of G/K vanishes[2] on $\bigwedge^2 TM$. Thus, a curved flat may be thought of as the enveloping submanifold of a congruence of flats — totally geodesic submanifolds — of the symmetric space. Taking a regular

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[1] Also, we would like to thank the referee for encouraging us to include the new material we developed after finishing the first version of this paper.
[2] Thus M is *curvature isotropic* in the sense of [9].

parametrization $\gamma : M \to G/K$ of a curved flat and a framing $F : M \to G$ of this parametrization, the Maurer-Cartan form $\Phi = F^{-1}dF$ of the framing has a natural decomposition $\Phi = \Phi_{\mathfrak{k}} + \Phi_{\mathfrak{p}}$ according to the symmetric decomposition[3] $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of the Lie algebra $\mathfrak{g}$. Now the condition for γ to parametrize a curved flat may be formulated as[4]

$$[[\Phi_{\mathfrak{p}} \wedge \Phi_{\mathfrak{p}}], \mathfrak{p}] \equiv 0 \ . \tag{1}$$

In case that G is semisimple, it is straightforward[5] to see that this is equivalent to

$$[\Phi_{\mathfrak{p}} \wedge \Phi_{\mathfrak{p}}] \equiv 0 \ . \tag{2}$$

To summarise, we have the

Definition of a curved flat: An immersion $\gamma : M \to G/K$ is said to parametrize a *curved flat*, if the $\mathfrak{p}$-part in the symmetric decomposition of the Maurer-Cartan form $F^{-1}dF = \Phi = \Phi_{\mathfrak{k}} + \Phi_{\mathfrak{p}}$ of a framing $F : M \to G$ of γ defines a congruence $p \mapsto \Phi_{\mathfrak{p}}|_p(T_pM)$ of abelian subalgebras of $\mathfrak{g}$.

At this point we should remark that curved flats naturally arise in one parameter families [9]: setting

$$\Phi_\lambda := \Phi_{\mathfrak{k}} + \lambda\Phi_{\mathfrak{p}} \tag{3}$$

the Maurer-Cartan equation $d\Phi_\lambda + \frac{1}{2}[\Phi_\lambda \wedge \Phi_\lambda] = 0$ for the loop $\lambda \mapsto \Phi_\lambda$ of forms splits into the three equations

$$\begin{aligned} 0 &= d\Phi_{\mathfrak{k}} + \tfrac{1}{2}[\Phi_{\mathfrak{k}} \wedge \Phi_{\mathfrak{k}}] \\ 0 &= d\Phi_{\mathfrak{p}} + [\Phi_{\mathfrak{k}} \wedge \Phi_{\mathfrak{p}}] \\ 0 &= [\Phi_{\mathfrak{p}} \wedge \Phi_{\mathfrak{p}}] \ , \end{aligned} \tag{4}$$

and hence the integrability of the loop $\lambda \mapsto \Phi_\lambda$ is equivalent to the forms Φ_λ being the Maurer-Cartan forms for some framings $F_\lambda : M \to G$ of curved flats $\gamma_\lambda : M \to G/K$. Thus integrable systems theory may be applied to produce examples.

Now we will consider the case leading to the theory of isothermic surfaces: let

$$G := O_1(5) \quad \text{and} \quad K := O(3) \times O_1(2) \ . \tag{5}$$

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[3] Thus $\mathfrak{k}$ and $\mathfrak{p}$ are the +1 and −1-eigenspaces, respectively, of the involution fixing $\mathfrak{k}$ and so satisfy the characteristic conditions

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}.$$

[4] The product

$$[\Phi \wedge \Psi](v, w) := [\Phi(v), \Psi(w)] - [\Phi(w), \Psi(v)]$$

defines a symmetric product on the space of Lie algebra valued 1-forms with values in the space of Lie algebra valued 2-forms.

[5] In fact, $[\mathfrak{p}, \mathfrak{p}] \oplus \mathfrak{p}$ is an ideal of $\mathfrak{g}$ so that we have a decomposition $\mathfrak{g} = \mathfrak{k}' \oplus [\mathfrak{p}, \mathfrak{p}] \oplus \mathfrak{p}$ where $\mathfrak{k}'$ is a complementary ideal commuting with $[\mathfrak{p}, \mathfrak{p}] \oplus \mathfrak{p}$. Thus, if $\mathfrak{a} \subset \mathfrak{p}$ satisfies $[[\mathfrak{a}, \mathfrak{a}], \mathfrak{p}] = 0$ we deduce that $[\mathfrak{a}, \mathfrak{a}]$ lies in the center of $\mathfrak{g}$ and so vanishes.

The coset space $G_+(5,3) = G/K$ of space-like 3-planes in the Minkowski space $\mathbb{R}^5_1$ becomes a six dimensional pseudo-Riemannian symmetric space of signature $(3,3)$ when endowed with the metric induced by the Killing form. We will consider two-dimensional curved flats

$$\gamma : M^2 \to G_+(5,3) \tag{6}$$

satisfying the regularity assumption that the metric on M^2 induced by γ is non-degenerate.

Fixing a pseudo orthonormal basis $(e_1, \ldots, e_5)$ of the Minkowski space $\mathbb{R}^5_1$ with

$$(\langle e_i, e_j \rangle)_{ij} = E_5 := \begin{pmatrix} I_3 & 0 \\ 0 & \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \end{pmatrix}, \tag{7}$$

we get the matrix representations

$$\begin{aligned} O_1(5) &= \{A \in Gl(5,\mathbb{R}) | A^t E_5 A = E_5\} \\ \mathfrak{o}_1(5) &= \{X \in \mathfrak{gl}(5,\mathbb{R}) | (E_5 X) + (E_5 X)^t = 0\}\ . \end{aligned} \tag{8}$$

The subalgebra $\mathfrak{k}$ and its complementary linear subspace $\mathfrak{p}$ in the symmetric decomposition of $\mathfrak{o}_1(5)$ are given by the $+1$- resp. -1-eigenspaces of the involutive automorphism $\mathrm{Ad}(Q) : \mathfrak{o}_1(5) \to \mathfrak{o}_1(5)$ with $Q = \begin{pmatrix} -I_3 & 0 \\ 0 & I_2 \end{pmatrix}$. Writing down the Maurer-Cartan form of a framing $F : M^2 \to O_1(5)$ of our curved flat $\gamma : M^2 \to G_+(5,3)$ with this notation we obtain

$$\begin{aligned} F^{-1}dF &= \Phi = \Phi_{\mathfrak{k}} + \Phi_{\mathfrak{p}} \quad \text{with} \\ \Phi_{\mathfrak{k}} &= \begin{pmatrix} \Omega & 0 \\ 0 & \mathcal{V} \end{pmatrix} : \quad TM \to \mathfrak{o}(3) \times \mathfrak{o}_1(2) \\ \Phi_{\mathfrak{p}} &= \begin{pmatrix} 0 & \eta \\ -E_2\eta^t & 0 \end{pmatrix} : \quad TM \to \mathfrak{p}\ . \end{aligned} \tag{9}$$

The image of $\Phi_{\mathfrak{p}}$ at each $p \in M^2$ is a 2-dimensional abelian subspace of $\mathfrak{p}$ on which the Killing form is non-degenerate. One can show that there are precisely two K-orbits of maximal abelian subspaces of $\mathfrak{p}$: one consists of 3-dimensional subspaces which are isotropic for the Killing form while the other consists of 2-dimensional subspaces on which the Killing form has signature $(1,1)$. We therefore conclude that the images of each $\Phi_{\mathfrak{p}}$ are maximal abelian and K-conjugate and so we can put η into the standard form

$$\eta = \begin{pmatrix} \omega_1 & -\omega_1 \\ \omega_2 & \omega_2 \\ 0 & 0 \end{pmatrix} \tag{10}$$

by applying a gauge transformation $F \;\; FH, H : M \to K$.

Calculating the Maurer-Cartan equation using the ansatz

$$\Omega = \begin{pmatrix} 0 & \omega & -\psi_1 \\ -\omega & 0 & -\psi_2 \\ \psi_1 & \psi_2 & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{V} = \begin{pmatrix} \nu & 0 \\ 0 & -\nu \end{pmatrix} \tag{11}$$

together with η given by (10), we see that

(12) $$d\omega_1 = d\omega_2 = 0\ .$$

So we are given canonical coordinates $(x, y) : M \to I\!R^2$ by integrating[6] the forms ω_1 and ω_2. Moreover, since we also have $d\nu = 0$, we may set $\nu = -du$ for a suitable function $u \in C^\infty(M)$ — this gives us $\omega = u_y dx - u_x dy$, where u_x and u_y denote the partial derivatives of u in x- resp. y-directions. Finally, the equations $\psi_1 \wedge \omega_1 = 0$ and $\psi_2 \wedge \omega_2 = 0$ show that $\psi_1 = e^u k_1 dx$ and $\psi_2 = e^u k_2 dy$ for two functions $k_i \in C^\infty(M)$.

We now perform a final $O_1(2)$-gauge $\begin{pmatrix} I_3 & 0 \\ 0 & \begin{matrix} e^u & 0 \\ 0 & e^{-u} \end{matrix} \end{pmatrix} : M \to O(3) \times O_1(2)$ and insert the spectral parameter λ to obtain the Maurer-Cartan form discussed in [6]:

(13) $$\Phi_\lambda = \begin{pmatrix} 0 & u_y dx - u_x dy & -e^u k_1 dx & \lambda e^u dx & -\lambda e^{-u} dx \\ -u_y dx + u_x dy & 0 & -e^u k_2 dy & \lambda e^u dy & \lambda e^{-u} dy \\ e^u k_1 dx & e^u k_2 dy & 0 & 0 & 0 \\ \lambda e^{-u} dx & -\lambda e^{-u} dy & 0 & 0 & 0 \\ -\lambda e^u dx & -\lambda e^u dy & 0 & 0 & 0 \end{pmatrix}.$$

We are now lead directly to the theory of

3. Isothermic Surfaces

In the context of Möbius geometry the three sphere S^3 is viewed as the projective light-cone PL^4 in $I\!R^5_1$ while the Lorentzian sphere S^4_1 should be interpreted as the space of (oriented) spheres in the three sphere[7]. Now, denoting by

(14) $$\begin{aligned} n &:= Fe_3 : M \to S^4_1 = \{v \in I\!R^5_1 | \langle v, v\rangle = 1\} \\ f &:= Fe_4 : M \to L^4 = \{v \in I\!R^5_1 | \langle v, v\rangle = 0\} \\ \hat{f} &:= Fe_5 : M \to L^4 \end{aligned}$$

one of the sphere congruences resp. the two immersions given by our frame F, we see that

Theorem: *The sphere congruence n given by our curved flat is a Ribaucour sphere congruence[8], which is enveloped by two isothermic immersions f and $\hat{f}$.*

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[6] Since our theory is local, all closed forms may be assumed to be exact.

[7] Or, equivalently, it may be interpreted as the space of (oriented) spheres and planes in Euclidean three space $I\!R^3$: the polar hyperplane to a vector v of the Lorentz sphere intersects the three sphere — thought of as the absolute quadric in projective four space — in a two sphere. Stereographic projection yields a sphere in $I\!R^3$ or, if the projection center lies on the sphere, a plane.

[8] **Recall:** *a sphere congruence is called "Ribaucour sphere congruence" if the curvature lines on its two envelopes do correspond.*

It is easy to see (from the Ricci equations) that a sphere congruence is Ribaucour iff the bundle defined by span$\{n, f, \hat{f}\}$ over M is flat (cf.(13)). Hence the map $p \mapsto d_p f(T_p M)$ defines a "normal congruence of circles" (cyclic system) [7]: for each $p \in M$

$$t \mapsto f_t(p) := \frac{1}{\sqrt{2}} \sin t \cdot n(p) + \frac{1}{2}(1 + \cos t) \cdot f(p) - \frac{1}{2}(1 - \cos t) \cdot \hat{f}(p)$$

Moreover $\hat f$ is a Darboux transform[9] *of* f. *Conversely, the "extended Gauß map"*

$$(15) \qquad \gamma : M^2 \to G_+(5,3)\,, \quad p \mapsto \mathrm{span}\{n(p), d_p n(T_p M)\}$$

of the Ribaucour congruence n *enveloped by an isothermic surface and one of its Darboux transforms is a curved flat.*
Proof: Since

$$(16) \qquad \begin{array}{lll} \langle f, n\rangle = 0 & \text{and} & \langle df, n\rangle \equiv 0, \\ \langle \hat f, n\rangle = 0 & \text{and} & \langle d\hat f, n\rangle \equiv 0\,, \end{array}$$

the immersions f and $\hat f$ do envelop the sphere congruence n and, since the bilinear forms

$$(17) \quad \langle df, dn\rangle = \lambda e^{2u}(k_1 dx^2 + k_2 dy^2)\,, \qquad \langle d\hat f, dn\rangle = \lambda(-k_1 dx^2 + k_2 dy^2)$$

are diagonal with respect to the induced metrics

$$(18) \quad \langle df, df\rangle = \lambda^2 e^{2u}(dx^2 + dy^2)\,, \qquad \langle d\hat f, d\hat f\rangle = \lambda^2 e^{-2u}(dx^2 + dy^2)\,,$$

our coordinates (x, y) are curvature line coordinates for both immersions f and $\hat f$; and hence the sphere congruence n is Ribaucour. Moreover the induced metrics (18) are both conformally equivalent to $dx^2 + dy^2$ which shows that f as well as $\hat f$ are isothermic and that each of them is a Darboux transform of the other [8] — here we should remark that from (13) we see that the Ribaucour congruence n does *not* belong to a fixed linear sphere complex.

To see the converse we remark first that n is Ribaucour and its two envelopes admit common isothermal curvature line coordinates $(x, y) : M^2 \to \mathbb{R}^2$ — hence the two isotropic normal fields f and $\hat f$ of n (in S^4_1) describing its two envelopes may be scaled such that

$$(19) \qquad f_x = \pm e^{2u}\hat f_x\,, \qquad f_y = \mp e^{2u}\hat f_y\,.$$

Here we have to have opposite signs since the Ribaucour congruence n is supposed to not belong to a fixed sphere complex [8] Consequently we may choose a framing $F = (s_1, s_2, n, f, \hat f) : M^2 \to O_1(5)$ of γ such that its connection form has the form (13) — this completes our proof. q.e.d.

To obtain the Euclidean representation of the Darboux transform (given for example in [1] or [6]) we use the following

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parametrizes the circle $(d_p f(T_p M))^\perp$ meeting the sphere $n(p)$ in $f(p)$ and $\hat f(p)$ orthogonal. Since n, f and $\hat f$ are parallel sections in this bundle, the maps $p \mapsto f_t(p)$ (which generically are not degenerate) parametrize the surfaces orthogonal to this congruence of circles.

In contrast to the theory of conformally flat hypersurfaces where *all* orthogonal hypersurfaces of an appropriate cyclic system are conformally flat [10], in our case the immersions f and $\hat f = f_\pi$ will generally be the only isothermic surfaces among the surfaces f_t.

[9] **Recall:** *if the correspondance between the two envelopes of a Ribaucour sphere congruence — which does not belong to fixed linear complex — is conformal then one envelope is called a "Darboux transform" of the other.* In this situation both envelopes will be isothermic surfaces [8]. On the other hand, given an isothermic surface, there exist always infinitely many Darboux transforms of this surface (see Darboux's theorem, p.205).

Lemma: *Let $s : M^2 \to S^4_1$ denote a sphere congruence enveloped by two immersions $f, \hat{f} : M^2 \to L^4$; and let $s_1, s_2 : M^2 \to S^4_1$ be two congruences of spheres, for each $p \in M^2$ $s_1(p)$ and $s_2(p)$ intersecting orthogonally in the circle orthogonal to $s(p)$ in $f(p)$ and $\hat{f}(p)$. After choosing "the point at infinity" $v_\infty \in L^4$ we may identify $I\!R^3 \cong \{v \in L^4 \mid \langle v, v_\infty\rangle = 1\}$ via an isometry*[10]*. Then*

$$\hat{f} \quad = \quad f + \tfrac{d}{2}(h_1 t_1 + h_2 t_2 + h\, t) \tag{20}$$

where $t_i(p)$ are the unit normal vectors (in $I\!R^3$) to the spheres $s_i(p)$ (and $s(p)$) in the point $f(p)$, and

$$h_i \quad = \quad \langle s_i, v_\infty\rangle \quad \text{and} \quad d \quad = \quad -\tfrac{2}{\langle f, v_\infty\rangle\langle \hat{f}, v_\infty\rangle} \tag{21}$$

are the curvatures of the spheres s_i and $\sqrt{d}$ is the distance of f and $\hat{f}$ in Euclidean space[11].

Proof: Without lost of generality we may assume $c = \langle f, v_\infty\rangle \equiv 1$. For the connection form

$$(s_1, s_2, s, f, \hat{f})^{-1} d(s_1, s_2, s, f, \hat{f}) = \begin{pmatrix} 0 & -\omega & -\psi_1 & \omega_1 & \chi_1 \\ \omega & 0 & -\psi_2 & \omega_2 & \chi_2 \\ \psi_1 & \psi_2 & 0 & 0 & 0 \\ -\chi_1 & -\chi_2 & 0 & \nu & 0 \\ -\omega_1 & -\omega_2 & 0 & 0 & -\nu \end{pmatrix} \tag{22}$$

of $(s_1, s_2, s, f, \hat{f}) : M^2 \to O_1(5)$ this yields $\nu = -h_1\omega_1 - h_2\omega_2$. Also writing down the connection form

$$(t_1, t_2, t)^{-1} d(t_1, t_2, t) = \begin{pmatrix} 0 & -\omega - h_2\omega_1 + h_1\omega_2 & -(\psi_1 + h\omega_1) \\ \omega + h_2\omega_1 - h_1\omega_2 & 0 & -(\psi_2 + h\omega_2) \\ \psi_1 + h\omega_1 & \psi_2 + h\omega_2 & 0 \end{pmatrix} \tag{23}$$

of the (Euclidean) framing $(t_1, t_2, t) : M^2 \to O(3)$ of f it is a straightforward calculation to see that the (Euclidean) normal field of $\hat{f}$ is given by $\hat{t} = t - \frac{d}{2}h(h_1 t_1 + h_2 t_2 + h\, t)$. Hence we have $\hat{f} + \frac{1}{h}\hat{t} = f + \frac{1}{h}t$ showing that the right hand side of (20) actually gives the second envelope of the sphere congruence s. q.e.d.

In case of a Darboux transform of an isothermic surface the coefficients in (20) satisfy — after a (constant) gauge transformation $(f, \hat{f})$ $(\frac{1}{\lambda}f, \lambda\hat{f})$ — the linear system

$$d(\tfrac{h_1}{c}, \tfrac{h_2}{c}, \tfrac{h}{c}, -\tfrac{1}{c}, \tfrac{2}{c\,d}) \quad = \quad (\tfrac{h_1}{c}, \tfrac{h_2}{c}, \tfrac{h}{c}, -\tfrac{1}{c}, \tfrac{2}{c\,d})\tilde{\Phi} \;, \qquad c \quad = \quad \langle f, v_\infty\rangle \;,$$

$$\tilde{\Phi} \quad = \quad \begin{pmatrix} 0 & \tilde{u}_y dx - \tilde{u}_x dy & -e^{\tilde{u}}\tilde{k}_1 dx & e^{\tilde{u}} dx & \lambda^2 e^{-\tilde{u}} dx \\ -\tilde{u}_y dx + \tilde{u}_x dy & 0 & -e^{\tilde{u}}\tilde{k}_2 dy & e^{\tilde{u}} dy & -\lambda^2 e^{-\tilde{u}} dy \\ e^{\tilde{u}}\tilde{k}_1 dx & e^{\tilde{u}}\tilde{k}_2 dy & 0 & 0 & 0 \\ -\lambda^2 e^{-\tilde{u}} dx & \lambda^2 e^{-\tilde{u}} dy & 0 & 0 & 0 \\ -e^{\tilde{u}} dx & -e^{\tilde{u}} dy & 0 & 0 & 0 \end{pmatrix} \tag{24}$$

[10] Which actually is unique up to motions of $I\!R^3$.

[11] Remark that in (20) f and $\hat{f}$ are considered as vectors in Euclidean three space $I\!R^3$ while in (21) they are considered as vectors in Minkowski space $I\!R^5_1$.

$e^{\tilde{u}} = \frac{1}{c}e^u$ and $\tilde{k}_i = c\,(k_i + \frac{h}{c})$ denoting the metric factor and the principal curvatures of f considered as an immersion into Euclidean space. Actually, this linear system is equivalent to Darboux's original system [8] (cf. [6] or [1]) — remark that we also have

$$h_1^2 + h_2^2 + h^2 + 2\,\frac{2}{d} \quad = \quad 0\ . \tag{25}$$

It is a straightforward calculation to see that

Theorem (Darboux). *Any solution of the system (24) satisfying (25) gives rise to a Darboux transform* $\hat{f} \;=\; f + \frac{d}{2}(h_1 t_1 + h_2 t_2 + h\,t)$ *of the isothermic surface* f.

Now, considering the *loop of curved flats* $\gamma_\lambda : M^2 \to G_+(5,3)$ with Maurer Cartan forms (13) we obtain an interesting "limit geometry" as $\lambda \to 0$: applying a (constant) conformal change (constant $O_1(2)$-gauge)

$$\begin{array}{llll} f \quad \frac{1}{\lambda}f & \text{and} & \hat{f} \quad \lambda\hat{f} & \text{or} \\ f \quad \lambda f & \text{and} & \hat{f} \quad \frac{1}{\lambda}\hat{f} & \end{array} \tag{26}$$

and sending $\lambda \to 0$, $\hat{f}$ resp. f becomes a constant vector — $\Phi_{\lambda=0}e_5$ resp. $\Phi_{\lambda=0}e_4$ vanishes. Obviously this constant light-like vector should be interpreted as the point at infinity and we therefore obtain an isothermic immersion $f : M \to \mathbb{R}^3$ with first and second fundamental forms

$$I = e^{2u}(dx^2 + dy^2)\,, \quad II = e^{2u}(k_1 dx^2 + k_2 dy^2) \tag{27}$$

resp. its Christoffel transform[12] $\hat{f} : M \to \mathbb{R}^3$ with first and second fundamental forms

$$\hat{I} = e^{-2u}(dx^2 + dy^2)\,, \quad \hat{II} = -k_1 dx^2 + k_2 dy^2\ . \tag{28}$$

We now recognise the remaining three equations from the Maurer-Cartan equation for Φ_λ

$$\begin{aligned} 0 &= \Delta u + e^{2u}k_1k_2 \\ 0 &= k_{1y} + (k_1 - k_2)u_y \\ 0 &= k_{2x} - (k_1 - k_2)u_x \end{aligned} \tag{29}$$

as the Gauß and Codazzi equations of the Euclidean immersion f resp. its Christoffel transform $\hat{f}$.

In this sense the Christoffel transform of an isothermic surface may be considered as a special kind of Darboux transformation.

Another way to obtain the Christoffel transform as a limiting case of the Darboux transform in the curved flat context is presented in [6]: applying Sym's formula to the associated family of frames $F = F(\lambda)$, we obtain a map

$$(\tfrac{\partial}{\partial\lambda}F)F^{-1}|_{\lambda=0} : M \to \mathfrak{p}\ ; \tag{30}$$

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[12] **Recall:** *the Christoffel transform (dual) of an isothermic surface is obtained by integrating the closed 1-form* $d\hat{f} := e^{-2u}(-f_x dx + f_y dy)$ — see [5] or [2].

When the normal congruence of circles mentioned in footnote 8 is projected to Euclidean three space $\mathbb{R}^3$, we see that, in the limit $\lambda \to 0$, the circles become straight lines — circles meeting the collapsed surface $\hat{f}$ resp. f in the point at infinity — while the Ribaucour sphere congruence enveloped by the two surfaces f and $\hat{f}$ becomes the congruence of tangent planes of f resp. $\hat{f}$.

interpreting $\mathfrak{p}$ as two copies of Euclidean three space[13] $\mathbb{R}^3$ this map gives us the immersion f, and in the other copy of $\mathbb{R}^3$, its dual $\hat{f}$: this can be seen by looking at the differential

$$d(\tfrac{\partial}{\partial\lambda}F)F^{-1}|_{\lambda=0} = F_0 \Phi_{\mathfrak{p}} F_0^{-1} \cong H_3 \begin{pmatrix} e^u dx & -e^{-u}dx \\ e^u dy & e^{-u}dy \\ 0 & 0 \end{pmatrix} . \tag{31}$$

Here $F_0 = \begin{pmatrix} H_3 & 0 \\ 0 & I_2 \end{pmatrix}$ solves the equation $F_0^{-1}dF_0 = \Phi_{\mathfrak{k}}$ and thus $H_3 : M \to O(3)$ may be viewed as a Euclidean framing of f resp. $\hat{f}$.

Summarizing these results we may formulate the following

Theorem. *In the limit $\lambda \to 0$ a loop of curved flats $\gamma_\lambda : M^2 \to G_+(5,3)$ with connection forms (13) gives us an isothermic surface $f : M^2 \to \mathbb{R}^3$ and its Christoffel transform (dual) $\hat{f} : M^2 \to \mathbb{R}^3$. Conversely, given an isothermic surface $f : M^2 \to \mathbb{R}^3$ (and its Christoffel transform[14]) we get a loop of curved flats $\gamma_\lambda : M^2 \to G_+(5,3)$ by integrating the loop of Maurer Cartan forms (13), which we are able to write down knowing the first and second fundamental forms of the immersions f and $\hat{f}$.*

There is another possibility for producing isothermal surfaces in Euclidean space $\mathbb{R}^3$ (or S^3): that is, by using a solution of

4. Calapso's equation

To understand this, we write down the Maurer-Cartan form of a frame $F : M \to O_1(5)$, which is Möbius-invariantly connected to a given immersion: taking $f = Fe_4$ an isometric lift of the isothermic immersion (which is unique up to a constant scaling) and $n = Fe_3$ the central sphere congruence (conformal Gauß map) of the immersion, the frame is determined by the assumption of being an adapted frame (i.e. $Fe_1 = f_x$ and $Fe_2 = f_y$ — $(x,y) : M^2 \to \mathbb{R}^2$ denoting the *isometric* principal curvature line coordinates for our lifting f). The associated Maurer-Cartan form will be

$$\Phi = \begin{pmatrix} 0 & 0 & kdx & dx & \chi_1 \\ 0 & 0 & -kdy & dy & \chi_2 \\ -kdx & kdy & 0 & 0 & \tau \\ -\chi_1 & -\chi_2 & -\tau & 0 & 0 \\ -dx & -dy & 0 & 0 & 0 \end{pmatrix} , \tag{32}$$

k^2 being the conformal factor relating the metric induced by the central sphere congruence to the isometric one, and the 1-forms χ_1, χ_2 and τ to be determined. From the Maurer-Cartan equation for this form we learn that

[13] Here the Euclidean metric is induced by the quadratic form $\frac{1}{2}\mathrm{tr}\Phi_{\mathfrak{p}}^t\Phi_{\mathfrak{p}}$ instead of the Killing form.

[14] We may think of this Christoffel transform as determining the Euclidean ambient space. Actually, this construction depends heavily on the Euclidean structure of the ambient space — consequently, we generally get a whole three parameter family of loops of curved flats from one isothermic surface: when viewing our given isothermic surface as a surface in the three sphere S^3, we may choose the point at infinity arbitrarily.

$$\tau = k_x dx - k_y dy$$
$$\chi_1 = (\tfrac{1}{2}k^2 - u)dx - \tfrac{k_{xy}}{k} dy \tag{33}$$
$$\chi_2 = -\tfrac{k_{xy}}{k} dx + (\tfrac{1}{2}k^2 + u)dy \; ,$$

where $u \in C^\infty(M)$ is a function satisfying the differential equation

$$du = -((\tfrac{k_{xy}}{k})_y + (k^2)_x)dx + ((\tfrac{k_{xy}}{k})_x + (k^2)_y)dy \tag{34}$$

— the integrability condition of this equation is a fourth order partial differential equation closely related to Calapso’s original equation [4]:

$$0 = \Delta(\tfrac{k_{xy}}{k}) + 2(k^2)_{xy} \tag{35}$$

This shows, that
Theorem. *Any isothermic surface gives rise to a solution of Calapso’s equation. Conversely, from a solution $k \in C^\infty(M)$ of Calapso’s equation we can construct a Möbius invariant frame of an isothermic surface by integrating the Maurer-Cartan form (32), where the function u is a solution of (34).*

Now, applying a conformal change $f \quad \frac{1}{k}f$ while fixing the central sphere congruence $n \quad n$, the Maurer-Cartan form of the associated frame becomes

$$\Phi = \begin{pmatrix} 0 & \omega & kdx & \frac{1}{k}dx & \chi_1 \\ -\omega & 0 & -kdy & \frac{1}{k}dy & \chi_2 \\ -kdx & kdy & 0 & 0 & 0 \\ -\chi_1 & -\chi_2 & 0 & 0 & 0 \\ -\frac{1}{k}dx & -\frac{1}{k}dy & 0 & 0 & 0 \end{pmatrix}, \tag{36}$$

where

$$\omega = -\tfrac{k_y}{k} dx + \tfrac{k_x}{k} dy$$
$$\chi_1 = k(\tfrac{k_{xx}}{k} - \tfrac{k_x^2 + k_y^2}{2k^2} + \tfrac{1}{2}k^2 - u)dx \quad . \tag{37}$$
$$\chi_2 = k(\tfrac{k_{yy}}{k} - \tfrac{k_x^2 + k_y^2}{2k^2} + \tfrac{1}{2}k^2 + u)dy$$

Here we see that the central sphere congruence of an isothermic surface is a Ribaucour sphere congruence, which actually is a characterisation of isothermic surfaces [8], and hence it has flat normal bundle as a codimension two surface in the Lorentz sphere S^4_1.

In general, the second enveloping surface of the central sphere congruence of an isothermic surface will not be an isothermic surface and there seems to be no natural relation between the curved flat framing and the Möbius invariant framing (32). We illustrate this with a simple

5. Example

Starting with an isothermic parametrization of a surface of revolution

$$f(x,y) = (r(x)\cos y, r(x)\sin y, z(x)) \; , \tag{38}$$

the functions r and z satisfying the differential equation

$$(39) \qquad r^2 = r'^2 + z'^2 \ ,$$

i.e. the curve (r, z) being parametrized by arc length (thought of as a curve in the Poincaré half plane), we may first write down the loop of Maurer-Cartan forms

$$(40) \qquad \Phi_\lambda = \begin{pmatrix} 0 & -\frac{r'}{r}dy & -\frac{r'z''-r''z'}{r^2}dx & \lambda r dx & -\frac{\lambda}{r}dx \\ \frac{r'}{r}dy & 0 & -\frac{z'}{r}dy & \lambda r dy & \frac{\lambda}{r}dy \\ \frac{r'z''-r''z'}{r^2}dx & \frac{z'}{r}dy & 0 & 0 & 0 \\ \frac{\lambda}{r}dx & -\frac{\lambda}{r}dy & 0 & 0 & 0 \\ -\lambda r dx & -\lambda r dy & 0 & 0 & 0 \end{pmatrix} ,$$

which gives us the immersion $f = f_0$ and its Christoffel transform $\hat{f}_0$ in the limit $\lambda \to 0$.

To understand the geometry of the two enveloping immersions $f_\lambda = F_\lambda e_4$ and $\hat{f}_\lambda = F_\lambda e_5$ given by (40) we remark that the two sphere congruences

$$(41) \qquad s_\lambda = F_\lambda(e_3 + \tfrac{z'}{r^2\lambda}e_4) \ , \qquad \hat{s}_\lambda = F_\lambda(e_3 + \tfrac{z'}{\lambda}e_5)$$

enveloped by the immersion f_λ resp. $\hat{f}_\lambda$ depend only on one parameter. Hence both surfaces are channel surfaces. Moreover all spheres s_λ as well as $\hat{s}_\lambda$ of the two one parameter families of spheres are perpendicular to the (for fixed λ) fixed circle given by

$$(42) \qquad c_\lambda = \text{span}\{F_\lambda e_2, \tfrac{1}{\sqrt{1+2\lambda^2}}F_\lambda(\tfrac{r'}{r}e_1 - \tfrac{z'}{r}e_3 + \tfrac{\lambda}{r}e_4 + r\lambda e_5)\}$$

Consequently, this fixed circle may be interpreted as the axis of rotation — and the surfaces f_λ and $\hat{f}_\lambda$ are surfaces of revolution (with the same axis of rotation) in Euclidean space after an appropriate stereographic projection.

Now, sending $\lambda \to 0$ after a gauge transformation $(f, \hat{f}) \quad (\frac{1}{\lambda}f, \lambda\hat{f})$, we easily read off the mean curvature $H = \frac{1}{2}(\frac{z'}{r^2} + \frac{r'z''-r''z'}{r^3})$ of our original surface f_0 from the connection form (40). So, its central sphere congruence will be given by $n_0 + H f_0$. The metric it induces has conformal factor k^2 (relative to the metric induced by f_0) given by

$$(43) \qquad k = \tfrac{1}{2r^2}(rz' - r'z'' + r''z') \ .$$

Since $k_y \equiv 0$, this is obviously a solution of Calapso's equation and a function u solving (34) is $u = \lambda^2 - k^2$. So the Maurer-Cartan form (32) becomes

$$(44) \quad \Phi_\lambda = \begin{pmatrix} 0 & 0 & kdx & dx & (\frac{3}{2}k^2 - \lambda^2)dx \\ 0 & 0 & -kdy & dy & (-\frac{1}{2}k^2 + \lambda^2)dy \\ -kdx & kdy & 0 & 0 & k_x dx \\ -(\frac{3}{2}k^2 - \lambda^2)dx & (\frac{1}{2}k^2 - \lambda^2)dy & -k_x dx & 0 & 0 \\ -dx & -dy & 0 & 0 & 0 \end{pmatrix} .$$

An appropriate change $n \quad n + kf$ of the sphere congruence enveloped by f — which will be hard to find in the general case — followed by an $O_1(2)$-gauge $f \quad \lambda f$ and $\hat{f} \quad \lambda^{-1}\hat{f}$ gives us the Maurer-Cartan forms

(45) $$\Phi_\lambda = \begin{pmatrix} 0 & 0 & 2kdx & \lambda dx & -\lambda dx \\ 0 & 0 & 0 & \lambda dy & \lambda dy \\ -2kdx & 0 & 0 & 0 & 0 \\ \lambda dx & -\lambda dy & 0 & 0 & 0 \\ -\lambda dx & -\lambda dy & 0 & 0 & 0 \end{pmatrix}.$$

of a loop of curved flats, quite different from (40): as before the immersions $f_\lambda = F_\lambda e_4$ and $\hat{f}_\lambda = F_\lambda e_5$ are channel surfaces, both enveloping the one parameter family of spheres $n_\lambda = F_\lambda e_3$. Again we have fixed circles given by $\mathrm{span}\{F_\lambda, \frac{1}{\sqrt{2}} F_\lambda(e_4 + e_5)\}$ which intersect all spheres n_λ of the one parameter families orthogonally. Hence our surfaces f_λ and $\hat{f}_\lambda$ are surfaces of revolution — but now they envelope just the same one parameter family of spheres. Moreover, the circles given by $\mathrm{span}\{F_\lambda(p)e_1, F_\lambda(p)e_2\}$ which intersect the spheres $n_\lambda(p)$ orthogonally in $f_\lambda(p)$ and $\hat{f}_\lambda(p)$ all meet the axis. Consequently, the immersions $\hat{f}_\lambda(p)$ are just axial reflections of the immersions f_λ parametrizing "opposite" pieces of the same surface of revolution.

Taking the limit $\lambda \to 0$, we obtain a cylinder resp. its Christoffel transform — which is an axial reflection of the original cylinder.

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