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The Incompressible Limit of the Non-Isentropic Euler Equations

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Abstract

We study the Euler equations for slightly compressible fluids, that is, after rescaling, the limits of the Euler equations of fluid dynamics as the Mach number tends to zero. In this paper, we consider the general non-isentropic equations and general data. We first prove the existence of classical solutions for a time independent of the small parameter. Then, on the whole space $\mathbb{R}^d$, we prove that the solution converges to the solution of the incompressible Euler equations.

1. Introduction

The nature of the incompressible limit of the Euler equations of fluid dynamics depends on several factors: the flow may be *isentropic* or *non-isentropic*. The initial data may be *prepared* to make the initial first time-derivatives uniformly bounded, or *general*. The domain may be *periodic* or the *whole space* $\mathbb{R}^d$.

Moreover, an analysis of the singular limit contains at least two parts: an existence and uniform boundedness result for a time independent of the small parameter appearing in the scaled equations, and a convergence result either to the fixed solution of a limit equation or to a limiting profile.

Solutions of the slightly compressible Euler equations are known to exist for a time independent of the small parameter in the equations, which is essentially the Mach number, whenever the flows are isentropic (see [KM1, KM2]) and whenever the data is prepared (see [Sch1]). Solutions converge to solutions of the corresponding incompressible Euler equations with the limit initial data whenever the initial data is prepared (see [KM1, KM2, Sch1]), while for the isentropic equations in $\mathbb{R}^d$ solutions tend to the solution of the incompressible equations whose initial data is the incompressible part of the original initial data, although this convergence is not uniform for times close to zero (see [Asa, Uka, Izo1–Izo3]). For the isentropic equations in a periodic domain the difference between solutions and appropriate profiles tends to zero (see [JMR, Sch3]).

These existence and convergence results cover all cases of the above-mentioned factors except for the non-isentropic equations with general initial data. The first main result in this paper is a uniform existence result for this case. The second main result is a convergence theorem for the non-isentropic equations in all of $\mathbb{R}^d$ with general initial data analogous to that for the isentropic equations. As in some of the previous work on the incompressible limit (see [KM1, Sch2, JMR, Sch3]), these results will be deduced as special cases of theorems about a class of equations. The limit of solutions to the non-isentropic equations with general initial data in a periodic domain will be considered in a separate paper.

The reason why the incompressible limit is more difficult to analyze in the non-isentropic case is that the matrix multiplying the time derivatives then depends strongly on the dependent variables. That is, the scaled equations have the form

$$B_0(U, \varepsilon U)\partial_t U + \frac{1}{\varepsilon}\mathcal{L}(\partial_x)U + B(U, \varepsilon, \partial_x)U = 0, \tag{1.1}$$

with B_0 depending on U as well as εU. Here ε is a small parameter, B_0 is a positive-definite symmetric matrix, and $\mathcal{L}$ and B are sums of first-order differential operators times symmetric matrices, which for $\mathcal{L}$ are constant so that the operator $\mathcal{L}$ is antisymmetric. Since (1.1) has symmetric-hyperbolic form, that system is well posed for fixed ε, so the main question is the behavior of solutions as the scaling parameter ε tends to zero. The scalar example

$$a(u)u_t + \frac{1}{\varepsilon}u_x = 0, \tag{1.2}$$

which can be solved by the method of characteristics, shows that in general the time of existence of the solution to the initial-value problem for (1.1) with fixed smooth initial data tends to zero with ε. Additional hypotheses must therefore be made on the structure of the equations or on the initial data in order to obtain a problem for which solutions exist for some time independent of ε.

If the initial data for (1.1) are restricted by the requirement that sufficiently many time derivatives of U are uniformly bounded at time zero, then solutions exist for a time independent of ε and converge as $\varepsilon \to 0$ to the solution of a limiting equation (see [BK]), as can be shown by estimating all space-time derivatives of the solution through some order. For (1.2) for example, the condition on the initial data implies that $u(0, x) = c(\varepsilon) + O(\varepsilon^2)$ for some constant $c(\varepsilon)$. Although this result applies to all systems of the form (1.1), less restrictive assumptions on the initial data suffice when the system has additional structure. In particular, when B_0 depends only on εU then no restriction is needed on the smooth initial data (see [KM1, KM2, Maj]); this yields a uniform existence result for the isentropic, slightly compressible Euler equations.

The key to both results just mentioned is that energy estimates for solutions and their derivatives are properly balanced in ε. This means that if we weight each $\partial_t^j\partial_x^\alpha U$ by the factor $\varepsilon^{p(j)}$ needed to make it uniformly bounded at time zero, then $\frac{d}{dt}\left(\varepsilon^{p(j)}\partial_t^j\partial_x^\alpha U, B_0\varepsilon^{p(j)}\partial_t^j\partial_x^\alpha U\right)$ is a uniformly bounded function of the set of $\varepsilon^{p(k)}\partial_t^k\partial_x^\beta U$. In particular, the large term $\frac{1}{\varepsilon}\mathcal{L}(\partial_x)$ disappears from these estimates because it is antisymmetric and commutes with derivatives. This yields uniform estimates for the $\varepsilon^{p(j)}\partial_t^j\partial_x^\alpha U$, which implies in particular that $\|U\|_{H^s}$ is uniformly bounded for some fixed time since $p(0) = 0$.

In contrast, when B_0 depends on U and only the initial first time-derivative of U is uniformly bounded, then direct energy estimates are unbalanced, because the equation for the time derivative of the supposedly bounded $\left(U_{tx}, B_0 U_{t,x}\right)$ includes the unbounded term $(U_{tx}, ((B_0)_U U_x) U_{tt})$. Nevertheless, the solution exists and is uniformly bounded for a time independent of ε provided that (see [Sch2])

$$\text{The dimension of } \ker \mathcal{L}(\xi) \text{ is constant for } \xi \in \mathbb{R} \setminus \{0\}. \tag{1.3}$$

For general initial data, the example in (1.2) shows that additional structural assumptions are needed in order to obtain uniform existence. The immediate point of that example is that since the term $(U, (\partial_t B_0)U)$ occurs in the equation for $\frac{d}{dt}(U, B_0 U)$, $\partial_t B_0$ must be uniformly bounded in order for even the basic L^2 estimate for U to be uniform. With Euler's equation in mind, we therefore restrict attention to those systems (1.1) having the additional structure

$$U = \begin{pmatrix} u \\ S \end{pmatrix}, \quad B_0(U) = \begin{pmatrix} E(S, \varepsilon u) & 0 \\ 0 & I \end{pmatrix}, \quad \mathcal{L}(\partial_x) = \begin{pmatrix} L(\partial_x) & 0 \\ 0 & 0 \end{pmatrix}, \tag{1.4}$$

so that the variables whose time derivatives are $O(\frac{1}{\varepsilon})$ occur in B_0 multiplied by a factor of ε. Although this ensures that $\partial_t B_0$ will be uniformly bounded whenever $\|U\|_{H^s}$ is, and hence that the L^2 energy estimate for U is balanced, energy estimates for first derivatives of U are not balanced, on account of the $O(\frac{1}{\varepsilon})$ term $((B_0)_S S_x)U_t$ in the equation for U_x. This imbalance is more severe than in the case of prepared data, in which it first occurred in the estimates for U_{tx}. In particular, since the equation satisfied by first derivatives is the linearization of the original system, that linearized system is not balanced, which means that small perturbations of the initial data might cause large changes in solutions.

In the light of this non-uniform linearized stability, it is not surprising that there exist solutions to systems (1.1) of form (1.4) obeying assumption (1.3) that are not uniformly bounded for a time independent of ε. An explicit example will be presented below.

Guided by the form of the scaled non-isentropic Euler equations, we will therefore make the additional hypotheses that

$$B(S, u, \partial_x) = B_0(U)(b(S, u) \cdot \nabla_x) \tag{1.5}$$

and that

$$\begin{gathered} E(S, \varepsilon u) \text{ commutes with the orthogonal projector on} \\ \ker L(\xi) \text{ for all } \xi. \end{gathered} \tag{1.6}$$

Together with (1.4), (1.5) implies that system (1.1) has the form

$$\begin{aligned} E(S, \varepsilon u)\,(\partial_t u + b(S, u) \cdot \nabla_x u) + \frac{1}{\varepsilon} L(\partial_x) u = 0, \\ \partial_t S + b(S, u) \cdot \nabla_x S = 0. \end{aligned} \tag{1.7}$$

The initial assumptions on system (1.1) together with conditions (1.3) and (1.6) can be expressed in terms of the functions appearing in (1.7) as

Assumption 1.1. (i) *We can write* $L(\partial_x) = \sum_{j=1}^{d} L_j \partial_{x_j}$, *where the* L_j *are constant real symmetric matrices, and the dimension of* $\ker L(\xi)$ *is constant for* $\xi \in \mathbb{R} \setminus \{0\}$.

(ii) *The matrix* $E(S, u)$ *is a real symmetric positive definite matrix that is a* C^∞ *function of* $(S, u) \in \mathbb{R} \times \mathbb{R}^N$ *and commutes with the orthogonal projection onto* $\ker L(\xi)$ *for all* ξ.

(iii) *Also,* $b(S, u) = (b_1, \dots, b_d) \in \mathbb{R}^d$ *is* C^∞ *in* $(S, u) \in \mathbb{R} \times \mathbb{R}^N$.

The following theorem shows that these hypotheses suffice to ensure uniform existence of solutions to (1.7) having general initial data in an appropriate Sobolev space $H^s(\mathbb{D})$, with the domain $\mathbb{D}$ being either the whole space $\mathbb{R}^d$ or the torus $\mathbb{T}^d$. Actually, in view of [Sch1] and [Rau], the existence proof does not depend significantly on the shape of the domain, so we could also consider bounded domains with appropriate boundary conditions.

Theorem 1.2. *Suppose that system* (1.7) *satisfies Assumption 1.1, and let* $s > 1 + d/2$ *be an integer. For all real* M_0, *there is a positive* T *such that for all* $\varepsilon \in]0, 1]$ *and all initial data* $(u_0, S_0) \in H^s(\mathbb{D})$ *satisfying*

$$\|(u_0, S_0)\|_{H^s(\mathbb{D})} \leqq M_0, \tag{1.8}$$

the Cauchy problem for (1.7) *has a unique solution* $(u, S) \in C^0([0, T]; H^s(\mathbb{D}))$.

Note that, in general, the family of mappings $(u_0, S_0) \mapsto (u, S)$ is *not* uniformly continuous with respect to ε. This means that Assumption 1.1 is sufficient to ensure uniform existence of solution but does not imply uniform stability of the linearized equations. This explains why the nonlinear energy estimates cannot be obtained from the L^2 esimates by an elementary argument using differentiation of the equations. We give below a simple example of system (1.7) satisfying Assumption 1.1 which illustrates this *instability*.

There are two different assumptions about the initial data of the non-isentropic Euler equations that allow them to be transformed so as to make Theorem 1.2 applicable. The original non-isentropic Euler equations are

$$\begin{aligned} \partial_t \rho + u \cdot \nabla \rho + \rho \nabla \cdot u &= 0, \\ \rho(\partial_t u + u \cdot \nabla u) + \nabla p &= 0, \\ \partial_t S + u \cdot \nabla S &= 0. \end{aligned} \tag{1.9}$$

Here u is the fluid velocity. The fluid density ρ, entropy S, and pressure p are related by an equation of state, which will be given here in the form $\rho = R(p, S)$, with R assumed to be defined for $p > 0$ and all S, to be smooth and positive, and to satisfy $\frac{\partial R}{\partial p} > 0$. For instance, for ideal fluids $\rho = p^{1/\gamma} e^{-S/\gamma}$. Because the incompressible limit can be understood as the limit in which the ratio of the fluid speed to the sound speed tends to zero, we begin by rescaling the fluid velocity u by $u = \varepsilon v$. Since the velocity is the time derivative of the position of a fluid particle, those particles will then travel a distance $O(\varepsilon)$ in times of order one and a distance of order one in time $O\left(\frac{1}{\varepsilon}\right)$, which suggests rescaling either the spatial variables x by $y = \frac{x}{\varepsilon}$ or else the time by $\tau = \varepsilon t$. Introducing either one of these rescalings, replacing the new variable name y or τ by the original name x or t, and using p, v and S as the dependent variables transforms the Euler equations into the form

$$\begin{aligned} A(\partial_t p + v \cdot \nabla p) + \nabla \cdot v &= 0, \\ \rho(\partial_t v + v \cdot \nabla v) + \frac{1}{\varepsilon^2} \nabla p &= 0, \\ \partial_t S + v \cdot \nabla S &= 0, \end{aligned}$$

where $A = \frac{1}{R(S,p)} \frac{\partial R(S,p)}{\partial p}$. In order to symmetrize these equations we need to transform the pressure by $p = \underline{p} + O(\varepsilon)$ for some constant $\underline{p}$. In order to avoid changing the domain of definition of $E(S, \varepsilon u)$ in Assumption 1.1 to take into account the positivity of p, let us use the transformation $p = \underline{p} e^{\varepsilon q}$, yielding

$$\begin{aligned} a(\partial_t q + v \cdot \nabla q) + \frac{1}{\varepsilon} \nabla \cdot v &= 0, \\ r(\partial_t v + v \cdot \nabla v) + \frac{1}{\varepsilon} \nabla q &= 0, \\ \partial_t S + v \cdot \nabla S &= 0, \end{aligned} \tag{1.10}$$

where

$$a = a(S, \varepsilon q) = A(S, \underline{p} e^{\varepsilon q}) \underline{p} e^{\varepsilon q}, \quad r = r(S, \varepsilon q) = \frac{R(S, \underline{p} e^{\varepsilon q})}{\underline{p} e^{\varepsilon q}}.$$

Equations (1.10) have the form (1.7) with

$$u = \begin{pmatrix} q \\ v \end{pmatrix}, \quad b = v, \quad E(S, \varepsilon u) = \begin{pmatrix} a(S, \varepsilon q) & 0 \\ 0 & r(S, \varepsilon q) I \end{pmatrix}, \quad L(\partial_x) = \begin{pmatrix} 0 & \nabla \cdot \\ \nabla & 0 \end{pmatrix}.$$

The orthogonal projector onto $\ker L(\xi)$ is

$$\begin{pmatrix} 0 & 0 \\ 0 & P_\perp(\xi) \end{pmatrix},$$

where $P_\perp(\xi)$ is the orthogonal projection on $\xi^\perp$. Since $P_\perp(\xi)$ has fixed rank for $\xi \neq 0$ and commutes with rI, Assumption 1.1 is satisfied, so Theorem 1.2 implies the following results for the Euler equations:

Theorem 1.3. *Let $s > 1 + d/2$ be an integer. For all positive $\underline{p}$ and M_0, there exists a positive T such that for k equal to zero or one, all ε in $(0, 1]$, and all initial data*

$$p_0 = \underline{p} e^{\varepsilon q_0(x/\varepsilon^k)}, \quad u_0 = \varepsilon v_0\left(\tfrac{x}{\varepsilon^k}\right), \quad S_0 = S_0\left(\tfrac{x}{\varepsilon^k}\right)$$

satisfying

$$\|(q_0(\cdot), v_0(\cdot), S_0(\cdot))\|_{H^s(\mathbb{D})} \leqq M_0, \tag{1.11}$$

the Cauchy problem for (1.9) has a unique solution for $0 \leqq t \leqq \frac{T}{\varepsilon^{1-k}}$ such that (q, v, S) are continuous with values in $H^s(\mathbb{D})$.

The case $k = 0$ is usually considered to be the case of slightly compressible fluids while the case $k = 1$ concerns large-amplitude high-frequency solutions. In the latter case the parameter ε has the meaning of a wavelength. As in [Ser, E, Hei], the entropy then has high-frequency waves of amplitude $O(1)$. However, the remaining hypotheses here are mutually exclusive with the hypotheses in those works. Here u and p are of order ε but those $O(\varepsilon)$ terms have arbitrary initial data and so include fast acoustic waves, whereas there u and p are $O(1)$ but have special initial data that excludes fast acoustic waves even of size $O(\varepsilon)$. In addition, [Ser, E, Hei] only treat problems in one space dimension but allow solutions to depend on x as well as $\frac{x}{\varepsilon}$. The reason solutions depending on x as well as $\frac{x}{\varepsilon}$ cannot be treated here is that assumption (1.3) would not be satisfied. An example will be presented below showing that when (1.3) does not hold then solutions of (1.7) need not be uniformly bounded for a time independent of ε, even when (1.6) holds.

We now turn to considering the limit of solutions of (1.10) in $\mathbb{R}^d$ as $\varepsilon \to 0$. When the initial data are prepared so that they obey the condition that $\nabla \cdot v_0^\varepsilon$ and ∇q_0^ε are $O(\varepsilon)$, then q^ε converges to zero and $(v^\varepsilon, S^\varepsilon)$ converge to the solution of the stratified incompressible Euler equations

$$\begin{aligned} \nabla \cdot v &= 0, \\ r(S,0)(\partial_t v + v \cdot \nabla v) + \nabla \pi &= 0, \\ \partial_t S + v \cdot \nabla S &= 0 \end{aligned} \tag{1.12}$$

with the limit initial data (see [Sch1]). For initial data not so constrained but still satisfying (1.11) plus an additional condition to be described shortly, we prove that $(v^\varepsilon, S^\varepsilon)$ converges strongly for $t > 0$ to the solution of (1.12) whose initial data is the incompressible part of the data in (1.11). More specifically:

Theorem 1.4. *Assume that* $(v^\varepsilon, q^\varepsilon, S^\varepsilon)$ *satisfy* (1.10) *and are uniformly bounded in* $C^0([0,T]; H^s(\mathbb{R}^d))$ *with* $s > 1 + d/2$ *and fixed* $T > 0$. *Suppose that the initial data* $(v_0^\varepsilon, S_0^\varepsilon)$ *converge in* $H^s(\mathbb{R}^d)$ *to* (v_0, S_0) *as* $\varepsilon \to 0$, *and that* S_0^ε *decays sufficiently rapidly at infinity in the sense that*

$$|S_0^\varepsilon(x)| \leqq C|x|^{-1-\delta}, \quad |\nabla S_0^\varepsilon(x)| \leqq C|x|^{-2-\delta} \tag{1.13}$$

for all ε *and some fixed* C *and* $\delta > 0$. *Then* $(v^\varepsilon, q^\varepsilon, S^\varepsilon)$ *converge weakly in* $L^\infty([0,T], H^s(\mathbb{R}^d))$ *and strongly in* $L^2([0,T], H^{s'}_{\mathrm{loc}}(\mathbb{R}^d))$ *for all* $s' < s$ *to a limit* $(v, 0, S)$. *Moreover,* (v, S) *is the unique solution in* $C^0([0,T]; H^s(\mathbb{R}^d))$ *of* (1.12) *with initial data* (w_0, S_0), *with* w_0 *being the unique solution in* $H^s(\mathbb{R}^d)$ *of*

$$\nabla \cdot w_0 = 0, \quad \operatorname{curl}(r_0 w_0) = \operatorname{curl}(r_0 v_0), \quad \textit{where} \quad r_0 = r(S_0, 0).$$

The incompressible pressure π *can be chosen in* $C^0([0,T] \times \mathbb{R}^d)$ *and satisfies*

$$\nabla \pi \in C^0([0,T]; H^{s-1}(\mathbb{R}^d)). \tag{1.14}$$

As usual for incompressible problems, we can show that (v, S) belongs to $C^0([0,T]; H^s)$ and satisfies

$$\begin{aligned} \nabla \cdot v &= 0, \\ \operatorname{curl}\big(r(S,0)(\partial_t v + v \cdot \nabla v)\big) &= 0, \\ \partial_t S + v \cdot \nabla S &= 0, \end{aligned} \tag{1.15}$$

which is formally equivalent to (1.12). This implies that $g := \big(r(S,0)(\partial_t v + v\cdot\nabla v)\big)$ belongs to $C^0([0,T]; H^{s-1}(\mathbb{R}^d)) \subset C^0([0,T]\times\mathbb{R}^d)$ and satisfies $\operatorname{curl} g = 0$. Thus, there is $\pi \in C^0([0,T]\times\mathbb{R}^d)$ such that $\nabla\pi = g$.

Note that Theorem 1.3 ensures the existence of solutions of (1.10) satisfying the first hypothesis of the theorem, provided the initial data satisfy (1.11) with fixed M_0.

In the isentropic case, this result has been obtained by [Asa] and [Uka]. The basic idea is that since the speed of propagation of the acoustic waves is of order $\frac{1}{\varepsilon}$, those waves radiate to spatial infinity after a small initial layer. The relevant wave equation is

$$\varepsilon^2\partial_t(a_0^\varepsilon\partial_t\varphi) - \nabla\cdot\left(\frac{1}{r_0^\varepsilon}\nabla\varphi\right) = 0, \tag{1.16}$$

where $a_0^\varepsilon = a(S^\varepsilon,0)$ and $r_0^\varepsilon = r(S^\varepsilon,0)$. In contrast to the isentropic case, this equation has unknown variable coefficients a_0^ε and r_0^ε. The new ingredient in the proof of Theorem 1.4 is a proof of the decay to zero of the local energy for the solutions of (1.16). Indeed, since estimates (1.13) are propagated in time, (1.16) appears as a time-dependent short-range perturbation of a constant-coefficient wave equation. Energy decay implies a form of local strong convergence for (q,v), which is the key to proving Theorem 1.4, since the absence of large terms in the equation for S easily implies that S also converges strongly.

As for the existence proof, we extend our analysis to systems of the form (1.7). In order to state our result for that case, we define $E_0(S) = E(S,0)$, where E is the coefficient appearing in the first equation in (1.7), and let $\Pi(D_x)$ be the operator defined by the Fourier multiplier $\Pi(\xi)$, which is the orthogonal projector on the kernel of $L(\xi)$.

Theorem 1.5. *Assume that* $(u^\varepsilon, S^\varepsilon)$ *satisfy* (1.7) *and are uniformly bounded in* $C^0([0,T]; H^s(\mathbb{R}^d))$ *with* $s > 1 + d/2$ *and fixed* $T > 0$. *Suppose that the initial data* $(S_0^\varepsilon, u_0^\varepsilon)$ *converge in* $H^s(\mathbb{R}^d)$ *to* (S_0, u_0). *Assume further that*

$$\begin{aligned}&\textit{for all } \sigma\in H^s(\mathbb{R}^d)\cap\mathbb{B} \textit{ and all } \tau\in\mathbb{R}\backslash\{0\}, \textit{ the kernel of}\\ &i\tau E_0(\sigma) + L(\partial_x) \textit{ in } L^2(\mathbb{R}^d) \textit{ is reduced to } \{0\},\end{aligned} \tag{1.17}$$

where $\mathbb{B}$ *is a Banach space in which* S^ε *is uniformly bounded and whose unit ball is closed under convergence in* $C^0([0,T], H^{s'}_{\mathrm{loc}}(\mathbb{R}^d))$ *for some* $s' < s$.

Then $(u^\varepsilon, S^\varepsilon)$ *converges weakly in* $L^\infty([0,T], H^s(\mathbb{R}^d))$ *and strongly in* $L^2([0,T]; H^{s'}_{\mathrm{loc}}(\mathbb{R}^d))$ *for all* $s' < s$. *The limit is the unique solution in* $C^0([0,T]; H^s(\mathbb{R}^d))$ *of*

$$\begin{aligned} L(\partial_x)u &= 0,\\ E_0(S)(\partial_t u + b(S,u)\cdot\nabla u) + L(\partial_x)\pi &= 0,\\ \partial_t S + b(S,u)\cdot\nabla S &= 0,\end{aligned} \tag{1.18}$$

satisfying the initial conditions

$$S_{|t=0} = S_0, \quad u_{|t=0} = v_0, \tag{1.19}$$

where $v_0 \in H^s(\mathbb{R}^d)$ *satisfies*

$$v_0 = \Pi(D_x)v_0, \quad \Pi(D_x)(E_0(S_0)v_0) = \Pi(D_x)(E_0(S_0)u_0). \tag{1.20}$$

The constraint variable $\pi \in C^0([0,T] \times \mathbb{R}^d)$ *and satisfies* (1.14).

As described above, the key step in the proof of Theorem 1.5 is the following:

Proposition 1.6. *Under the assumptions of Theorem 1.5,* u^ε *converges strongly to* u *in* $L^2([0,T], H^{s'}_{\mathrm{loc}}(\mathbb{R}^d))$ *for* $s' < s$.

The hypothesis that $S^\varepsilon(t)$ lies in $\mathbb{B}$ is included to allow for the possibility that the conclusion of assumption (1.17) may hold only for σ in some appropriate subspace of H^s. This hypothesis therefore plays no role in the proof of Theorem 1.5 itself, but only in the proof that the theorem applies to the slightly compressible Euler equations. Indeed, in order to apply Theorem 1.5 to those equations so as to obtain Theorem 1.4, it suffices to show that assumption (1.17) is satisfied. The Banach space $\mathbb{B}$ will be taken to be the space of functions satisfying (1.13), and assumption (1.13) will be utilized in order to show that coefficients $a = a_0^\varepsilon$ and $b = b_0^\varepsilon := 1/r_0^\varepsilon$ from (1.16) satisfy the first hypothesis of the following theorem, which implies (1.17).

Theorem 1.7. *Suppose that the coefficients* a *and* b *satisfy*

$$\begin{aligned} |a(x) - \underline{a}| &\leqq C(1+|x|)^{-1-\delta}, \quad |\nabla a(x)| \leqq C(1+|x|)^{-2-\delta}, \\ |b(x) - \underline{b}| &\leqq C(1+|x|)^{-1-\delta}, \quad |\nabla b(x)| \leqq C(1+|x|)^{-2-\delta}, \end{aligned} \tag{1.21}$$

and

$$\forall x \in \mathbb{R}^d, \quad a(x) \geqq c \quad \text{and} \quad b(x) \geqq c, \tag{1.22}$$

for some positive constants c, C, $\underline{a}$, *and* $\underline{b}$. *Then, for all* $\tau \in \mathbb{R}$, *the kernel of* $a\tau^2 + \nabla \cdot (b\nabla)$ *in* $L^2(\mathbb{R}^d)$ *is reduced to* $\{0\}$.

Theorem 1.2 is proved in Section 2. In Section 3 we prove Theorem 1.5 assuming Proposition 1.6, and then use it to prove Theorem 1.4 assuming Theorem 1.7. Finally, Proposition 1.6 and Theorem 1.7 are proved in Sections 4 and 5 respectively. We end this introduction with the examples promised above showing the necessity of various combinations of the assumptions included in the structure in (1.7) and in Assumption 1.1. We also give an example showing that when Assumption 1.1 is satisfied, the Cauchy problem for (1.7) is still non-uniformly stable.

Example 1. The system

$$S^{-1}(u_t + u_x) + \frac{1}{\varepsilon} u_y = 0, \quad S_t + S_x = 0,$$

has the structure of (1.7) and satisfies (1.5) and (1.6) but not (1.3). Assume for simplicity that the initial data for S depends only on x. Solving first the equation for S to obtain $S(t,x,y) = S^0(x-t)$, and then the equation for u to obtain $u(t,x,y) = u^0\big(x-t, y - \frac{t}{\varepsilon} S^0(x-t)\big)$, we find that

$$u_x(t,x,y) = u_x^0\Big(x-t, y - \frac{t}{\varepsilon}S^0(x-t)\Big) - \frac{t}{\varepsilon} S_x^0(x-t) u_y^0\Big(x-t, y-\frac{t}{\varepsilon}S^0(x-t)\Big),$$

which is not uniformly bounded.

Example 2. The system

$$u_t - u_x + \frac{1}{\varepsilon} u_x = 0, \quad v_t + \frac{1}{\varepsilon} S u_x = 0, \quad S_t - S_x = 0$$

can be put into the form (1.1), (1.4) by multiplying the equations for u and v by the matrix

$$E = \begin{pmatrix} a & -cS \\ -cS & c(1-\varepsilon) \end{pmatrix}$$

where a and c are positive functions of S satisfying $a - cS^2 \geqq k > 0$. However, the resulting equations do not satisfy condition (1.5) because there is no term $-v_x$ in the equation for v. Those equations also fail to satisfy condition (1.6), although (1.3) does hold.

Since the system is well posed at least non-uniformly in ε, what we are trying to show is that the H^s norm of the solution with some fixed H^s initial data is not uniformly bounded for some time $T(\varepsilon) = o(1)$.

In the periodic case, the solutions of the equations for u and S can be written in the form

$$u(t,x) = \sum_{k\in\mathbb{Z}} u_k e^{ik(x+t-t/\varepsilon)}, \quad S(t,x) = \sum_{k\in\mathbb{Z}} S_k e^{ik(x+t)}.$$

Upon plugging the form $v(t,x) = \sum_{k\in\mathbb{Z}} v_k(t) e^{inx}$ into the equation for v and solving the resulting ordinary differential equation for v_n we find that

$$v_n = v_n^0 + \sum_{k\in\mathbb{Z}} \frac{k}{k - \varepsilon n} S_{n-k} u_k \big(e^{it(n-k/\varepsilon)} - 1\big). \tag{1.23}$$

In order to simplify the computations, make the following assumptions: $u_{\pm 1} = 1$ and $u_k = 0$ for $k \neq \pm 1$; $S_k = \frac{1}{|k|^p}$ when k is a multiple of four and vanishes otherwise; $v^0 \equiv 0$. Then v_n is nonzero only for odd n, in which case the sum in (1.23) contains only a single nonzero term. In particular, for $n = \big[\frac{1}{\varepsilon}\big] + m$ equalling one more than a multiple of four,

$$v_n = \frac{e^{i(m+O(1))t} - 1}{\varepsilon(m + O(1))\big(\frac{1}{\varepsilon} + m + O(1)\big)^p}.$$

In order that m should be small compared to $\frac{1}{\varepsilon}$, let us restrict consideration to m such that $|m| \leqq \frac{2}{\sqrt{\varepsilon}}$. In order that $|e^{i(m+O(1))t} - 1|$ should not be close to zero even for t close to zero, let us restrict consideration to m satisfying $|m| \geqq \frac{1}{\sqrt{\varepsilon}}$. Furthermore, in order to avoid the necessity of determining where that expression is not close to zero, let us consider the average of $\|v(t)\|^2_{H^s}$ over $0 \leqq t \leqq \varepsilon^{1/4}$. In order to show that the H^s norm is not uniformly bounded on this interval it suffices to show that its average over the interval is not uniformly bounded.

Since $\varepsilon^{-1/4} \int_0^{\varepsilon^{1/4}} e^{cit/\sqrt{\varepsilon}} = O(\varepsilon^{1/4})$, which is negligible compared to the average of 1 over the same interval, the contribution to average norm squared from the set of v_n of the above form with m in the above range is $O\Big(\big(\frac{1}{\varepsilon}\big)^{2(s-p)+3/2}\Big)$.

Comparing this to the condition $2(s-p)+1<0$ for the initial data of S to lie in H^s, we see that if $s+\frac{1}{2}<p<s+\frac{3}{4}$ then the initial data will belong to H^s but the time-averaged H^s norm of the solution will tend to infinity as $\varepsilon \to 0$.

Example 3. The system

$$a(S)u_t + \frac{1}{\varepsilon}u_x = 0, \quad S_t = 0,$$

satisfies Assumption 1.1. When S_0 is constant, the solution is $u(t,x) = u_0\big(x - t/\varepsilon a(S_0)\big)$. Thus, a small perturbation of S_0 induces in time $O(\varepsilon)$ a perturbation of u which is not small. This is a typical example of instability.

2. Proof of the existence theorem

Consider a system of the form (1.7) and assume that Assumption 1.1 is satisfied. The system is symmetric hyperbolic. Therefore, for all fixed $\varepsilon > 0$ there is $T = T(\varepsilon, M_0) > 0$ such that for all initial data which satisfy (1.8), the Cauchy problem has a unique solution on $C^0([0,T]; H^s(\mathbb{D}))$. Moreover, if $T^*(\varepsilon)$, the maximal time of existence of such a smooth solution, is finite, then

$$\limsup_{t\to T^*(\varepsilon)} \|(u(t),S(t))\|_{W^{1,\infty}(\mathbb{D})} = \infty, \tag{2.1}$$

(see, e.g., [Maj]). In particular, this implies that the H^s norm of $(u(t),S(t))$ is unbounded as t tends to $T^*(\varepsilon)$ if $T^*(\varepsilon)$ is finite.

In view of this preliminary remark, Theorem 1.2 is a consequence of the following estimates.

Proposition 2.1. *Given $s > 1 + d/2$ and M_0, there is a constant C_0 and a function $C(\cdot)$ from $[0,\infty[$ to $[0,\infty[$, such that for all $T \in]0,1]$, $\varepsilon \in]0,1]$ and $(u,S) \in C^0([0,T]; H^s(\mathbb{D}))$ solution of (1.7) with initial conditions satisfying (1.8), the norm*

$$M := \sup_{t\in[0,T]} \|(u(t),S(t))\|_{H^s(\mathbb{D})} \tag{2.2}$$

satisfies the estimate

$$M \leqq C_0 + (T+\varepsilon)C(M). \tag{2.3}$$

To see this, choose first $M_1 > C_0$ and next $\varepsilon_1 > 0$ and $T_1 \in]0,1]$ such that $C_0 + (T_1+\varepsilon_1)C(M_1) < M_1$. Consider initial data satisfying (1.8). Let $T^*(\varepsilon)$ denote the upper bound of the $T > 0$ such that the Cauchy problem has a solution in $C^0([0,T]; H^s(\mathbb{D}))$. The classical existence result for symmetric hyperbolic equations implies that $T^*(\varepsilon) > 0$ and that $T^*(\varepsilon)$ is bounded from below by $T_2 > 0$ when $\varepsilon \geqq \varepsilon_1$. For $t < T^*(\varepsilon)$, denote by $M(t)$ the norm (2.2) defined on $[0,t]$. When $\varepsilon \leqq \varepsilon_1$, (2.3) implies that $M(t) < M_1$ for $t < \min(T_1, T^*(\varepsilon))$. Therefore (2.1) implies that $T^*(\varepsilon) > T_1$ for all $\varepsilon \leqq \varepsilon_1$. This shows that Proposition 2.1 implies Theorem 1.2.

From now on, we consider a solution $(u, S) \in C^0([0,T]; H^s(\mathbb{D}))$ of (1.7), with initial data satisfying (1.8). We denote by M the norm defined in (2.2). To simplify notation, let $\|\cdot\|_s$ denote the norm in $H^s(\mathbb{D})$. To prove (2.3), we first give an estimate for the entropy S.

Lemma 2.2. *There are a constant C_0 and a function $C(\cdot)$, depending only on M_0, such that*

$$\forall t \in [0,T], \quad \|S(t)\|_s \leqq C_0 + tC(M). \tag{2.4}$$

Proof. We use the following well-known nonlinear estimates. For $k \geqq 0$, $l \geqq 0$, $k+l \leqq \sigma$ and $\sigma > d/2$, the product maps continuously $H^{\sigma-k}(\mathbb{D}) \times H^{\sigma-l}(\mathbb{D})$ to $H^{\sigma-k-l}(\mathbb{D})$ and

$$\|uv\|_{\sigma-k-l} \leqq C\|u\|_{\sigma-k}\|v\|_{\sigma-l}. \tag{2.5}$$

Similarly, if F is a smooth function such that $F(0) = 0$ and $u \in H^\sigma(\mathbb{D})$, then $F(u) \in H^\sigma(\mathbb{D})$ and its norm is bounded by

$$\|F(u)\|_\sigma \leqq C(\|u\|_\sigma) \tag{2.6}$$

where $C(\cdot)$ is independent of u and maps $[0,\infty[$ into $[0,\infty[$.

For $\alpha \in \mathbb{N}^d$, $|\alpha| \leqq s$, introduce $S_\alpha = \partial_x^\alpha S$. Then

$$(\partial_t + b(S,u)\cdot\partial_x)S_\alpha = h_\alpha := -\big[\partial_x^\alpha, (\partial_t + b(S,u)\cdot\partial_x)\big]S. \tag{2.7}$$

The nonlinear estimate (2.6) implies that $b'(t) := b\big(S(t), u(t)\big) - b(0,0) \in H^s(\mathbb{D})$ with norm less than $C(M)$. Since the commutator h_α is a sum of terms $\partial_x^\beta b' \partial^\gamma S$ with $|\beta| + |\gamma| \leqq s+1$, $\beta > 0$ and $\gamma > 0$, the rule (2.5) applied with $\sigma = s - 1 > d/2$, implies that

$$\|h_\alpha(t)\|_0 \leqq C(M). \tag{2.8}$$

Since $s > 1 + d/2$, the first x-derivatives of the coefficient $b(S,u)$ are C^1 with norm bounded by $C(M)$. The usual L^2 energy estimate for the transport equation (2.7) implies that

$$\|S_\alpha(t)\|_0 \;\leqq\; e^{tC(M)}\|S_\alpha(0)\|_0 + \int_0^t e^{(t-t')C(M)}\|h_\alpha(t')\|_0\,dt'.$$

The lemma follows by adding up these estimates for $|\alpha| \leqq s$ and using (2.8) together with the elementary inequality

$$e^{tC(M)} \leqq 1 + t\widetilde{C}(M)$$

for nonnegative t less than some arbitrary fixed T.

In order to prove a bound analogous to (2.4) for u, the first step is to obtain L^2 estimates for the partially linearized equations

$$E(\partial_t\dot{u} + b\cdot\nabla\dot{u}) + \frac{1}{\varepsilon}L(\partial_x)\dot{u} = \dot{f}, \tag{2.9}$$

with $E = E(S, \varepsilon u)$ and $b = R(S,u)$.

Lemma 2.3. *There are C_0 and $C(\cdot)$, depending only on M_0, such that for all $\dot u \in C^0([0,T]; L^2(\mathbb{D}))$ and $\dot f \in C^0([0,T]; L^2(\mathbb{D}))$ satisfying* (2.9),

$$\|\dot u(t)\|_0 \leqq C_0 e^{tC(M)} \|\dot u(0)\|_0 + C(M) \int_0^t e^{(t-t')C(M)} \|\dot f(t')\|_0 \, dt'. \tag{2.10}$$

Proof. First, since $s - 1 > d/2$, using (1.7) and properties (2.5), (2.6), we remark that

$$\|\partial_t S(t)\|_{s-1} \leqq C(M), \quad \|\varepsilon \partial_t u(t)\|_{s-1} \leqq C(M). \tag{2.11}$$

In particular, this implies that $E = E(S, \varepsilon u)$ satisfies

$$\|\partial_{t,x} E(t)\|_{L^\infty(\mathbb{D})} \leqq C\big(\|E(t)\|_s + \|\partial_t E(t)\|_{s-1}\big) \leqq C(M). \tag{2.12}$$

The inverse matrix $E^{-1}(t)$ satisfies similar estimates. Moreover, (2.6) implies that $b = b(S, u)$ satisfies

$$\|\partial_x b(t)\|_{L^\infty(\mathbb{D})} \leqq C\|b(t)\|_s \leqq C(M). \tag{2.13}$$

It is sufficient to prove (2.10) for H^1 functions. Multiply the equations in (2.9) by $\dot u$ and integrate over $[0, t] \times \mathbb{D}$. Because $L(\partial_x)$ is skew-adjoint, the terms in $1/\varepsilon$ cancel out; as for the other terms, the derivatives of the coefficients involve $\partial_t E$, $\partial_x E$ and $\partial_x b$, which are estimated in L^∞ by $C(M)$. Using the symmetry of the matrices $E(t)$, this implies that

$$\begin{aligned}\big(E(t)\dot u(t), \dot u(t)\big)_0 &\leqq \big(E(0)\dot u(0), \dot u(0)\big)_0 \\ &\quad + \int_0^t \|\dot f(t')\|_0 \|\dot u(t')\|_0 \, dt' \;+\; C(M) \int_0^t \|\dot u(t')\|_0^2 \, dt',\end{aligned}$$

where $(\cdot,\cdot)_0$ denotes the scalar product in $L^2(\mathbb{D})$. Because E is positive definite,

$$\|\dot u(t)\|_0^2 \leqq \|E^{-1}(t)\|_{L^\infty} \big(E(t)\dot u(t), \dot u(t)\big)_0.$$

These estimates, Gronwall's lemma and the bounds

$$\|E^{-1}(t)\|_{L^\infty} \leqq \|E^{-1}(0)\|_{L^\infty} + t\|\partial_t E^{-1}(t)\|_{L^\infty} \leqq C_0 + tC(M),$$

imply (2.10), and the lemma is proved.

We first use Lemma 2.3 to give estimates for

$$u_k := \big(E^{-1}(S, u) L(\partial_x)\big)^k u, \quad k \in \{0, \dots s\}. \tag{2.14}$$

Lemma 2.4. *There are C_0 and $C(\cdot)$, depending only on M_0, such that for all $k \leqq s$ and $t \in [0, T]$,*

$$\|u_k(t)\|_0 \leqq C_0 + tC(M). \tag{2.15}$$

Proof. For $k = 0$, this is an immediate consequence of Lemma 2.3. Introduce $L_E(\partial_x) := E^{-1}L(\partial_x)$. This operator is bounded from $C^0(H^\sigma)$ to $C^0(H^{\sigma-1})$ for $\sigma \in \{0, \dots, s+1\}$. For $k \geqq 1$, we commute L_E^k with the equation, premultiplied by E^{-1}. Next, we multiply the result by E. This yields the equation

$$E(\partial_t + b\partial_x)u_k + \frac{1}{\varepsilon}L(\partial_x)u_k = Ef_k, \quad f_k := \big[\partial_t + b\partial_x, L_E^k\big]u.$$

Using Lemma 2.3, to prove (2.15) it is sufficient to show that

$$\|f_k(t)\|_0 \leqq C(M). \tag{2.16}$$

We have

$$\big[\partial_t + b\partial_x, L_E^k\big] = \sum_{l=0}^{k-1} L_E^j\big[\partial_t + b\partial_x, L_E\big]L_E^{k-j-1}, \tag{2.17}$$

and

$$\big[\partial_t + b\partial_x, L_E\big] = \sum_{j=1}^{d} C_j\partial_{x_j},$$

where the C_j are sums of bilinear functions of b and $\partial_{t,x}E^{-1}$ or $\partial_x b$ and E^{-1}. The key point is that C_j does not involve the time derivatives of b. Therefore, (2.12) and (2.13) imply that

$$\|C_j(t)\|_{s-1} \leqq C(M). \tag{2.18}$$

The identity (2.17) implies that the components of f_k are finite sums of terms of the form

$$(\partial_x^{\beta_1}e_1)\dots(\partial_x^{\beta_k}e_k)\partial_x^\gamma c\partial_x^\alpha u_m$$

with $|\beta_1|+\ldots+|\beta_k|+|\gamma|+|\alpha| \leqq k \leqq s$, $|\alpha| > 0$ and thus $|\gamma| \leqq k-1 \leqq s-1$. In this formula, $(e_1, \dots, e_k)$, c and u_m denote coefficients of E^{-1}, C_j and u respectively. The multiplicative property (2.5) and the estimates (2.12) and (2.18) imply that the L^2 norm of each term is bounded by $C(M)$, so (2.16) and the lemma follows.

Now we really use the special structure of the equations, i.e., (1.5) and part (ii) of Assumption 1.1. Introduce the matrix-valued functions

$$\begin{aligned}
E_0(S) &= E(S, 0),\\
F(S, u) &= E(S, u)E_0^{-1}(S) = Id + F'(S, u),\\
F^{-1}(S, u) &= E_0(S)E^{-1}(S, u) = Id - F''(S, u).
\end{aligned} \tag{2.19}$$

With little risk of confusion, to shorten notation, we also denote by $E_0, E, F, F' \dots,$ the function $E_0(S(t,x))$, $E\big(S(t,x), \varepsilon u(t,x)\big) \dots$. We can factor out εu in F' and F''. In particular, we can write

$$F'' = \varepsilon G \quad \text{with} \quad \|G(t)\|_s \leqq C(M). \tag{2.20}$$

Because E_0 depends only on S, the equation for S implies that $(\partial_t + b\cdot\partial_x)E_0 = 0$, so the equation for u is equivalent to

$$(\partial_t + b\cdot\partial_x)(E_0u) + \frac{1}{\varepsilon}L(\partial_x)u = GL(\partial_x)u. \tag{2.21}$$

Introduce the orthogonal projector $\Pi(\xi)$ onto the kernel of $L(\xi)$. Because the kernel has constant dimension, Π is a real analytic function of ξ for $\xi \neq 0$, homogeneous of degree zero. In the periodic case $\mathbb{D} = \mathbb{T}^d$, $\xi \in \mathbb{Z}^d$ and one needs the definition of Π at the origin, $\Pi(0) = Id$. We denote by $\Pi(D_x)$, or simply by Π, the convolution operator associated with the Fourier multiplier $\Pi(\xi)$. The operator $\Pi(D_x)$ is bounded from $H^\sigma(\mathbb{D})$ to itself for all σ and $\Pi(D_x)L(\partial_x) = 0$. Therefore, $\Pi E_0 u$ satisfies

$$\begin{aligned}(\partial_t + b\cdot\partial_x)(\Pi(D_x)E_0u) = \big[b\partial_x, \Pi(D_x)\big](E_0u)\\ + \big[\Pi(D_x), G\big]L(\partial_x)u.\end{aligned} \tag{2.22}$$

To estimate the right-hand side f, we use the following result.

Lemma 2.5. *Suppose that G is a matrix with coefficients in $H^s(\mathbb{D})$ with $s > 1+d/2$, such that $G(x)$ and $\Pi(\xi)$ commute for all x and ξ. Then, for all $\sigma \in \{0, \dots, s-1\}$ and $v \in H^\sigma(\mathbb{D})$,*

$$\|[\Pi(D_x), G]v\|_{\sigma+1} \leqq C\|G\|_s\|v\|_\sigma. \tag{2.23}$$

Proof. This is a classical result about the commutation of a pseudodifferential operator $\Pi(D_x)$ and the multiplication by a function $G \in H^s$. We briefly recall a sketch of proof. When $\mathbb{D} = \mathbb{R}^d$, we can use the paradifferential calculus of BONY [Bo]. Denoting by $\mathcal{T}_G$ the operator of para-multiplication by G and more generally by $\mathcal{T}_A$ the paradifferential operator of symbol $A(x,\xi)$, we have

$$\begin{aligned}\|\mathcal{T}_G v - Gv\|_{\sigma+1} &\leqq \|G\|_s\|v\|_\sigma,\\ \|\Pi(D_x)\mathcal{T}_G v - \mathcal{T}_{\Pi G}v\|_{\sigma+1} &\leqq \|G\|_s\|v\|_\sigma,\\ \mathcal{T}_G\Pi(D_x)v &= \mathcal{T}_{G\Pi}v.\end{aligned}$$

Since the symbols $G(x)\Pi(\xi)$ and $\Pi(\xi)G(x)$ are equal, (2.23) follows.

When $\mathbb{D} = \mathbb{T}^d$, expanding the functions into their Fourier series, the ν-th Fourier coefficient of $w := [\Pi(D_x), G]v$ is

$$w_\nu = \sum_\mu \big(\Pi(\nu)G_{\nu-\mu} - G_{\nu-\mu}\Pi(\mu)\big)v_\mu.$$

When $|\nu-\mu| < |\mu|/2$, we use the fact that the matrices G_μ and $\Pi(\nu)$ commute and that $\Pi(\nu) - \Pi(\mu) = O|(|\mu-\nu|/|\nu|\big)$ to obtain

$$\big|\Pi(\nu)G_{\nu-\mu} - G_{\nu-\mu}\Pi(\mu)\big| \leqq C|\mu-\nu||G_{\nu-\mu}|/|\nu|.$$

The factor $\frac{|\mu-\nu|}{|\nu|}$ has the effect of making one "derivative", i.e., factor of $|\nu|$, in the expression $\sum_\nu(1+|\nu|)^{2\sigma+2}|w_\nu|^2$ apply only to G, not to v, thereby yielding an estimate in which at most σ derivatives are applied to v, as desired.

When $|\nu-\mu| \geqq |\mu|/2$ it suffices to use the estimate $\big|\Pi(\nu)G_{\nu-\mu} - G_{\nu-\mu}\Pi(\mu)\big| \leqq 2|G_{\nu-\mu}|$, since then $|\nu| \leqq 3|\nu-\mu|$, which allows all the factors of $|\nu|$ to be converted to factors of $|\nu-\mu|$ that apply only to G. Hence, for for $s > 1+d/2$, this implies that

$$\sum_\nu(1+|\nu|)^{2\sigma+2}|w_\nu|^2 \leqq C\Big(\sum_\nu(1+|\nu|)^{2s}|G_\nu|^2\Big)\Big(\sum_\nu(1+|\nu|)^{2\sigma}|v_\nu|^2\Big),$$

which is (2.23). This proof can also been carried over $\mathbb{R}^d$, replacing the Fourier series by Fourier integrals, but the splitting of the frequencies sketched above is exactly what the paradifferential calculus does.

Condition (1.6) implies that $E(t,x)$, $E_0(t,x)$, $F(t,x)$, and hence also $G(t,x)$, commute with $\Pi(\xi)$. Lemma 2.5 and the estimate (2.20) therefore imply that the right-hand side f of equation (2.22) satisfies

$$\|f(t)\|_s \leqq C(M)$$

Repeating the proof of Lemma 2.2 for the transport equation (2.22), we obtain

Corollary 2.6. *There are C_0 and $C(\cdot)$, depending only on M_0, such that*

$$\forall t \in [0,T], \quad \|\Pi(D_x)E_0 u(t)\|_s \leqq C_0 + tC(M). \tag{2.24}$$

Having estimated $\Pi(D_x)E_0 u$ and $L_E^k u$, we can now estimate u. The idea is that the system $(L_E^s, |D_x|^s \Pi(D_x)E_0)$ is elliptic in x. We start with the following estimate.

Lemma 2.7. *There exists a K, and there exist C_0 and $C(\cdot)$ depending only on M_0, such that for $\sigma \in \{1, \dots, s\}$, $t \in [0,T]$ and $v \in H^\sigma(\mathbb{D})$,*

$$\|v\|_\sigma \leqq K\|L(\partial_x)v\|_{\sigma-1} + \widetilde{C}\big(\|\Pi(D_x)E_0(t)v\|_\sigma + \|v\|_{\sigma-1}\big), \tag{2.25}$$

and

$$\|v\|_\sigma \leqq \widetilde{C}\big(\|(L_{E(t)}(\partial_x))^\sigma v\|_0 + \|\Pi(D_x)E_0(t)v\|_\sigma + \|v\|_{\sigma-1}\big), \tag{2.26}$$

with $\widetilde{C} := C_0 + (t+\varepsilon)C(M)$.

Proof. We start from the estimates

$$\|v\|_\sigma \leqq K\big(\|L(\partial_x)v\|_{\sigma-1} + \|\Pi(D_x)v\|_\sigma + \|v\|_{\sigma-1}\big), \tag{2.27}$$

which are immediate using Fourier transforms of Fourier series expansions. By (2.5),

$$\|\Pi(D_x)v\|_\sigma \leqq K\|E_0^{-1}(t)\|_s\|E_0(t)\Pi(D_x)v\|_\sigma.$$

Lemma 2.5 implies that

$$\|E_0(t)\Pi(D_x)v - \Pi(D_x)E_0(t)v\|_\sigma \leqq \|E_0(t)\|_s\|v\|_{\sigma-1}.$$

Thus (2.25) follows from (2.27) and Lemma 2.2, which implies that there are C_0 and $C(\cdot)$ such that

$$\|E_0^{-1}(t)\|_s + \|E_0(t)\|_s \leqq C_0 + tC(M). \tag{2.28}$$

Next, we prove (2.26) by induction on σ. We have

$$\|L(\partial_x)v\|_0 \leqq \|E(t)\|_{L^\infty}\|L_{E(t)}(\partial_x)v\|_0.$$

Recall that $E = E_0 + F'E_0$, and that F' satisfies estimates similar to (2.20). Therefore, there are C_0 and $C(\cdot)$ such that

$$\|E(t)\|_s \leqq C_0 + tC(M) + \varepsilon C(M). \tag{2.29}$$

Therefore, for $\sigma = 1$, (2.26) immediately follows from (2.25) and (2.29).

If (2.26) is satisfied at the order $\sigma < s$, then

$$\|L_{E(t)}v\|_\sigma \leqq \widetilde{C}\big(\|L_{E(t)}^{\sigma+1}v\|_0 + \|\Pi(D_x)E_0(t)L_{E(t)}v\|_\sigma + \|L_{E(t)}v\|_{\sigma-1}\big).$$

By (2.29),

$$\|L_{E(t)}v\|_{\sigma-1} \leqq \widetilde{C}\|v\|_\sigma.$$

Moreover, $E_0(t)L_{E(t)}(\partial_x) = F^{-1}L(\partial_x)$ and $\Pi(D_x)L(\partial_x) = 0$. Thus, Lemma 2.5 and (2.20) imply that

$$\|\Pi(D_x)E_0(t)L_{E(t)}v\|_\sigma \leqq \varepsilon C(M)\|v\|_\sigma.$$

This implies that there are C_0 and $C(\cdot)$ such that

$$\|L_{E(t)}v\|_\sigma \leqq \widetilde{C}\big(\|L_{E(t)}^{\sigma+1}v\|_0 + \|v\|_\sigma\big).$$

Substituting in (2.25) at the order $\sigma + 1$, implies (2.26) at the same order, and the proof of Lemma 2.7 is now complete.

The next estimate finishes the proof of Proposition 2.1.

Lemma 2.8. *There are C_0 and $C(\cdot)$ which depend only on M_0, such that for $t \in [0, T]$ one has*

$$\|u(t)\|_s \leqq KC_0 + (t + \varepsilon)C(M).$$

Proof. The L^2 norm of $u(t)$ is estimated in (2.15), taking $k = 0$. Next, Lemma 2.4, Corollary 2.6 and Lemma 2.7 imply by induction on $\sigma \in \{0, \dots, s\}$ that there are C_0 and $C(\cdot)$ such that

$$\|u(t)\|_\sigma \leqq KC_0 + (t + \varepsilon)C(M). \tag{2.30}$$

The last estimate, with $\sigma = s$ gives Lemma 2.8.

3. The incompressible limit

Consider a family of solutions $(u^\varepsilon, S^\varepsilon)$ of (1.7), uniformly bounded in $C^0([0, T]; H^s(\mathbb{R}^d))$ with $s > 1 + d/2$ and fixed $T > 0$. In this section,we study the limit of $(u^\varepsilon, S^\varepsilon)$ as ε tends to zero. We first list the convergences which follow directly from the bounds and next use Proposition 1.6 to finish the proof of Theorem 1.5.

The equation for S implies that $\partial_t S^\varepsilon$ is bounded in $C^0([0, T]; H^{s-1}(\mathbb{R}^d))$ (see (2.11)). Therefore, after extracting a subsequence, we can assume that

$$S^\varepsilon \to S \quad \text{strongly in } C^0([0, T]; H^{s'}_{\mathrm{loc}}(\mathbb{R}^d)) \tag{3.1}$$

for all $s' < s$. The limit S belongs to the space $C^0_w([0,T];H^s)$ of functions in $L^\infty([0,T];H^s)$ which are continuous for the weak topology of H^s. In particular, $S \in C^0([0,T];H^{s'}_{\mathrm{loc}}(\mathbb{R}^d))$. In addition, $S \in L^\infty([0,T];H^s(\mathbb{R}^d)\cap \mathbb{B})$.

Extracting further subsequences, we can also assume that

$$u^\varepsilon \;\to\; u \quad \text{weakly} * \ \text{in } L^\infty([0,T],H^s(\mathbb{R}^d)). \tag{3.2}$$

Moreover, $w^\varepsilon := \Pi(D_x)(E_0(S^\varepsilon)u^\varepsilon)$ is bounded in $C^0([0,T];H^s(\mathbb{R}^d))$ and satisfies the transport equation (2.22)

$$(\partial_t + b(S^\varepsilon,u^\varepsilon)\cdot\nabla)w^\varepsilon = f^\varepsilon,$$

where f^ε is bounded in $C^0([0,T];H^s(\mathbb{R}^d))$ by Lemma 2.5. Therefore, the family $\partial_t w^\varepsilon$ is bounded in $C^0([0,T],H^{s-1})$. By (3.1), (3.2), $\Pi(D_x)(E_0(S^\varepsilon)u^\varepsilon)$ converges weakly to $\Pi(D_x)(E_0(S)u)$. Therefore, the uniform estimates of w^ε and $\partial_t w^\varepsilon$ imply that for $s' < s$,

$$\begin{aligned}\Pi(D_x)(E_0(S^\varepsilon)u^\varepsilon) \;\to\; &\Pi(D_x)(E_0(S)u)\\ &\text{strongly in } C^0([0,T];H^{s'}_{\mathrm{loc}}(\mathbb{R}^d)).\end{aligned} \tag{3.3}$$

Equation (2.21) implies that

$$\varepsilon\partial_t(E_0(S^\varepsilon)u^\varepsilon) + L(\partial_x)u^\varepsilon = \varepsilon g^\varepsilon \tag{3.4}$$

where g^ε is bounded in $C^0([0,T];H^{s-1})$. Moreover, $E_0(S^\varepsilon)u^\varepsilon$ converges weakly, and thus its time derivative converges in the sense of distributions. Therefore $\varepsilon\partial_t(E_0(S^\varepsilon)u^\varepsilon)$ converges to zero in that sense. Since $L(\partial_x)u^\varepsilon$ converges weakly to $L(\partial_x)u$, we conclude that

$$L(\partial_x)u = 0, \quad \text{or equivalently,}\;\; \Pi(D_x)u = u. \tag{3.5}$$

Proof of Theorem 1.5 (given Proposition1.6)**.**

Step 1. We first show that some subsequence of the $(u^\varepsilon, S^\varepsilon)$ converges and that the limit (u,S) satisfies the version of (1.18) where the constraint π is eliminated.

We now make use of the assumption that Proposition 1.6 holds: Together with (3.1) and (3.2), Proposition 1.6 implies that, for all bounded open sets $\Omega \subset \mathbb{R}^d$, we have on $[0,T]\times\Omega$

$$\begin{aligned} b(S^\varepsilon,u^\varepsilon) &\to b(S,u) && \text{in } L^2,\\ \nabla u^\varepsilon &\to \nabla u && \text{in } L^2,\\ \nabla S^\varepsilon &\to \nabla S && \text{in } L^2,\\ E(S^\varepsilon,\varepsilon u^\varepsilon) &\to E_0(S) && \text{in } L^\infty.\end{aligned}$$

In addition, Proposition 1.6 together with the equation for u in (1.7) plus (3.5) imply that

$$\varepsilon\partial_t u^\varepsilon \to 0 \quad \text{in } L^2.$$

Since $E - E_0 = O(\varepsilon)$, this shows that

$$\left\{E(S^\varepsilon, \varepsilon u^\varepsilon) - E_0(S^\varepsilon)\right\}\left(\partial_t u^\varepsilon + b(S^\varepsilon, u^\varepsilon)\cdot\nabla u^\varepsilon\right) \to 0.$$

Therefore

$$\begin{aligned}
E(S^\varepsilon, \varepsilon u^\varepsilon)&\big(\partial_t u^\varepsilon + b(S^\varepsilon, u^\varepsilon)\cdot\nabla u^\varepsilon\big)\\
&= \left\{E(S^\varepsilon, \varepsilon u^\varepsilon) - E_0(S^\varepsilon)\right\}\left(\partial_t u^\varepsilon + b(S^\varepsilon, u^\varepsilon)\cdot\nabla u^\varepsilon\right)\\
&\quad + \partial_t\big(E_0(S^\varepsilon)u^\varepsilon\big) + b(S^\varepsilon, u^\varepsilon)\cdot\nabla\big(E_0(S^\varepsilon)u^\varepsilon\big)\\
\to \partial_t\big(E_0(S)u\big) &+ b(S,u)\cdot\nabla\big(E_0(S)u\big)\\
&= E_0(S)\big(\partial_t u + b(S,u)\cdot\nabla u\big)
\end{aligned}$$

in the sense of distributions. Applying $\Pi(D_x)$ to the first equation in (1.7) implies that

$$\Pi(D_x)\Big(E_0(S)\big(\partial_t u + b(S,u)\cdot\nabla u\big)\Big) = 0. \tag{3.6}$$

In addition, passing to the limit in the equation of S^ε implies that the limit S satisfies

$$\partial_t S + b(S,u)\cdot\nabla S = 0.$$

Thus (u, S) satisfies

$$\begin{aligned}
L(\partial_x)u &= 0,\\
\Pi(D_x)\Big(E_0(S)(\partial_t u + b(S,u)\cdot\nabla u)\Big) &= 0,\\
(\partial_t S + b(S,u)\cdot\nabla S) &= 0.
\end{aligned}\tag{3.7}$$

Step 2. Next, we prove that $u \in C^0_w([0,T]; H^s(\mathbb{R}^d))$ and that (u,S) satisfy the initial condition (1.19), (1.20). The convergence (3.1) implies that $S_{|t=0}$ is the limit S_0 of the initial data $S^\varepsilon_{|t=0}$. Similarly, the uniform bounds and the convergences (3.2), (3.3) and (3.5) imply that

$$\begin{aligned}
u = \Pi(D_x)u &\in L^\infty([0,T]; H^s(\mathbb{R}^d)),\\
f := \Pi(D_x)(E_0(S)u) &\in C^0_w([0,T]; H^s).
\end{aligned}\tag{3.8}$$

Moreover, $\big(E_0(S^\varepsilon)u^\varepsilon\big)_{|t=0}$ converges to $E_0(S_0)u_0$ in $H^s(\mathbb{R}^d)$. Together with (3.3), this implies

$$(\Pi(D_x)E_0(S)u)_{|t=0} = \Pi(D_x)E_0(S_0)u_0. \tag{3.9}$$

Introduce the space $H^s_\flat$ of functions $v \in H^s$ such that $v = \Pi(D_x)v$. For $t \in [0,T]$, let $K(t)$ denote the operator

$$v \mapsto \Pi(D_x)E_0(S(t))\Pi(D_x)v$$

from $H^s_\flat$ to itself. Because $E_0(S(t)) \in H^s$ is positive definite, one proves by induction on $k \leqq s$, that there is C such that

$$\frac{1}{C}\|v\|_k^2 \leqq \big(K(t)v, v\big)_{H^k} \leqq C\|v\|_k^2.$$

Therefore, $K(t)$ is an isomorphism from $H^s_\flat$ onto itself. In particular, this implies that (1.20) uniquely determines v_0. Moreover, (3.8) implies that for almost all $t \in [0, T]$,

$$u(t) = K(t)^{-1} f(t). \tag{3.10}$$

We show that $u \in C^0_w([0, T]; L^2(\mathbb{R}^d))$. Fix $t_0 \in [0, T]$ and $\phi \in L^2_\flat(\mathbb{R}^d)$. Consider $\psi := K(t_0)^{-1}\phi \in L^2_\flat(\mathbb{R}^d)$. Then, because $E_0(S)$ is uniformly bounded in L^∞ and is continuous in times on compact sets, $E_0(S(t))\psi \in C^0([0, T]; L^2(\mathbb{R}^d))$ and $\phi(t) := \Pi(E_0(t)\psi) \in C^0([0, T]; L^2(\mathbb{R}^d))$. By (3.10), we have

$$\begin{aligned}\big(\Pi u(t), \phi\big)_0 &= \big(\Pi u(t), \phi(t)\big)_0 + \big(\Pi u(t), \phi - \phi(t)\big)_0 \\ &= \big(f(t), \psi\big)_0 + \big(\Pi u(t), \phi - \phi(t)\big)_0.\end{aligned}$$

Because u is uniformly bounded in L^2 and $\phi(t)$ is continuous with values in L^2, the last term tends to zero when $t \to t_0$. With (3.8), this shows that $u(t)$ is weakly continuous at t_0. Because of the uniform bounds, this implies that $u \in C^0_w([0, T]; H^s(\mathbb{R}^d))$. With (3.10) and (3.9), one sees that the initial condition $u_{|t=0}$ satisfies (1.19).

Step 3. The usual iterative method shows that equations (3.7) with initial data (1.19) have a unique solution (u^*, S^*) in $C^0([0, T]; H^s(\mathbb{R}^d)) \cap C^1([0, T]; H^{s-1}(\mathbb{R}^d))$.

Because $u \in C^0_w([0, T]; H^s(\mathbb{R}^d))$, (3.7) implies that

$$\partial_t(I - \Pi(D_x))u = 0, \quad \Pi(D_x)(E_0\partial_t u) \in C^0_w([0, T]; H^{s-1}(\mathbb{R}^d)).$$

With Lemma 2.7, this implies that $\partial_t u \in C^0_w([0, T]; H^{s-1}(\mathbb{R}^d))$.

Thus, we can estimate the difference $(u - u^*, S - S^*)$ in L^2, implying that $u = u^*$ and $S = S^*$.

The uniqueness of the limit implies that the full sequence $(u^\varepsilon, S^\varepsilon)$ converges to (u, S).

Step 4. It remains to show that (3.7) implies (1.18). Define

$$g = -\big(E_0(S)(\partial_t u + b(S, u) \cdot \nabla u)\big).$$

Because $u \in C^0([0, T]; H^s(\mathbb{R}^d))$, using (3.7) and Lemma 2.7 implies that $\partial_t u$ and thus g belong to $C^0([0, T]; H^{s-1}(\mathbb{R}^d))$. Moreover, $\Pi(D_x)g = 0$. We want to solve the equation

$$L(\partial_x)\pi = g. \tag{3.11}$$

Introduce $M(\xi)$ the partial inverse of $L(\xi)$ such that

$$M(\xi)L(\xi) = L(\xi)M(\xi) = I - \Pi(\xi), \quad M(\xi)\Pi(\xi) = 0;$$

M is C^∞ on $\mathbb{R}^d \backslash \{0\}$ and homogeneous of order -1. The operator $M(D_x)$ defined by the Fourier multiplier $M(\xi)$is therefore well defined on H^s when $d \geqq 3$ and $\pi = -iM(D_x)g$ satisfies (3.11) and (1.14).

In general, $M(D_x)$ is defined on the space of functions φ in the Schwartz' class $\mathcal{S}$ such that the Fourier transform $\widehat{\varphi}$ vanishes at the origin. Introduce $\theta \in \mathcal{S}$

such that $\widehat{\theta}$ has compact support and is equal to 1 for $|\xi| \leqq 1$. For $\varphi \in \mathcal{S}$ define $\varphi^\sharp := \varphi - \widehat{\varphi}(0)\theta \in \mathcal{S}$. For $g \in L^2(\mathbb{R}^d)$, the formula

$$\langle \pi, \varphi \rangle = -i \langle g, M(D_x)\varphi^\sharp \rangle$$

defines a distribution $\pi = M^\sharp g \in \mathcal{S}'$. Its Fourier transform is given by

$$\langle \widehat{\pi}, \psi \rangle = -i \int \Big\langle \widehat{g}(\xi), M(\xi)\big(\psi(\xi) - \psi(0)\widehat{\theta}(\xi)\big)\Big\rangle d\xi.$$

In particular, $\widehat{\pi}(\xi) = -iM(\xi)\widehat{g}(\xi)$ on $\mathbb{R}^d \backslash \{0\}$. Thus, $\pi \in H^1 + C^\infty$ and $\pi \in H^s + C^\infty$ when $g \in H^{s-1}$. Note that in dimension $d \geqq 3$, $M^\sharp g = M(D_x)g$ since $M\widehat{g} \in L^2 + L^p$ for $1/2 + 1/d < p < 1$ and the functions $\psi \in \mathcal{S}$ such that $\psi(0) = 0$ are dense in $L^{p'}$ for $p' > 0$.

The definition implies that

$$\begin{aligned} \langle L(\partial_x)\pi, \varphi \rangle &= i \langle g, M(D_x)(L(\partial_x)\varphi)^\sharp \rangle \\ &= i \langle g, M(D_x)(L(\partial_x)\varphi) \rangle = \langle g, (I - \Pi(D_x))\varphi \rangle. \end{aligned}$$

Thus $L(\partial_x)\pi = (I - \Pi(D_x))g$. Similarly, one shows that $\partial_j \pi = M_j(D_x)g$ where $M_j(D_x)$ is the convolution operator associated with the Fourier multiplier $i\xi_j M(\xi) \in L^\infty$. Thus, $\partial_j \pi \in H^{s-1}$ whenever $g \in H^{s-1}$.

Knowing that $g \in C^0([0, T]; H^{s-1}(\mathbb{R}^d))$ and $\Pi(D_x)g = 0$, $\pi(t) = M^\sharp g(t) \in C^0([0, T] \times \mathbb{R}^d)$ satisfies (3.11) and (1.14).

Proof of Theorem 1.4 (given Theorem 1.7)**.** Consider the Banach space $\mathbb{B}$ of the functions which satisfy (1.13). In the transport equation $\partial_t S^\varepsilon + u^\varepsilon \cdot \nabla S^\varepsilon$, the speed u^ε is uniformly bounded. Thus, the decay assumptions (1.13) for $S_0(x)$ are propagated and the solutions S^ε are also uniformly bounded in $L^\infty([0, T]; \mathbb{B})$. It only remains to show that assumption (1.17) is satisfied by the system (1.10). The operator $i\tau E_0(S) + L(\partial_x)$ is

$$\begin{pmatrix} i\tau A_0(S) & \nabla\cdot \\ \nabla & i\tau R_0(S) \end{pmatrix} \tag{3.12}$$

where $A_0 = A(S, 0)$ and $R_0 = R(S, 0)$. For $\tau \neq 0$ and $t \in [0, T]$, (q, u) is in the kernel of $i\tau E_0(S(t)) + L(\partial_x)$ if and only if

$$\begin{aligned} a\tau^2 q + \nabla \cdot (b\nabla q) &= 0 \\ u &= ib\nabla q, \end{aligned} \tag{3.13}$$

where $a(x) = A(S(t, x), 0)$ and $b(x) = 1/R(S(t, x), 0)$. Because $S(t) \in \mathbb{B}$, the hypotheses of Theorem 1.7 hold. The equivalence of (3.12) and (3.13) means that Theorem 1.7 then implies that assumption (1.17) holds.

4. Decay of the local energy

In this section we prove Proposition 1.6. Consider a family of solutions of (1.7), $(u^\varepsilon, S^\varepsilon)$. It is assumed to be bounded in $C^0([0,T]; H^s(\mathbb{R}^d))$ with $s > 1 + d/2$ and $T > 0$, independent of ε. Moreover S^ε is bounded in $L^\infty([0,T]; \mathbb{B})$. As explained at the beginning of Section 3, we can extract a sub-sequence such that S^ε converge strongly to S in $C^0([0,T]; H^{s'}_{\mathrm{loc}})$ for all $s' < s$, u^ε converges to u weakly$\star$ in $L^\infty([0,T], H^s)$ and $\Pi(D_x)(E_0(S^\varepsilon)u^\varepsilon)$ converges strongly to $\Pi(D_x)(E_0(S)u)$ in $C^0([0,T]; H^{s'}_{\mathrm{loc}})$ for all $s' < s$.

The main step in the proof of Proposition 1.6 is to prove the strong convergence of $(I - \Pi)u^\varepsilon$.

Proposition 4.1. *$(I - \Pi(D_x))u^\varepsilon$ converges to 0 for the strong topology in $L^2([0,T], H^{s'}_{\mathrm{loc}}(\mathbb{R}^d))$ for all $s' < s$.*

The strategy of the proof is very simple. In the spirit of P. Gérard ([Gér]), we introduce the microlocal defect measures of subsequences of u^ε. They are measures $\mathcal{M}$, on $\mathbb{R}_t \times \mathbb{R}_\tau$ valued in the space $\mathcal{L}$ of trace class operators on $L^2(\mathbb{R}^d)$. They can be written

$$\mathcal{M}(dt, d\tau) = M(t,\tau)\mu(dt, d\tau), \tag{4.1}$$

where μ is a scalar nonnegative Radon measure and M is an integrable function with respect to μ with values in $\mathcal{L}$. The usual feature of defect measures is that they are supported in the characteristic variety of the equation. In our case, this means that for μ-almost all (t,τ), $M(t,\tau)$ is valued in $H^1(\mathbb{R}^d)$ and

$$\big(iE_0(t)\tau + L(\partial_x)\big)M(t,\tau) = 0. \tag{4.2}$$

Assumption (1.17) then implies that $M(t,\tau) = 0$ for μ-almost all t and $\tau \neq 0$. Thus, M is supported in $\tau = 0$ so (4.2) implies that

$$L(\partial_x)M(t,\tau) = 0 \quad \text{or} \quad (I - \Pi(D_x))M(t,\tau) = 0, \quad \mu\text{-a.e.} \tag{4.3}$$

As a corollary, the microlocal defect measure of $(I - \Pi(D_x))u^\varepsilon$ vanishes and, together with the uniform bounds in H^s, this implies Proposition 4.1.

We now proceed to the details.

As noticed in (2.11), equation (1.7) implies that $\varepsilon\partial_t u^\varepsilon$ is bounded in $C^0([0,T]; H^{s-1})$. As in (2.21), equation (1.7) implies that

$$\varepsilon E_0(S^\varepsilon)\partial_t u^\varepsilon + L(\partial_x)u^\varepsilon = \varepsilon f^\varepsilon, \tag{4.4}$$

where f^ε is bounded in $C^0([0,T]; H^{s-1})$. To avoid boundary terms in the integrations by parts, we extend the functions to $t \in \mathbb{R}$: First, choose extensions $\widetilde{S}^\varepsilon$ of S^ε, supported in $\{-1 \leqq t \leqq T+1\}$, uniformly bounded in $C^0_0(\mathbb{R}; H^s(\mathbb{R}^d) \cap \mathbb{B})$ and converging to $\widetilde{S}$ in $C^0_0(\mathbb{R}; H^{s'}_{\mathrm{loc}}(\mathbb{R}^d))$. Next, introduce a family of functions $\chi_\varepsilon \in C^\infty_0(]0,T[)$ such that

$$\chi_\varepsilon(t) = 1, \quad \text{for } t \in [\varepsilon^{1/2}, T - \varepsilon^{1/2}],$$
$$\|\varepsilon\partial_t\chi_\varepsilon\|_{L^\infty} \leqq 2\varepsilon^{1/2}.$$

Then $\widetilde{u}^\varepsilon = \chi_\varepsilon u^\varepsilon$ satisfies

$$\varepsilon E_0(\widetilde{S}^\varepsilon)\partial_t \widetilde{u}^\varepsilon + L(\partial_x)\widetilde{u}^\varepsilon = \widetilde{f}^\varepsilon, \tag{4.5}$$

where $\widetilde{f}^\varepsilon$ tends to zero in $C^0(\mathbb{R}; H^{s-1})$.

Introduce the wave-packets operator

$$W^\varepsilon v(t,\tau,x) := c\varepsilon^{-3/4} \int_{\mathbb{R}} e^{\left(i(t-s)\tau-(t-s)^2\right)/\varepsilon} v(s,x)\, ds, \tag{4.6}$$

with $c = 1/(2\pi^3)^{-1/4}$. The operator W^ε is an isometry from $L^2(\mathbb{R}^{1+d})$ to $L^2(\mathbb{R}^{2+d})$

$$\|W^\varepsilon v\|_{L^2(\mathbb{R}^{2+d})} = \|v\|_{L^2(\mathbb{R}^{1+d})}. \tag{4.7}$$

Lemma 4.2. *The wave packets $U^\varepsilon := W^\varepsilon \widetilde{u}^\varepsilon$ satisfy*

$$\sup_{\varepsilon\in]0,1]} \sum_{j+|\alpha|\leqq 1} \|\tau^j \partial_x^\alpha U^\varepsilon\|_{L^2(\mathbb{R}^{2+d})} < +\infty, \tag{4.8}$$

$$F^\varepsilon := \left(i\tau E_0(\widetilde{S}^\varepsilon(t)) + L(\partial_x)\right)U^\varepsilon \to 0 \quad \textit{in}\ \ L^2(\mathbb{R}^{2+d}) \quad \textit{as}\ \ \varepsilon \to 0. \tag{4.9}$$

Proof. The operator W^ε commutes with ∂_x and thus preserves smoothness in x. Therefore, U^ε is a bounded family in $L^2([0,T]\times\mathbb{R}; H^s(\mathbb{R}^d))$, implying (4.8) when $j = 0$.

Moreover, if $\varepsilon\partial_t v \in L^2$, then

$$W^\varepsilon(\varepsilon\partial_t v) - i\tau W^\varepsilon v = 2c\varepsilon^{-3/4} \int_{\mathbb{R}} e^{\left(i(t-s)\tau-(t-s)^2\right)/\varepsilon}(s-t)v(s,x)\, ds.$$

Hence,

$$\|W^\varepsilon(\varepsilon\partial_t u) - i\tau W^\varepsilon u\|_{L^2} \leqq C\sqrt{\varepsilon}\|u\|_{L^2}. \tag{4.10}$$

This implies (4.8) for $j = 1$. Similarly, if $a(t,x) \in C^1 \cap W^{1,\infty}(\mathbb{R}^{1+d})$,

$$aW^\varepsilon u - W^\varepsilon(au) = c\varepsilon^{-3/4} \int_{\mathbb{R}} e^{\left(i(t-s)\tau-(t-s)^2\right)/\varepsilon}\left(a(t,x)-a(s,x)\right)u^\varepsilon(s,x)\, ds$$

and since $|a(t,x) - a(s,x)| \leqq |t-s|\|\partial_t a\|_{L^\infty}$, we have

$$\|W^\varepsilon(au) - aW^\varepsilon u\|_{L^2} \leqq \sqrt{\varepsilon}\ \|\partial_t a\|_{L^\infty}\|u\|_{L^2}. \tag{4.11}$$

Equation (4.5) implies that $\left(i\tau E_0(\widetilde{S}^\varepsilon) + L(\partial_x)\right)U^\varepsilon$ is the sum of $W^\varepsilon \widetilde{f}^\varepsilon$ and errors terms which are dominated by (4.10), (4.11). Since S^ε is uniformly bounded in $W^{1,\infty}$, this implies (4.9).

Following P. GÉRARD ([Gér]), we introduce next the microlocal defect measures of u^ε. We denote by $\mathcal{K}$ [or $\mathcal{L}$] the space of compact operators [resp., trace class operators] in $L^2(\mathbb{R}^d)$ and by $\mathcal{K}_+$ [or $\mathcal{L}_+$] the subclass of nonnegative self adjoint operators in $\mathcal{K}$ [resp., $\mathcal{L}$]. The space $\mathcal{K}$ is equipped with the norm of bounded operators in $L^2(\mathbb{R}^d)$. The space $\mathcal{L}$ can be identified with the dual space of $\mathcal{K}$, with the duality bracket $\operatorname{tr}(KL)$. Note that $\Phi \in C_0^0(\mathbb{R}^2, \mathcal{K})$ acts in $L^2(\mathbb{R}^{2+d})$, by the obvious formula

$$(\Phi U)(t,\tau,x) = \left(\Phi(t,\tau)U(t,\tau,\cdot)\right)(x).$$

Lemma 4.3. *For all bounded family, U^ε in $L^2(\mathbb{R}^{2+d})$, there is a subsequence such that there is a finite nonnegative Borel measure μ on $\mathbb{R}^2$ and there is $M \in L^1(\mathbb{R}^2, \mathcal{L}_+, \mu)$ such that for all $\Phi \in C_0^0(\mathbb{R}^2; \mathcal{K})$,*

$$\int_{\mathbb{R}^{2+d}} (\Phi U^\varepsilon)(t,\tau,x) \,.\, \overline{U^\varepsilon(t,\tau,x)}\, dt\, d\tau\, dx \longrightarrow \int_{\mathbb{R}^2} \mathrm{tr}\big(\Phi(t,\tau) M(t,\tau)\big)\mu(dt, d\tau) \tag{4.12}$$

as ε tends to 0 in the subsequence.

Moreover, if U^ε satisfies (4.8) and (4.9), then, for almost all $(t,\tau) \in [0,T]\times\mathbb{R}$, $M(t,\tau)$ is a bounded operator from $L^2(\mathbb{R}^d)$ to $H^1(\mathbb{R}^d)$ and

$$\big(i\tau E_0(\widetilde{S}) + L(\partial_x)\big) M(t,\tau) = 0 \quad \mu\text{-a.e.} \tag{4.13}$$

Proof. (See [Gér].)

Step 1. Extracting a subsequence, we can assume that

$$\|U^\varepsilon(t,\tau)\|^2_{L^2(\mathbb{R}^d)} \;\to\; \mu \tag{4.14}$$

in the vague topology, where μ is a nonnegative bounded Borel measure on $\mathbb{R}^2$.

Introduce a countable orthonormal basis ϕ_j of $L^2(\mathbb{R}^d)$. Let $K_{j,k}$ be the operator $v \mapsto K_{j,k}v = \big(v, \phi_j\big)_0 \phi_k$. Then, extracting subsequences, we can assume that for all j and k,

$$\big(K_{j,k}U^\varepsilon(t,\tau), U^\varepsilon(t,\tau)\big)_0 = \big(U^\varepsilon(t,\tau), \phi_j\big)_0 \big(\phi_k, U^\varepsilon(t,\tau)\big)_0 \;\to\; \mu_{j,k} = m_{j,k}\mu, \tag{4.15}$$

where $\mu_{j,k}$ is a bounded Borel measure on Ω, which is absolutely continuous with respect to μ, hence of the form $m_{j,k}\mu$ with $m_{j,k} \in L^1(\mathbb{R}^2; \mu)$. Note that $m_{k,j} = \overline{m_{j,k}}$ and that the matrices $\{m_{j,k}\}_{1\leqq j,k\leqq n}$ are nonnegative. In addition, because

$$\sum_{1\leqq j\leqq n} \big|\big(U^\varepsilon(t,\tau), \phi_j\big)_0\big|^2 \;\leqq\; \|U^\varepsilon(t,\tau)\|_0^2,$$

we have

$$\sum_{1\leqq j\leqq n} m_{j,j}(t,\tau) \leqq 1 \quad \mu\text{-a.e.} \tag{4.16}$$

Introduce the operators in $L^2(\mathbb{R}^d)$:

$$M_n(t,\tau)v = \sum_{1\leqq j,k\leqq n} m_{j,k}(t,\tau)\big(v, \phi_k\big)_0 \phi_j. \tag{4.17}$$

They are bounded for μ-almost all (t,τ). They are hermitian symmetric and nonnegative. Moreover, the sequence M_n is nondecreasing. With (4.16), this implies that for almost all (t,τ) the sequence $M_n(t,\tau)$ converges in the trace class norm, the limit $M(t,\tau) \in \mathcal{L}_+$ and

$$\mathrm{tr} M(t,\tau) \leqq 1 \quad \mu\text{-a.e.} \tag{4.18}$$

Because $\operatorname{tr}(K_{j,k}M(t,\tau)) = m_{j,k}(t,\tau)$, (4.15) means that the convergence (4.12) holds for $\Phi(t,\tau) = \varphi(t,\tau)K_{j,k}$ for all indices j and k and all $\varphi \in C_0^0(\mathbb{R}^2)$. Therefore, it extends by linearity and density to all $\Phi(t,\tau) = \varphi(t,\tau)K$ with $K \in \mathcal{K}$ and next to all $\Phi \in C_0^0(\mathbb{R}^2; \mathcal{K})$.

By construction, note that

$$\begin{aligned}\int_{\mathbb{R}^2} \|M(t,\tau)\|_{\mathcal{L}}\mu(dt,d\tau) &= \int_{\mathbb{R}^2} \operatorname{tr}\big(M(t,\tau)\big)\mu(dt,d\tau) \\ &\leqq \limsup_{\varepsilon\to 0} \|U^\varepsilon\|^2_{L^2(\mathbb{R}^{2+d})}.\end{aligned} \tag{4.19}$$

Step 2. Suppose now that U^ε is bounded in $L^2(\mathbb{R}^2; H^1(\mathbb{R}^d))$. For $\delta \geqq 0$, introduce the operators

$$P_\delta = (Id - \delta\Delta_x)^{-1/2}(Id - \Delta_x)^{1/2}.$$

They are bounded in L^2 for $\delta > 0$ and uniformly bounded from H^1 to L^2 for $\delta \geqq 0$. They are self-adjoint and nonnegative and $P_\delta \geqq P_{\delta'}$ if $\delta \leqq \delta'$, as is easily seen using the Fourier transform on $\mathbb{R}^d$. Applying (4.12) to the test operator $P_\delta\Phi(t,\tau)P_\delta$, shows that the measure associated with $P_\delta U^\varepsilon$ is $P_\delta M(t,\tau)P_\delta\mu(dt,d\tau)$. The uniform boundedness of $P_\delta U^\varepsilon$ in L^2 and (4.19) imply that there is C such that

$$\forall\delta \in]0,1], \qquad \int \operatorname{tr}\big(P_\delta M(t,\tau)P_\delta\big)\mu(dt,d\tau) \leqq C.$$

Since the family P_δ is non-increasing, the norms

$$\|P_\delta M(t,\tau)P_\delta\|_{\mathcal{L}} = \operatorname{tr}\big(P_\delta M(t,\tau)P_\delta\big)$$

are non-increasing functions of δ. Therefore, the estimate above and Fatou's Lemma imply that

$$\sup_{\delta\in]0,1]} \|P_\delta M(t,\tau)P_\delta\|_{\mathcal{L}} \in L^1(\mathbb{R}^2,\mu) \tag{4.20}$$

and therefore

$$\sup_{\delta\in]0,1]} \|P_\delta M(t,\tau)P_\delta\|_{\mathcal{L}} < \infty \quad \mu\text{-a.e.} \tag{4.21}$$

In particular, for μ-almost all (t,τ), the operators $P_\delta M(t,\tau)P_\delta$ are uniformly bounded from L^2 to L^2; hence $M(t,\tau)$ extends as a bounded operator from H^{-1} to H^1. In addition, (4.18) implies that $P_0M(t,\tau)P_0 \in \mathcal{L}_+$ and (4.19) implies that $P_0MP_0 \in L^1(\mathbb{R}^2; \mathcal{L},\mu)$. In particular,

$$\partial_{x_j} M \ \in\ L^1(\mathbb{R}^2; \mathcal{L},\mu). \tag{4.22}$$

Using (4.12) with the test operators $\Phi(1-\delta\Delta_x)^{-1/2}\partial_{x_j}$ and letting δ tend to zero, implies that, for all $\Phi \in C_0^0(\mathbb{R}^2;\mathcal{K})$, we have

$$\begin{aligned}&\int_{\mathbb{R}^2} \Big(\Phi(t,\tau)\partial_{x_j}U^\varepsilon)(t,\tau), U^\varepsilon(t,\tau)\Big)_0 \,dt\,d\tau \\ &\qquad\longrightarrow \int_{\mathbb{R}^2} \operatorname{tr}\big(\Phi(t,\tau)\partial_{x_j}M(t,\tau)\big)\mu(dt,d\tau).\end{aligned} \tag{4.23}$$

Step 3. Next, we show that if U^ε satisfies (4.8) and (4.9), then (4.13) holds. The convergence (4.9) and the estimate (4.8) imply that for all $\Phi \in C_0^0(\mathbb{R}^2; \mathcal{K})$,

$$\int \Big(\Phi(t,\tau)F^\varepsilon, U^\varepsilon(t,\tau)\Big)_0 \, dt\, d\tau \to 0.$$

Moreover, the local strong convergence of $\widetilde{S}^\varepsilon$ implies that for all compact operators $K \in \mathcal{K}$,

$$K\big(E_0(\widetilde{S}^\varepsilon) - E_0(\widetilde{S})\big)U^\varepsilon \longrightarrow 0 \quad \text{in } L^2.$$

Using (4.12) and (4.23), this implies that

$$\int \operatorname{tr}\Big(\Phi(t,\tau)\big(i\tau E_0(\widetilde{S}(t)) + L(\partial_x)\big)M(t,\tau)\Big)\mu(dt, d\tau) = 0.$$

Because Φ is arbitrary, (4.13) follows.

Proof of Proposition 4.1. Assumption (1.17) plus (4.13) imply that $M(t,\tau) = 0$ for $\tau \neq 0$, μ-almost everywhere. Thus, $\tau M = 0$, and by using (4.13) again, we find that $L(\partial_x)M = 0$ μ-almost everywhere. This is equivalent to

$$(I - \Pi(D_x))M(t,\tau) = 0 \quad \mu\text{-a.e.}$$

This implies that for all $\varphi \in C_0^0(\mathbb{R}^2)$ and all $K \in \mathcal{K}$, we have

$$\int_{\mathbb{R}^2} \varphi(t,\tau)\Big(K(I-\Pi)U^\varepsilon(t,\tau), K(I-\Pi)U^\varepsilon(t,\tau)\Big)_0 \, dt\, d\tau \longrightarrow 0. \tag{4.24}$$

Because $\widetilde{u}^\varepsilon$ is supported in $\{0 \leqq t \leqq T\}$,

$$\int_{\{t \leqq -1\} \cup \{t \geqq T+1\}} \|U^\varepsilon(t,\tau)\|_0^2 \, dt\, d\tau \longrightarrow 0.$$

By (4.8), τU^ε is bounded in L^2. Thus, with (4.24), we see that for all $K \in \mathcal{K}$,

$$\int_{\mathbb{R}^2} \big\|K(I-\Pi)U^\varepsilon(t,\tau)\big\|_0^2 \, dt\, d\tau \longrightarrow 0$$

when $\varepsilon \to 0$, in the subsequence extracted in Lemma 4.3. Because

$$K(I-\Pi)U^\varepsilon = W^\varepsilon\big(K(I-\Pi)\widetilde{u}^\varepsilon\big),$$

this means that $K(I-\Pi)\widetilde{u}^\varepsilon$ tends to zero in $L^2(\mathbb{R}^{1+d})$ (see (4.7)). Since u^ε is bounded in $C^0([0,T]; L^2)$, $\widetilde{u}^\varepsilon - u^\varepsilon$ converges to zero in $L^2(\mathbb{R}^{1+d})$, and therefore we have proved that

$$\forall K \in \mathcal{K}, \quad \big\|K(I-\Pi)u^\varepsilon\big\|_{L^2([0,T]\times\mathbb{R}^d)} \longrightarrow 0. \tag{4.25}$$

Since the limit is zero, no extraction of subsequence is necessary for this result and the convergence holds for the given family u^ε.

Given that $(I-\Pi)u^\varepsilon$ is uniformly bounded in $C^0([0,T]; H^s(\mathbb{R}^d))$, (4.25) implies and is equivalent to the convergence of $(I-\Pi)u^\varepsilon$ to zero in $L^2([0,T]; H^{s'}_{\mathrm{loc}}(\mathbb{R}^d))$ for all $s' < s$. This completes the proof of Proposition 4.1.

To prove Proposition 1.6, we show that the defect measure of $u^\varepsilon - u$ vanishes, where u denotes the weak limit of u^ε. Repeating the proof of Lemma 4.3, we can show that

Lemma 4.4. *There are a subsequence $\varepsilon_n \to 0$, a finite nonnegative Borel measure μ_* on $[0,T]$, and $M_* \in L^1([0,T], \mathcal{L}_+, \mu_*)$, such that for all $\Phi \in C^0([0,T]; \mathcal{K})$,*

$$\int_{[0,T]} \Big(\Phi(t)(u^\varepsilon - u)(t), (u^\varepsilon - u)(t) \Big)_0 \, dt \longrightarrow \int_{[0,T]} \mathrm{tr}\big(\Phi(t) M_*(t)\big) \mu_*(dt) \tag{4.26}$$

as ε tends to 0 in the subsequence.

Proof of Proposition 1.6. Because S^ε converges locally uniformly to S, and because of the uniform estimates on u^ε, we have

$$\big\| \phi\big(E_0(S^\varepsilon) - E_0(S)\big) u^\varepsilon \big\|_{L^2([0,T]\times\mathbb{R}^d)} \longrightarrow 0.$$

for all $\phi \in C_0^0(\mathbb{R}^d)$. Thus

$$\big\| K\big(E_0(S^\varepsilon) - E_0(S)\big) u^\varepsilon \big\|_{L^2([0,T]\times\mathbb{R}^d)} \longrightarrow 0.$$

for all $K \in \mathcal{K}$. Together with the strong convergence (3.3), this implies that

$$\big\| K \Pi(D_x) E_0(S)(u^\varepsilon - u) \big\|_{L^2([0,T]\times\mathbb{R}^d)} \longrightarrow 0.$$

This implies that

$$\Pi(D_x) E_0(S(t)) M_*(t) = 0 \quad \mu_*\text{-a.e.} \tag{4.27}$$

Since $(I - \Pi)u = 0$, (4.25) which is equivalent to the conclusion of Proposition 4.1, implies that

$$(I - \Pi(D_x)) M_*(t) = 0, \quad \mu_*\text{-a.e.} \tag{4.28}$$

Thus, for μ_*-almost all t, $M_*(t)$ is valued in the space $L^2_\flat = \ker(I - \Pi) \cap L^2$ and (4.27) implies that $\Pi E_0(S(t)) \Pi M_*(t) = 0$. Taking the scalar product with M_* (as in Step 2 in the proof of Theorem 1.5), implies that

$$\big(E_0(S(t)) \Pi M_*(t) \cdot, \Pi M_*(t) \cdot \big) = 0$$

for μ_*-almost all t. Therefore, $\Pi M_*(t) = M_*(t) = 0$ and the definition of M_* implies that for all $K \in \mathcal{K}$,

$$\big\| K(u^\varepsilon - u) \big\|_{L^2([0,T]\times\mathbb{R}^d)} \longrightarrow 0 \tag{4.29}$$

as $\varepsilon \to 0$ in the subsequence. Since the limit is zero, the entire family converges. Given the uniform bounds for u^ε, (4.29) implies that $u^\varepsilon - u$ tends to zero in $L^2([0,T]; H^{s'}_{\mathrm{loc}}(\mathbb{R}^d))$ for all $s' < s$ and Proposition 1.6 is proved.

5. Absence of eigenvalues

In this section, we prove Theorem 1.7. We assume that the coefficients a and b satisfy (1.21). For $\tau \in \mathbb{R}$, we introduce the operator

$$Pu := a\tau^2 u + \nabla \cdot (b\nabla u). \tag{5.1}$$

It it clear that if $u \in L^2(\mathbb{R}^d)$ and $Pu = 0$, then $u \in H^1(\mathbb{R}^d)$ and

$$\big(b\nabla u, \nabla u\big)_0 = \tau^2 \big(au, u\big)_0.$$

In particular, when $\tau = 0$ this implies that $u = 0$. Thus in the remaining part of the section, we assume that $\tau^2 > 0$. Moreover, if $Pu = 0$, then $\Delta u = -(a\tau^2 u + \nabla b.\nabla u)/b \in L^2$ and thus $u \in H^2(\mathbb{R}^d)$.

The proof of Theorem 1.7 is very classical (see [RS] and [Hö] for example). We first show that the solutions of $Pu = 0$ are rapidly decreasing, and next we use the strong uniqueness theorem for second order elliptic equation to conclude that u vanishes on a neighborhood of infinity and hence that u vanishes identically.

Lemma 5.1. *If $u \in H^2(\mathbb{R}^d)$ satisfies $Pu = 0$, then for all $n \in \mathbb{N}$, $|x|^n u$ and $|x|^n \nabla u$ are square integrable on $\mathbb{R}^d$.*

This is well known and follows from much more precise results when the coefficients are smooth (see e.g. Corollary 14.5.6 and Theorem 30.2.10 of [Hö]). For the sake of completeness, we sketch a direct proof of the result that applies to C^1 coefficients. A similar proof for operators of the form $\Delta + V(x)$ can be found in [RS].

Proof. To simplify notation, we can assume without restriction that $\tau^2 \underline{a}/\underline{b} = 1$. Then

$$\frac{1}{b} Pu = \Delta u + u + V(x, D_x)u = 0, \tag{5.2}$$

where

$$V(x, D_x)u = \frac{1}{b}\nabla b \cdot \nabla u + \left(\frac{a\tau^2}{b} - 1\right) u. \tag{5.3}$$

Consider even functions ψ and φ in $C^\infty(\mathbb{R})$, to be chosen later on. Introduce the multiplier

$$Mu = 2A \cdot \nabla u + Bu \quad \text{with} \quad \begin{cases} A(x) = \psi(|x|)x, \\ B(x) = \nabla \cdot A(x) - \varphi(|x|). \end{cases} \tag{5.4}$$

We assume that ψ and its derivatives up to order four are $O(1/|x|)$ at infinity and that φ and its derivatives up to order two are bounded. Thus, A, B, and their derivatives up to order 2 are bounded. Since $u \in H^2$, integration by parts yields

$$\begin{aligned} -\int (\Delta + 1)u.(2A \cdot \nabla + B)u\, dx = &\int C|u|^2\, dx \\ &+ \int \sum_{1 \leqq j,k \leqq d} E_{j,k}\partial_j u \partial_k u\, dx \end{aligned} \tag{5.5}$$

with

$$\begin{aligned} C &= (\nabla\cdot A - B - \Delta B/2) &&= \varphi - \Delta B/2, \\ E_{j,k} &= (B - \nabla\cdot A)\delta_{j,k} + 2\partial_j A_k &&= (2\psi - \varphi)\delta_{j,k} + 2\psi' x_j x_k/|x|. \end{aligned}$$

With $r = |x|$, the quadratic form $E = (E_{j,k})$ is bounded from below by $\theta(r)$ where

$$\theta := \min\left\{2\psi - \varphi, 2(\psi + r\psi') - \varphi\right\}.$$

Substituting (5.2) plus this bound for E into (5.5) yields

$$\begin{aligned} &\int \varphi|u|^2\,dx + \int \theta|\nabla u|^2\,dx \\ &\qquad \leqq \int |V(x, D_x)u||Mu|\,dx + \tfrac{1}{2}\int \Delta B|u|^2\,dx. \end{aligned} \tag{5.6}$$

For $\alpha \geqq 0$ and $\varepsilon > 0$ we choose

$$\psi = \psi_{\alpha,\varepsilon}(r) := \frac{(1+r^2)^\alpha}{(1+\varepsilon r^2)^{\alpha+1/2}}, \quad r := |x|, \tag{5.7}$$

which converges to $(1+r^2)^\alpha$ when $\varepsilon \to 0$. Note that

$$r\psi'_{\alpha,r}(r) + \psi_{\alpha,\varepsilon}(r) = \frac{2\alpha+1}{1+\varepsilon r^2}\,\psi_{\alpha,\varepsilon} - \frac{2\alpha}{1+r^2}\psi_{\alpha,\varepsilon}.$$

Next we we choose

$$\varphi = \varphi_{\alpha,\varepsilon}(r) := \frac{2\alpha+1}{2\alpha+1+\varepsilon r^2}\,\psi_{\alpha,\varepsilon}. \tag{5.8}$$

Then $\varphi \leqq \min\{\psi, r\psi' + \psi + 2\psi_{\alpha,\varepsilon}/(1+r^2)\}$ and thus $\theta \geqq \varphi - 4\psi_{\alpha,\varepsilon}/(1+r^2)\}$. Note that for any fixed $\varepsilon > 0$, $\psi_{\alpha,\varepsilon}$ and $\varphi_{\alpha,\varepsilon}$ are $O(1/r)$ and their derivatives of order k are $O(1/r^{1+k})$ so that the energy estimate (5.6) holds for this choice of weights. Thus, for all $\alpha \geqq 0$ and $\varepsilon > 0$, we have

$$\begin{aligned} \int \varphi_{\alpha,\varepsilon}\big(|u|^2 + |\nabla u|^2\big)\,dx \leqq &\int |V(x, D_x)u||M_{\alpha,\varepsilon}u|\,dx \\ &+ \int 4\frac{\psi_{\alpha,\varepsilon}}{(1+r^2)}|\nabla u|^2\,dx + \frac{1}{2}\int \Delta B_{\alpha,\varepsilon}|u|^2\,dx, \end{aligned} \tag{5.9}$$

where $M_{\alpha,\varepsilon}$ and $B_{\alpha,\varepsilon}$ are associated with $\psi_{\alpha,\varepsilon}$ as indicated in (5.4).

Now, we show by induction on n, that $(1+|x|)^{n\delta/2}(|u| + |\nabla u|)$ is in L^2. This is true by assumption for $n = 0$ since $u \in H^1$. Assume that this property holds up to $n-1$. Introduce $\alpha = n\delta$ and use (5.9). We use the induction hypothesis to bound the right-hand side. We have $A_{\alpha,\varepsilon} = \psi_{\alpha,\varepsilon}x$ and $B_{\alpha,\varepsilon} = d\psi_{\alpha,\varepsilon} + r\psi'_{\alpha,\varepsilon} - \varphi_{\alpha,\varepsilon}$, thus

$$\begin{aligned} &|\psi_{\alpha,\varepsilon}(x)| \leqq C(1+|x|)^{2\alpha}, \quad |A_{\alpha,\varepsilon}(x)| \leqq C(1+|x|)^{2\alpha+1}, \\ &|B_{\alpha,\varepsilon}(x)| \leqq C(1+|x|)^{2\alpha}, \quad |\Delta B_{\alpha,\varepsilon}(x)| \leqq C(1+|x|)^{2\alpha-2}, \end{aligned}$$

where C is independent of ε. Thus the last two terms in (5.9) are uniformly dominated by the norm of $(1+|x|)^{\alpha-1}(|u|+|\nabla u|)$ in L^2. Moreover,

$$|Mu(x)| \leqq C\psi_{\alpha,\varepsilon}\big(|x||\nabla u| + |u(x)|\big) \leqq C\ (1+|x|)^{2\alpha+1}\big(|\nabla u| + |u(x)|\big),$$

and the assumptions (1.21) imply that

$$|V(x, D_x)u(x)| \leqq (1+|x|)^{-1-\delta}\big(|\nabla u(x)| + |u(x)|\big).$$

Therefore, the first term in the right-hand side of (5.9) is dominated by the norm of $(1+|x|)^{\alpha-\delta/2}(|u|+|\nabla u|)$ in L^2. Thus, by the induction hypothesis, the right-hand side of (5.9) is bounded by a constant independent of ε, and letting ε tend to zero, Fatou's Lemma implies that $(1+|x|)^{\alpha}(|u|+|\nabla u|)$ is in L^2. Therefore the induction hypothesis is satisfied for all n and the Lemma is proved.

Proof of Theorem 1.7. The conditions (1.21) imply that the assumptions of Theorem 17.2.8 in [Hö] are satisfied. Therefore, any solution $u \in H^2(\mathbb{R}^2)$ of $Pu = 0$ which satisfies $(1+|x|)^n(|u|+|\nabla u|) \in L^2$ for all $n \geqq 0$, is identically zero. With Lemma 5.1, this proves that the equation $Pu = 0$ has no nontrivial solution in H^2.

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