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Stability of spectra of Hodge-de Rham laplacians

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1 Introduction

In this paper we investigate the stability of eigenspaces of the Laplace operator acting on differential forms satisfying relative or absolute boundary conditions on a compact, oriented, Riemannian manifold with boundary (this includes, in particular, both Neumann and Dirichlet conditions for the Laplace-Beltrami operator on functions). More precisely, our main result is that the gap between corresponding eigenspaces (precise definition will be recalled below) measured using the L_∞ norm, converges to zero when smooth metrics g converge to g_0 in the $\mathscr{C}^1$ topology. It is quite well known (cf. [3] or [14]) that the eigenvalues of the Laplacian vary continuously under $\mathscr{C}^0$-continuous perturbations of the metric. It is perhaps less well known, but implicit in the work of Cheeger [3], that eigenspaces vary continuously as subspaces of L_2 when the metric is perturbed $\mathscr{C}^0$-continuously. We reprove this $\mathscr{C}^0$ - L_2 stability in Sect. 4 for completeness and in order to be able to use certain notation, conventions and partial results in the proof of $\mathscr{C}^1$ - L_∞ stability in Sect. 5.

The second section of the paper contains a review of the Hodge theory for the Laplace operator with absolute and relative boundary conditions. We also state here the Sobolev embedding theorems and the basic *a priori* estimates for the square root $d + \delta$ of the Laplacian Δ. We need to work with $d + \delta$ rather than $\Delta = (d + \delta)^2$ since the coefficients of Δ depend on the second derivatives of the metric tensor and we allow only $\mathscr{C}^1$-continuous perturbations of the metric and do not assume any bounds on the second derivatives. In the third section we review following Kato [12] and Osborn [16] general results from functional analysis concerning perturbation theory for compact operators on Banach spaces, that reduce proving convergence of eigenvalues and eigenspaces

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to proving convergence of Green's operators. Osborn's paper, in certain special cases, extends Kato's general perturbation theory and relates estimates of rates of convergence of Green's operators to *a priori* estimates.

A natural way to obtain uniform bounds for solutions of elliptic equations (e.g. eigenforms) is to apply elliptic estimates to powers of the operator and then use Sobolev embedding theorems. We cannot do this since the coefficients of high powers of $d + \delta$ depend on many derivatives of the metric. We use L_p estimates, $p >> 1$, for $d + \delta$ and the bounded embedding $H_{1,p} \subset \mathscr{C}^0$ instead.

It is important to realize that, in the case of the Laplacian on functions, one can define the Laplacian if the metric is merely measurable, and compare the Laplacians for the measurable metric and a smooth Lipschitz equivalent metric (cf. [5]). To study the Laplacian on forms some additional regularity of the metric is necessary (cf. [15]). In view of this, it is natural for us to consider $\mathscr{C}^1$-perturbations of the metric. We remark here that the constants involved in our estimates depend only on the *bounds* of the first derivatives of the metric. It is thus conceivable that some version of our results will hold for Lipschitz manifolds. We recall here that Teleman [20, 19] proved that, on a compact Lipschitz manifold equipped with a Lipschitz Riemannian metric, the signature operator $d + \delta$ has compact resolvent. Thus $d + \delta$ on such manifolds has discrete spectrum. One has in this context eigenvalues (squares of eigenvalues of $d + \delta$) and eigenforms (homogeneous components of eigenforms of $d + \delta$) of the Laplacian although the Laplacian itself is defined only in the distributional sense. In view of spectacular applications of analysis on manifolds of low smoothness (i.e. with Lipschitz, quasiconformal or PL structure) to topology [18, 7] spectral geometry in this context appears to be a promising area of further study and we anticipate that the techniques of this paper may be of use.

We remark further that there has been a great deal of work concerning variation of eigenvalues and eigenspaces or certain functions, e.g. determinants, of various elliptic operators. We give a very incomplete list of examples. Variations of harmonic forms appear in the work of Donaldson [6], in his theory of 4-manifolds, and more recently Kronheimer and Mrowka [13] in their proof of the Thom conjecture. Forman [8] studied the behavior of harmonic forms and eigenforms with small eigenvalues under "adiabatic limit" to derive a Hodge theoretic version of Leray spectral sequence. We also mention the papers of Hejhal [9], Wolpert [21] and Ji [11] studying spectral invariants as functions on moduli spaces of Riemann surfaces. Deformations of metrics in all these cases were of very special kinds and smoothness of the metric was not an issue. We decided that it was worthwhile to provide reasonable, i.e. not too stringent, general conditions under which one obtains continuous variation of eigenspaces under deformations of the metric. We obtain in particular estimates for the gaps between eigenspaces for the Laplacian of the base metric and the perturbed metric in terms of appropriate distances between metrics.

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main result, and for many thoughtful comments that, in particular, were useful in bridging that gap.

2 Hodge theory and a priori estimates for $d + \delta$

We begin by reviewing the Hodge theory for manifolds with boundary [4], [15, Chapter 7], [17]. Let M be a $\mathscr{C}^\infty$, oriented, compact manifold of m dimensions, with boundary $\Gamma = \Gamma_1 \cup \Gamma_2$, where Γ_1 and Γ_2 are two closed disjoint smooth submanifolds of Γ. We allow the possibility that either Γ_1 or Γ_2 or both are empty. Let $\mathscr{D}(M) = \oplus_{q=0}^m \mathscr{D}^q(M)$, where $\mathscr{D}^q(M)$ denotes the space of $\mathscr{C}^\infty$ differential forms on M of degree q with complex coefficients. A Riemannian metric g on M induces the Hodge star operator $* : \mathscr{D}^q(M) \to \mathscr{D}^{m-q}(M)$ and the inner product in $\mathscr{D}^q(M)$ for every q given by

$$(\phi, \psi) = \int_M \phi \wedge *\bar{\psi}. \tag{2.1}$$

We extend this inner product to $\mathscr{D}(M)$ by requiring forms of different degrees to be orthogonal. The formal adjoint δ of the exterior derivative d is defined on forms of degree q as

$$\delta = (-1)^{mq+m+1} * d * . \tag{2.2}$$

We consider $d + \delta$ as an operator on $\mathscr{D}(M)$. Then $\Delta = (d + \delta)^2 = \delta d + d\delta$ is the usual Laplacian and, in particular, it preserves the degree of differential forms. We now state for reference the local coordinate expressions for the metric, the exterior derivative operator, and $*$ [10, §27.2]. The summation convention is used throughout. Thus

$$g = g_{ij}\, dx^i dx^j . \tag{2.3}$$

A differential form of degree q can be written locally as

$$f = \frac{1}{q!} f_{i_1,\dots,i_q} dx^{i_1} \wedge \dots \wedge dx^{i_q},$$

where the summation is extended over all sequences $i_1, \dots, i_q$ and $f_{i_1,\dots,i_q}$ is skew-symmetric in its indices. Then,

$$(df)_{i_1,\dots,i_{q+1}} = \sum_{l=1}^{q+1} (-1)^{l-1} \frac{\partial f_{i_1\dots i_{l-1} i_{l+1} \dots i_{q+1}}}{\partial x^{i_l}} \tag{2.4}$$

and

$$(*f)_{i_1,\dots,i_{m-q}} = \frac{1}{q!} \sqrt{\det(g_{rs})}\, \epsilon(j_1, \dots, j_q, i_1, \dots, i_{m-q})\, g^{j_1 k_1} \dots g^{j_q k_q} f_{k_1,\dots,k_q}, \tag{2.5}$$

where $(g^{ij}) = (g_{ij})^{-1}$ and $\epsilon(j_1, \dots, j_q, i_1, \dots, i_{m-q})$ denotes the sign of the permutation $j_1, \dots, j_q, i_1, \dots, i_{m-q}$ of $1, 2, \dots, m$.

By Stokes' theorem

$$(d\phi,\psi) - (\phi,\delta\psi) = \int_{\Gamma_1} \phi \wedge *\bar{\psi} + \int_{\Gamma_2} \phi \wedge *\bar{\psi}. \tag{2.6}$$

We introduce boundary conditions that make the right-hand side of this formula vanish. Every, possibly inhomogeneous, differential form $f \in \mathscr{D}(M)$ has, at every boundary point of M, a natural decomposition $f = f_t + f_n$, into its tangential part f_t and the normal part f_n. We note that the condition $f_t = 0$ is defined independently of the Riemannian metric g and is equivalent to the vanishing of the pullback of f to ∂M via the inclusion mapping. In addition, $*^2 = (-1)^{q(m-q)}$ and

$$(*f)_n = *(f_t), \quad (*f)_t = *(f_n), \quad *d = (-1)^{q+1}\delta *, \quad *\delta = (-1)^q d\, * \tag{2.7}$$

for all forms f, the first two equalities holding at all points of $\Gamma = \partial M$, and the last two at all points of M. Thus, for example, if $\phi_t = 0$ on Γ_1 and $\psi_n = 0$ on Γ_2 then the right-hand side of (2.6) vanishes. We consider the following two boundary value problems.

$$\begin{aligned} (d+\delta)u &= f \\ u_t &= 0 \quad \text{on} \quad \Gamma_1 \\ u_n &= 0 \quad \text{on} \quad \Gamma_2 \end{aligned} \tag{2.8}$$

$$\begin{aligned} \Delta u &= f \\ u_t &= 0, \quad (\delta u)_t = 0 \quad \text{on} \quad \Gamma_1 \\ u_n &= 0, \quad (du)_n = 0 \quad \text{on} \quad \Gamma_2 \end{aligned} \tag{2.9}$$

We shall also have to consider another boundary value problem related to (2.8). Namely, suppose that u is a solution of (2.8). Then $v = *\tau u$ is a solution of the system

$$\begin{aligned} (d-\delta)v &= *f \\ v_n &= 0 \quad \text{on} \quad \Gamma_1 \\ v_t &= 0 \quad \text{on} \quad \Gamma_2 \end{aligned} \tag{2.10}$$

where τ is a linear operator which acts on forms of degree q by multiplication by $(-1)^q$. This follows by applying $*$ to (2.8) and using (2.7). Note that the boundary conditions on Γ_1 and Γ_2 are interchanged in the process.

All three boundary value problems above are elliptic in the sense of [1]. Since our main result relies in an essential way on estimates from that paper as applied to (2.8) and (2.10), we include here the verification that the boundary conditions in these first order systems satisfy the Complementing Condition of [1]. To state this condition for a general elliptic system one considers a neighborhood U of a boundary point p with local coordinates chosen so that the boundary is given by $t = x^n = 0$, p is the origin, and the half-space $t \geq 0$ contains U. Consider the homogeneous problem, i.e. $f = 0$, in this half-space to which the given problem reduces when all leading coefficients (of equations and the boundary conditions) are fixed to their values at p and the lower order terms are set equal to zero. Let $x = (x^1, \ldots, x^{n-1})$ be the coordinates in the plane $t = 0$ and let ξ be an arbitrary nonzero vector in this plane. Consider solutions of the problem introduced above of the form $e^{ix\cdot\xi}w(t)$. The complementing condition is satisfied if for every $\xi \neq 0$ every such solution with bounded w is identically zero. We choose our coordinates so that the vector $\partial/\partial t$ is the unit normal to the boundary at p. Then, in our case (problems (2.8) and (2.10)), the auxiliary problem introduced above amounts to the identical homogeneous problem for the standard flat metric on the half-space. Until further notice we use only the standard metric and coordinates. We consider first the problem (2.8) and a point $p \in \Gamma_2$. Let $u = e^{ix\cdot\xi}w(t) = e^{ix\cdot\xi}(f_1(t) \wedge dt + f_2(t))$ be a solution of the auxiliary problem with f_j's bounded. The condition that the normal component vanishes on the boundary, the second condition of (2.8), requires that $f_1(0) = 0$. Here f_1 and f_2 denote forms on the upper half-space with coefficients depending only on t and not containing dt when expressed in terms of the coordinates.

Since $(d + \delta)u = 0$, $\Delta u = 0$. The Euclidean Laplacian is just the negative of the sum of second derivatives applied to components of u. It follows that

$$\Delta u = e^{ix\cdot\xi}\left(|\xi|^2(f_1 \wedge dt + f_2) - \frac{\partial^2 f_1}{\partial t^2} \wedge dt - \frac{\partial^2 f_2}{\partial t^2}\right),$$

i.e. that f_j satisfy

$$f_j'' - |\xi|^2 f_j = 0.$$

Thus, since u is bounded, $f_j = \alpha_j e^{-|\xi|t}$. However $\alpha_1 = f_1(0) = 0$, so that $f_1 \equiv 0$ and $u = e^{ix\cdot\xi}f_2(t)$. A computation using (2.2) and (2.4) for the Euclidean metric shows that that the coefficient of dt in $(d+\delta)u = 0$ is equal up to sign to $e^{ix\cdot\xi}f_2'(t)$. Thus $f_2 = \alpha_2 e^{-|\xi|t}$ is constant and hence equal to zero.

The verification that the boundary conditions in (2.8) are complementing at points of Γ_1 is completely analogous with the roles of f_1 and f_2 interchanged. Similarly, the conditions in the problem (2.10) are complementing, since bounded solutions of the auxiliary problem can be obtained from bounded solutions for the auxiliary problem for (2.8) by applying $*\tau$ for the Euclidean metric.

We now go back to the general metric on M and introduce some Sobolev spaces of differential forms. Define $\mathscr{D}_1(M) = \oplus_{q=0}^m \mathscr{D}_1^q(M)$ to be the space of $\mathscr{C}^\infty$ forms satisfying the boundary conditions in (2.8) and $\mathscr{D}_2(M) = \oplus_{q=0}^m \mathscr{D}_2^q(M)$ as the space of forms satisfying the boundary conditions of (2.9). Denote the pointwise norm of a differential form f by $|f|$ and let $L_p\mathscr{D}(M)$ be the completion of $\mathscr{D}(M)$ with respect to the norm

$$\| f \|_p = \left(\int_M |f|^p dV\right)^{1/p}, \tag{2.11}$$

$1 \le p < \infty$. Similarly, let $H_{k,p}\mathscr{D}(M)$ be the Sobolev space of forms whose derivatives up to order k are in L_p, i.e. the completion of $\mathscr{D}(M)$ in the norm

$$\| f \|_{k,p} = \left(\sum_{l=0}^{k} \int_M |\nabla^l f|^p dV \right)^{1/p}, \tag{2.12}$$

where $\nabla^l f$ is the l-th covariant derivative of the form f. Note that $L_p\mathscr{D}(M) = H_{0,p}\mathscr{D}(M)$. We also define $H_{k,p}\mathscr{D}_i(M)$ as the closure of $\mathscr{D}_i(M)$ in $H_{k,p}\mathscr{D}(M)$ and $\mathscr{H}(M) = \oplus_{q=0}^{m} \mathscr{H}^q(M)$ as the space of harmonic forms, i.e. forms $h \in \mathscr{D}_1(M)$ satisfying $dh = 0$ and $\delta h = 0$. Since (2.8) is an elliptic problem $\mathscr{H}(M)$ is finite dimensional and by the de Rham theorem $\mathscr{H}^q(M)$ is isomorphic to $H^q(M, \Gamma_1)$ the q-th relative cohomology group of (M, Γ_1) with complex coefficients. We denote by H the orthogonal projection of $L_2\mathscr{D}(M)$ onto $\mathscr{H}(M)$. The Green's operator for the problem (2.8) will be denoted by $\mathscr{G}^{(1)}$. For $\phi \in \mathscr{D}(M)$, $u = \mathscr{G}^{(1)}\phi \in \mathscr{D}_1(M)$ is the unique form orthogonal to $\mathscr{H}(M)$ and satisfying

$$(d + \delta)u = \phi - H\phi. \tag{2.13}$$

The Green's operator for the problem (2.9) is given by $\mathscr{G}^{(2)} = \mathscr{G}^{(1)} \circ \mathscr{G}^{(1)}$. It maps $\mathscr{D}(M)$ into $\mathscr{D}_2(M)$ and $\mathscr{G}^{(2)}\phi$ can be characterized as the unique form $u \in \mathscr{D}_2(M)$ orthogonal to $\mathscr{H}(M)$ and satisfying

$$\Delta u = \phi - H\phi. \tag{2.14}$$

$\mathscr{G}^{(1)}$ and $\mathscr{G}^{(2)}$ extend to bounded operators from $L_2\mathscr{D}(M)$ into $H_{1,2}\mathscr{D}_1(M)$ and $H_{2,2}\mathscr{D}_2(M)$ respectively. Moreover, if $\lambda > 0$ is the smallest positive eigenvalue of Δ for the boundary conditions in (2.9), then

$$\begin{aligned} \| \mathscr{G}^{(2)}\phi \|_2 &\le \lambda^{-1} \| \phi \|_2 \\ \| \mathscr{G}^{(1)}\phi \|_2 &\le \lambda^{-1/2} \| \phi \|_2 . \end{aligned} \tag{2.15}$$

The Hodge decomposition of an L_2 form ϕ can be obtained as follows.

$$\begin{aligned} \phi &= H\phi + d\mathscr{G}^{(1)}\phi + \delta\mathscr{G}^{(1)}\phi \\ &= H\phi + d\delta\mathscr{G}^{(2)}\phi + \delta d\mathscr{G}^{(2)}\phi \end{aligned} \tag{2.16}$$

and, by uniqueness the Green's operators satisfy

$$d\mathscr{G}^{(1)} = \mathscr{G}^{(1)}\delta, \qquad \delta\mathscr{G}^{(1)} = \mathscr{G}^{(1)}d, \tag{2.17}$$

and

$$d\mathscr{G}^{(2)} = \mathscr{G}^{(2)}d, \qquad \delta\mathscr{G}^{(2)} = \mathscr{G}^{(2)}\delta, \tag{2.18}$$

when applied to forms which are sufficiently smooth and satisfy appropriate boundary conditions.

We define

$$T_d = d\delta\mathscr{G}^{(2)} = d\mathscr{G}^{(1)} \qquad \text{and} \qquad T_\delta = \delta d\mathscr{G}^{(2)} = \delta\mathscr{G}^{(1)}. \tag{2.19}$$

These are orthogonal projections of $L_2\mathscr{D}(M)$ onto $\overline{d\mathscr{D}_1(M)}$ and $\overline{\delta\mathscr{D}_1(M)}$ respectively. This gives rise to L_2 orthogonal Hodge decompositions

$$\begin{aligned} \mathscr{D}(M) &= d\mathscr{D}_1(M) \oplus \mathscr{H}(M) \oplus \delta\mathscr{D}_1(M) \\ L_2\mathscr{D}(M) &= \overline{d\mathscr{D}_1(M)} \oplus \mathscr{H}(M) \oplus \overline{\delta\mathscr{D}_1(M)} \\ &= T_d L_2\mathscr{D}(M) \oplus \mathscr{H}(M) \oplus T_\delta L_2\mathscr{D}(M). \end{aligned} \tag{2.20}$$

We make the following useful observation.

Lemma 2.21 *The space of exact forms $T_d L_2\mathscr{D}(M)$ and the space of closed forms $T_d L_2\mathscr{D}(M) \oplus \mathscr{H}(M) = (T_\delta L_2\mathscr{D}(M))^\perp$ are independent of the Riemannian metric on M.*

Proof. Suppose $\eta = d\beta$ and $\beta \in \mathscr{D}(M)$ satisfies $\beta_t = 0$ on Γ_1. Then η is orthogonal to $\mathscr{H}(M)$ and to $\delta\mathscr{D}_1(M)$ by (2.6). It follows that

$$\begin{aligned} T_d L_2\mathscr{D}(M) &= \overline{d\mathscr{D}_1(M)} \\ &\subset \overline{\{\eta \in \mathscr{D}(M) \mid \eta = d\beta, \beta \in \mathscr{D}(M), \beta_t = 0 \text{ on } \Gamma_1\}} \\ &\subset \overline{d\mathscr{D}_1(M)}. \end{aligned}$$

Since the topology of $L_2\mathscr{D}(M)$ is independent of the metric and so is the condition $\beta_t = 0$ on Γ_1, it follows that $T_d L_2\mathscr{D}(M) = \overline{d\mathscr{D}_1(M)}$ is independent of the metric. Similarly,

$$d\mathscr{D}_1(M) \oplus \mathscr{H}(M) \subset \{\eta \in \mathscr{D}(M) \mid d\eta = 0, \eta_t = 0 \text{ on } \Gamma_1\} \subset \left(\delta\mathscr{D}_1(M)\right)^\perp.$$

Taking the L_2 closures we see that $(T_\delta L_2\mathscr{D}(M))^\perp$ is independent of the metric as well. □

Define the L_∞ norm, for a form ϕ with measurable coefficients, in the usual way as

$$\| \phi \|_\infty = \operatorname*{ess\,sup}_{x\in M} |\phi(x)|$$

and let $\mathscr{C}^0\mathscr{D}(M)$ denotes the space of forms with continuous coefficients equipped with this norm.

We have the following special cases of the Sobolev embedding theorem (cf. [2]).

Theorem 2.22 *(i) For positive integers $k, k' \geq 1$ and real $p, p' \in [1, \infty)$ $H_{k,p}\mathscr{D}(M) \subset H_{k',p'}\mathscr{D}(M)$ provided $k - m/p \geq k' - m/p'$. The inclusion is compact if $k - m/p > k' - m/p'$ and $k > k'$.*

(ii) If $p > m$ then $H_{1,p}\mathscr{D}(M) \subset \mathscr{C}^0\mathscr{D}(M)$ and the inclusion is compact.

The following is our basic *a priori* estimate. It is a special case of Theorem 10.5 of [1].

Theorem 2.23 *Suppose $u \in H_{1,p}\mathscr{D}(M)$, $p \geq 2$, is a solution of (2.8) or (2.10). Then*

$$\| u \|_{1,p} \leq C(\| f \|_p + \| u \|_2)$$

with the constant C depending only on p, m, and the bounds of the components of the metric tensor and their first derivatives.

This follows from Theorem 10.5 of [1] since, by (2.2), (2.3), (2.4), and (2.5), the coefficients of $d+\delta$ depend only on the metric tensor and the first derivatives of its components.

Remarks. 1. Since the boundary conditions in (2.8) and (2.10) are homogeneous, our bounds for $\| u \|_{1,p}$ do not contain the norms of functions appearing in the right-hand side of the equations defining boundary conditions present in [1, (10.7)], which covers general inhomogenuous boundary conditions.

2. The estimate of [1], which yields (2.23), contains the L_1 norm of u. We chose the formulation with $\| u \|_2$, which is a consequence, since we use only the L_2 norm on the right hand side.

3. In the *a priori* estimates above, the term $\| u \|_2$ can be dropped altogether provided u is L_2-orthogonal to $\mathscr{H}(M)$. However the constant C has to be changed and its dependence on the metric becomes much more delicate (cf. (2.15)). For this reason we will use such improved inequalities only for a *fixed* metric.

We now review the spectral decomposition of Δ. For a number $\lambda \geq 0$, let $\mathscr{E}(\lambda) \subset \mathscr{D}_2(M)$ be the linear space of forms $\phi \in \mathscr{D}_2(M)$ satisfying $\Delta\phi = \lambda\phi$. Clearly $\mathscr{E}(0) = \mathscr{H}(M)$. For every $\lambda \geq 0$, $\dim \mathscr{E}(\lambda) < \infty$ and there exists a sequence $0 = \mu_0 \leq \mu_1 \ldots \to \infty$ such that $L_2\mathscr{D}(M)$ is the Hilbert space direct sum

$$L_2\mathscr{D}(M) = \bigoplus_{i=0}^{\infty} \mathscr{E}(\mu_i).$$

Since the Laplacian preserves the degree of a form and commutes with d and δ every eigenspace decomposes further as follows.

$$\begin{aligned} \mathscr{E}(\mu_i) &= \bigoplus_{q=0}^{m} \mathscr{E}^q(\mu_i) \\ \mathscr{E}^q(\mu_i) &= \mathscr{E}^q_d(\mu_i) \oplus \mathscr{E}^q_\delta(\mu_i) \end{aligned}$$

where $\mathscr{E}^q(\mu_i)$ denotes $\mathscr{E}(\mu_i) \cap \mathscr{D}^q(M)$ and $\mathscr{E}^q_d(\mu_i) = \mathscr{E}^q(\mu_i) \cap d\mathscr{D}_2(M)$, $\mathscr{E}^q_\delta(\mu_i) = \mathscr{E}^q(\mu_i) \cap \delta\mathscr{D}_2(M)$.

The Green's operator $\mathscr{G}^{(2)}$ has $\mathscr{E}(0) = \mathscr{H}(M)$ as its kernel and its restriction to every eigenspace of Δ belonging to a positive eigenvalue λ acts as the multiplication by λ^{-1}. Therefore to study the dependence of eigenfunctions and eigenvalues on the Riemannian metric, it will suffice to do so for $\mathscr{G}^{(2)}$ rather than Δ. This is technically easier since $\mathscr{G}^{(2)}$ is a bounded compact operator on several of the spaces introduced above. Basic facts from perturbation theory needed to carry this out are reviewed below.

3 Perturbation theory for semisimple compact operators on Banach spaces

The general reference for the material in this section is [12, Chapter 4, §3]. We discovered that these techniques were useful for the study of eigenspaces of the Laplace operator by reading the paper [16] of Osborn who studied a somewhat more general situation than needed for our purposes. In addition, Osborn discusses applications to approximation theory which are very close in spirit to what we do in Sects. 4 and 5.

Let $G : X \to X$ be a compact linear operator on a complex Banach space $(X, \| \cdot \|)$. Let $\sigma(G)$ be the spectrum of G and $\rho(G)$ the resolvent set of G. Assume that G is semisimple i.e. that for every $\lambda \in \sigma(G)$, every $k = 1, 2, \ldots$ and every $v \in X$

$$(G - \lambda I)^k v = 0$$

implies that

$$(G - \lambda I)v = 0.$$

All operators considered here will satisfy this condition. For $z \in \rho(G)$ denote the resolvent operator evaluated at z by $R_z(G) = (G - zI)^{-1}$. Now consider a family $\{G\}$ of operators as above converging to an operator G_0 in the operator norm topology. G_0 has a countable spectrum of which 0 is the only accumulation point. Suppose ν is a nonzero element of $\sigma(G_0)$. Then ν is an eigenvalue of G_0 of finite multiplicity l. Denote by γ_ν a circle centered at ν which lies in $\rho(G_0)$ and which encloses no points of $\sigma(G_0) \setminus \{\nu\}$. The spectral projection associated with ν and G_0 is defined by

$$E_{0,\nu} = \frac{1}{2\pi i} \int_{\gamma_\nu} R_z(G_0)\, dz\,.$$

It is well known that if the operator norm $\| G - G_0 \|$ is sufficiently small then $\gamma_\nu \subset \rho(G)$ and the corresponding spectral projection E_ν for G is well defined by

$$E_\nu = \frac{1}{2\pi i} \int_{\gamma_\nu} R_z(G)\, dz\,.$$

The collection of operators E_ν converges to $E_{0,\nu}$ in norm if $\| G - G_0 \| \to 0$. The projection E_ν is in fact the spectral projection associated with G and those eigenvalues of G which belong to the open disk bounded by γ_ν. Furthermore E_ν maps X onto the direct sum of eigenspaces corresponding to these eigenvalues. Suppose $\nu_1, \nu_2, \ldots \nu_j$ are the distinct eigenvalues of G within γ_ν. Let $\mathscr{F}(\nu_i)$, $i = 1, 2, \ldots j$, be the eigenspaces of G corresponding to ν_i and let $\mathscr{F}_0(\nu)$ be the eigenspace of G_0 belonging to ν. If $m_i = \dim \mathscr{F}(\nu_i)$, the multiplicity of ν_i as an eigenvalue of G, then $l = \sum_i m_i$ and the range of E_ν is equal to $\oplus_{i=1}^{j} \mathscr{F}(\nu_i)$.

Since the radius of γ_ν can be chosen arbitrarily small, we see that when $\| G - G_0 \| \to 0$, G will have precisely l eigenvalues (counting according to multiplicity) converging to ν. To formulate convergence of eigenspaces we recall the notion of the gap between two closed subspaces of X. Given two closed linear subspaces $A, B \subset X$ we define

$$\vartheta(A,B) = \sup\{\operatorname{dist}(x,B) \mid x \in A, \| x \| = 1\}$$

and

$$\hat{\vartheta}(A,B) = \max\{\vartheta(A,B), \vartheta(B,A)\}. \tag{3.1}$$

$\hat{\vartheta}(A,B)$ is called the gap between A and B. The following theorem is a special case of Theorem 3.16, Chapter 4 of [12].

Theorem 3.2 *There exists a constant C_1 depending only on G_0 such that, if $\| G - G_0 \|$ is sufficiently small, then*

$$\hat{\vartheta}\left(\bigoplus_{i=1}^{j} \mathscr{F}(\nu_i)\,,\,\mathscr{F}_0(\nu)\right) \leq C_1 \| G - G_0 \| \,.$$

Remarks. 1.) Osborne [16] proves a somewhat more general version of this theorem.
2.) $\| G - G_0 \|$ in the estimate above can be replaced by $\sup\{ \| (G - G_0)v \| \mid v \in \mathscr{F}_0(\nu), \| v \| = 1\}$.

4 L^2 continuity of Green's operators

In this section we consider a family of smooth metrics on M, with typical element g, converging in $\mathscr{C}^0$ topology to a fixed metric g_0. We wish to show that the Green's operators $\mathscr{G}^{(2)}$ associated to the metric g converge to the Green's operator $\mathscr{G}_0^{(2)}$ for the metric g_0 in the operator norm topology in the space of bounded operators on $L_2\mathscr{D}(M)$. In view of the factorization $\mathscr{G}^{(2)} = \mathscr{G}^{(1)} \circ \mathscr{G}^{(1)}$ it will suffice to investigate the behavior of $\mathscr{G}^{(1)}$. In general, an object associated with the metric g_0 will have a subscript 0 to distinguish it from an analogous object for a metric g in our family. As the measure of closeness of g to g_0 we can take $\epsilon(g, g_0)$ defined by

$$\epsilon(g,g_0)^2 = \sup\{ |g(v,v) - g_0(v,v)| \mid x \in M\,,\ v \in T_x(M)\,,\ g_0(v,v) = 1\}.$$

The differences of the components (cf. (2.3)) of the two metrics in local coordinates can be estimated in terms of $\epsilon(g, g_0)$ in an obvious way.

In the remainder of this section $C(\epsilon)$ will denote a function satisfying $C(\epsilon) = O(\epsilon)$ with the constant implicit in $O(\epsilon)$ independent of the metric g but not necessarily equal in different inequalities.

We first establish some lemmata.

Lemma 4.1 *Suppose $0 < \epsilon(g, g_0) = \epsilon < \bar{\epsilon}$. Then for every $\phi, \psi \in L_2\mathscr{D}(M)$*

$$|(\phi,\psi) - (\phi,\psi)_0| < C(\epsilon)^2 \| \phi \|_0 \cdot \| \psi \|_0 \,.$$

Proof. This becomes obvious when the integrand of $(\phi, \psi) = \int_M \phi \wedge *\overline{\psi}$ is written in local coordinates using (2.5). □

Lemma 4.2 *Let P, P_0 denote the projections of $L_2\mathscr{D}(M)$ onto the space, by (2.21) independent of the metric, of closed forms with respect to the inner product induced by the metrics g and g_0 respectively. Then, if $0 < \epsilon(g, g_0) = \epsilon$ is sufficiently small,*

$$\| Pf - P_0 f \|_0 \leq C(\epsilon) \| f \|_0,$$

where $\| \cdot \|_0$ is the L_2 norm induced by the metric g_0.

Proof. By the Pythagorean theorem

$$\| Pf - P_0 f \|_0^2 = \| f - Pf \|_0^2 - \| f - P_0 f \|_0^2 \tag{4.3}$$

because both Pf and $P_0 f$ are closed. Now

$$\| f - Pf \|_0^2 \leq (1 + C(\epsilon)^2) \| f - Pf \|^2 \leq (1 + C(\epsilon)^2) \| f - P_0 f \|^2$$

since the two inner products are very close by Lemma 4.1 and the norm $\| f - Pf \|$ minimizes $\| f - \omega \|$ over all closed forms ω. We now go back to the norm $\| \cdot \|_0$.

$$\| f - Pf \|_0^2 \leq (1 + C(\epsilon)^2) \| f - P_0 f \|_0^2$$

and substitute this into (4.3) to obtain

$$\| Pf - P_0 f \|_0^2 \leq C(\epsilon)^2 \| f - P_0 f \|_0^2 \leq C(\epsilon)^2 \| f \|_0^2 . \qquad \square$$

A similar estimate holds for the harmonic projections.

Lemma 4.4 *Suppose $0 < \epsilon(g, g_0) = \epsilon$ is sufficiently small. Then*

$$\| Hf - H_0 f \|_0 \leq C(\epsilon) \| f \|_0 .$$

Proof. Note first that $Hf = HPf$ for every $f \in L_2\mathscr{D}(M)$. Therefore, in view of Lemma 4.2, it suffices to prove the estimate assuming that the form f is closed. In this case $f = Pf = P_0 f$. We use the characterization, which follows easily from the Hodge decomposition (2.20), of harmonic forms as the minima of norm in their cohomology classes and the fact that the harmonic projection preserves the cohomology class of a closed form. Thus $Hf - H_0 f$ is exact and therefore perpendicular to $\mathscr{H}_0(M)$. Therefore

$$\begin{aligned} \| Hf - H_0 f \|_0^2 &= \| Hf \|_0^2 - \| H_0 f \|_0^2 \\ &\leq (1 + C(\epsilon)^2) \| Hf \|^2 - \| H_0 f \|_0^2 \\ &\leq (1 + C(\epsilon)^2) \| H_0 f \|^2 - \| H_0 f \|_0^2 \\ &\leq C(\epsilon)^2 \| f \|_0^2, \end{aligned}$$

where the first inequality follows from Lemma 4.1 and the second one from the minimizing property of Hf. $\square$

We remark that estimates analogous to those contained in Lemmas 4.2 and 4.4 hold as a consequence of these Lemmas for orthogonal projections onto spaces of exact and coexact differential forms.

We are now ready to prove the main theorem of this section.

Theorem 4.5 *For every* $f \in L_2\mathscr{D}(M)$

$$\| \mathscr{G}^{(2)}f - \mathscr{G}_0^{(2)}f \|_0 \leq C(\epsilon) \| f \|_0$$

provided $0 < \epsilon = \epsilon(g, g_0)$ *is sufficiently small.*

Proof. We will first establish the estimate for $\mathscr{G}^{(1)}$. The inequality for $\mathscr{G}^{(2)}$ follows since $\mathscr{G}^{(2)} = \mathscr{G}^{(1)} \circ \mathscr{G}^{(1)}$. Thus consider the equation

$$(d + \delta)u = f - Hf \tag{4.6}$$

for an arbitrary $f \in L_2\mathscr{D}(M)$ and its unique solution $u = \mathscr{G}^{(1)}f \in H_{1,2}\mathscr{D}_1(M)$ perpendicular to $\mathscr{H}(M)$ and the analogous equation for the metric g_0 with the solution $u_0 = \mathscr{G}_0^{(1)}f$. We have to estimate the L_2 norm of $u - u_0$. We will use the same conventions as above, however all Sobolev norms that we shall use are defined using the metric g_0 and its Levi-Civita connection. In addition, C_1 will denote a positive constant that does not depend on the metric g provided $\epsilon(g, g_0) \leq \bar{\epsilon}$; i.e. C_1 may depend on $\bar{\epsilon}$. We begin with a calculation of $(u - u_0, \phi)_0$ for an arbitrary L_2 form $\phi = h + (d + \delta_0)v$, $h \in \mathscr{H}_0(M)$, $v = \mathscr{G}_0^{(1)}\phi$. The method of estimating will be to use $(\cdot\,,\cdot)$ or $(\cdot\,,\cdot)_0$ as convenient compensating whenever necessary with terms which are small when g is close to g_0. Thus

$$(u - u_0, \phi)_0 = (u, h)_0 + (u, dv)_0 + (u, \delta_0 v)_0 - (u_0, dv)_0 - (u_0, \delta_0 v)_0. \tag{4.7}$$

The last two terms above yield after integration by parts $-(f - H_0 f, v)_0$. The third term is equal to $(du, v)_0$. The second term in (4.7) can be written as

$$\begin{aligned}
(u, dv)_0 &= (u, dv) + [(u, dv)_0 - (u, dv)] \\
&= (\delta u, v) + [(u, dv)_0 - (u, dv)] \\
&+ (\delta u, v)_0 + [(\delta u, v) - (\delta u, v)_0] + [(u, dv)_0 - (u, dv)]
\end{aligned}$$

We will show that the terms in square brackets are small when $\epsilon(g, g_0)$ is small. Observe that the sum of the second and third terms in (4.7) amounts to

$$\begin{aligned}
&((d + \delta)u, v)_0 + [\cdots] + [\cdots] = \\
&\quad (f - Hf, v)_0 + [\cdots] + [\cdots]
\end{aligned}$$

where we used the equations satisfied by u. It follows that the right hand side of (4.7) can be written as

$$-(Hf - H_0 f, v)_0 + (u, h)_0 + [\cdots] + [\cdots]. \tag{4.8}$$

We estimate each term of this sum separately.

$$|(Hf - H_0 f, v)_0| \leq C(\epsilon) \| f \|_0 \cdot \| v \|_0 \leq C(\epsilon) \| f \|_0 \cdot \| \phi \|_0 \tag{4.9}$$

by Lemma 4.4 and the bound (2.15) for the metric g_0. Note that

$$\| u \|_0 \leq \| u - u_0 \|_0 + \| u_0 \|_0 \leq \| u - u_0 \|_0 + C_1 \| f \|_0 \tag{4.10}$$

using (2.15) for g_0 again. We now estimate the second term in (4.8) using Lemma 4.1, equivalence of norms $\|\ \ \|$ and $\|\ \ \|_0$, the inequality $\| h \|_0 \leq \| \phi \|_0$, and the fact that u is perpendicular to $\mathscr{H}(M)$ with respect to the inner product induced by g. Using (4.10) we obtain the following inequality.

$$\begin{aligned} |(u,h)_0| &= |(u,(H_0-H)h)_0 + (u,Hh)_0| \\ &= |(u,(H_0-H)h)_0 + (u,Hh)_0 - (u,Hh)| \\ &\leq C(\epsilon) \| u \|_0 \cdot \| h \|_0 \\ &\leq C(\epsilon) \left(\| u-u_0 \|_0 + C_1 \| f \|_0\right) \cdot \| \phi \|_0 \ . \end{aligned} \tag{4.11}$$

Similarly,

$$\begin{aligned} |(\delta u, v) - (\delta u, v)_0| &\leq C(\epsilon) \| u \|_{1,2} \cdot \| v \|_0 \\ &\leq C(\epsilon) \left(\| f - Hf \|_0 + \| u \|_0\right) \cdot \| \phi \|_0 \\ &\leq C(\epsilon) \left(\| u-u_0 \|_0 + C_1 \| f \|_0\right) \cdot \| \phi \|_0, \end{aligned} \tag{4.12}$$

where we used Theorem 2.23, (4.10), and (2.15) for the metric g_0. To estimate the fourth term in (4.8) we note that the definition of v and the fact that the images of d and δ_0 are perpendicular with respect to $(\cdot\,,\cdot)_0$ imply that

$$\| dv \|_0^2 \leq \| (\delta_0 + d)v \|_0^2 \leq C_1 \| \phi \|_0^2 \ .$$

Thus, using (4.10),

$$\begin{aligned} |(u,dv)_0 - (u,dv)| &\leq C(\epsilon) \| u \|_0 \cdot \| dv \|_0 \\ &\leq C(\epsilon) \left(\| u-u_0 \|_0 + \| f \|_0\right) \cdot \| \phi \|_0 \ . \end{aligned} \tag{4.13}$$

We now take $\phi = (u-u_0)/\| u-u_0 \|_0$ in (4.7) and collect the estimates (4.9), (4.11), (4.12), (4.13) to obtain

$$\| u-u_0 \|_0 \leq C(\epsilon) \| f \|_0 + C(\epsilon) \| u-u_0 \|_0 \ .$$

This proves the theorem since $C(\epsilon) \leq C_1\epsilon$ and the last term on the right can be absorbed in the left-hand side. □

As a corollary we deduce convergence of eigenvalues and eigenspaces of the Laplacian as $\epsilon(g,g_0) \to 0$. In the theorem below the gap $\hat{\vartheta}$ is as defined in (3.1) using the norm $\| \cdot \|_0$ on $L_2\mathscr{D}(M)$.

Theorem 4.14 *Suppose $\epsilon(g,g_0) \to 0$. Then the gap $\hat{\vartheta}(\mathscr{H}(M),\mathscr{H}^0(M))$ tends to zero. Let $\lambda > 0$ be an eigenvalue of the Laplacian Δ_0 acting on forms of degree q for the boundary condition in (2.9) and $\mathscr{E}_0^q(\lambda) = \mathscr{E}_{0,d}^q(\lambda) \oplus \mathscr{E}_{0,\delta}^q(\lambda)$. Let $k = \dim \mathscr{E}_{0,d}^q(\lambda)$, $l = \dim \mathscr{E}_{0,\delta}^q(\lambda)$ and let I_λ be an open interval in $\mathbb{R}$ around λ such that $I_\lambda \cap \sigma(\Delta_0) = \{\lambda\}$. Then, for $\epsilon(g,g_0)$ sufficiently small, the Laplacian Δ acting on exact (respectively coexact) forms of degree q has, when counted with multiplicities, exactly k (respectively l) eigenvalues in I_λ. Let $\mu_1,\ldots,\mu_{k_1} \in I_\lambda$ be the distinct eigenvalues of Δ corresponding to the exact eigenforms of degree q and let $\nu_1,\ldots\nu_{l_1} \in I_\lambda$ correspond to coexact ones. Then*

$$\lim_{\epsilon\to 0}\mu_i = \lambda, \qquad \lim_{\epsilon\to 0}\nu_i = \lambda$$

and

$$\hat{\vartheta}\left(\bigoplus_{i=0}^{k_1}\mathscr{E}_d^q(\mu_i)\,,\ \mathscr{E}_{0,d}^q(\lambda)\right) \quad\rightarrow\quad 0$$

$$\hat{\vartheta}\left(\bigoplus_{i=0}^{l_1}\mathscr{E}_\delta^q(\nu_i)\,,\ \mathscr{E}_{0,\delta}^q(\lambda)\right) \quad\rightarrow\quad 0$$

when $\epsilon(g, g_0)$ *tends to zero.*

Proof. The statement concerning harmonic forms follows from Lemma 4.4. The rest is a formal consequence of Theorem 4.5. Namely $\mathscr{G}^{(2)} = \mathscr{G}^{(1)}\circ\mathscr{G}^{(1)} \to \mathscr{G}_0^{(1)}\circ\mathscr{G}_0^{(1)} = \mathscr{G}_0^{(2)}$ in the operator norm topology by Theorem 4.5. Green's operator preserves the degree of a form and, since $d\circ\mathscr{G}^{(2)} = \mathscr{G}^{(2)}\circ d$, $\mathscr{G}^{(2)}$ maps $T_d L_2\mathscr{D}^q(M)$ into itself. Denote by A (respectively A_0) the restriction of the Green's operator for the metric g (respectively g_0) to the space of exact forms of degree q. Clearly $A \to A_0$ and the eigenspaces of A for a positive eigenvalue κ are precisely the eigenspaces of Δ on exact forms of degree q corresponding to the eigenvalue $\lambda = 1/\kappa$. Thus the statement about exact eigenspaces and corresponding eigenvalues follows from the convergence $A \to A_0$ via Theorem 3.2. To obtain an analogous statement for coexact eigenspaces we note that, by (2.7), $*\mathscr{E}_\delta^q(\lambda) = \mathscr{F}_d^{m-q}(\lambda)$, where $\mathscr{F}_d^{m-q}(\lambda)$ denotes the space of exact eigenforms belonging to $\lambda > 0$ for the boundary value problem

$$\begin{aligned}
\Delta u &= f \\
u_t &= 0, \quad (\delta u)_t = 0 \qquad \text{on} \qquad \Gamma_2 \\
u_n &= 0, \quad (du)_n = 0 \qquad \text{on} \qquad \Gamma_1.
\end{aligned}$$

This is the same as the problem (2.9) with Γ_1 and Γ_2 interchanged. By the argument above

$$\hat{\vartheta}\left(\bigoplus_{i=0}^{l_1}\mathscr{F}_d^{m-q}(\nu_i)\,,\ \mathscr{F}_{0,d}^{m-q}(\lambda)\right) \to 0$$

and therefore, since $*$ is an isometry,

$$\hat{\vartheta}\left(\bigoplus_{i=0}^{l_1}\mathscr{E}_\delta^q(\nu_i)\,,\ \mathscr{E}_{0,\delta}^q(\lambda)\right) \to 0.$$

This finishes the proof. □

5 $\mathscr{C}^1$ - $L\infty$ stability of eigenspaces

We consider metrics $g \to g_0$ in the $\mathscr{C}^1$ topology and prove that in this context Theorem 4.14 holds with $\hat{\vartheta}$ replaced by $\hat{\vartheta}_\infty$, the gap between subspaces of $\mathscr{C}^0\mathscr{D}(M)$ equipped with the L_∞ norm. Define $\eta(g, g_0)$ by

$$\eta(g, g_0)^2 = \epsilon(g, g_0)^2 + \sup_{x \in M} |\nabla_0 g(x)|^2$$

where $\nabla_0 g$ denotes the covariant derivative, with respect to the Levi-Civita connection for the metric g_0, of g considered as a tensor on M. $\eta(g, g_0)$ controls the differences of the components of the metrics and derivatives of these differences in terms of local coordinates. In particular, $\eta(g, g_0) \to 0$ if and only if g approaches g_0 in the $\mathscr{C}^1$ topology. All norms considered in this section, i.e. $\| \cdot \|_p$ and $\| \cdot \|_{1,p}$, are computed using the base metric g_0. They are clearly comparable with constants depending only on $\overline{\eta}$ for $\eta(g, g_0) < \overline{\eta}$ to corresponding norms for the metric g.

We shall need the following lemma.

Lemma 5.1 *Suppose $\eta = \eta(g, g_0) < \overline{\eta}$. There exists a constant $C_1 > 0$ depending only on $p \geq 2$ and $\overline{\eta}$ so that*

$$\| Hf \|_{1,p} \leq C_1 \| f \|_p$$

for all forms in $L_p\mathscr{D}(M)$. In particular, H is a bounded operator on $L_p\mathscr{D}(M)$.

Proof. By Theorem 2.23

$$\| Hf \|_{1,p} \leq C \left(\| (d + \delta)Hf \|_p + \| Hf \|_2 \right) \leq C \| f \|_2 \leq C_1 \| f \|_p . \qquad \square$$

Since $\mathscr{C}^0\mathscr{D}(M)$ and $L_p\mathscr{D}(M)$ for $p \geq 2$ are contained in $L_2\mathscr{D}(M)$ the Green's operator $\mathscr{G}^{(1)}$ is defined on these spaces. It follows easily from Theorem 2.23 and from Lemma 5.1 that $\mathscr{G}^{(1)}$ is a bounded operator from $L_p\mathscr{D}(M)$ to $H_{1,p}\mathscr{D}_1(M)$ for $p > 2$ and that for $\eta(g, g_0) < \overline{\eta}$ its norm is uniformly bounded. Finally (cf. (2.22)), the compactness of the inclusion $H_{1,p}\mathscr{D}(M) \subset \mathscr{C}^0\mathscr{D}(M)$ for $p > m$ shows that $\mathscr{G}^{(1)}$ restricts to a compact operator on $\mathscr{C}^0\mathscr{D}(M)$. The same is true as a consequence for $\mathscr{G}^{(2)} = \mathscr{G}^{(1)} \circ \mathscr{G}^{(1)}$. It follows from the elliptic regularity that $\mathscr{G}^{(2)}$ considered as an operator on $\mathscr{C}^0\mathscr{D}(M)$ has exactly the same spectrum and eigenspaces as $\mathscr{G}^{(2)}$ on $L_2\mathscr{D}(M)$ and that it is semisimple as an operator on $\mathscr{C}^0\mathscr{D}(M)$. We will establish the stability of eigenspaces as a consequence of the convergence of Green's operators in the operator norm on the space of bounded operators on $\mathscr{C}^0\mathscr{D}(M)$. To this end we shall estimate the norm of $\mathscr{G}^{(1)} - \mathscr{G}_0^{(1)}$ as the operator from $L_p\mathscr{D}(M)$ into $H_{1,p}\mathscr{D}(M)$ for a fixed $p > m$. As in the previous section $C(\eta)$ will be equal to $O(\eta)$ so that the constant implied depends only on p and the upper bound $\overline{\eta}$ of $\eta(g, g_0) = \eta$. We need two more lemmata.

Lemma 5.2 *For $v \in H_{1,p}\mathscr{D}(M)$,*

$$\| (\delta - \delta_0)v \|_p \leq C(\eta) \| v \|_{1,p}$$

and

$$\| (* - *_0)v \|_{1,p} \leq C(\eta) \| v \|_{1,p} .$$

Proof. This follows from (2.2), (2.5), and (2.4). $\square$

Lemma 5.3 *For every $p > m = \dim M$ and every $f \in L_p\mathscr{D}(M)$*

$$\| Hf - H_0 f \|_{1,p} \leq C(\eta) \| f \|_p .$$

Proof. Choose nonnegative smooth functions ϕ_1 and ϕ_2 such that $\phi_1 + \phi_2 = 1$ and $\phi_i = 1$ near Γ_i for $i = 1, 2$. We would like to apply the basic elliptic estimate (2.23) to $\omega = Hf - H_0 f$. This cannot be done directly since ω does not satisfy the condition $\omega_n = 0$ on Γ_2 neither with respect to the metric g nor with respect to g_0. We note however that $\omega_t = 0$ on Γ_1 is satisfied independently of any metric. It follows that the estimate is applicable to $\phi_1\omega$. We use this together with the fact that $*_0$ is a covariant constant for the metric g_0 and that $*$ interchanges the two boundary conditions to prove the lemma. Thus

$$\begin{aligned}
\| \omega \|_{1,p} &\leq \| \phi_1\omega \|_{1,p} + \| \phi_2\omega \|_{1,p} \\
&\leq \| \phi_1\omega \|_{1,p} + \| \phi_2 * (Hf - H_0 f) \|_{1,p} \\
&\leq \| \phi_1\omega \|_{1,p} + \| \phi_2(*Hf - *_0 H_0 f) \|_{1,p} + \| \phi_2(*_0 - *)H_0 f \|_{1,p} .
\end{aligned} \tag{5.4}$$

In the sequel, we use C, possibly with subscripts, to denote a constant depending only on $\overline{\eta}$, the functions ϕ_i, p, m but independent of the metric g. The value of C in different inequalities need not be the same.

The last term of the inequality above is estimated using Lemmas 5.1 and 5.2 as follows.

$$\| \phi_2(*_0 - *)H_0 f \|_{1,p} \leq C(\eta) \| H_0 f \|_{1,p} \leq C(\eta) \| f \|_p . \tag{5.5}$$

We now apply the estimate in Theorem 2.23 to $\phi_1\omega$, which satisfies the boundary conditions of (2.8), to obtain

$$\begin{aligned}
\| \phi_1\omega \|_{1,p} &\leq C(\| (d + \delta_0)\phi_1\omega \|_p + \| \phi_1\omega \|_2) \\
&\leq C(\| (d + \delta_0)\phi_1\omega \|_p + C(\epsilon) \| f \|_2) \\
&\leq C(\| (d + \delta_0)\omega \|_p + \| \omega \|_p + C(\eta) \| f \|_p).
\end{aligned} \tag{5.6}$$

Lemma 4.4 was used above to bound $\| \phi_1\omega \|_2 \leq \| Hf - H_0 f \|_2$. Now $(d + \delta_0)\omega = \delta_0 Hf = (\delta - \delta_0)Hf$. It follows from Lemmas 5.2 and 5.1 that

$$\| (d + \delta_0)\omega \|_p \leq C(\eta) \| f \|_p . \tag{5.7}$$

Next we estimate the L_p norm of ω. We use the L_2 estimate provided by Lemma 4.4 and the Sobolev inequality of Theorem 2.22. Let $\alpha > 0$ be a parameter whose value will be fixed below and set $r = p/(p-2)$, $t = p/2$.

$$\begin{aligned}
\| \omega \|_p &= \left(\int_M |\omega|^p \right)^{1/p} \\
&\leq C \| \omega \|_\infty^{\frac{p-2}{p}} \| \omega \|_2^{\frac{2}{p}} \\
&\leq C_1\alpha^r \| \omega \|_\infty + C_2\alpha^{-t} \| \omega \|_2 \\
&\leq C_1\alpha^r \| \omega \|_{1,p} + C(\epsilon)\alpha^{-t} \| f \|_p \\
&\leq \frac{1}{4} \| \omega \|_{1,p} + C(\eta) \| f \|_p
\end{aligned} \tag{5.8}$$

provided α is chosen so that $C_1\alpha^r \leq 1/4$.

Combining (5.6), (5.7) and (5.8) we see that

$$\| \phi_1\omega \|_{1,p} \quad \leq \quad \frac{1}{4} \| \omega \|_{1,p} + C(\eta) \| f \|_p \, .$$

An analogous estimate holds for the norm of $\phi_2(*Hf - *_0H_0f) = \phi_2(H*f - H_0*_0 f)$. It is proved in a similar way using the fact that this form satisfies the boundary conditions of (2.8) with Γ_1 and Γ_2 interchanged, so that the main apriori estimate applies, and that $(d+\delta_0)(*Hf - *_0H_0f) = \delta_0 * Hf = (\delta_0 - \delta) * Hf$. We do not repeat the details but state the resulting inequality.

$$\| \phi_2(*Hf - *_0H_0f) \|_{1,p} \quad \leq \quad \frac{1}{4} \| \omega \|_{1,p} + C(\eta) \| f \|_p$$

The last two inequalities and (5.5) yield the Lemma when substituted into (5.4). □

Theorem 5.9 *Suppose* $\eta(g, g_0) < \bar{\eta}$ *and* $p > m$ *are fixed. For every* $f \in L_p\mathscr{D}(M)$,

$$\| \mathscr{G}^{(1)}f - \mathscr{G}_0^{(1)}f \|_{1,p} \leq C(\eta) \| f \|_p \, .$$

Proof. We use a scheme similar to the proof of Theorem 4.5. Let $u = \mathscr{G}^{(1)}f$, $u_0 = \mathscr{G}_0^{(1)}f$, $f \in L_p\mathscr{D}(M)$. Then u and u_0 are perpendicular to $\mathscr{H}(M)$ and $\mathscr{H}_0(M)$ respectively for the inner products defined respectively by g and g_0. u satisfies the equation (4.6) and u_0 is the solution of the analogous equation for g_0. As in the proof of Lemma 5.3 we use cutoff functions ϕ_j, $j = 1, 2$ to write $\zeta = u - u_0$ as the sum $\phi_1\zeta + \phi_2\zeta$ of forms supported in neighborhoods of Γ_1 and Γ_2 respectively.

We remark that this sort of localization would be unnecessary if either $\Gamma_1 = \emptyset$ or $\Gamma_2 = \emptyset$. For example, if $\Gamma_2 = \emptyset$, then $u - u_0$ satisfies the boundary condition $(u - u_0)_t = 0$ for all metrics and the main apriori estimate of Theorem 2.23 can be applied. If, on the other hand, $\Gamma_1 = \emptyset$, the Theorem 5.14 could be deduced *a posteriori* from the case when $\Gamma_2 = \emptyset$ by applying $*$ to all eigenfunctions. For some applications ([17]) one has to consider the more general case of different boundary conditions on different parts of the boundary and for this reason we do this as well.

We furthermore write $*_0\tau\phi_2\zeta = \phi_2 *_0 \tau(u - u_0) = \phi_2(*\tau u - *_0\tau u_0) + \phi_2(*_0\tau u - *\tau u)$ where τ is the algebraic operator introduced in connection with (2.10). We observe that $\phi_1\zeta$ satisfies the boundary conditions of (2.8) and that $\phi_2(*\tau u - *_0\tau u_0)$ has vanishing tangential component on Γ_2 and is identically zero near Γ_1, i.e. satisfies the boundary conditions in (2.10). Clearly

$$\| \zeta \|_{1,p} \quad \leq \quad \| \phi_1\zeta \|_{1,p} + \| \phi_2(*\tau u - *_0\tau u_0) \|_{1,p} + \| \phi_2(*_0\tau u - *\tau u) \|_{1,p} \, . \tag{5.10}$$

The last term can be estimated in terms of $C(\eta) \| f \|_p$ using Lemma 5.2 and the uniform boundedness of the Green's operators. The norms of ϕ_1 *zeta* and

$\phi_2(*\tau u - *_0\tau u_0)$ can be estimated in a way similar to how (5.6) was treated in the proof of Lemma 5.3. We will prove only the estimate for $\phi_2(*\tau u - *_0\tau u_0) = \phi_2\kappa$ since the argument for $\phi_1\zeta$ is very similar. ¿From Theorem 2.23

$$\begin{aligned} \| \phi_2\kappa \|_{1,p} &\leq C \left(\| (d - \delta_0)\phi_2\kappa \|_p + \| \phi_2\kappa_2 \|_2\right) \\ &\leq C \left(\| (d - \delta_0)\kappa \|_{1,p} + \| \kappa \|_p + \| \kappa \|_2\right) . \end{aligned} \tag{5.11}$$

Using the definition of τ and (2.7) we see that $\| (d-\delta_0)\kappa \|_p = \| (d+\delta_0)(u-u_0) \|_p$. In addition, $\kappa = *_0\tau(u-u_0)+(*-*_0)\tau u$ so that $\| \kappa \|_q \leq \| u-u_0 \|_q + \| (*-*_0)u \|_q$ for all $q \geq 1$. Thus (5.11) implies that

$$\| \phi_2\kappa \|_{1,p} \leq C \left(\| (d + \delta_0)(u - u_0) \|_{1,p} + \| u - u_0 \|_p + \| u - u_0 \|_2\right) . \tag{5.12}$$

Now $(d+\delta_0)(u-u_0) = H_0f - Hf + (\delta-\delta_0)u$. so that the first term on the right-hand side of (5.12) can be bounded by $C(\eta) \| f \|_p$ in view of Lemmas 5.3 and 5.2. The second term above is estimated exactly as in (5.8) using Theorem 4.5. The bound $C(\eta) \| f \|_p$ for the third term follows from Theorem 4.5 as well. We thus obtain the following inequality.

$$\| \phi_2\kappa \|_{1,p} \leq \frac{1}{4} \| u - u_0 \|_{1,p} + C(\eta) \| f \|_p$$

As remarked above an analogous estimate holds for $\| \phi_1\zeta \|_p$ which proves the theorem in view of (5.10). □

Corollary 5.13 *If $\eta(g, g_0) \leq \overline{\eta}$, then*

$$\| \mathscr{G}^{(1)}f - \mathscr{G}_0^{(1)}f \|_\infty \leq C(\eta) \| f \|_\infty$$

and

$$\| Hf - H_0f \|_\infty \leq C(\eta) \| f \|_\infty$$

for every $f \in \mathscr{C}^0\mathscr{D}(M)$.

Proof. The first inequality follows from the Sobolev embedding theorem (2.22) and Theorem 5.9 above since

$$\| u - u_0 \|_\infty \leq C_1 \| u - u_0 \|_{1,p} \leq C(\eta) \| f \|_p \leq C(\eta) \| f \|_\infty .$$

The second assertion follows in a similar way from Lemma 5.3. □

Finally, we can state the main theorem. It follows from the corollary the same way as Theorem 4.14 follows from Theorem 4.5. We will not repeat the argument.

Theorem 5.14 *Exact and coexact eigenspaces of the Laplace operator Δ for the boundary conditions in (2.9) converge, when $\eta(g, g_0) \to 0$, to corresponding eigenspaces of Δ_0 as subspaces of $\mathscr{C}^0\mathscr{D}(M)$. More precisely the conclusion of Theorem 4.14 holds with the gap $\hat{\vartheta}$ replaced by the gap $\hat{\vartheta}_\infty$ based on the L_∞ norm.*

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