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Formal Néron models and Weil restriction

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In the present article we deal with problems of base change for Néron models within the context of formal and rigid geometry. Formal Néron models were first introduced in [BS] in order to study uniformization phenomena of abelian varieties on the level of their Néron models. In standard cases, one can pass from ordinary Néron models to formal ones by formal completion along the special fibre. By their definition, Néron models are compatible with étale base change; i. e., if R'/R is an étale extension of discrete valuation rings with corresponding extension of fields of fractions K'/K, and if $\mathcal{U}_R$ (resp. $\mathcal{U}'_{R'}$) is a Néron model of a smooth K-scheme $\mathcal{X}_K$ (resp. $\mathcal{X}_{K'} = \mathcal{X}_K \times_K K'$), the canonical base change morphism $\mathcal{U}_R \times_R R' \longrightarrow \mathcal{U}'_{R'}$ is an isomorphism. In the case of a (finite) ramified extension K'/K, the relationship between $\mathcal{U}_R$ and $\mathcal{U}'_{R'}$ is much more complicated. If $\mathcal{X}_K$ is a group scheme one shows that the Weil restriction $\mathfrak{R}_{R'/R}(\mathcal{U}'_{R'})$ is a Néron model of its generic fibre $\mathfrak{R}_{K'/K}(\mathcal{X}_{K'})$ and that the Néron model $\mathcal{U}_R$ can be interpreted as a group smoothening of the schematic closure of $\mathcal{X}_K \hookrightarrow \mathfrak{R}_{K'/K}(\mathcal{X}_{K'})$ in $\mathfrak{R}_{R'/R}(\mathcal{U}'_{R'})$; cf. [BLR], 7.2/4. In particular, if $\mathcal{X}_K$ is an abelian variety whose semi-stable reduction is known over some finite ramified extension of K, the Néron model of $\mathcal{X}_K$ over R can be obtained by this process. Furthermore, Edixhoven has shown in [Ed] that, for tamely ramified extensions K'/K, the schematic closure of $\mathcal{X}_K$ in $\mathfrak{R}_{R'/R}(\mathcal{U}'_{R'})$ is already smooth so that the smoothening process is unnecessary in this case.

In the following we will deal with the same problem, but now for formal Néron models. To give a few details, let K' be any finite field extension of K. Starting out from a smooth rigid K-group X_K and a formal Néron model $U'_{R'}$ of its K'-extension $X_{K'} = X_K \times_K K'$, we show as main result that a formal Néron model U_R of X_K exists as a formal R-group scheme and, similarly as in

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the scheme case, is obtained as a group smoothening of the schematic closure of

$$q\colon X_K \times_{\mathfrak{R}_{K'/K}(X_{K'})} \mathfrak{R}_{R'/R}\left(U'_{R'}\right)_K \hookrightarrow \mathfrak{R}_{R'/R}\left(U'_{R'}\right)_K$$

in the Weil restriction $\mathfrak{R}_{R'/R}(U'_{R'})$. So, again, the notion of Weil restriction plays an important role in handling the base change problem. It is needed both on the level of formal schemes and rigid spaces, and this is why a substantial part of the present paper is devoted to develop the theory of Weil restrictions of formal schemes and of rigid spaces. The formal scheme case is not very complicated; it is closely related to the scheme case by interpreting formal schemes as limits of ordinary ones. However, the case of rigid spaces is more difficult to handle and we have to face phenomena which do not show up in the classical scheme case. For example, given any affinoid K'-space $Y_{K'}$, the Weil restriction $\mathfrak{R}_{K'/K}(Y_{K'})$ exists, but is not necessarily affinoid. Indeed it may not even be quasi-compact again. Nevertheless, we can show that, in principle, the same representability result as in the scheme situation holds, if one looks at rigid spaces from the point of view of Zariski-Riemann spaces.

Now, in order to derive the above result on formal Néron models and base change, we have to settle several problems. One of these is that the formation of Weil restriction of formal schemes is *not* compatible with passing to generic fibres; this is somehow related to the fact that, when passing from a rigid group to its formal Néron model, in general, a part of the generic fibre is lost. As a consequence, the morphism q is more complicated than the analogous one in the scheme situation. Furthermore, in spite of the above representability result for the rigid Weil restriction, we know very little about its representability on rigid groups. This fact implies that a priori q is only a morphism of functors. Once we have shown that the morphism q is indeed representable, it follows from the universal properties of group smoothening, Weil restriction and schematic closure that the formal scheme obtained as a group smoothening of the schematic closure of $X_K \times_{\mathfrak{R}_{K'/K}(X_{K'})} \mathfrak{R}_{R'/R}(U'_{R'})_K$ in $\mathfrak{R}_{R'/R}(U'_{R'})$ is a formal Néron model of X_K. Finally, we show that the work of Edixhoven on tamely ramified extensions can be adapted to our situation without problems.

1. Weil restriction

In this section R will be a complete valuation ring of height 1 and π an element of $K = \mathrm{Fr}(R)$ with positive norm smaller then 1. All formal R-schemes are supposed to be locally of topologically finite presentation (see [FI]). For results about rigid K-spaces we refer to the monograph [BGR].

We recall the definition of the Weil restriction. Let $\mathfrak{C}$ be a category with fibre products (we will only consider schemes or formal R-schemes or rigid K-spaces) and S an object of $\mathfrak{C}$. Denote by $\mathfrak{C}_S$ the category whose objects are those of $\mathfrak{C}$ above S and the morphisms the S-morphisms, and by $\mathfrak{F}_S$ the category of contravariant functors from $\mathfrak{C}_S$ to (Sets).

Definition 1.1. *Let $h : S' \to S$ be a morphism in $\mathfrak{C}$ and X' an object of $\mathfrak{C}_{S'}$. Consider the following functor*

$$\mathfrak{R}_{S'/S}(X') : (\mathfrak{C}_S)^o \longrightarrow \text{(Sets)}$$

$$T \longrightarrow \mathrm{Mor}_{\mathfrak{C}_{S'}}\left(T \times_S S', X'\right)$$

which is called the Weil restriction functor. If it is representable, the corresponding object will be denoted by $\mathfrak{R}_{S'/S}(X')$ as well. It is called the Weil restriction of X' with respect to h.

More generally, given a functor F' in $\mathfrak{F}_{S'}$ one can consider the push forward functor $h_*F' : (\mathfrak{C}_S)^o \longrightarrow$ (Sets) defined via $h_*F'(T) = F'(T \times_S S')$. It is clear that the functors h_*F' and $\mathfrak{R}_{S'/S}(X')$ coincide in the case where F' is represented by an object X' in $\mathfrak{C}_{S'}$. We list here some properties of the Weil restriction.

(W_1) The functor $\mathfrak{R}_{S'/S}(X')$ is represented by an object Y of $\mathfrak{C}_S$ if and only if there exists an S'-morphism $\Phi_{X'}\colon Y \times_S S' \to X'$ such that for any object T in $\mathfrak{C}_S$ the map $\mathrm{Mor}_{\mathfrak{C}_S}(T, Y) \to \mathrm{Mor}_{\mathfrak{C}_{S'}}(T \times_S S', X')$ given by $f \to \Phi_{X'} \circ (f \times \mathrm{id}_{S'})$ is a bijection. In particular, $\Phi_{X'}$ is associated to id_Y.

(W_2) Given a morphism $\alpha\colon T \to S$ in $\mathfrak{C}$, let $T' = T \times_S S'$, $h' : T' \to T$ be the projection morphism and X' an object of $\mathfrak{C}_{S'}$. There exists an isomorphism

$$\mathfrak{R}_{T'/T}\left(X' \times_{S'} T'\right) \cong \mathfrak{R}_{S'/S}(X') \times_S T$$

where the functor of points $\mathrm{Mor}_{\mathfrak{C}_S}(-, T)$ is still denoted by T. More generally, given morphisms $F' \to G'$ and $T' \to G'$ in $\mathfrak{F}_{S'}$ we have an isomorphism of functors in $\mathfrak{F}_T$

$$h'_*\left(F' \times_{G'} T'\right) \cong h_*(F') \times_{h_*(G')} T.$$

By definition $h_*(F') \times_{h_*(G')} T$ (resp. $F' \times_{G'} T'$) is an object of $\mathfrak{F}_S$ (resp. $\mathfrak{F}_{S'}$). In the formula above we have used the same notation for the corresponding functor in $\mathfrak{F}_T$ (resp. $\mathfrak{F}_{T'}$); i.e., given a functor H in $\mathfrak{F}_S$ and a morphism $\phi : H \to T$, we denote by H also the functor H_T in $\mathfrak{F}_T$ defined via

$$H_T\left(Q \xrightarrow{\eta} T\right) = \left\{\zeta \in H\left(Q \xrightarrow{\alpha\circ\eta} S\right) = \mathrm{Mor}_{\mathfrak{F}_S}(Q, H) \mid \phi \circ \zeta = \eta\right\}.$$

This makes sense because the functor H_T is represented by an object Y, more precisely $Y \to T$, in $\mathfrak{C}_T$ if and only if Y viewed via α as an object of $\mathfrak{C}_S$ represents H.

(W$_3$) If $F' \to H'$ and $G' \to H'$ are morphisms in $\mathfrak{F}_{S'}$, then

$$h_* \left(F' \times_{H'} G'\right) \cong h_* F' \times_{h_* H'} h_*(G').$$

Properties of the scheme-theoretical Weil restriction functors are discussed in [BLR] §7.6. Before proceeding to study the formal and rigid ones, we fix some notation. We will use letters X_K, Y_K, .. for rigid K-spaces, X_R, Y_R, .. for formal R-schemes and calligraphic letters $\mathcal{X}, \mathcal{Y}$, .. for schemes. As an exception, we will write simply X_n for the n-levels of a formal R-scheme X_R and $\mathcal{X}^{\text{an}}$ for the rigid K-space associated to a scheme $\mathcal{X}$ locally of finite type over an affinoid K-algebra. Given any formal R-scheme (locally of topologically finite presentation) X_R, we will denote by X_K the associated rigid K-space. If K' is a finite field extension of K with valuation ring R', we will write $X_{R'}$ for the fibre product $X_R \times_R R'$ and $Y_{K'}$ for the rigid K'-space $Y_K \times_K K'$. In the same way, given an R-algebra A we will shortly write A_K for $A \otimes_R K$. We denote $R_n = R/(\pi)^{n+1}R$.

It is not surprising that the representability of a Weil restriction functor in the formal setting can be deduced from the representability of related scheme-theoretical Weil restriction functors. One has to remember that we can associate to any formal R-scheme X_R a family $X_n = (X_R, \mathcal{O}_n)$ of R_n-schemes having as topological space the same space as X_R and having as canonical sheaf $\mathcal{O}_n = \mathcal{O}_{X_R}/(\pi)^{n+1}\mathcal{O}_{X_R}$. We will call X_n the the n-level of X_R. The scheme X_m can be identified with the fibre product $X_n \times_{R_n} R_m$ for $m \leq n$. For any formal R-scheme X_R, we will call ρ_{mn} the injection morphism $X_m \to X_n$. Given a morphism of formal R-schemes $f : X_R \to Y_R$, it determines a family of morphisms $f_n : X_n \to Y_n$ such that $\rho_{mn} \circ f_m = f_n \circ \rho_{mn}$. Furthermore, given X_R, Y_R and Z_R formal R-schemes there exists a canonical bijection (see [EGA I] 10.6.9)

$$\operatorname{Hom}_{Z_R}(Y_R, X_R) \xrightarrow{\sim} \varprojlim \operatorname{Hom}_{Z_n}(Y_n, X_n).$$

We then start with the representability of the formal Weil restriction functor in the affine case.

Lemma 1.2. *Let A and A' be R-algebras of topologically finite presentation, where A' is a free A-module with base $e_1, \cdots, e_n$. Then*

$$\mathfrak{R}_{A'/A}\left(\operatorname{Spf}\left(\frac{A'\langle x_1, \cdots, x_m\rangle}{\mathfrak{a}}\right)\right) \cong \operatorname{Spf}\left(\frac{A\langle x_{11}, \cdots, x_{1n}, \cdots, x_{mn}\rangle}{\mathfrak{a}^{\text{co}}}\right)$$

where $\mathfrak{a}^{\text{co}}$ is the ideal of coefficients of $\mathfrak{a}$ via the homomorphism $\phi^* : A'\langle x_.\rangle \to A'\langle x_{ji}\rangle$ defined by $\phi^*(x_j) = \sum_{i=1}^n x_{ji} e_i$.

We recall that the ideal $\mathfrak{a}^{\text{co}}$ is generated by the coefficients $f_j \in A\langle x_{ji}\rangle$ of $\phi^*(f) = \sum_{i=1}^n f_i e_i$ as f varies over $\mathfrak{a}$. The proof of the lemma above is simply a translation of the analogous statement in [BLR].

One can also see that Weil restriction (or more generally the push forward) functors are compatible with open and closed immersions for a large family of morphisms h.

Proposition 1.3. *Let $h : S'_R \to S_R$ be a proper morphism of formal R-schemes. Given functors $F', G' : (\mathrm{For}/S'_R)^o \longrightarrow (\mathrm{Sets})$ and a functorial morphism $u' : F' \to G'$, let $h_*(u') : h_*(F') \longrightarrow h_*(G')$ be the canonical morphism associated to u'.*
i) Assume that u' is an open immersion, then $h_*(u')$ is an open immersion.
ii) Assume that u' is a closed immersion and h is finite and locally free. Then the morphism $h_*(u')$ is a closed immersion.

This proposition can also be proved by translating the arguments in [BLR] 7.6/2. We recall that a morphism $F \to G$ of functors $F, G : (\mathrm{For}/S_R)^o \longrightarrow (\mathrm{Sets})$ is called an open (resp. closed) immersion if for every functorial morphism $T_R \to G$, where T_R is an arbitrary formal S_R-scheme, the morphism $F \times_G T_R \to T_R$ obtained by base change with T_R over G is an open (resp. closed) immersion of formal R-schemes.

Using result 1.3 it is now possible to prove the representability of some formal Weil restriction functors in a constructive way, simply by patching the formal schemes which locally represent the Weil restriction functor. We have to suppose that the morphism h is finite and locally free and that the formal R-scheme X'_R satisfies a certain property $(\mathcal{P}_{\mathrm{For}})$. In particular, if the special fibre of X'_R satisfies an analogous property $(\mathcal{P}_{\mathrm{Sch}})$, formal Weil restriction can be described in terms of its n-levels.

Given a scheme $\mathcal{X}$ we will denote by $(\mathcal{P}_{\mathrm{Sch}})$ the following property:

$(\mathcal{P}_{\mathrm{Sch}})$: *Each finite set of points of $\mathcal{X}$ is contained in an open affine subscheme.*

In the same way, given a formal R-scheme X_R we will denote by $(\mathcal{P}_{\mathrm{For}})$ the following property:

$(\mathcal{P}_{\mathrm{For}})$: *Each finite set of points of X_R is contained in an open affine formal subscheme of X_R.*

It is clear that a formal R-scheme X_R satisfies $(\mathcal{P}_{\mathrm{For}})$ if and only if one n-level (and hence each n-level) satisfies $(\mathcal{P}_{\mathrm{Sch}})$.

Theorem 1.4. *Let $h : S'_R \to S_R$ be a finite, locally free morphism of formal R-schemes and X'_R a formal R-scheme over S'_R. If the functor $\Re_{S'_n/S_n}(X'_n)$ is representable for all n, then $\Re_{S'_R/S_R}(X'_R)$ is represented by $\varinjlim \Re_{S'_n/S_n}(X'_n)$.*

In particular, this is true if the formal R-scheme X'_R satisfies property $(\mathcal{P}_{\mathrm{For}})$.

Proof.. All schemes S'_n and X'_n are locally of finite presentation. Then the morphisms $X'_n \to S'_n$ are locally of finite presentation ([EGA I] 6.2.6). If all functors $\Re_{S'_n/S_n}(X'_n)$ are representable, then the corresponding S_n-schemes are locally of finite presentation ([BLR] 7.6/5). Let $n \geq m$. The schemes $\Re_{S'_m/S_m}(X'_m)$ and $\Re_{S'_n/S_n}(X'_n) \times_{S_n} S_m$ are canonically isomorphic and $\varinjlim \Re_{S'_n/S_n}(X'_n)$ is a formal S_R-scheme ([EGA I] 10.6.3). The limit is locally of topologically finite (tf) presentation because all n-levels are locally of finite presentation ([Bo] 1.1.8).

Given any formal S_R-scheme $Y_R = \varinjlim Y_n$ we have the following bijections $\mathrm{Hom}_{S'_R}(Y_R \times_{S_R} S'_R, X'_R) \simeq \varprojlim \mathrm{Hom}_{S'_n}(Y_n \times_{S_n} S'_n, X'_n) \simeq \varprojlim \mathrm{Hom}_{S_n}(Y_n, \Re_{S'_n/S_n}(X'_n)) \simeq \mathrm{Hom}_{S_R}(Y_R, \varinjlim \Re_{S'_n/S_n}(X'_n))$ from which it follows that the formal R-scheme $\varinjlim \Re_{S'_n/S_n}(X'_n)$ represents $\Re_{S'_R/S_R}(X'_R)$.

For the last assertion: the formal scheme X_R satisfies $(\mathcal{P}_{\mathrm{For}})$ if and only if all n-levels satisfy property $(\mathcal{P}_{\mathrm{Sch}})$. Then all $\Re_{S'_n/S_n}(X'_n)$ are representable because of [BLR] 7.6/4. □

We want to list here some properties which are preserved by formal Weil restriction.

Proposition 1.5. *Let X'_R and h be as in the previous theorem. Suppose that the morphism $X'_R \to S'_R$ is*

a) topologically of finite presentation (= quasi-compact)

b) separated

c) smooth

then the same is true for the morphism $\Re_{S'_R/S_R}(X'_R) \to S_R$.
Suppose that X'_R is a formal S'_R-group. Then $\Re_{S'_R/S_R}(X'_R)$ inherits an S_R-group structure.
Suppose that $X'_R = X_R \times_{S_R} S'_R$ for some separated formal S_R-scheme X_R. Then the canonical morphism $\Psi : X_R \to \Re_{S'_R/S_R}(X'_R)$ is a closed immersion.

Proof.. Any morphism of formal R-schemes (locally of tf presentation) is locally of tf presentation ([EGA I] §6.3). Then the condition of tf presentation is equivalent to quasi-compactness. The latter is a topological property and we can argue on n-levels. Assertion $a)$ follows from [BLR] 7.6/5 e), as all morphisms $X_n \to S_n$ are of finite presentation and, hence, quasi-compact. The separatedness and smoothness can be checked on n-levels applying [BLR] 7.6/5 b), h).

The group structure on $\Re_{S'_R/S_R}(X'_R)$ descends from the definition of Weil restriction and property (W_3).

Suppose now that there exists a separated formal S_R-scheme X_R such that $X'_R = X_R \times_{S_R} S'_R$. All its n-levels are separated and the morphisms $\psi_n : X_n \to \Re_{S'_n/S_n}(X'_n)$ are closed immersions. This is sufficient to conclude that ψ is also a closed immersion. □

It is more difficult to analyse the representability of the Weil restriction in the rigid context. We start with an easy example: let K'/K be a finite extension and $e_1, \cdots, e_n$ a fixed base of K' over K. Recall the scheme case $\Re_{K'/K}(\mathrm{Spec}(K'[x])) \cong \mathrm{Spec}(K[x_1, \cdots, x_n])$ (see [BLR] ♧7.6). Given a K-algebra A and a homomorphism

$$f \in \mathrm{Hom}_{K'}\left(K'[x], A \otimes_K K'\right), \quad f(x) = \sum a_i e_i, \ a_i \in A$$

the universal property of Weil restriction associates to f a homomorphism

$$g \in \mathrm{Hom}_K(K[x_1, \cdots, x_n], A), \qquad g(x_i) = a_i.$$

Given now an affinoid K-algebra B and a homomorphism

$$f \in \mathrm{Hom}_{K'}\left(K'\langle x\rangle, B \otimes_K K'\right), \qquad f(x) = \sum b_i e_i,\ b_i \in B$$

it is not possible in general to assign a homomorphism

$$g \in \mathrm{Hom}_K(K\langle x_1, \cdots, x_n\rangle\rangle, B), \qquad g(x_i) = b_i.$$

In fact, any homomorphism of K-affinoid algebras has to be a contraction with respect to the supremum semi-norm ([BGR] 6.2.2/1) but we can have $\| \sum b_i e_i \|_{\sup} \leq 1$ with some $\|b_i\|_{\sup} > 1$. Moreover, if the extension is inseparable we can have $\| \sum b_i e_i \|_{\sup} = 0$ and $\|b_i\|_{\sup}$ not a priori bounded. All this says that the n-dimensional rigid ball does not in general represent the functor $\Re_{K'/K}(\mathrm{Sp}(K'\langle x\rangle)$. In particular, Weil restriction does not commute with formation of generic fibres. In fact, we have already seen that if R' is finite and free over R, then $\Re_{R'/R}(\mathrm{Spf}(R'\langle x\rangle)$ is represented by $\mathrm{Spf}(R\langle x_1, \cdots, x_n\rangle)$.

To find a good candidate to represent the functor $\Re_{K'/K}(\mathrm{Sp}(K'\langle x\rangle)$ we have to characterize the condition $\| \sum b_i e_i \|_{\sup} \leq 1$ in terms of the b_i. This can be done using the coefficients of the characteristic polynomial of $\sum b_i e_i$. These coefficients are obtained canonically from the coefficients $c_i(x_.) \in B[x_1, \cdots, x_n]$ of the characteristic polynomial of $\sum_i x_i e_i \in B'[x_1, \cdots, x_n]$ via the substitution $x_i \to b_i$.

We recall that the spectral value of a polynomial $p(z) = z^n + c_1 z^{n-1} + \cdots + c_n$ with coefficients in a semi-normed ring $(C, \| \ \|)$ is defined as the real number $\sigma(p(z)) = \max_i \|c_i\|^{1/i}$. From now on, we will denote by $\| \ \|$ the supremum semi-norm on any affinoid K-algebra.

Lemma 1.6. *Let B and B' be affinoid K-algebras with $B' = \oplus_{i=1}^n B e_i$ a free B-module. Suppose given an element $b = \sum_i b_i e_i$ of B' with characteristic polynomial $p(b, z) = z^n + c_1(b_.)z^{n-1} + \cdots + c_n(b_.)$. Then $\|b\|_{\sup} = \sigma(p(b, z))$. In particular, $\|b\|_{\sup} \leq 1$ if and only if $\|c_j(b_.)\|_{\sup} \leq 1$ for every $j \in \{1, \cdots, n\}$.*

Proof.. Suppose B to be a finite field extension of K and $b = \sum_i b_i e_i \in B'$ with $p(b, z) = \prod p_i(z)^{\alpha_i}$ and $p_i(z)$ irreducible polynomials in $B[z]$. Let $p_m(b, z) = \prod p_i(z)^{\beta_i}$ with $\beta_i \leq \alpha_i$ be the minimal polynomial of b. Then $\|b\| = \sigma(p_m(b, z)) = \max_i \ \sigma(p_i(z)) = \sigma(p(b, z))$ ([BGR] 6.2.2./2 and 1.5.4/1).

Let now B be a more general affinoid K-algebra. Then

$$\begin{aligned}
\|b\| &= \left\| \sum_i b_i e_i \right\| = \sup_{\mathrm{x}\in \mathrm{Max} B'} \left\| \left(\sum_i b_i e_i \right)(\mathrm{x}) \right\| \\
&= \sup_{\mathrm{y}\in \mathrm{Max} B} \left\{ \max_{\substack{\mathrm{x}\in \mathrm{Max} B' \\ \mathrm{x}\cap B=\mathrm{y}}} \left\| \left(\sum_i b_i e_i \right)(\mathrm{x}) \right\| \right\} \\
&= \sup_{\mathrm{y}\in \mathrm{Max} B} \left\| \sum_i b_i(\mathrm{y}) e_i \right\| = \sup_{\mathrm{y}\in \mathrm{Max} B} \sigma \left(p \left(\sum_i b_i(\mathrm{y}) e_i, z \right) \right) \\
&= \sup_{\mathrm{y}\in \mathrm{Max} B} \left\{ \max_i \|c_i(b.(\mathrm{y}))\|^{\frac{1}{i}} \right\} = \max_i \|c_i(b.)\|^{\frac{1}{i}} = \sigma(p(b,z)).
\end{aligned}$$

In the fourth equality we have applied lemma 3.8.1/5 in [BGR] and in the fifth the result already obtained for field extensions. As usual $(\sum_i b_i e_i)(\mathrm{x})$ denotes the image of $\sum_i b_i e_i$ in the field B'/x and $b_i(\mathrm{y})$ the image of b_i in B/y. □

It is then possible to control the supremum semi-norm after tensor product and hence to describe the rigid Weil restriction in the affinoid case.

Proposition 1.7. *Let A and A' be affinoid K-algebras with A' a free A-module of base $e_1 \cdots e_n$. Then*

$$\Re_{A'/A}(\mathrm{Sp}(A'\langle x\rangle)) \cong \varinjlim \mathrm{Sp}(C_\lambda)$$

with λ integers, $\lambda \geq 0$ and $C_\lambda = A\langle \pi^\lambda x_1, \cdots, \pi^\lambda x_n\rangle\langle c_1(x_.), \cdots, c_n(x_.)\rangle$, where $c_j(x_.) \in A[x_1, \cdots x_n]$ are the coefficients of the characteristic polynomial of $\sum_i x_i e_i \in A'[x_1, \cdots x_n]$.

The space $\Re_{A'/A}(\mathrm{Sp}(A'\langle x\rangle))$ can be viewed as an open subspace of the affine n-space over A, explicitly $\{\mathrm{y} \in (\mathrm{Spec}(A[x_1, \cdots, x_n]))^{\mathrm{an}}$ such that $| c_j(x_.)(\mathrm{y}) | \leq 1$ for all integers $1 \leq j \leq n \}$. It is an increasing union of affinoid spaces with $j_{\lambda\mu} : \mathrm{Sp}(C_\lambda) \to \mathrm{Sp}(C_\mu)$ $(\mu \geq \lambda)$ the canonical open immersions.

Proof.. Denote by $j^*_{\lambda\mu}$ the homomorphism associated to $j_{\lambda\mu}$ and by $X_K^{(\lambda)}$ the affinoid K-space $\mathrm{Sp}(C_\lambda)$. For any integer $\lambda \geq 0$ consider the homomorphism

$$\Phi^*_\lambda : A'\langle x\rangle \to C_\lambda \otimes_A A', \qquad \Phi^*_\lambda(x) = \sum_{i=1}^n x_i e_i.$$

It is well-defined as $\| \sum_i x_i e_i \| \leq 1$ if and only if $\|c_j(x_.)\| \leq 1$ for all indices j. As $(j^*_{\lambda\mu} \otimes \mathrm{id}_{A'}) \circ \Phi^*_\mu = \Phi^*_\lambda$ the family $(\Phi^*_\lambda)_{\lambda\geq 0}$ defines a morphism $\Phi \in \mathrm{Hom}_{A'}(\varinjlim X_K^{(\lambda)} \times_A A', \mathrm{Sp}(A'\langle x\rangle))$.

To prove that $\varinjlim X_K^{(\lambda)}$ represents the Weil restriction functor we have to prove that for any K-space Y_K over $\mathrm{Sp}(A)$ the map

$$\begin{aligned}\mathrm{Hom}_A\left(Y_K, \varinjlim X_K^{(\lambda)}\right) &\longrightarrow \mathrm{Hom}_{A'}\left(Y_K \times_A A', \mathrm{Sp}(A'\langle x\rangle)\right)\\ \psi &\longrightarrow \Phi\circ(\psi\times\mathrm{id})\end{aligned}$$

is bijective. It is sufficient to show it for $Y_K = \mathrm{Sp}(B)$ affinoid.

For the injectivity: let ψ_1 and ψ_2 be morphisms in $\mathrm{Hom}_A(Y_K, \varinjlim X_K^{(\lambda)})$ such that $\Phi\circ(\psi_1\times\mathrm{id}) = \Phi\circ(\psi_2\times\mathrm{id})$. There exists a positive integer μ such that both ψ_1 and ψ_2 factor through $X_K^{(\mu)}$. If $\psi_1^*, \psi_2^*\colon C_\mu \to B$ are the corresponding homomorphisms, $\Phi\circ(\psi_1\times\mathrm{id}) = \Phi\circ(\psi_2\times\mathrm{id})$ implies $(\psi_1^*\otimes\mathrm{id}_{A'})\circ\Phi_\mu^* = (\psi_2^*\otimes\mathrm{id}_{A'})\circ\Phi_\mu^*$. As the $e_1,\cdots,e_n$ are free generators, the equalities

$$\begin{aligned}\sum_i \psi_1^*(x_i)e_i &= \left(\left(\psi_1^*\otimes\mathrm{id}_{A'}\right)\circ\Phi_\mu^*\right)(x) = \left(\left(\psi_2^*\otimes\mathrm{id}_{A'}\right)\circ\Phi_\mu^*\right)(x)\\ &= \sum_i \psi_2^*(x_i)e_i\end{aligned}$$

give $\psi_1^*(x_i) = \psi_2^*(x_i)$ for any index $i\in\{1,\cdots,n\}$ and hence $\psi_1=\psi_2$.

For the surjectivity: let φ' be a morphism in $\mathrm{Hom}_{A'}(Y_K\times_A A', \mathrm{Sp}(A'\langle x\rangle))$. It corresponds to a homomorphism $\varphi'^*: A'\langle x\rangle \longrightarrow B\otimes_A A' = \oplus_{i=1}^n Be_i$ given by $\varphi'^*(x) = \sum_i b_ie_i$ with $b_i\in B$ and $\|\sum_i b_ie_i\|\le 1$. Let $\tilde\lambda$ be the minimal non-negative integer such that $\|\pi^{\tilde\lambda}b_i\|\le 1$ for all i. For any $\lambda\ge\tilde\lambda$ define $\varphi_\lambda^*:\ C_\lambda\to B$ with $\varphi_\lambda^*(x_i) = b_i$. It is well-defined by 1.6. The family $(\varphi_\lambda^*)_{\lambda\ge\tilde\lambda}$ gives a morphism $\varphi\in\mathrm{Hom}_A(Y_K,\varinjlim X_K^{(\lambda)})$ such that $\Phi\circ(\varphi\times\mathrm{id}) = \varphi'$. In particular, Φ is the unique morphism associated to $\mathrm{id}_{\Re_{A'/A}(\mathrm{Sp}(A'\langle x\rangle))}$. □

We can now consider more general affinoid spaces.

Proposition 1.8. *Let A and A' be affinoid K-algebras with A' a free A-module of base $e_1,\cdots,e_n$. Define $D_\lambda = C_\lambda^{\hat\otimes m}$ as the complete tensor product of m copies of the affinoid K-algebra C_λ defined in 1.7. Explicitly*

$$D_\lambda = A\langle\pi^\lambda x_{11},\cdots,\pi^\lambda x_{1n},\cdots,\pi^\lambda x_{mn}\rangle\langle c_1(x_{1.}),\cdots,c_n(x_{1.}),c_1(x_{2.}),\cdots,c_n(x_{m.})\rangle.$$

Then we can prove the following:

i) $$\Re_{A'/A}(\mathrm{Sp}(A'\langle x_1,\cdots,x_m\rangle)) \cong \prod_{i=1}^m \Re_{A'/A}\left(\mathrm{Sp}\left(A'\langle x_i\rangle\right)\right) \cong \varinjlim \mathrm{Sp}(D_\lambda).$$

The homomorphisms $\Phi_\lambda^*: A'\langle x_1,\cdots,x_m\rangle\to D_\lambda\otimes_A A$ defined via $\Phi_\lambda^*(x_h) = \sum_i x_{hi}e_i$ give the unique morphism Φ associated to $\mathrm{id}_{\Re_{A'/A}(\mathrm{Sp}(A'\langle x_1,\cdots,x_m\rangle))}$.

ii) Given an ideal $\mathfrak{a} = (f_1,\cdots,f_r)$ in $A'\langle x_1,\cdots,x_m\rangle$

$$\Re_{A'/A}\left(\mathrm{Sp}\left(\frac{A'\langle x_1,\cdots,x_m\rangle}{\mathfrak{a}}\right)\right)\cong\varinjlim\mathrm{Sp}\left(\frac{D_\lambda}{\mathfrak{a}^{\mathrm{co}}}\right)$$

where $\mathfrak{a}^{co} = (f_{11}, \cdots, f_{1n}, f_{21}, \cdots, f_{rn})$ *is generated by the coefficients of* $\Phi_\lambda^*(f_j) = \sum_i f_{ji} e_i$ *in* D_λ.
iii) If K'/K *is a finite separable field extension and* $A' = A \otimes_K K'$*, there exists an integer* $\mu \geq 0$ *such that*

$$\Re_{A'/A}\left(\mathrm{Sp}\left(\frac{A'\langle x_1, \cdots, x_m\rangle}{\mathfrak{a}}\right)\right) \cong \mathrm{Sp}\left(\frac{D_\mu}{\mathfrak{a}^{co}}\right).$$

iv) If all generators f_j *of* $\mathfrak{a}$ *are elements of* $A\langle x_1, \cdots, x_m\rangle$*, the unique morphism* $\Psi_{\mathfrak{a}}$ *associated to* $\mathrm{id}_{\mathrm{Sp}(A'\langle x_1, \cdots, x_m\rangle/\mathfrak{a})}$ *is a closed immersion.*

Proof.. The assertion in $i)$ descends from the fact (see (W_3)) that Weil restriction commutes with fibre products. Hence

$$\Re_{A'/A}(\mathrm{Sp}(A'\langle x_1, \cdots, x_m\rangle)) \cong \Re_{A'/A}\left(\prod_{i=1}^m \mathrm{Sp}\left(A'\langle x_i\rangle\right)\right) \cong \prod_{i=1}^m \left(\varinjlim \mathrm{Sp}(C_\lambda)\right)_i.$$

As any morphism $T_K \to \varinjlim \mathrm{Sp}(C_\lambda)$, with T_K an affinoid K-space, factors through an affinoid space $\mathrm{Sp}(C_\lambda)$ for an index λ large enough, we have also $\prod \varinjlim \mathrm{Sp}(C_\lambda) \cong \varinjlim \prod \mathrm{Sp}(C_\lambda) \cong \varinjlim \mathrm{Sp}(D_\lambda)$.

For the second assertion one repeats the arguments in the proof of proposition 1.7. This time we have to consider also n-dimensional Tate algebras modulo an ideal. The ideal of coefficients $\mathfrak{a}^{co}$ we have introduced is defined in the same way as in [BLR] 7.6/4 (or in lemma 1.2).

For $iii)$: the functor $\Re_{K'/K}(\mathrm{Sp}(K'\langle x_.\rangle))$ is represented by a quasi-compact rigid K-space because after base extension $\mathrm{Sp}(K'') \to \mathrm{Sp}(K)$, it becomes isomorphic to the product of n-copies of $\mathrm{Sp}(K''\langle x_.\rangle))$, where K'' is the smallest normal extension of K containing K' (in a fixed separable closure of K') and n the degree of K'/K. Hence the functor $\Re_{K'/K}(\mathrm{Sp}(K'\langle x_.\rangle)) \times_K A \cong \Re_{A'/A}(\mathrm{Sp}(A'\langle x_.\rangle))$ is represented by a quasi-compact rigid space over $\mathrm{Sp}(A)$. This means that $\varinjlim \mathrm{Sp}(D_\lambda)$ must coincide with $\mathrm{Sp}(D_\mu)$ for some integer μ and that $\Re_{A'/A}(\mathrm{Sp}(A'\langle x_.\rangle/\mathfrak{a}))$ is represented by $\mathrm{Sp}(D_\mu/\mathfrak{a}^{co})$.

The fourth assertion can be proved as in the classical scheme situation. We recall only the definition of the morphism $\Psi_{\mathfrak{a}}$. If $\Gamma_i \in A$ are the coefficients of $1 = \sum_i \Gamma_i e_i$ in A' and $\bar{\lambda} \geq 0$ is the minimal non-negative integer such that $\max_i \|\pi^{\bar{\lambda}} \Gamma_i\| \leq 1$, we can consider the morphism $\Psi : \mathrm{Sp}(A\langle x_.\rangle) \to \varinjlim \mathrm{Sp}(D_\lambda)$ given by the family of surjective homomorphisms

$$\Psi_\lambda^* : D_\lambda \to A\langle x_.\rangle, \qquad \Psi_\lambda^*(x_{ji}) = \Gamma_i x_j \quad \text{for } \lambda \geq \bar{\lambda}.$$

As $(\Psi_\lambda^* \otimes \mathrm{id}_{A'}) \circ \Phi_\lambda^* = \mathrm{id}_{A'\langle x_.\rangle}$, the morphism Ψ is the unique morphism such that $\mathrm{id}_{\mathrm{Sp}(A'\langle x_.\rangle)} = \Phi \circ (\Psi \times \mathrm{id})$. The morphism Ψ is a closed immersion. Calling $\Phi_{\mathfrak{a}}$ the morphism induced by Φ and $\Psi_{\mathfrak{a}} : \mathrm{Sp}(A\langle x_.\rangle/\mathfrak{a}) \to \varinjlim \mathrm{Sp}(D_\lambda/\mathfrak{a}^{co})$

the one induced by Ψ. Then also the morphism $\Psi_{\mathfrak{a}}$ is a closed immersion and $\mathrm{id}_{\mathrm{Sp}(A'\langle x_\cdot\rangle/\mathfrak{a})} = \Phi_{\mathfrak{a}} \circ (\Psi_{\mathfrak{a}} \times \mathrm{id})$. □

At this point one would like to proceed by glueing the spaces which represent locally the Weil restriction functor and then find a sufficient condition such that this glued space represents our functor. What has to be proved is that Weil restriction is compatible with open immersions, at least for h a finite and locally free morphism. As in the formal scheme setting, we can prove it in functorial language.

Proposition 1.9. *Let $h : S'_K \to S_K$ be a proper and flat morphism of rigid K-spaces. Given functors $F', G' : (\mathrm{Rig}/S'_K)^o \longrightarrow (\mathrm{Sets})$ and a functorial morphism $u' : F' \to G'$, let $h_*(u') : h_*(F') \longrightarrow h_*(G')$ be the canonical morphism associated to u'.*
i) Assume that u' is an open immersion. Then $h_*(u')$ is an open immersion.
ii) Assume that u' is a closed immersion and h is finite and locally free. Then $h_*(u')$ is a closed immersion.

As immediate consequences we have:

Corollary 1.10. *Let h be as above. Suppose that $u : U_K \to V_K$ is an open (resp. closed) immersion of S'_K-spaces and $\mathfrak{R}_{S'_K/S_K}(V_K)$ is representable. Then $\mathfrak{R}_{S'_K/S_K}(U_K)$ is represented by an open (resp. closed) subspace of $\mathfrak{R}_{S'_K/S_K}(V_K)$.*

Proposition 1.9/ii) implies also an assertion about separatedness. In fact, (W_3) gives an isomorphism $\mathfrak{R}_{S'_K/S_K}(X'_K) \times_{S_K} \mathfrak{R}_{S'_K/S_K}(X'_K) \cong \mathfrak{R}_{S'_K/S_K}(X'_K \times_{S'_K} X'_K)$.

Corollary 1.11. *Let $h : S'_K \to S_K$ be a finite and locally free morphism of rigid K-spaces and X'_K a separated rigid S'_K-space. If $\mathfrak{R}_{S'_K/S_K}(X'_K)$ is representable then the corresponding S_K-space is separated.*

Let us now prove proposition 1.9.

Proof.. Consider a rigid S_K-space T_K and a functorial morphism $T_K \to h_*(G')$. Define $F'_{T'_K} = F' \times_{G'} T'_K$ and $F_{T_K} = h_*F \times_{h_*G} T_K$, where T'_K is the space $T_K \times_{S_K} S'_K$. Let h' be the projection morphism $T'_K \to T_K$ and consider the following diagram

$$\begin{array}{ccc} F' \times_{G'} T'_K = F'_{T'_K} & \longrightarrow & T'_K \\ & & \big\downarrow h' \\ h_*F \times_{h_*G} T_K = F_{T_K} & \longrightarrow & T_K \end{array}$$

The first arrow is an open (resp. closed) immersion by hypothesis. We have to prove that the second arrow is an open (resp. closed) immersion as well. Write W_K for the open (resp. closed) subspace of T'_K such that $F'_{T'_K}$ is represented by W_K. We have the following isomorphisms of functors from $(\mathrm{Rig}/T_K)^o$ to (Sets)

$$\begin{aligned}\Re_{T'_K/T_K}(W_K) &= h'_*(W_K) = h'_*\left(F'_{T'_K}\right)\\ &= h'_*\left(F' \times_{G'} T'_K\right) \overset{(\mathrm{W}_2)}{\cong} h_*(F') \times_{h_*G'} T_K = F_{T_K}.\end{aligned}$$

If we can prove that the functor $\Re_{T'_K/T_K}(W_K)$ is represented by an open (resp. closed) subspace of T_K, the same rigid space represents also F_{T_K}. The lemma below gives the desired result. □

Lemma 1.12. *Let $h' : T'_K \to T_K$ be a proper and flat (resp. finite and locally free) morphism of rigid K-spaces. For any open (resp. closed) subspace U'_K of T'_K the functor $\Re_{T'_K/T_K}(U'_K)$ is represented by an open (resp. closed) subspace of T_K.*

Proof.. For the statement about closed subspaces we reduce to 1.8. Let $\{V_{K,i}\}_{i\in I}$ be an admissible covering of T_K by open affinoid subspaces and denote by $V'_{K,i}$ the open subspace $(h')^{-1}(V_{K,i})$ of T'_K. By property (W_2) we have isomorphisms $\Re_{T'_K/T_K}(U'_K) \times_{T_K} V_{K,i} \cong \Re_{V'_{K,i}/V_{K,i}}(U'_K \times_{T'_K} V'_{K,i})$. These allow us to restrict the proof to the affinoid case $T_K = \mathrm{Sp}(A)$ and $T'_K = \mathrm{Sp}(A')$ with A' a finite and free A-module. Let $e_1, \cdots, e_n$ be a family of free generators of A' over A. The closed subspace U'_K is of the form $\mathrm{Sp}(A'/\mathfrak{a})$ for some ideal $\mathfrak{a} = (f_1, \cdots, f_r)$. From 1.8 (case m=0) we know that the functor $\Re_{T'_K/T_K}(U'_K) = \Re_{A'/A}(\mathrm{Sp}(A'/\mathfrak{a}))$ is represented by the closed subspace $\mathrm{Sp}(A/\mathfrak{a}^{\mathrm{co}})$ of T_K with $\mathfrak{a}^{\mathrm{co}} = (f_{ij})$ and f_{ij} coefficients of $f_i = \sum_j f_{ij} e_j$ in A.

The assertion about open subspaces requires more work. We can suppose again that T_K is an affinoid K-space. Consider the set $R(U'_K)$ of all points of T_K whose h'-fibre is contained in U'_K. If we can prove that $R(U'_K)$ is admissible open, it is immediate to check that $R(U'_K)$, with the canonical structure of rigid space induced by T_K, represents $\Re_{T'_K/T_K}(U'_K)$.

If U'_K is a Zariski-open subset of T'_K and we denote by U'^c_K the (closed) analytic subset $T'_K - U'_K$, then $h'(U^c_K)$ is a closed analytic subset of T_K and its complement is $R(U'_K)$. For more general admissible open subsets, it is no more possible to proceed in this way because the complement of an admissible open subset is not in general an analytic subset. To overcome this problem, we will switch from the rigid to the formal level. It is immediate to see that we can restrict to the case where all spaces $U'_K \hookrightarrow T'_K \to T_K$ are separated and quasi-compact. By [Ra], we can find morphisms of admissible formal R-schemes $i : U'_R \to T'_R$ and $\psi : T'_R \to T_R$ such that the associated rigid morphisms are the ones above. In particular, we can choose as i an open immersion ([FI] 4.4) and ψ flat ([FII]

5.2). The morphism ψ is automatically proper as h' is proper. This is the easy direction in the equivalence between properness in the rigid and formal setting.

We want to prove that the quasi-compact open subspace $W_K = (\psi(U_R'^c)^c)_K$ of T_K, i.e. the rigid space associated to the complement of the image of the closed subset $T_R' - U_R'$, represents $\Re_{T_K'/T_K}(U_K')$.

It is clear that $W_K \subseteq R(U_K')$ as the fibre in T_K' of any rigid point of W_K is contained in U_K'. For the converse: let $x \in R(U_K')$ be a rigid point of T_K whose fibre is contained in U_K' and call $\bar{x}$ its specialization in T_R. The point x corresponds to a closed immersion $\mathrm{Sp}(L) \to T_K$ where L is a finite field extension of K or, in the same way, to a closed immersion $\mathrm{Spf}(B) \to T_R$ where B is an integral local ring with quotient field L. Then, the point $\bar{x}$ is the image of the closed point of $\mathrm{Spf}(B)$. The fibre product $\mathrm{Spf}(B) \times_{T_R} T_R'$ is an admissible model of $\mathrm{Sp}(L) \times_{T_K} T_K'$, i.e. of the fibre of x in T_K'. Usually one has to divide out the π-torsion but in this case the flatness of ψ implies that $\mathrm{Spf}(B) \times_{T_R} T_R'$ already has no π-torsion. Hence any closed point of T_R' above the closed point $\bar{x}$ is the specialization of some point of T_K' above x. As the fibre of x is all contained in U_K', the fibre of $\bar{x}$ is all contained in U_R'. This says that $\bar{x}$ is a point of $\psi(U_R'^c)^c$ and that $x \in W_K$. □

We are now allowed to glue what we obtained locally. Even if we have proved the representability in 1.8 only for S_K and S_K' affinoid, we are interested in morphisms h which involve not only affinoid K-spaces. Therefore, we introduce suitable coverings $\mathfrak{S}$ of S_K and $\mathfrak{U}$ of X_K' which permit to use locally the results in 1.8. If S_K and S_K' are affinoid and h is free then the covering $\mathfrak{U}$ is simply the covering of X_K' given by all its open affinoid subsets.

Theorem 1.13. *Let $h\colon S_K' \to S_K$ be a finite and locally free morphism of rigid K-spaces and X_K' a rigid S_K'-space. Let furthermore $\mathfrak{S}$ be the covering of S_K given by all open affinoid subspaces V_K such that $h^{-1}(V_K)$ is finite and free over V_K and $\mathfrak{U}$ the set of all open affinoid subspaces of $X_K' \times_{S_K} V_K$ as V_K varies over $\mathfrak{S}$. It is possible to glue the rigid Weil restrictions $\Re_{S_K'/S_K}(U_K')$ as U_K' varies over $\mathfrak{U}$ obtaining a rigid S_K-space $R_{S_K'/S_K}(X_K')$. If X_K' is separated then $R_{S_K'/S_K}(X_K')$ is separated.*

Proof.. First of all observe that, if V_K is in $\mathfrak{S}$ and we denote by V_K' the open subspace $h^{-1}(V_K)$ of S_K', then $\Re_{S_K'/S_K}(U_K') \cong \Re_{V_K'/V_K}(U_K')$ for all open subspaces U_K' of $X_K' \times_{S_K'} V_K'$. This descends from property (W_2). Then the functor $\Re_{S_K'/S_K}(U_K')$ is representable for all affinoid spaces U_K' in $\mathfrak{U}$.

For the glueing we apply proposition 9.3.2./1 in [BGR] with:

- $X_i = \Re_{S_K'/S_K}(U_K^i)$ as U_K^i varies over $\mathfrak{U}$.
- $X_{ij} = \Re_{S_K'/S_K}(U_K^i \cap U_K^j)$, $X_{ij} \hookrightarrow X_i$, the canonical open immersion (1.9).
- $\varphi_{ij} : X_{ij} \to X_{ji}$ the obvious isomorphism.

The unique point that has really to be checked is the cocycle condition: $\varphi_{ijl} : X_{ij} \cap X_{il} \to X_{ji} \cap X_{jl}$ (induced by φ_{ij}) satisfy $\varphi_{ijl} = \varphi_{lji} \circ \varphi_{ilj}$. This follows from

$$\begin{aligned}
X_{ij} \cap X_{il} &\cong \mathfrak{R}_{S'_K/S_K}\left(U^i_K \cap U^j_K\right) \times_{\mathfrak{R}_{S'_K/S_K}(U^i_K)} \mathfrak{R}_{S'_K/S_K}\left(U^i_K \cap U^l_K\right) \\
&\cong \left(\mathfrak{R}_{S'_K/S_K}\left(U^i_K\right) \times_{\mathfrak{R}_{\cdot}(X'_K)} \mathfrak{R}_{S'_K/S_K}\left(U^j_K\right)\right) \times_{\mathfrak{R}_{\cdot}(U^i_K)} \left(\mathfrak{R}_{S'_K/S_K}\left(U^i_K\right)\right. \\
&\qquad \left.\times_{\mathfrak{R}_{\cdot}(X'_K)} \mathfrak{R}_{S'_K/S_K}\left(U^l_K\right)\right) \\
&\cong \mathfrak{R}_{S'_K/S_K}\left(U^i_K\right) \times_{\mathfrak{R}_{\cdot}(X'_K)} \mathfrak{R}_{S'_K/S_K}\left(U^j_K\right) \times_{\mathfrak{R}_{\cdot}(X'_K)} \mathfrak{R}_{S'_K/S_K}\left(U^l_K\right) \\
&\cong \mathfrak{R}_{S'_K/S_K}\left(U^i_K \cap U^j_K \cap U^l_K\right)
\end{aligned}$$

where we have applied several times property (W_3), reading the intersections in terms of fibre products.

There exists furthermore a morphism

$$\Lambda : R_{S'_K/S_K}\left(X'_K\right) \times_{S_K} S'_K \longrightarrow X'_K$$

obtained by glueing the $\Phi_{U^i_K} : \mathfrak{R}_{S'_K/S_K}(U^i_K) \times_{S_K} S'_K \to U^i_K$, where $\Phi_{U^i_K}$ is the unique morphism associated to $\mathrm{id}_{\mathfrak{R}_{S'_K/S_K}(U^i_K)}$ ([BGR] 9.3.2/1).

Suppose that X'_K is a separated S'_K-space. This means that the diagonal morphism $X'_K \to X'_K \times_{S'_K} X'_K$ is a closed immersion. Its associated morphism

$$\mathfrak{R}_{S'_K/S_K}\left(X'_K\right) \longrightarrow \mathfrak{R}_{S'_K/S_K}\left(X'_K \times_{S'_K} X'_K\right) \overset{(\mathrm{W}_3)}{\cong} \mathfrak{R}_{S'_K/S_K}\left(X'_K\right) \times_{S_K} \mathfrak{R}_{S'_K/S_K}\left(X'_K\right)$$

is a closed immersion (of functors) by 1.9. In particular,

$$\mathfrak{R}_{S'_K/S_K}(U_K \cap V_K) \longrightarrow \mathfrak{R}_{S'_K/S_K}(U_K) \times_{S_K} \mathfrak{R}_{S'_K/S_K}(V_K)$$

is a closed immersion for any U_K, V_K open affinoid subspaces of X'_K in $\mathfrak{U}$. By definition of $R_{S'_K/S_K}(X'_K)$,

$$\mathfrak{R}_{S'_K/S_K}(U_K \cap V_K) \cong \mathfrak{R}_{S'_K/S_K}(U_K) \times_{R_{S'_K/S_K}(X'_K)} \mathfrak{R}_{S'_K/S_K}(V_K)$$

and then the diagonal morphism $R_{S'_K/S_K}(X'_K) \to R_{S'_K/S_K}(X'_K) \times_{S_K} R_{S'_K/S_K}(X'_K)$ is a closed immersion too. □

We will now give a condition for the representability of rigid Weil restriction functors. It is easy to see that if h is a finite and free morphism of affinoid spaces and X'_K is an affinoid space or a rigid Stein space, then X'_K satisfies it.

Proposition 1.14. *Let the hypothesis be as in 1.13. The rigid space $R_{S'_K/S_K}(X'_K)$ represents $\mathfrak{R}_{S'_K/S_K}(X'_K)$ if and only if for any (affinoid) S_K-space T_K and any S'_K-morphism $\phi : T_K \times_{S_K} S'_K = T'_K \to X'_K$ the covering $\phi^*\mathfrak{U} = \{\phi^{-1}(U_K)\}_{U_K \in \mathfrak{U}}$ admits a refinement $\mathfrak{V}' = \{W_K \times_{S_K} S'_K\}_{W_K \in \mathfrak{V}}$ with $\mathfrak{V}$ an admissible covering of T_K.*

Proof.. Suppose that the condition on the covering $\mathfrak{U}$ is satisfied and let the space T_K, the morphism ϕ and the covering $\mathfrak{V}$ be as in the hypothesis. For any $W_K \in \mathfrak{V}$ there exists an element $U_K^{(W)} \in \mathfrak{U}$ with $\phi(W_K \times_{S_K} S'_K) \subset U_K^{(W)}$ and a unique morphism $\psi_W : W_K \to \mathfrak{R}_{S'_K/S_K}(U_K^{(W)})$ such that $\phi_{|W_K \times_{S_K} S'_K} = \Lambda \circ (\psi_W \times \mathrm{id}_{S'_K})$, where Λ was defined in 1.13/proof. The unique morphism $\psi : T_K \to R_{S'_K/S_K}(X'_K)$ such that $\phi = \Lambda \circ (\psi \times \mathrm{id}_{S'_K})$ is simply given by the glueing of the ψ_W. By property (W_1), this is sufficient to conclude that $R_{S'_K/S}(X'_K)$ represents $\mathfrak{R}_{S'_K/S_K}(X'_K)$ and $\Lambda = \Phi_{X'_K}$.

For the converse: suppose that the space $R_{S'_K/S_K}(X'_K)$ represents $\mathfrak{R}_{S'_K/S_K}(X'_K)$ with $\Lambda = \Phi_{X'_K}$. Choose a rigid K-space T_K and a morphism $\phi : T'_K \to X'_K$ as in the hypothesis. There exists a unique $\psi : T_K \to R_{S'_K/S_K}(X'_K)$ which satisfies $\phi = \Lambda \circ (\psi \times \mathrm{id}_{S'_K})$. The extension to T'_K of the admissible covering $\{\psi^{-1}(\mathfrak{R}_{S'_K/S_K}(U_K)\}_{U_K \in \mathfrak{U}}$ of T_K is trivially a refinement of $\phi^*\mathfrak{U}$. □

The condition above is not easy to check. We look for a weaker but sufficient condition. We could try to translate properties ($\mathcal{P}_{\mathrm{Sch}}$) and ($\mathcal{P}_{\mathrm{For}}$) with the following

"Any finite set of rigid points is contained in some open affinoid subspace".

This is already a step in the right direction. The following example shows that if a rigid K-space does not satisfy the property above, it is not to be expected that the space introduced in 1.13 represents Weil restriction.

Example 1.15. Let $S_K = \mathrm{Sp}(K)$ and $S'_K = \mathrm{Sp}(K')$ with K' a finite Galois extension of K of degree $n \geq 2$. Consider the disc $D = \mathrm{Sp}(K'\langle x \rangle)$ and choose a K'-valued point as center O. Let X'_K be the glueing of two copies of D along the open subspace $D - \{O\}$. X'_K is a disc with a double center O_1 and O_2. It is obviously not quasi-separated and there is no open affinoid subspace of X'_K containing both O_1 and O_2. It is evident that $R_{S'_K/S_K}(X'_K)$ can not represent $\mathfrak{R}_{S'_K/S_K}(X'_K)$. If this was the case then any morphism $\mathrm{Sp}(K' \otimes_K K') \to X_K$ whose image contains the points O_1 and O_2 would factor through an open affinoid subspace of X'_K and this is absurd.

Unfortunately the property above is not sufficient to say that the glueing space in 1.13 represents the rigid Weil restriction. This comes from the fact that even if $\mathfrak{R}_{S'_K/S_K}(X'_K)$ is representable and the morphism $\lambda : R_{S'_K/S_K}(X'_K) \to \mathfrak{R}_{S'_K/S_K}(X'_K)$ is locally an open immersion and a bijection between the rigid points, it may happen that λ is not an isomorphism. We can explain this fact saying that there are not sufficiently many rigid points to characterize their rigid space. For this reason we introduce the *Zariski-Riemann space*. We will consider only rigid K-spaces which have an admissible model, i.e. a flat formal R-scheme whose rigid fibre is the space we are considering. This is the case if the rigid space is quasi-separated and quasi-compact ([Ra]) or more generally paracompact ([Bo]). For properties of Zariski-Riemann spaces we refer to [Fu] or [Bo]. We recall that given a flat formal R-scheme X_R then the ZR-space associated to X_R is

$$<X> = \varprojlim X_{\mathcal{A}}$$

where $X_{\mathcal{A}}$ runs over all admissible blowing-ups $\phi_{\mathcal{A}} : X_{\mathcal{A}} \to X_R$ of X_R with respect to an open coherent ideal $\mathcal{A} \subseteq \mathcal{O}_{X_R}$. It is clear that there exists a (unique) X_R-morphism $\phi_{\mathcal{A}\mathcal{B}} : X_{\mathcal{B}} \to X_{\mathcal{A}}$ if and only if $\mathcal{A}\mathcal{O}_{X_{\mathcal{B}}}$ is invertible in $X_{\mathcal{B}}$ and the morphism $\phi_{\mathcal{A}\mathcal{B}}$ is necessarily the blowing-up with respect to $\mathcal{B}\mathcal{O}_{X_{\mathcal{A}}}$. This construction works for R any complete valuation ring, not necessarily of height 1. In our case, there is a specialization map

$$\mathrm{sp} : X_K \longrightarrow <X>$$

which associates to any rigid point $x \in X_K$ the family $(x_{\mathcal{A}})_{\mathcal{A} \subseteq \mathcal{O}_{X_R}}$ of projections of x. It is injective and has dense image for the constructible topology on $<X>$ ([Bo] 2.1.5). There exists also a map (see [Bo] for a precise definition)

$$\theta^{-1} : \{\mathrm{G-topology\ on\ } X_K\} \longrightarrow \{\mathrm{topology\ on\ } <X>\}.$$

This associates to the generic fibre of an open formal subscheme $U_{\mathcal{A}}$ of some $X_{\mathcal{A}}$ the open subset $\pi_{\mathcal{A}}^{-1}(U_{\mathcal{A}}) = <U>$ with $\pi_{\mathcal{A}} : <X> \to X_{\mathcal{A}}$ the canonical projection. Observe also that the ZR-space depends only on the rigid space X_K and not on the particular admissible R-model chosen to construct it. One of the advantages one has working with ZR-spaces is that they permit to describe admissible rigid coverings in terms of their topology. If X_K is an affinoid K-space and $\mathfrak{V}$ a family of open admissible subsets of X_K then $\mathfrak{V}$ is an admissible covering of X_K if and only if $\{\theta^{-1}(V_K)\}_{V_K \in \mathfrak{V}}$ is a covering of $<X>$.
We will say that a rigid K-space X_K satisfies property $(\mathcal{P}_{\mathrm{Rig}})$ if

$(\mathcal{P}_{\mathrm{Rig}})$: *$X_K$ has an admissible R-model and given a finite set of points I of the Zariski-Riemann space $<X>$, there exists an open affinoid subspace U_K of X_K such that $I \subseteq <U> \subseteq <X>$.*

This property is trivially fulfilled by affinoid K-spaces and it is a local property. This means that any open quasi-compact subspace of X_K satisfies $(\mathcal{P}_{\mathrm{Rig}})$ if X_K satisfies $(\mathcal{P}_{\mathrm{Rig}})$. Property $(\mathcal{P}_{\mathrm{Rig}})$ is of some use if given a finite locally free morphism $\psi_K : Z_K \to Y_K$ the corresponding map $<\psi> : <Z> \to <Y>$ between ZR-spaces is surjective and with finite fibres. To prove this fact, we can work locally and suppose that $\psi_K : \mathrm{Sp}(C_K) \to \mathrm{Sp}(B_K)$ is a finite free morphism of affinoid K-spaces. It is surjective because of [FII] 5.11 and [BGR] 9.6/3. As K-affinoid algebras are Jacobson rings, we have surjectivity between the usual affine spectra. Points of the Zariski-Riemann space $<\mathrm{Sp}(B_K)>$ correspond to pairs $(\mathfrak{p}, V)$ with $\mathfrak{p}$ a prime ideal in B_K and V a valuation in the field of fractions of $B_K/\mathfrak{p}$ satisfying certain conditions (cf. [Bo], Introduction, [PS]). Similarly for points in $<\mathrm{Sp}(C_K)>$. Moreover $(\mathfrak{q}, W)$ in $<\mathrm{Sp}(C_K)>$ is above $(\mathfrak{p}, V)$ in $<\mathrm{Sp}(B_K)>$ if and only if $\mathfrak{q} \cap B_K = \mathfrak{p}$ and W extends V. Now, there are finitely many prime ideals $\mathfrak{q}_i$ in C_K above any $\mathfrak{p}$ in B_K and the extension of fields of fractions $\mathrm{Fr}(C_K/\mathfrak{q}_i)/\mathrm{Fr}(B_K/\mathfrak{p})$ is finite. Hence the conclusion.

We can now prove that property $(\mathcal{P}_{\mathrm{Rig}})$ is sufficient for the representability of the rigid Weil restriction.

Theorem 1.16. *Let $h : S'_K \to S_K$ be a finite and locally free morphism of rigid K-spaces and X'_K a formal rigid S'_K-space. Suppose that X'_K satisfies $(\mathcal{P}_{\mathrm{Rig}})$. Then the functor $\mathfrak{R}_{S'_K/S_K}(X'_K)$ is represented by the space $R_{S'_K/S_K}(X'_K)$ defined in 1.13.*

Proof.. We will check that the condition on 1.14 is satisfied. Let T_K be an affinoid S_K-space, $\phi : T_K \times_{S_K} S'_K = T'_K \to X'_K$ a given S'_K-morphism and $\mathfrak{U}$ the covering of X'_K defined in 1.13. It suffices to prove that $\{\mathfrak{R}_{T'_K/T_K}(\phi^{-1}(V'_K))\}_{V'_K \in \mathfrak{U}}$ is an admissible covering of T_K. In fact, its extension to T'_K is a refinement of $\phi^*\mathfrak{U}$. This is equivalent to show that any point in the ZR-space $<T>$ is contained in some open subset of the form $\theta^{-1}(\mathfrak{R}_{T'_K/T_K}(\phi^{-1}(V'_K)))$. It is sufficient to prove it for closed points as $<T>$ is a Jacobson space.

As we will work locally, we can assume that S_K is affinoid and h is free. This is possible because property $(\mathcal{P}_{\mathrm{Rig}})$ is preserved. Let $\tilde{t}$ be a closed point of $<T>$. Its fibre in $<T'>$ is finite and is contained in some open subset of the form $\theta^{-1}(\phi^{-1}(V'_K))$ as X'_K satisfies $(\mathcal{P}_{\mathrm{Rig}})$. It might be that $\phi^{-1}(V'_K)$ is not quasi-compact, so we choose a quasi-compact open subspace U'_K of $\phi^{-1}(V'_K)$ such that $<U'>$ contains the fibre of $\tilde{t}$. Recall now the construction used in the proof of 1.12. There exist morphisms $i : U'_R \to T'_R$ and $\psi : T'_R \to T_R$ with i an open immersion and ψ proper and faithfully flat such that the associated rigid morphisms are $U'_K \hookrightarrow T'_K \to T_K$. Then $\mathfrak{R}_{T'_K/T_K}(U'_K)$ is the rigid space associated to the open formal subscheme $\psi(U'^c_R)^c$ of T_R. Call t the projection of $\tilde{t}$ on T_R. With arguments similar to those in [Bo] 3.2.2, we can see that all closed points of T'_R above t are projections of points of $<T'>$ above $\tilde{t}$. This implies that the fibre of t is totally contained in U'_R. Hence the point t is in $\psi(U'^c_R)^c$ and $\tilde{t} \in \theta^{-1}(\mathfrak{R}_{T'_K/T_K}(U'_K)) \subseteq \theta^{-1}(\mathfrak{R}_{T'_K/T_K}(\phi^{-1}(V'_K)))$. □

Condition $(\mathcal{P}_{\mathrm{Rig}})$ is easier to handle than the one in 1.14. It has the advantage that one can check it looking at admissible R-models. Suppose that a rigid K-space X_K has an admissible R-model which satisfies $(\mathcal{P}_{\mathrm{For}})$. It follows from the definition of the ZR-space that the rigid space X_K satisfies $(\mathcal{P}_{\mathrm{Rig}})$. More generally, this is true even if the formal R-model is not flat.

In any case, even if X_K satisfies $(\mathcal{P}_{\mathrm{Rig}})$, we are forced to consider almost all open affinoid subsets of X_K (at least those in $\mathfrak{U}$ as defined in 1.12) in order to describe an admissible covering of the rigid Weil restriction. It is sometimes useful to consider particular admissible coverings of X_K coming from coverings of an admissible R-model, and see if they induce not only a covering of the formal Weil restriction but also an admissible covering of the rigid Weil restriction. So we change condition $(\mathcal{P}_{\mathrm{Rig}})$ and make it depend on coverings as follows:

Let X_K be a rigid K-space which has an admissible formal R-model. Suppose given an admissible covering $\mathfrak{V}$ of X_K by open affinoid subspaces. We will say that the covering $\mathfrak{V}$ satisfies property $(\mathcal{P}_{\mathrm{Rig}})_{|\mathfrak{V}}$ if:

$(\mathcal{P}_{\mathrm{Rig}})_{|\mathfrak{V}}$: *Given any finite set of points I of the Zariski-Riemann space $<X>$, there exists a $U_K \in \mathfrak{V}$ with $I \subset \theta^{-1}(U_K) =<U>\subset<X>$.*

In particular X_K satisfies $(\mathcal{P}_{\mathrm{Rig}})$. Hence $(\mathcal{P}_{\mathrm{Rig}})_{|\mathfrak{V}}$ is another sufficient condition for the representability of rigid Weil restriction functors.

Proposition 1.17. *Let $h : S'_K \to S_K$ be a finite and locally free morphism of affinoid K-spaces and let X'_K be a rigid S'_K-space which admits a covering $\mathfrak{V}$ by open affinoid subspaces satisfying $(\mathcal{P}_{\mathrm{Rig}})_{|\mathfrak{V}}$. Then $\mathfrak{R}_{S'_K/S_K}(X'_K)$ is representable and an admissible covering of the representing space is given by the rigid spaces $\mathfrak{R}_{S'_K/S_K}(V_K)$, as V_K varies over $\mathfrak{V}$.*

Proof.. It is clear that $(\mathcal{P}_{\mathrm{Rig}})_{|\mathfrak{V}}$ implies $(\mathcal{P}_{\mathrm{Rig}})$ and hence the functor $\mathfrak{R}_{S'_K/S_K}(X'_K)$ is represented by the glueing space $R_{S'_K/S_K}(X'_K)$ defined in 1.13. Proceeding as in 1.13 one proves that the rigid spaces $\mathfrak{R}_{S'_K/S_K}(V_K)$ give an admissible covering of $\mathfrak{R}_{S'_K/S_K}(X'_K)$ if and only if for any (affinoid) S_K-space T_K and any S'_K-morphism $\phi\colon T_K \times_{S_K} S'_K = T'_K \to X'_K$ the covering $\phi^*\mathfrak{V} = \{\phi^{-1}V_K\}_{V_K\in\mathfrak{V}}$ admits a refinement which descends to an admissible covering of T_K. To conclude, we can repeat what we have done in 1.16 with $(\mathcal{P}_{\mathrm{Rig}})_{|\mathfrak{V}}$ in place of $(\mathcal{P}_{\mathrm{Rig}})$ and the glueing of $\mathfrak{R}_{S'_K/S_K}(V_K)$, $V_K \in \mathfrak{V}$, in place of $R_{S'_K/S_K}(X'_K)$. □

If X'_K has a "good" admissible model X'_R we can consider only open affinoid subspaces coming from open formal subschemes of X'_R to describe the rigid Weil restriction.

Corollary 1.18. *Let h be as in Proposition 1.17. Suppose that the S'_K-space X'_K has an admissible R-model X'_R which satisfies property $(\mathcal{P}_{\mathrm{For}})$. Then the functor $\mathfrak{R}_{S'_K/S_K}(X'_K)$ is representable and the spaces $\mathfrak{R}_{S'_K/S_K}(V_K^{(i)})$, as $V_R^{(i)}$ varies over the open affine formal subschemes of X'_R, give an admissible covering of $\mathfrak{R}_{S'_K/S_K}(X'_K)$.*

Proof.. As X'_R satisfies $(\mathcal{P}_{\mathrm{For}})$, the covering $\{V_K^{(i)}\}_{i\in I}$ described above satisfies $(\mathcal{P}_{\mathrm{Rig}})_{|\{V_K^{(i)}\}_{i\in I}}$. To check this, it is sufficient to recall the definition of Zariski-Riemann spaces. Any point $\tilde{x} \in<X'>$ has a projection $x \in X'_R$. It is clear that $x \in V_R^{(i)}$ if and only if $\tilde{x} \in<V^{(i)}>$. □

We conclude this section with comparing Weil restriction on schemes, formal R-schemes and rigid K-spaces via analytification, formal completion and Raynaud functors. At first we consider the analytification functor. For more details about this functor we suggest [Kö]. We are interested in schemes $\mathcal{X}'$ which satisfy $(\mathcal{P}_{\mathrm{Sch}})$. This permits to describe a covering of $\mathfrak{R}_{.}(\mathcal{X}')$ in terms of affine subschemes of $\mathcal{X}'$.

Proposition 1.19. *Let A and A' be affinoid K-algebras, A' a finite and free A-module and $\mathcal{X}'$ an A'-scheme locally of finite type. Suppose that $\mathcal{X}'$ satisfies $(\mathcal{P}_{\mathrm{Sch}})$. Then the functor $\mathfrak{R}_{A'/A}(\mathcal{X}')$ is represented by an A-scheme locally of finite type and the rigid space $\mathfrak{R}_{A'/A}(\mathcal{X}')^{\mathrm{an}} \cong R_{A'/A}(\mathcal{X}'^{\mathrm{an}})$ represents $\mathfrak{R}_{A'/A}(\mathcal{X}'^{\mathrm{an}})$.*

Proof.. Let $\mathcal{U}'$ be an affine A'-scheme of finite type, $\mathcal{T} = \mathrm{Spec}(B)$ with B an affinoid K-algebra over A and $\mathcal{T}' = \mathcal{T} \times_A A'$. By [Kö] 1.1 & 1.2 we have canonical bijections $\mathrm{Hom}_A(\mathrm{Sp}(B), \mathfrak{R}_{A'/A}(\mathcal{U}')^{\mathrm{an}}) \xrightarrow{\sim} \mathrm{Hom}_A(\mathcal{T}, \mathfrak{R}_{A'/A}(\mathcal{U}')) \xrightarrow{\sim} \mathrm{Hom}_{A'}(\mathcal{T}', \mathcal{U}') \xrightarrow{\sim} \mathrm{Hom}_{A'}(\mathrm{Sp}(B \otimes_A A'), \mathcal{U}'^{\mathrm{an}})$. This implies that $\mathfrak{R}_{A'/A}(\mathcal{U}'^{\mathrm{an}}) \cong \mathfrak{R}_{A'/A}(\mathcal{U}')^{\mathrm{an}}$.

In the general case: since $\mathcal{X}'$ satisfies $(\mathcal{P}_{\mathrm{Sch}})$ the functor $\mathfrak{R}_{A'/A}(\mathcal{X}')$ is represented by an A-scheme locally of finite type (7.6/4 and 7.6/5 in [BLR]) and $\mathfrak{R}_{A'/A}(\mathcal{X}')^{\mathrm{an}}$ is defined. Moreover, there exists a covering $\mathfrak{V}$ of $\mathcal{X}'$ by open affine subschemes such that $\mathfrak{R}_{A'/A}(\mathcal{X}')$ is covered by $\mathfrak{R}_{A'/A}(\mathcal{U}_j)$ as $\mathcal{U}_j$ varies over $\mathfrak{V}$. Let T_K be an affinoid A-space and $\psi : T'_K = T_K \times_A A' \to \mathcal{X}'^{\mathrm{an}}$ an A'-morphism. Then $\psi^{-1}(\mathcal{U}_j^{\mathrm{an}})$ are Zariski-open subsets of T'_K and also $\mathfrak{R}_{T'_K/T_K}(\psi^{-1}(\mathcal{U}_j^{\mathrm{an}}))$ are Zariski-open subsets of T_K. In fact, if we denote by p the projection morphism $T'_K \to T_K$ then $\mathfrak{R}_{A'/A}(\psi^{-1}(\mathcal{U}_j^{\mathrm{an}}))$ is simply obtained as the complement of $p(C_j)$ where C_j is the closed analytic subset $T'_K - \psi^{-1}(\mathcal{U}_j^{\mathrm{an}})$. The open subspaces $\mathfrak{R}_{T'_K/T_K}(\psi^{-1}(\mathcal{U}_j^{\mathrm{an}}))$ cover T_K as $\mathcal{X}'$ satisfies $(\mathcal{P}_{\mathrm{Sch}})$. In fact any point in T_K has finite fibre in T'_K and its image is contained in some $\mathcal{U}_j^{\mathrm{an}}$. They then give an admissible covering of T_K ([BGR] 9.1.4/7). Applying (W_1) it is immediate to see that the glueing of $\mathfrak{R}_{A'/A}(\mathcal{U}_j^{\mathrm{an}}) \cong \mathfrak{R}_{A'/A}(\mathcal{U}'_j)^{\mathrm{an}}$, i.e. $\mathfrak{R}_{A'/A}(\mathcal{X}')^{\mathrm{an}}$, represents the functor $\mathfrak{R}_{A'/A}(\mathcal{X}'^{\mathrm{an}})$.

The space $R_{A'/A}(\mathcal{X}'^{\mathrm{an}})$ is defined and there exists a locally open immersion $\lambda : R_{A'/A}(\mathcal{X}'^{\mathrm{an}}) \to \mathfrak{R}_{A'/A}(\mathcal{X}'^{\mathrm{an}})$ such that $\Lambda = \Phi_{\mathcal{X}'^{\mathrm{an}}} \circ (\lambda \times \mathrm{id})$. We recall that the morphism Λ was defined in 1.13/proof. Let $\{\mathcal{U}_j\}_{j \in J}$ be the affine covering of $\mathcal{X}'$ previously considered. Each $\mathcal{U}_j^{\mathrm{an}}$ is isomorphic to some $\varinjlim \mathcal{U}_{j,h}$, $h \in \mathbb{N}$, with $\mathcal{U}_{j,h}$ open affinoid subspaces of $\mathcal{X}'^{\mathrm{an}}$ and $\mathcal{U}_{j,h} \hookrightarrow \mathcal{U}_{j,h+1}$ open immersions. It is easy to check that the functor $\mathfrak{R}_{A'/A}(\mathcal{U}_j^{\mathrm{an}})$ is represented by $\varinjlim \mathfrak{R}_{A'/A}(\mathcal{U}_{j,h})$. This implies that the covering $\{\mathfrak{R}_{A'/A}(\mathcal{U}_{j,h})\}_{(j,h) \in J \times \mathbb{N}}$ of the rigid space $\mathfrak{R}_{A'/A}(\mathcal{X}'^{\mathrm{an}})$ is admissible. The spaces $\mathfrak{R}_{A'/A}(\mathcal{U}_{j,h})$ are also open subspaces of $R_{A'/A}(\mathcal{X}'^{\mathrm{an}})$ and they give an admissible covering via λ. Then the morphism λ is indeed an isomorphism. □

Let now $\mathcal{X}$ be an R-scheme locally of finite presentation. Its completion X_R along the closed subscheme given by the ideal $\pi\mathcal{O}_{\mathcal{X}}$ is a formal R-scheme locally of topologically finite presentation. Working on special fibres one can see that if the scheme $\mathcal{X}$ satisfies property $(\mathcal{P}_{\mathrm{Sch}})$ then the formal scheme X_R satisfies $(\mathcal{P}_{\mathrm{For}})$. We can then give another comparison result.

Proposition 1.20. *Let $h : \mathrm{Spec}(A') \to \mathrm{Spec}(A)$ be a finite and free morphism with A' and A two R-algebras of finite presentation. Let $\mathcal{X}'$ be an A'-scheme locally of finite presentation and suppose that it satisfies $(\mathcal{P}_{\mathrm{Sch}})$. If we denote by X'_R its π-adic completion and by $\hat{A}'$ (resp. $\hat{A}$) the π-adic completion of A' (resp. A), then the functor $\Re_{A'/A}(\mathcal{X}')$ is represented by an A-scheme locally of finite presentation whose π-adic completion represents $\Re_{\hat{A}'/\hat{A}}(X'_R)$.*

Proof.. If $\mathcal{X}'$ satisfies $(\mathcal{P}_{\mathrm{Sch}})$ then X'_R satisfies $(\mathcal{P}_{\mathrm{For}})$ because the special fibre $\mathcal{X}'_k$ satisfies $(\mathcal{P}_{\mathrm{Sch}})$ as well. It is then sufficient to work locally on open affine subscheme of $\mathcal{X}'$ and then the result follows comparing 1.2 with the analogous result for schemes. □

At this point it remains to compare the formal and rigid Weil restriction. We have already seen that the formal one preserves quasi-compactness even when the rigid one does not. Remember the case of the unit rigid ball and the affine formal line. So we have no possibility to prove that the Weil restriction commutes with the formation of generic fibres. The best we can prove is that we obtain canonical open immersions.

Lemma 1.21. *Let $h : \mathrm{Spf}(A') \to \mathrm{Spf}(A)$ be a finite and free morphism of affine formal R-schemes (of tf presentation) and X'_R an affine formal $\mathrm{Spf}(A')$-scheme. The space $\Re_{A'/A}(X'_R)_K$ is canonically isomorphic to an open affinoid subspace of $\Re_{A'_K/A_K}(X'_K)$.*

Proof.. Let $X'_R = \mathrm{Spf}(A'\langle x_.\rangle/\mathfrak{a})$ with $x_.$ a set of indeterminates $x_1, \cdots, x_m$ and $\mathfrak{a} = (f_1, \cdots, f_r)$. We have seen in 1.2 that

$$\Re_{A'/A}\left(\mathrm{Spf}\left(\frac{A'\langle x_.\rangle}{\mathfrak{a}}\right)\right) = \mathrm{Spf}\left(\frac{A\langle x_{11}, \cdots, x_{1n}, \cdots, x_{mn}\rangle}{\mathfrak{a}^{\mathrm{co}}}\right).$$

Recall from 1.8 the definition of D_0. Then

$$\Re_{A'/A}(X')_K \cong \mathrm{Sp}\left(\frac{A_K\langle x_{11}, \cdots, x_{1n}, \cdots, x_{mn}\rangle}{\mathfrak{a}^{\mathrm{co}}}\right) \cong \mathrm{Sp}\left(\frac{D_0}{\mathfrak{a}^{\mathrm{co}}}\right)$$

is an open affinoid subspace of $\Re_{A'_K/A_K}(X'_K)$. For the last isomorphism one uses the fact that the elements $e_1, \cdots, e_n$ are free generators of A' over A and hence $\|\sum_i x_i e_i\| \leq 1$ is trivially true in $A'_K\langle x_{11}, \cdots, x_{mn}\rangle$. □

This lemma can be generalized.

Proposition 1.22. *Let $h : S'_R \to S_R$ be a finite and locally free morphism of admissible formal R-schemes and X'_R a formal S'_R-scheme. Suppose that X'_R satisfies $(\mathcal{P}_{\mathrm{For}})$. Then both $\Re_{S'_R/S_R}(X'_R)$ and $\Re_{S'_K/S_K}(X'_K)$ are representable and the canonical morphism*

$$\xi : \Re_{S'_R/S_R}\left(X'_R\right)_K \longrightarrow \Re_{S'_K/S_K}\left(X'_K\right)$$

is an open immersion.

Proof.. We can restrict to the case where h is a finite and free morphism of affine formal schemes and then check that

$$\xi^{-1}\left(\Re_{S'_K/S_K}\left(W'_K\right)\right) = \Re_{S'_R/S_R}\left(W'_R\right)_K$$

for any open affine formal subscheme W'_R of X'_R. Hence the assertion is a consequence of lemma 1.21. □

Among the formal R-schemes those which are smooth groups behave particular well with respect to Weil restriction functors.

Proposition 1.23. *Let $h : R \to R'$ be a finite and free morphism of complete valuation rings and $X'_{R'}$ a smooth formal R'-group scheme. Then*
a) The functor $\Re_{R'/R}(X'_{R'})$ is representable. The corresponding formal scheme $\Re_{R'/R}(X'_{R'})$ is covered by the open affine formal subschemes $\Re_{R'/R}(U'_{R'})$ as $U'_{R'}$ varies over the open affine formal subschemes of $X'_{R'}$.
b) The functor $\Re_{K'/K}(X'_{K'})$ is representable. The rigid spaces $\Re_{K'/K}(U'_{K'})$, as $U'_{R'}$ varies over the open affine formal subschemes of $X'_{R'}$, give an admissible covering of the representing space.
c) The canonical morphism

$$\xi : \Re_{R'/R}\left(X'_{R'}\right)_K \longrightarrow \Re_{K'/K}\left(X'_{K'}\right)$$

is an open immersion.

Proof.. The formal R'-scheme $X'_{R'}$ satisfies $(\mathcal{P}_{\mathrm{For}})$ because the identity component of its special fibre is a quasi-projective variety. Then a) follows from 1.4, b) from 1.18 and c) is 1.22. □

2. Formal Néron models

Let in the following R be a complete discrete valuation ring. We recall that given a smooth rigid K-space X_K a *formal Néron model* of X_K over R is a smooth formal R-scheme U_R, whose generic fibre U_K is an open rigid subspace of X_K and which satisfies the following universal property:

(N) *Given a smooth formal R-scheme Z_R and a morphism of rigid K-spaces $u_K\colon Z_K \to X_K$, u extends uniquely to a morphism $u : Z_R \to U_R$ of formal R-schemes.*

A formal Néron model U_R is unique up to unique isomorphism. If X_K is separated, U_R will be separated and if X_K is a rigid K-group, U_R inherits a group structure. For more details about formal Néron models we refer to [BS].
It is immediate to see that the formation of formal Néron models is compatible with étale base change. Let now K'/K be a finite field extension. Suppose there exists a formal Néron model $U'_{R'}$ of $X_{K'} = X_K \times_K K'$ over $\mathrm{Spf}(R')$. One can ask if a formal Néron model U_R of X_K exists and, in the affirmative case, look for a relation between U_R and $U'_{R'}$. To do this we have to deal with formal and rigid Weil restrictions and with smoothening processes. The smoothening process in the formal context involves (admissible) formal blowing-ups with centers in the special fibre. More precisely, if $X'_R \to X_R$ is an admissible formal blowing-up of a formal R-scheme with center $Y_k \subset X_k$ and $\mathcal{I} \subset \mathcal{O}_{X_R}$ is the corresponding (open) ideal, the open formal subscheme of X'_R where $\mathcal{I}\mathcal{O}_{X'_R}$ is generated by the uniformizing parameter $\pi \in R$ is called the *dilatation* of Y_k on X_R and denoted by $X'_{R,\pi}$. It is flat and it satisfies the following universal property:

(D) *If Z_R is a flat formal R-scheme and $v : Z_R \to X_R$ is an R-morphism such that v_k factors through Y_k, then v lifts uniquely to a morphism of formal R-schemes $Z_R \to X'_{R,\pi}$.*

Dilatations commute with products. This implies that the dilatation of a formal R-group scheme with center in a subgroup Y_k of X_k is a formal group scheme and the canonical map $X'_{R,\pi} \to X_R$ is a group homomorphism. The smoothening process in the formal context is described in [BS] §3. It is a process introduced to deal with situations where a formal group may have smooth generic fibre without being smooth.
If G_R is a formal R-group such that its generic fibre is smooth a *group smoothening of G_R* will be a morphism $G'_R \to G_R$ of formal R-groups such that G'_R is smooth and each R-morphism $Z_R \to G_R$ with Z_R smooth admits a unique factorization through G'_R. Given an R^{sh}-valued point a of G_R one defines $\delta(a)$ as the length of the torsion part of $a^*\Omega_{G_R/R}$. It measures the defect of smoothness at a. The key result in order to prove that group smoothenings indeed exist, is the following:

Lemma 2.1. *Let G_R be a formal R-group such that its generic fibre is smooth. Denote by F_k the Zariski closure in G_k of the set of the k_s-valued points which lift to R^{sh}-valued points of G_R. Then F_k provided with its canonical reduced structure is a closed subgroup scheme of G_k. Let $u : G'_{R,\pi} \to G_R$ be the dilatation of F_k in G. We have*

$$\delta(a') \leq max\{0, \delta(a) - 1\}$$

for each R^{sh}-valued point a of G_R and its unique lifting a' to $G'_{R,\pi}$. Furthermore: For any smooth formal R-scheme Z_R and each R-morphism $v : Z_R \to G_R$ with Z_R smooth, v admits a unique factorization through $G'_{R,\pi}$.

The proof of this fact is a translation of the analogous result 7.1/4 in [BLR] using lemma 3.4 in [BS].

Proposition 2.2. *Let G_R be a formal R-group with smooth generic fibre. Then G_R admits a group smoothening.*

Proof.. As the identity component G_R^0 of G_R is quasi-compact the function δ is bounded on G_R^0 by an integer, say m. Applying finitely many times (at most m) the previous lemma, we obtain a formal R-group G'_R which is flat and smooth at all R^{sh}-valued points of the identity component. In particular, it is smooth at the origin and then G'_R is smooth ([SGA 3] VI_A 1.3.1 and [FII] 1.2). By construction it satisfies the desired lifting property. □

We now go back to the initial problem on formal Néron models. It is easy to prove the existence of U_R if the Néron model $U'_{R'}$ is quasi-compact. One simply applies 1.2 in [BS]. We have, however, no description of U_R in terms of $U'_{R'}$. Furthermore, in the non quasi-compact case, it was not known that the existence of $U'_{R'}$ implies the existence of U_R. In the following theorem we describe the relation between the formal Néron models $U'_{R'}$ and U_R by means of formal and rigid Weil restriction functors and we prove the existence of U_R even in the non quasi-compact case.

Theorem 2.3. *Let X_K be a smooth rigid K-group. Suppose that the rigid K'-group $X_{K'} = X_K \times_K K'$ admits a formal Néron model $U'_{R'}$. Then a formal Néron model U_R of X_K can be obtained as the group smoothening of the schematic closure of $H_K = X_K \times_{\mathfrak{R}_{K'/K}(X_{K'})} \mathfrak{R}_{R'/R}(U'_{R'})_K$ in the formal Weil restriction $\mathfrak{R}_{R'/R}(U'_{R'})$.*

Proof.. Observe that we have not proved the representability of $\mathfrak{R}_{K'/K}(X_{K'})$ for $X_{K'}$ a rigid K'-group and we are apparently forced to work with functors instead of spaces. First of all we have to see that H_K is represented by an open subgroup of X_K, still denoted by H_K. Consider the following diagram of contravariant functors on (Rig/K):

$$
\begin{array}{ccccc}
H_K & & \mathfrak{R}_{X_{K'}/X_K}(U'_{K'}) & \xrightarrow{\zeta'} & X_K \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & \square & \downarrow{\scriptstyle\Psi} \\
\mathfrak{R}_{R'/R}(U'_{R'})_K & \xrightarrow{\xi} & \mathfrak{R}_{K'/K}(U'_{K'}) & \xrightarrow{\zeta} & \mathfrak{R}_{K'/K}(X_{K'})
\end{array}
$$

The morphism ξ is an open immersion and both functors are representable (1.23). The morphism ζ is also an open immersion by 1.9. Furthermore, the representability of $\mathfrak{R}_{X_{K'}/X_K}(U'_{K'})$ and property (W_2) say that the diagram on the right is cartesian. Hence H_K can also be defined as the fibre product

$$\mathfrak{R}_{X_{K'}/X_K}\left(U'_{K'}\right) \times_{\mathfrak{R}_{K'/K}\left(U'_{K'}\right)} \mathfrak{R}_{R'/R}\left(U'_{R'}\right)_K .$$

This implies that H_K is representable because all the functors above are representable. Moreover, the canonical morphism $H_K \to X_K$ is an open immersion which factors through the open subspace $\mathfrak{R}_{X_{K'}/X_K}\left(U'_{K'}\right)$ of X_K. In particular, H_K is represented by a (smooth) subgroup of X_K because X_K, $\mathfrak{R}_{K'/K}(X_{K'})$ and $\mathfrak{R}_{R'/R}(U'_{R'})_K$ are group functors.
To prove that the morphism α is a closed immersion, it is sufficient to see that β is a closed immersion. This fact would be clear if $\mathfrak{R}_{K'/K}(X_{K'})$ were representable. In fact, as X_K is separated, the morphism $\Psi\colon X_K \to \mathfrak{R}_{K'/K}(X_{K'})$ would be a closed immersion. In our case, we have no information about the representability of such a functor. We only know that there exists a rigid K-space $R_{K'/K}(X_{K'})$ which almost represents it and that the canonical morphism $\psi\colon X_K \to R_{K'/K}(X_{K'})$ is a closed immersion (this can be proved in the same way as one would prove that Ψ is a closed immersion if representable). Now, there is a locally open immersion of functors $\lambda\colon R_{K'/K}(X_{K'}) \to \mathfrak{R}_{K'/K}(X_{K'})$ and ζ factors through λ. In fact, given any open affinoid subspace $W_{K'}$ of $X_{K'}$, the space $\mathfrak{R}_{K'/K}(W_{K'} \cap U'_{K'})$ is open both in $\mathfrak{R}_{K'/K}(U'_{K'})$ and in $R_{K'/K}(X_{K'})$. Hence the morphism β is a closed immersion because the diagram above remains cartesian if we write ψ in place of Ψ. In particular, if $U'_{R'}$ is quasi-compact then so are $\mathfrak{R}_{R'/R}(U'_{R'})$ and H_K.
It is immediate to see, using the definition of H_K as a fibre product, that a formal Néron model of H_K will be a formal Néron model of X_K. To construct it, let H_R be the schematic closure of H_K in $\mathfrak{R}_{R'/R}(U'_{R'})$. Locally on an affine open formal subscheme $Z_R = \mathrm{Spf}(A)$ of $\mathfrak{R}_{R'/R}(U'_{R'})$ it is defined as $\mathrm{Spf}(A/\mathrm{Ker}(\rho))$ with $\rho\colon A \to A_K \to A_K/I$ and $I \subset A_K$ the ideal which defines $\alpha^{-1}(Z_K)$ as a closed analytic subspace of H_K. As $\mathfrak{R}_{R'/R}(U'_{R'})$ is flat, H_R exists, is flat and satisfies the following universal property:

(S) *For any flat formal R-scheme Y_R and any morphism $\phi_R\colon Y_R \to \mathfrak{R}_{R'/R}(U'_{R'})$ such that ϕ_K factors through H_K, ϕ_R factors through H_R.*

The formal scheme H_R inherits also an R-group structure. It satisfies the Néron mapping property (N) but it might not be smooth. Consider then a group smoothening of H_R, say H_R^{sm}. It exists by 2.2. It is a smooth, formal R-group and it satisfies property (N) with respect to H_K. Hence it is a formal Néron model of H_K and then of X_K. □

Although the theorem above is a complete answer to our problem, we are disturbed by the smoothening process which is not easy to control. We will see

that if the extension K'/K is tamely ramified, we can use what was done by Edixhoven in [Ed] to prove that the formal R-group scheme H_R obtained as the schematic closure of H_K in $\mathfrak{R}_{R'/R}(U'_{R'})$ is already smooth and hence a formal Néron model of X_K over R. To do this, we will divide the proof in several steps. This is necessary because after base change we may lose the existence of formal Néron models.
It is known that formal Néron models are compatible with étale base change. We want to see that in this case the schematic closure of H_K in $\mathfrak{R}_{R'/R}(U'_{R'})$ is isomorphic to U_R.

Lemma 2.4. *Let the rigid spaces X_K and $X_{K'}$ and the formal Néron models U_R and $U'_{R'}$ be as in 2.3. Suppose furthermore that the finite field extension K'/K is unramified. Then U_R is isomorphic to H_R, the schematic closure of $X_K \times_{\mathfrak{R}_{K'/K}(X_{K'})} \mathfrak{R}_{R'/R}(U'_{R'})_K$ in the formal Weil restriction $\mathfrak{R}_{R'/R}(U'_{R'})$.*

Proof.. It is clear that $U'_{R'} = U_{R'}$ as $R \to R'$ is étale and $\delta_R : U_R \to \mathfrak{R}_{R'/R}(U'_{R'})$ is a closed immersion because any formal R-group is separated. In particular, U_R is the schematic closure of U_K in $\mathfrak{R}_{R'/R}(U'_{R'})$. But in this case U_K is nothing else than $\mathfrak{R}_{X'_{K}/X_K}(U'_{K'})$. If we look at the diagram in 2.3/proof, we see that $\beta = \xi \circ \delta_K$ and $H_K = U_K \times_{\mathfrak{R}_{K'/K}(U'_{K'})} \mathfrak{R}_{R'/R}(U'_{R'})_K$ is indeed isomorphic to U_K. As both U_R and H_R are flat closed formal subschemes of $\mathfrak{R}_{R'/R}(U')$, this is sufficient to conclude. □

Suppose now that the extension K'/K is finite, Galois with group G. The group G acts on $X_{K'}$ but it acts also on $U'_{R'}$. In fact, G acts on R' and we can apply the universal property of formal Néron models. This action induces an action on $\mathfrak{R}_{R'/R}(U'_{R'})$ described as follows:
Let T_R be a formal R-scheme and $\tau \in \mathfrak{R}_{R'/R}(U'_{R'})(T_R) = \mathrm{Hom}_{R'}(T_{R'}, U'_{R'})$. Then

$$\tau \cdot g = \rho_{U'_{R'}}(g) \circ \tau \circ \rho_{T_{R'}}(g)^{-1}$$

where $\rho_{U'_{R'}}(g)$ is the automorphism of $U'_{R'}$ induced by g. The same for $T_{R'}$.
Chosen a formal R-scheme Y_R on which G acts, we will consider the functor of fixed points (see [Ed] in the scheme-theoretic setting)

$$Y^G : (\mathrm{For}/R)^o \longrightarrow (\mathrm{Sets})$$

$$T_R \longrightarrow Y_R(T_R)^G$$

In our case $Y = \mathfrak{R}_{R'/R}(U'_{R'})$. We can consider the infinitesimal n-levels Y_n on which G acts as well and apply [Ed] 3.1. Then Y_R^G is represented by a closed formal subscheme of Y_R as the formal scheme Y_R is separated. In particular

$$\mathfrak{R}_{R'/R}\left(U'_{R'}\right)^G = \varinjlim \mathfrak{R}_{R'_n/R_n}\left(U'_n\right)^G .$$

If we suppose that the order of G is prime to the characteristic of the residue field, then we can apply [Ed] 3.4 to each n-level and recognize that $\mathfrak{R}_{R'/R}(U'_{R'})^G$ is smooth.

Lemma 2.5. *Let the notation be as in 2.3 and K'/K a finite Galois extension with group G, with the order of G prime to the characteristic of the residue field k. Then $\mathfrak{R}_{R'/R}(U'_{R'})^G$ is a formal Néron model of X_K over R. In particular, it is the schematic closure of H_K in $\mathfrak{R}_{R'/R}(U'_{R'})$.*

Proof.. We have already seen that $\mathfrak{R}_{R'/R}(U'_{R'})^G$ is represented by a smooth formal R-scheme. It remains to prove that it is isomorphic to H_R. We have closed immersions of flat formal R-schemes

$$\mathfrak{R}_{R'/R}\left(U'_{R'}\right)^G \xrightarrow{\eta} \mathfrak{R}_{R'/R}\left(U'_{R'}\right) \xleftarrow{\xi} H_R$$

It is then sufficient to prove that the corresponding rigid fibres are isomorphic as closed subspaces of $\mathfrak{R}_{R'/R}(U'_{R'})_K$. First we observe that the Galois action on $X_{K'}$ induces a Galois action on U'_K. This rigid space descends to the open subspace $Z_K = \mathfrak{R}_{X_{K'}/X_K}(U'_K)$ of X_K. There is also a closed immersion $\beta\colon Z_K \to \mathfrak{R}_{K'/K}(U'_{K'})$ (see 2.3/proof) and Z_K represents $\mathfrak{R}_{K'/K}(U'_{K'})^G$. The action of G on $\mathfrak{R}_{K'/K}(U'_{K'})$ and the fix-point functors are defined in the analogous way as for the formal Weil restriction. It is easy to see, simply working on points, that

$$\left(\mathfrak{R}_{R'/R}\left(U'_{R'}\right)^G\right)_K = \mathfrak{R}_{R'/R}\left(U'_{R'}\right)_K \times_{\mathfrak{R}_{K'/K}\left(U'_{K'}\right)} Z_K.$$

We are somehow proving that the fix-point functor commutes with the formation of generic fibres. The formulation is complicated by the fact that Weil restriction does not commute with the formation of generic fibre and in the best case we can only expect an open immersion $\mathfrak{R}_{R'/R}(U'_{R'})_K \hookrightarrow \mathfrak{R}_{K'/K}(U'_{K'})$. The fibre product on the right is H_K. Hence the conclusion. □

Till now we have considered only particular field extensions. We want to see that it is possible to add or forget unramified extensions without loosing property $C(K'; K)$. This is a short notation to say that a formal Néron model of the smooth rigid K-group X_K is given by the schematic closure of

$$X_K \times_{\mathfrak{R}_{K'/K}(X_{K'})} \mathfrak{R}_{R'/R}\left(U'_{R'}\right)_K \qquad \text{in} \qquad \mathfrak{R}_{R'/R}\left(U'_{R'}\right).$$

Applying 2.4 and the universal properties of formal Néron models and Weil restriction, we can prove the following fact:

Lemma 2.6. *Let K'/K be a finite extension, K^u an unramified extension of K in K' and X_K a smooth rigid K-group. Suppose that $X_{K'}$ admits a formal Néron model $U'_{R'}$. Then $C(K'; K)$ holds if $C(K'; K^u)$ holds.*

It is also possible to add unramified extensions above.

Lemma 2.7. *Let notations be as in the previous lemma. Suppose that K^t is an intermediate extension with K'/K^t unramified. Then $C(K^t; K)$ holds if $C(K'; K)$ holds.*

We can now collect all these partial results and prove that $C(K'; K)$ holds by tamely ramified extension.

Proposition 2.8. *Let X_K be a smooth rigid K-group and K'/K a tamely ramified extension. Suppose that the K'-group $X_{K'}$ admits a formal Néron model $U'_{R'}$. Then the schematic closure of $X_K \times_{\mathfrak{R}_{K'/K}(X_{K'})} \mathfrak{R}_{R'/R}(U'_{R'})_K$ in the formal Weil restriction $\mathfrak{R}_{R'/R}(U'_{R'})$ is a formal Néron model of X_K.*

Proof.. We have to prove that property $C(K'; K)$ holds. By 2.6 we can suppose that K'/K is totally ramified and tame. Let $\widetilde{K}$ the smallest normal extension of K, containing K', in a fixed separable closure of K'. The extension $\widetilde{K}/K'$ is unramified and hence a formal Néron model of $X_{\widetilde{K}}$ exists. By 2.7 we can reduce to the case $K' = \widetilde{K}$. Consider then the maximal unramified extension of K in K', say K^{nr}. Again by 2.6 we can suppose that $K^{\mathrm{nr}} = K$ and hence K'/K will be totally ramified, tame and Galois. In particular the order of $\mathrm{Gal}(K'/K)$ is prime to the characteristic of the residue field and the conclusion follows from lemma 2.5. □

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