Arch. Rational Mech. Anal. 157 (2001) 91–163
Digital Object Identifier (DOI) 10.1007/s002050100131

Travelling Fronts and Entire Solutions of the Fisher-KPP Equation in $\mathbb{R}^N$

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Communicated by V. Šverák

Abstract

This paper is devoted to time-global solutions of the Fisher-KPP equation in $\mathbb{R}^N$:

$$u_t = \Delta u + f(u), \quad 0 < u(x,t) < 1, \quad x \in \mathbb{R}^N,\ t \in \mathbb{R}$$

where f is a C^2 concave function on $[0, 1]$ such that $f(0) = f(1) = 0$ and $f > 0$ on $(0, 1)$. It is well known that this equation admits a finite-dimensional manifold of planar travelling-fronts solutions. By considering the mixing of any density of travelling fronts, we prove the existence of an infinite-dimensional manifold of solutions. In particular, there are infinite-dimensional manifolds of (nonplanar) travelling fronts and radial solutions. Furthermore, up to an additional assumption, a given solution u can be represented in terms of such a mixing of travelling fronts.

1. Introduction and main results

This paper is devoted to the question of the description of the set of the solutions $u(x,t)$, defined for all time, of the Fisher-KPP equation

$$u_t = \Delta u + f(u), \quad 0 < u(x,t) < 1, \quad x \in \mathbb{R}^N,\ t \in \mathbb{R}. \tag{1}$$

We deal with the solutions that are defined for all time and for all points $x \in \mathbb{R}^N$, and which we call “entire”. We assume that the nonlinearity f satisfies: $f(0) = f(1) = 0$, $f'(0) > 0$, $f'(1) < 0$ and $f(u) > 0$ for any $0 < u < 1$. We also assume that f is a concave function of class C^2 in $[0, 1]$. An example of such a function f is the quadratic nonlinearity $f(u) = u(1-u)$ considered by Kolmogorov, Petrovsky & Piskunov in their pioneering paper [20]. We refer to Aronson & Weinberger [2], Barenblatt & Zeldovich [3], Fife [9], Fisher [11], Freidlin [12], Murray [28], Rothe [33] or Stokes [35] for a derivation of this equation in models for population dynamics (like models for the spread of advantageous genetic traits in a population) and other biological models.

Because of the strong parabolic maximum principle, a solution u of $u_t = \Delta u + f(u)$ that is defined for all $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ and satisfies $0 \leqq u \leqq 1$, is either identically equal to 0, 1, or is in the range $0 < u(x,t) < 1$ for all (x,t). We only deal here with the case $0 < u < 1$.

Problem (1) clearly admits solutions $u(t)$ that depend on time only, namely, u solves $u'(t) = f(u),\ 0 < u < 1, t \in \mathbb{R}$. These solutions $u(t)$ are increasing in t, they satisfy $u(t) \to 0$ as $t \to -\infty$ and $u(t) \to 1$ as $t \to +\infty$. Furthermore, they are unique up to translation in time. It is convenient for what follows to define $\xi(t)$ as the only solution of that type such that

$$\xi(t) \sim e^{f'(0)t} \text{ as } t \to -\infty. \tag{2}$$

The set of all the solutions $u(t)$ of (1) is equal to the 1-dimensional manifold $\{t \mapsto \xi(t+h),\ h \in \mathbb{R}\}$.

It is well known that problem (1) also has, in dimension $N \geqq 2$, an $(N+1)$-dimensional manifold of entire solutions of planar travelling waves type, namely $u_{\nu,c,h}(x,t) = \varphi_c(x \cdot \nu + ct + h)$ where ν varies in the unit sphere S^{N-1}, h varies in $\mathbb{R}$ and c varies in $[c^*, +\infty[$ with $c^* = 2\sqrt{f'(0)} > 0$. In space dimension $N = 1$, there are two 2-dimensional manifolds of travelling-waves solutions: $u^+_{c,h}(x,t) = \varphi_c(x + ct + h)$ and $u^-_{c,h}(x,t) = \varphi_c(-x + ct + h)$ (see for instance ARONSON & WEINBERGER [2], BRAMSON [6], FIFE [9], FREIDLIN [12], HADELER & ROTHE [15], KANEL' [18], ROTHE [33], STOKES [35]). For any $c \geqq c^*$, the function φ_c satisfies

$$\varphi_c'' - c\varphi_c' + f(\varphi_c) = 0 \text{ in } \mathbb{R}, \quad \varphi_c(-\infty) = 0 \text{ and } \varphi_c(+\infty) = 1.$$

The function φ_c is increasing, unique up to translation. For each $c \geqq c^*$, let λ_c be the positive real number defined by

$$\lambda_c = \frac{c - \sqrt{c^2 - 4f'(0)}}{2} = \frac{c - \sqrt{c^2 - c^{*2}}}{2} > 0 \tag{3}$$

(λ_c satisfies $\lambda_c^2 - c\lambda_c + f'(0) = 0$). For any $c > c^*$, it is known that $\varphi_c(s)e^{-\lambda_c s}$ goes to a finite positive limit as $s \to -\infty$. Up to translation, we can then assume that

$$\forall\, c > c^*, \quad \varphi_c(s) \sim e^{\lambda_c s} \text{ as } s \to -\infty. \tag{4}$$

For the minimal speed $c = c^* = 2\sqrt{f'(0)}$, we have, up to translation,

$$\varphi_{c^*}(s) \sim |s|e^{\lambda^* s} \text{ as } s \to -\infty, \quad \lambda^* = \lambda_{c^*} = \sqrt{f'(0)} = c^*/2 \tag{5}$$

(see AGMON & NIRENBERG [1], BERESTYCKI & NIRENBERG [4], BRAMSON [6], CODDINGTON & LEVINSON [8], HADELER & ROTHE [15], KAMETAKA [17], PAZY [29], UCHIYAMA [37]).

Many works have been devoted to the question of the behavior for large time and the convergence to the travelling waves for the solutions of the Cauchy problem for (1), especially in dimension 1, under a wide class of initial conditions (ARONSON & WEINBERGER [2], BRAMSON [6,7], FREIDLIN [12], KAMETAKA [17], KANEL' [18], KOLMOGOROV, PETROVSKY & PISKUNOV [20], LARSON [21], LAU [22], MCKEAN [24], MOET [26], ROTHE [34], UCHIYAMA [37], VAN SAARLOS [38]). Other stability results have been obtained for the KPP equation in straight infinite cylinders (BERESTYCKI & NIRENBERG [4], MALLORDY & ROQUEJOFFRE [23], ROQUEJOFFRE [32]) and for a larger class of KPP type equations (BIRO & KERSNER [5], PELETIER & TROY [30,31], VAN SAARLOS [38], ZHAO [40]) as well as under other restrictions of the function f (see ROTHE [33], STOKES [35,36] if $c^* > 2\sqrt{f'(0)}$, or ARONSON & WEINBERGER [2], FIFE & MCLEOD [10], KANEL' [18,19] if f is of the "bistable" type).

The entire solutions of (1) can be viewed as orbits $\{u(\cdot, t),\ t \in \mathbb{R}\}$ lying in the space of the functions $\psi \in C^2(\mathbb{R}^N_x)$ such that $0 < \psi < 1$. The goal of this paper is then to describe the set of the orbits for (1) and the qualitative properties of these orbits. The difficulty is that we have to deal both with a direct *well-posed* Cauchy problem and an inverse *ill-posed* Cauchy problem for a nonlinear heat equation.

The question of the existence of entire solutions of (1) other than the solutions independent of x and the travelling-waves solutions has been answered in the case of planar solutions (solutions which depend only on time and on one space variable) in an earlier paper [16]. In dimension $N = 1$, 4 other manifolds of entire solutions of (1) have been constructed: one of these manifolds is 5-dimensional, one is 4-dimensional and two are 3-dimensional. Furthermore, the 4- and the 3-dimensional manifolds, as well as the travelling-waves solutions and the solutions $t \mapsto \xi(t + h)$, are on the boundary of that 5-dimensional manifold of entire solutions of (1) (see [16]).

One of the basic ideas in [16] for constructing new entire solutions of the KPP equation (1) in dimension 1 consists in considering two travelling waves $\varphi_{c'}(-x + c't + h')$ and $\varphi_c(x + ct + h)$ with speeds $c, c' > c^*$, one coming from the left side and the other from the right side of the real axis and mixing.

In Section 1.1, we shall show how this mixing procedure can be extended, in any space dimension $\mathbb{R}^N$, by allowing both for the mixing of any finite number of travelling waves (Theorem 1.1) and for the mixing of an integrable sum of travelling waves, each of them being characterized by its direction and its speed. That leads to the existence of an infinite-dimensional manifold of solutions of (1) (Theorem 1.2). In Section 1.2, we state an "almost-uniqueness" result (Theorem 1.4): namely, up to an additional assumption that is almost generically satisfied, each entire solution of (1) belongs to the infinite-dimensional manifold of solutions constructed in Theorem 1.2. Furthermore, we give an easy characterization of the entire solutions of (1) that only depend on time (Theorem 1.5). Lastly, in Section 1.3, as a consequence of the results in Sections 1.1 and 1.2, we get the existence of an infinite-dimensional manifold of nonplanar travelling waves and of radial solutions of (1) (Theorems 1.7 and 1.8).

1.1. Existence of an infinite-dimensional manifold of entire solutions

In [16], in the 1-dimensional case, we showed how two travelling waves with speeds greater than the minimal speed c^* and coming from opposite sides of the real axis could mix together and give rise to an entire solution of (1); moreover, the so-built entire solution behaves like each of these two travelling waves on each side of the real axis as the time goes to $-\infty$.

In the following theorem, in any dimension N, we generalize that mixing procedure by considering any finite number of travelling waves coming from directions ν_i with speeds $c_i \geqq c^*$ and mixing. We also allow both the mixing of travelling waves coming from the same direction with different speeds and the mixing of travelling waves with solutions of the type $t \mapsto \xi(t+h)$. In statements (6)–(9) below, we show the relationship between the so-built entire solutions u and the travelling waves from which they originated. We shall see that property (10) below characterizes each of these new entire solutions u.

Theorem 1.1 (Mixing a finite number of travelling waves**).** *Let p be a positive integer. For each $i = 1, \cdots, p$, let ν_i be in the unit sphere S^{N-1}, let $c_i \in [c^*, +\infty]$ and let $h_i \in \mathbb{R}$. Assume that $c_i \neq c_j$ as soon as $\nu_i = \nu_j$ with $i \neq j$. Furthermore, assume that at most one c_i takes the value $+\infty$. Then there exists an entire solution $u(x,t) = u_{(\nu_i, c_i, h_i;\ i=1,\cdots,p)}(x,t)$ of (1) such that*

$$\forall i, \quad \begin{array}{ll} u(x,t) \geqq \varphi_{c_i}(x\cdot\nu_i + c_i t + h_i) & \text{if } c^* \leqq c_i < +\infty \\ u(x,t) \geqq \xi(t+h_i) & \text{if } c_i = +\infty, \end{array} \tag{6}$$

$$u(x,t) \leqq \sum_{i,\ c_i<\infty} \varphi_{c_i}(x\cdot\nu_i + c_i t + h_i) \; + \sum_{i,\ c_i=\infty} \xi(t+h_i). \tag{7}$$

For any $(\nu, c) \in S^{N-1} \times [c^*, +\infty[$,

$$\begin{array}{l} \text{if } c\nu\cdot\nu_j < c_j \text{ for all } j, \text{ then } u(-ct\,\nu + x, t) \underset{t\to-\infty}{\longrightarrow} 0, \\ \text{if } \exists\, i,\ c\nu\cdot\nu_i = c_i,\ c\nu\cdot\nu_j < c_j\ \forall\, j\neq i, \\ \qquad\qquad \text{then } u(-ct\,\nu + x, t) \underset{t\to-\infty}{\longrightarrow} \varphi_{c_i}(x\cdot\nu_i + h_i), \\ \text{if } c\nu\cdot\nu_i > c_i \text{ for some } i, \text{ then } u(-ct\,\nu + x, t) \underset{t\to-\infty}{\longrightarrow} 1, \end{array} \tag{8}$$

$$\begin{array}{l} \text{if } c\nu\cdot\nu_j > c_j \text{ for all } j, \text{ then } u(-ct\,\nu + x, t) \underset{t\to+\infty}{\longrightarrow} 0, \\ \text{if } \exists\, i,\ c\nu\cdot\nu_i = c_i,\ c\nu\cdot\nu_j > c_j\ \forall\, j\neq i, \\ \qquad\qquad \text{then } u(-ct\,\nu + x, t) \underset{t\to+\infty}{\longrightarrow} \varphi_{c_i}(x\cdot\nu_i + h_i), \\ \text{if } c\nu\cdot\nu_i < c_i \text{ for some } i, \text{ then } u(-ct\,\nu + x, t) \underset{t\to+\infty}{\longrightarrow} 1. \end{array} \tag{9}$$

Moreover, as $t \to -\infty$:

$$
\begin{aligned}
&u(x,t)e^{-f'(0)t} \longrightarrow e^{f'(0)h_i} \text{ if } \exists\, i,\ c_i = +\infty, \quad u(x,t)e^{-f'(0)t} \to 0 \text{ otherwise},\\
&\forall\, z \in \mathbb{R}^N,\ 0 < |z| < c^* = 2\sqrt{f'(0)},\\
&u(-zt + x, t)\, e^{-\frac{1}{4}(c^{*2}-|z|^2)t} \to e^{\frac{1}{2}|z|h_i} e^{\frac{1}{2}z\cdot x} \text{ if } \exists\, i,\ c_i < +\infty,\ 2\lambda_{c_i}\nu_i = z,\\
&u(-zt + x, t)\, e^{-\frac{1}{4}(c^{*2}-|z|^2)t} \to 0 \text{ otherwise},\\
&\forall\, \nu \in S^{N-1},\\
&u(-c^*t\ \nu + x, t) \to \varphi_{c^*}(x\cdot\nu + h_i) \text{ if } \exists\, i, (\nu, c^*) = (\nu_i, c_i),\\
&u(-c^*t\ \nu + x, t) \to 0 \text{ otherwise}.
\end{aligned}
\tag{10}
$$

All the above convergences hold in $C^2_{\text{loc}}(\mathbb{R}^N_x)$.

Lastly, the set of the solutions u of that type contains the planar travelling waves, the functions of the type $t \mapsto \xi(t+h)$ and the planar solutions constructed in [16].

In the second statement of (8), if we take $(\nu, c) = (\nu_i, c_i)$, then the convergence $u(-c_i t\ \nu_i + x, t) \longrightarrow \varphi_{c_i}(x\cdot\nu_i + h_i)$ as $t \to -\infty$ holds at least for the smallest c_i's but it does not hold in general for all the c_i's. Roughly speaking, that means that only some fronts, those with small speeds, can be "viewed" as $t \to -\infty$, the other ones being "hidden". More restrictive conditions are required for some of the travelling fronts to be seen as $t \to +\infty$: indeed, for a given i, the convergence $u(-c_i t\ \nu_i + x, t) \longrightarrow \varphi_{c_i}(x\cdot\nu_i + h_i)$ in (9) requires especially that $\nu_i \cdot \nu_j > 0$ for all $j \neq i$; the latter may not be satisfied in general.

The property (10) deals with the behavior of the function u along the rays $\frac{z}{|z|}$ as $t \to -\infty$ with $|z| \leqq c^*$. Notice that, from (10), $u(-zt, t) \to 0$ as $t \to -\infty$ if $|z| < c^*$ (the latter actually holds for each entire solution of (1), see (16) and more comments after Theorem 1.2 below). Finally, notice that, unlike properties (8) or (9), the asymptotic behavior (10) easily implies that the so-built finite-mixing-type entire solutions u are different from each other.

After the mixing of any finite number of travelling waves coming from any directions, it is natural to wonder if a integrable sum of travelling waves (with respect to a measure supported on $S^{N-1} \times [c^*, +\infty]$) can mix. The answer is yes, and it will be the subject of Theorem 1.2 below. Before stating this theorem, we introduce some notation. Let $B(0, c^*) = B\left(0, 2\sqrt{f'(0)}\right) = \{z \in \mathbb{R}^N,\ |z| < c^*\}$ be the open ball of $\mathbb{R}^N$ with center 0 and radius c^*. Let us define the topological spaces

$$
\begin{aligned}
X &= S^{N-1} \times [c^*, +\infty)\ \cup\ \{\infty\},\\
\hat{X} &= S^{N-1} \times (c^*, +\infty)\ \cup\ \{\infty\}\ = X \setminus S^{N-1} \times \{c^*\}
\end{aligned}
$$

as follows: we use on the set $S^{N-1} \times [c^*, +\infty)$ (resp., $S^{N-1} \times (c^*, +\infty)$) the topology induced by the Euclidean structure of $\mathbb{R}^N$ and, on the other hand, we say that a set $\mathcal{A}$ is a neighborhood of ∞ in X (and $\hat{X}$) if and only if $\infty \in \mathcal{A}$ and if there exists a real number $c_0 \geqq c^*$ such that $(\nu, c) \in \mathcal{A}$ for all $\nu \in S^{N-1}$ and $c \geqq c_0$.

The set X is compact and it can also be viewed as the set $\{x \in \mathbb{R}^N,\ |x| \geqq c^*\}$ to which we add a point at infinity, which can be thought of as an infinite speed.

Let $\mathcal{M}$ be the set of all nonnegative and nonzero Radon-measures μ on X ($0 < \mu(X) < +\infty$), such that the restriction μ^* of μ on the sphere $S^{N-1} \times \{c^*\}$ can be written as a finite sum of Dirac distributions:

$$\mu^* = \sum_{1 \leqq i \leqq k} m_i\ \delta_{(\nu_i, c^*)}$$

for some integer $k \geqq 0$, some directions $\nu_i \in S^{N-1}$ different from each other and some positive real numbers m_i. In particular, the set $\mathcal{M}$ contains all the nonnegative Radon measures whose support is compactly included in $S^{N-1} \times (c^*, +\infty)$.

For any $\mu \in \mathcal{M}$, we denote $\hat{\mu}$ the restriction of μ on the set $\hat{X}$ and $\Phi_*\hat{\mu}$ the image of $\hat{\mu}$ by the continuous, one-to-one and onto map

$$\begin{aligned} \Phi : \hat{X} = S^{N-1} \times (c^*, +\infty) \cup \{\infty\} &\longrightarrow B(0, c^*), \\ (\nu, c) \neq \infty &\longmapsto z = 2\lambda_c \nu = (c - \sqrt{c^2 - c^{*2}})\ \nu, \\ \infty &\longmapsto 0. \end{aligned}$$

Let $\hat{\mathcal{M}}$ be the set of measures $\mu \in \mathcal{M}$ such that $\mu^* = 0$ (i.e., $k = 0$). We say that a sequence of measures $\mu^n \in \hat{\mathcal{M}}$ converges to a measure $\mu \in \hat{\mathcal{M}}$ if: (a) $\int_{\hat{X}} f d\hat{\mu}^n \to \int_{\hat{X}} f d\hat{\mu}$ for each continuous function f on $\hat{X}$ such that $f \equiv 0$ on $S^{N-1} \times (c^*, c^* + \varepsilon)$ for some $\varepsilon > 0$, (b) $\mu^n(\hat{X}) \to \mu(\hat{X})$ and (c) $\mu^n(\infty) \to \mu(\infty)$.

Let $\mathcal{E}$ be the set of all entire solutions of (1). We say that some functions $u^n \in \mathcal{E}$ approach a function $u \in \mathcal{E}$ in the sense of the topology $\mathcal{T}$ if the functions u^n go to u in $C^1_{\text{loc}}(\mathbb{R}_t)$ and $C^2_{\text{loc}}(\mathbb{R}^N_x)$.

The following theorem provides the existence of an entire solution u_μ for each measure $\mu \in \mathcal{M}$ and, generalizing the property (10) in Theorem 1.1, we give an interpretation, in terms of the measure μ, of the asymptotic behavior of u_μ as $t \to -\infty$ along the rays ν if one moves with speeds less than c^*.

Theorem 1.2 (Main existence theorem**).** *For any $N \geqq 1$, there exists an infinite-dimensional manifold of entire solutions of* (1). *Namely, there exists a one-to-one map, $\mu \mapsto u_\mu$, from $\mathcal{M}$ to $\mathcal{E}$, which is continuous on $\hat{\mathcal{M}}$. Moreover, given a measure $\mu \in \mathcal{M}$, the entire solution u_μ satisfies the following properties:*

(i) (behavior as $t \to -\infty$).

$$\begin{aligned} &u_\mu(-c^* t\ \nu + x, t) \underset{t \to -\infty}{\longrightarrow} \varphi_{c^*}(x \cdot \nu + c^* \ln m_i) \ \textit{in}\ C^2_{\text{loc}}(\mathbb{R}^N_x)\ \textit{if}\ \nu = \nu_i\ \textit{for some}\ i, \\ &u_\mu(-c^* t\ \nu + x, t) \underset{t \to -\infty}{\longrightarrow} 0 \qquad\qquad \textit{otherwise} \end{aligned} \tag{11}$$

and, for any sequence $t_n \to -\infty$,

$$\left(\frac{|t_n|}{4\pi}\right)^{N/2} u_\mu(-t_n z + x, t_n + t)\ e^{-\frac{1}{4}((c^*)^2 - |z|^2)t_n}\ dz \underset{t_n \to -\infty}{\rightharpoonup} e^{(f'(0) + \frac{1}{4}|z|^2)(t + \ln \hat{M}) + \frac{1}{2} x \cdot z}\ \frac{1}{\hat{M}}\ \Phi_*\hat{\mu}(dz) \tag{12}$$

in $C_c(B(0,c^*))'$, under the convention that the right-hand side is zero if $\hat{M}=0$; namely, for any continuous function $\psi(z)$ with compact support on $B(0,c^*)$,

$$\int_{B(0,c^*)}\left(\frac{|t_n|}{4\pi}\right)^{N/2} u_\mu(-t_n z+x, t_n+t)\, e^{-\frac{1}{4}(c^{*2}-|z|^2)t_n}\,\psi(z)\,dz \underset{t_n\to-\infty}{\longrightarrow} \int_{B(0,c^*)} e^{(f'(0)+\frac{1}{4}|z|^2)(t+\ln\hat{M})+\frac{1}{2}x\cdot z}\,\psi(z)\,\frac{1}{\hat{M}}\,\Phi_*\hat{\mu}(dz) \quad (13)$$

in the sense of the topology $\mathcal{T}$.

(ii) (monotonicity in time). *The function u_μ is increasing in time t.*

(iii) (multiplication of μ by positive constants). *For each positive real number α, $u_{\alpha\mu}(x,t)=u_\mu(x,t+\ln\alpha)$ for all $(x,t)\in\mathbb{R}^N\times\mathbb{R}$; furthermore, $u_{\alpha\mu}\to 1$ (or 0) as $\alpha\to+\infty$ (resp. 0^+) in the sense of $\mathcal{T}$.*

(iv) (case of absolutely continuous measures with respect to $d\nu\times dc$). *If $\mu\in\hat{\mathcal{M}}$ (i.e., $\mu(S^{N-1}\times\{c^*\})=0$, i.e., $k=0$) and if the restriction $\tilde{\mu}$ of μ on the set $S^{N-1}\times(c^*,+\infty)$ is absolutely continuous with respect to the Lebesgue measure $d\nu\times dc$, then*

$$\begin{aligned}&\forall\,\nu\in S^{N-1},\ \forall\,c\geqq c^*,\ \forall\,h\in\mathbb{R},\\ &u_\mu(-ct\,\nu+x,t)\not\to\varphi_c(x\cdot\nu+h)\ \text{ as } t\to\pm\infty.\end{aligned} \quad (14)$$

Lastly, the set of the solutions of the type u_μ contains the planar travelling waves, the solutions $t\mapsto\xi(t+h)$, as well as the other planar solutions constructed in [16] *and the finite-mixing-type solutions of Theorem* 1.1. *The solutions in Theorem* 1.1 *correspond to measures which can be written as finite sums of Dirac distributions.*

For each solution u_μ of (1), the asymptotic behavior (11), (12) is a consequence of the construction of suitable sub- and super-solutions for u_μ (see the lower and upper bounds (30) in Section 3 below). Note that, unlike the asymptotic behavior of the function u_μ as $t\to-\infty$ along the rays zt with $|z|\geqq c^*$, the asymptotic behavior (11), (12) along the rays of the "inner" cone $\mathcal{C}=\{(zt,t),\ t\leqq 0,\ z\in\mathbb{R}^N,\ |z|\leqq c^*\}$ characterizes each entire solution of the type u_μ, in the sense that if $\mu_1\neq\mu_2$, then $u_1\neq u_2$ (which is proved in Lemma 3.5, Section 3.5). Let us now comment on this formula (12) more thoroughly. First, the following fact, known as the "hair-trigger" effect (see Aronson & Weinberger [2]), holds for any solution u of (1):

$$\forall\,0\leqq c<c^*,\qquad \min_{|x|\leqq ct} u(x,t)\to 1\ \text{ as } t\to+\infty. \quad (15)$$

Notice here that this fact immediately implies that there are no stationary or time-periodic solutions of (1). It follows that

$$\forall\ 0\leqq c<c^*,\qquad \max_{|x|\leqq c|t|} u(x,t)\to 0\ \text{ as } t\to-\infty \quad (16)$$

for each solution u of (1) (see Lemma 4.1 for more details). It is then not surprising that, in the left-hand side of (12) (as in the first two statements of (10) in Theorem 1.1, or in (25) in Section 2), the terms $u_\mu(-zt_n+x,t_n+t)$, with $|z|<c^*$ and $t_n\to-\infty$,

have to be renormalized by asymptotically small factors. These asymptotically small terms in (12) are of the type $(|t_n|/4\pi)^{-N/2}e^{\frac{1}{4}((c^*)^2-|z|^2)t_n}$. On the other hand, in the right-hand side of (12), each term $e^{(f'(0)+\frac{1}{4}|z|^2)(t+\ln\hat{M})+\frac{1}{2}x\cdot z}\frac{1}{\hat{M}}$ is a solution of the linearized heat equation around $u=0$:

$$\partial_t U = \Delta U + f'(0)U.$$

Putting these together, the asymptotic behavior (12) can then be thought of as a spectral decomposition of the function u_μ as $t \to -\infty$ along the rays $|z| < c^*$ in terms of pure exponential solutions of the linearized heat equation balanced by the measure $\Phi_*\hat{\mu}(dz)$, the function u_μ being itself suitably renormalized by the exponentially decaying weights $e^{\frac{1}{4}(c^{*2}-|z|^2)t_n}(|t_n|/4\pi)^{-N/2}$ which are larger and larger as $|z|$ approaches c^*.

Property (iv) implies that if the measure μ is absolutely continuous with respect to $d\nu \times dc$ on $S^{N-1} \times (c^*, +\infty)$ and if the restriction μ^* of μ on $S^{N-1} \times \{c^*\}$ is zero, then the function u_μ does not converge as $t \to -\infty$ (nor as $t \to +\infty$) to any travelling front along any ray ν if the frame moves with any speed greater than or equal to the minimal speed. (Let us also mention that some non-convergence results more general than property (iv) are proved in Section 3.8.) On the contrary, for each entire solution obtained from the mixing of a finite number of planar travelling waves (Theorem 1.1), there exists at least one direction ν_i, one speed $c_i \geqq c^*$ and one real number $h_i \in \mathbb{R}$ such that $u(-c_i t\ \nu_i + x) \to \varphi_{c_i}(x\cdot\nu_i + h_i)$ as $t \to -\infty$. Theorem 1.2 provides then the existence of entire solutions that are different from those obtained from the finite mixing of travelling waves. But, by definition, the manifold of the solutions u_μ, which is infinite-dimensional, is actually much bigger than the countably-many finite-dimensional manifolds of solutions obtained from the mixing of a finite number of travelling waves.

Lastly, property (iii) simply says that multiplying a measure μ by a positive constant is the same as shifting u_μ in time.

Remark 1.3 (Behavior when $t \to +\infty$**).** As far as the asymptotic behavior of u_μ as $t \to +\infty$ is concerned, it is known from [2] that $\min_{|x|\leqq ct} u_\mu(x,t) \to 1$ as $t \to +\infty$, as soon as $0 \leqq c < c^*$. We give here a sufficient (and almost necessary) condition, which has an easy geometric interpretation, for a solution u_μ to converge uniformly to 1 as $t \to +\infty$. Namely, as proved in Section 3.4,

– if, for all $\nu_0 \in S^{N-1}$, there exists $\varepsilon > 0$ such that $\mu(\{c^* \leqq c < \infty,\ \nu\cdot\nu_0 \geqq \varepsilon\} \cup \{\infty\}) > 0$, then $\inf_{\mathbb{R}^N} u_\mu(\cdot,t) > 0$ for all $t \in \mathbb{R}$ and $\inf_{\mathbb{R}^N} u_\mu(\cdot,t) \to 1$ as $t \to +\infty$,
– if there exists $\nu_0 \in S^{N-1}$ such that $\mu(\{c^* \leqq c < \infty,\ \nu\cdot\nu_0 \geqq 0\} \cup \{\infty\}) = 0$, then $\inf_{\mathbb{R}^N} u_\mu(\cdot,t) = 0$ for all $t \in \mathbb{R}$.

As a consequence, in dimension $N = 1$, a solution $u_\mu(x,t)$ of (1) converges to 1 uniformly in $x \in \mathbb{R}$ as $t \to +\infty$ if and only if $\mu(\{c^* \leqq c < +\infty,\ \nu = \nu_\pm\} \cup \{\infty\}) > 0$ for each $\nu_\pm = \pm 1$. Otherwise, $\inf_{\mathbb{R}} u_\mu(\cdot,t) = 0$ for all $t \in \mathbb{R}$.

Notice here that we shall see in Theorem 1.5 below that, when $t \to -\infty$, a solution u of (1) in $\mathbb{R}^N$ cannot converge to 0 uniformly in x as $t \to -\infty$, unless u depends on t only.

1.2. Two partial uniqueness results

As already mentioned in the previous section, each solution $u(x,t)$ of (1) satisfies (16), namely,

$$\forall\ 0 \leqq c < c^*, \quad \max_{|x| \leqq c|t|} u(x,t) \to 0 \ \text{ as } t \to -\infty.$$

We shall see later (Lemma 4.7 and Remark 4.8) that if a measure $\mu \in \mathcal{M}$ is such that $\mu(S^{N-1} \times [c^*, \overline{c}]) = 0$ for some $\overline{c} \in [c^*, +\infty[$, then $\max_{|x| \leqq \overline{c}|t|} u_\mu(x,t) \to 0$ as $t \to -\infty$.

Conversely, we can actually characterize all the solutions u of (1) satisfying such a property with $\overline{c} > c^*$, that is to say, that u satisfies a slightly stronger assumption than (16):

Theorem 1.4 (Partial uniqueness result**).** *Let $u(x,t)$ be a solution of* (1)*. If there exists $\varepsilon > 0$ such that*

$$\max_{|x| \leqq (c^*+\varepsilon)|t|} u(x,t) \to 0 \ \textit{ as } t \to -\infty,$$

then $u = u_\mu$ for some (unique) measure $\mu \in \mathcal{M}$. Therefore, u satisfies all properties (i)–(iv) *of Theorem* 1.2*. Moreover, μ is concentrated on the set $S^{N-1} \times [c^* + \varepsilon, +\infty) \cup \{\infty\}$.*

The next theorem, which can be proved from Theorem 1.4, gives an easy characterization of the functions depending only on time t among all the entire solutions of (1):

Theorem 1.5 (Uniqueness in the class of solutions bounded away from 1**).** *Let $u(x,t)$ be a solution of* (1)*. Then,*

either $$\forall t \in \mathbb{R}, \quad \sup_{x \in \mathbb{R}^N} u(x,t) = 1$$

or $$u(x,t) \equiv u(t).$$

As a consequence, any solution u_μ of (1) *is such that* $\sup u_\mu(\cdot,t) = 1$ *for all $t \in \mathbb{R}$ as soon as μ is not concentrated on the single point $\{\infty\}$, i.e., as soon as $\mu \not\equiv 0$ on $S^{N-1} \times [c^*, +\infty)$.*

This means that if a solution u of (1) is such that the function $u(\cdot, t_0)$ is bounded away from 1 at some time t_0, then u is independent of x for all time. In particular, there are no "pulse-like" solutions of (1), i.e., solutions such that $u(x,t_0) \to 0$ as $|x| \to +\infty$ at some time t_0 (see similar results for entire solutions of another class of parabolic equations in [25]).

Having (16) and Theorems 1.2 and 1.4 in mind, we now formulate the following

Conjecture 1.6 (Uniqueness**).** *The set $\mathcal{E}$ of all entire solutions of* (1)*, such that $0 \leqq u \leqq 1$, is the closure, in the sense of the topology $\mathcal{T}$, of the set of the solutions u_μ.*

If this conjecture were true, that would mean that all the solutions of (1) could be described, in a certain sense, from the travelling waves and from the solutions $t \mapsto \xi(t+h)$, which could also be thought of as travelling waves with an infinite speed. By analogy, the travelling waves, with finite or infinite speeds, would then play the role of a basis of eigenfunctions for this nonlinear problem, as do some pure exponential functions for the heat equation $\partial_t v = \Delta v$ in $\mathbb{R}^N \times \mathbb{R}$ (see WIDDER [39]).

1.3. Applications to travelling waves and radial solutions

As said earlier, there is a finite-dimensional manifold of planar travelling waves for equation (1). Each planar travelling wave can be written as $\varphi_c(x \cdot \nu + ct + h)$ for some direction $\nu \in S^{N-1}$, some speed $c \geqq c^*$ and some real number $h \in \mathbb{R}$. Such a travelling wave $\varphi_c(x \cdot \nu + ct + h)$ propagates in the direction $-\nu$ with the speed c.

We can now ask ourselves if there are nonplanar travelling waves for (1). By a travelling wave for (1), we understand a solution $u(x,t)$ such that

$$\forall\,(x,t) \in \mathbb{R}^N \times \mathbb{R},\ \forall\,\tau \in \mathbb{R}, \quad u(x, t+\tau) = u(x + c_0 \tau \nu_0, t) \tag{17}$$

for some direction $\nu_0 \in S^{N-1}$ and some speed $c_0 \geqq 0$ (up to a change $\nu_0 \to -\nu_0$, we can always assume $c_0 \geqq 0$). Such a wave is propagating in the direction $-\nu_0$ with the speed c_0. The function u can be written as

$$u(x,t) = v(x + c_0 t \nu_0) \tag{18}$$

where v is (uniquely) defined by $v(y) = u(y, 0)$ for all $y \in \mathbb{R}^N$. The function v is such that $0 < v(y) < 1$ for all $y \in \mathbb{R}^N$ and it satisfies the elliptic equation

$$\Delta v - c_0 \partial_{\nu_0} v + f(v) = 0 \text{ in } \mathbb{R}^N \tag{19}$$

where $\partial_{\nu_0} v = \nu_0 \cdot \nabla v$. Conversely, any solution $0 < v < 1$ of (19) gives rise to a travelling wave $u(x,t) = v(x + c_0 t \nu_0)$ for (1), which propagates in the direction $-\nu_0$ with the speed c_0.

For each couple $(\nu_0, c_0) \in S^{N-1} \times [0, +\infty)$, set

$$S_{(\nu_0, c_0)} = \left\{(\nu, c) \in S^{N-1} \times [c^*, +\infty),\ c_0 \nu_0 \cdot \nu = c\right\}$$

($= S(c_0\nu_0/2, c_0/2) \backslash B(0, c^*)$) where $S(c_0\nu_0/2, c_0/2)$ is the sphere with center $c_0\nu_0/2$ and radius $c_0/2$, and $B(0, c^*)$ is the open ball centered at the origin and with radius c^*. Note that $S_{(\nu_0, c_0)}$ is empty as soon as $0 \leqq c_0 < c^*$, and that, in dimension $N = 1$, $S_{(\nu_0, c_0)}$ reduces to the single point (ν_0, c_0) if $c_0 \geqq c^*$. Finally, let $\mathcal{M}_{TW}$ be the subset of $\mathcal{M}$ defined by

$$\mathcal{M}_{TW} = \left\{\mu \in \mathcal{M},\ \exists\,(\nu_0, c_0) \in S^{N-1} \times [0, +\infty),\ \mu \text{ is concentrated on } S_{(\nu_0, c_0)}\right\}.$$

Theorem 1.7 (Travelling waves**).** (i) *Let u be a travelling wave for* (1) *and assume that u satisfies* (17), *namely, that u propagates in direction $-\nu_0$ with speed c_0. Then,* (i-a) $c_0 \geqq c^*$;

(i-b) *the function* v *defined by* (18) *is increasing in each direction* $\nu \in S^{N-1}$ *such that* $\nu \cdot \nu_0 > \cos(\arcsin(\frac{c^*}{c_0}))$, *namely,* ν *belongs to the open cone directed by* ν_0 *with angle* $\arcsin(\frac{c^*}{c_0})$. *Furthermore, for each such* ν, $\lim_{s\to-\infty} v(a + s\nu) = 0$ *and* $\lim_{s\to+\infty} v(a + s\nu) = 1$ *for all vector* $a \in \mathbb{R}^N$;

(i-c) *if* $c_0 = c^*$, *then* u *is a planar travelling wave with speed* c^*, *namely,* $u(x, t) = \varphi_{c^*}(x\cdot\nu_0 + c^*t + h)$ *for some* $h \in \mathbb{R}$. *In other words, if* $0 < v < 1$ *is a solution of* (19) *for* $c_0 = c^*$ *and for some* $\nu_0 \in S^{N-1}$, *then* $v(y) = \varphi_{c^*}(y \cdot \nu_0 + h)$ *for some* $h \in \mathbb{R}$.

(ii-a) *In dimension* $N \geqq 2$, *there exists an infinite-dimensional manifold of travelling waves for* (1). *Namely, the restriction of the map* $\mu \mapsto u_\mu$ *on* $\mathcal{M}_{TW}$ *ranges in the set of travelling waves for* (1), *and it is one-to-one on* $\mathcal{M}_{TW}$ *and continuous on* $\mathcal{M}_{TW} \cap \hat{\mathcal{M}}$. *If*

$$\mu = \sum_{i=1}^{k} m_i \delta_{(\nu_i, c^*)} + \hat{\mu} \quad \in \mathcal{M}$$

is concentrated on $S_{(\nu_0, c_0)}$ *for some* (ν_0, c_0), *then* u_μ *is a travelling wave satisfying* (17). *Furthermore,* $v_\mu(y) = u_\mu(y, 0)$ *is the smallest solution of* (19) *such that*

$$v_\mu(y) \geqq \max\Bigg(\max_{1\leqq i\leqq k} \varphi_{c^*}(y\cdot\nu_i + c^*\ln m_i), \int_{S^{N-1}\times(c^*,+\infty)} \varphi_c(y\cdot\nu + c\ln\hat{M})\frac{1}{\hat{M}}d\tilde{\mu}\Bigg) \tag{20}$$

for all $y \in \mathbb{R}^N$, *where* $\hat{M} = \mu(\hat{X})$ *(if* $\hat{M} = 0$, *then the second term in the right-hand side of the above inequality drops out);*

(ii-b) *In dimension* $N \geqq 2$, *for each* $c_0 > c^*$ *and for each* $\nu_0 \in S^{N-1}$, *there exists an infinite-dimensional manifold of solutions* $v(y)$, $0 < v < 1$, *of the elliptic equation* (19);

(ii-c) *Let* $u(x, t)$ *be a travelling wave of* (1) *satisfying* (17). *If* u *is of the type* u_μ *for some* $\mu \in \mathcal{M}$, *then* μ *is concentrated on* $S_{(\nu_0, c_0)}$.

(iii) *Let* u *be a travelling wave for* (1) *satisfying* (17) *and let* v *be defined by* (18). *Then,*

(iii-a) $\forall\, 0 \leqq c < c^*, \quad \max_{|y|\leqq c|s|} v(c_0\nu_0 s + y) \to 0$ *as* $s \to -\infty$;

(iii-b) *If there exists* $\varepsilon > 0$ *such that*

$$\max_{|y|\leqq(c^*+\varepsilon)|s|} v(c_0\nu_0 s + y) \to 0 \text{ as } s \to -\infty,$$

then $u = u_\mu$ *for some measure* $\mu \in \mathcal{M}_{TW}$ *concentrated on* $S_{(\nu_0, c_0)} \cap \{c \geqq c^* + \varepsilon\}$ *and* u *satisfies all properties* (i) *and* (ii) *above.*

Let us now consider the case of radial solutions of (1). We say that a solution $u(x,t)$ of (1) is radially symmetric, or radial, if there exists a point $a \in \mathbb{R}^N$ such that u can be written as

$$u(x,t) = v(|x-a|, t)$$

for all $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$. The function $v = v(r,t)$ satisfies

$$\begin{cases} v_t = v_{rr} + \dfrac{N-1}{r}\, v_r + f(v), \quad r > 0,\ t \in \mathbb{R}, \\ v(r,t) \text{is } C^2 \text{in } r \in [0,+\infty[\text{and } C^1 \text{in } t, \text{ and}, \forall\, t \in \mathbb{R},\ v_r(0,t) = 0. \end{cases} \tag{21}$$

Note that the set of the solutions of (1) which are radially symmetric with respect to a point $a \in \mathbb{R}^N$ is the set of functions $\{(x,t) \mapsto u(x-a,t)\}$ where u is radially symmetric with respect to the origin.

We can now ask ourselves if there are radial solutions of (1) and, if yes, what is the size of the set of such solutions. Before answering this question in the next theorem, let us define the set

$$\mathcal{M}_R = \{\mu \in \mathcal{M},\ \ \forall\, \rho \in SO(N),\ \forall \text{Borel } A \subset X,\ \ \mu(\rho(A)) = \mu(A)\}.$$

The set $\mathcal{M}_R$ is the set of the measures $\mu \in \mathcal{M}$ that are rotationaly invariant. Since the restriction of any measure $\mu \in \mathcal{M}$ on the set $S^{N-1} \times \{c^*\}$ is a finite sum of Dirac masses, it follows that, for each measure $\mu \in \mathcal{M}_R$, we have $\mu^* = 0$. In other words, $\mathcal{M}_R \subset \hat{\mathcal{M}}$.

Theorem 1.8 (Radial solutions**).** (i-a) *There exists an infinite-dimensional manifold of radial solutions of* (1). *Namely, the map*

$$\begin{aligned} \mathcal{M}_R \times \mathbb{R}^N &\to \mathcal{E} \\ (\mu, a) &\mapsto u_{\mu,a} = u_\mu(\cdot - a, \cdot) \end{aligned}$$

ranges in the set of radial solutions of (1) *. This map is continuous and its restriction to the set of measures $\mu \in \mathcal{M}_R$ which are not concentrated on the single point $\{\infty\}$, is one-to-one. Furthermore, for each given $(\mu, a) \in \mathcal{M}_R \times \mathbb{R}^N$, the function $u_{\mu,a}$ is radially symmetric with respect to the point a and the function v defined by $u_{\mu,a}(x,t) = v(|x-a|,t)$ solves* (21)*, and it is such that $v(r,t) \to 1$ as $r \to +\infty$ for all $t \in \mathbb{R}$, provided μ is not concentrated on $\{\infty\}$.*

(i-b) *There exists an infinite-dimensional manifold of solutions v of* (21).
(ii) *Each solution v of* (21) *is such that*

$$\forall\, 0 \leqq c < c^*, \qquad \max_{0 \leqq r \leqq c|t|} v(r,t) \to 0 \text{ as } t \to -\infty.$$

Furthermore, if v is a solution of (21) *such that*

$$\max_{0 \leqq r \leqq (c^*+\varepsilon)|t|} v(r,t) \to 0 \text{ as } t \to -\infty$$

for some $\varepsilon > 0$, then there exists a measure $\mu \in \mathcal{M}_R$ such that $v(|x|,t) = u_\mu(x,t)$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$.

Structure of the paper. The rest of the paper is organized as follows: Section 2 is devoted to the construction of solutions that are obtained from the mixing of a finite number of travelling waves (Theorem 1.1). These solutions are constructed from a sequence of Cauchy problems starting at times $-n \to -\infty$. Section 3 deals with the proof of Theorem 1.2 about the existence of an infinite-dimensional manifold of solutions of (1). Section 4 is devoted to the proof of partial uniqueness results (Theorems 1.4 and 1.5). Lastly, Section 5 deals with the cases of (nonplanar) travelling waves and radial solutions of (1).

2. Construction of entire solutions from the mixing of a finite number of travelling waves (Theorem 1.1)

This section is devoted to the proof of Theorem 1.1. Let p be a positive integer $p \geqq 1$ and for each $i = 1, \cdots, p$, let ν_i, c_i, h_i be such that $\nu_i \in S^{N-1}$, $c^* \leqq c_i \leqq +\infty$, $h_i \in \mathbb{R}$. Assume that $c_i \neq c_j$ if $\nu_i = \nu_j$ and assume that there exists at most one index i such that $c_i = +\infty$. Our goal is to prove that there exists an entire solution u of (1) satisfying properties (6)–(10) stated in Theorem 1.1.

Consider the case where $k := \#\{i,\ c_i = c^*\} \geqq 1$ and $\#\{i,\ c_i = +\infty\} = 1$ (the cases $\#\{i,\ c_i = c^*\} = 0$ or $\#\{i,\ c_i = +\infty\} = 0$ are similar and even easier to deal with). Up to a renumbering, we can then assume that

$$c_1 = \cdots = c_k = c^* < c_{k+1} \leqq \cdots \leqq c_{p-1} < +\infty = c_p.$$

For each $n \in \mathbb{N}$, let $U_n(x,t)$ be the solution of the Cauchy problem

$$(U_n)_t = \Delta U_n + f(U_n), \quad x \in \mathbb{R}^N,\ t > -n$$
$$U_n(x,-n) = \max\Big(\max_{1\leqq i\leqq p-1} \varphi_{c_i}(x\cdot\nu_i - c_i n + h_i),\ \xi(-n+h_p)\Big),$$

where $0 \leqq U_n(x,-n) \leqq 1$. This Cauchy problem is well posed and the maximum principle yields

$$0 \leqq \max\left(\max_{1\leqq i\leqq p-1} \varphi_{c_i}(x\cdot\nu_i + c_i t + h_i),\ \xi(t+h_p)\right) \leqq U_n(x,t) \leqq 1 \qquad (22)$$

for all $x \in \mathbb{R}^N$ and $t \geqq -n$. Another application of the maximum principle shows that the functions $(U_n(x,t))_n$ are nondecreasing with respect to n. Indeed, for each $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, if $n' > n > |t|$, then $U_{n'}(\cdot,-n) \geqq U_n(\cdot,-n)$, whence $U_{n'}(x,t) \geqq U_n(x,t)$. Eventually, there exists a function $u(x,t)$ such that $0 \leqq u(x,t) \leqq 1$ and $U_n(x,t) \to u(x,t)$ for each $(x,t) \in \mathbb{R}^N \times \mathbb{R}$. Furthermore, from standard parabolic estimates and Sobolev's injections, the function u is an entire solution of (1). Let us now prove that u satisfies all properties (6)–(10).

Proof of (6). It follows immediately from (22). □

Proof of (7). It follows from the following result due to Bramson; this result resorts to the concavity of the function f and to the maximum principle. □

Lemma 2.1 (Bramson [6]**).** *Let us extend the function f by 0 on the interval $[1,+\infty)$. Let $u_{i,0}(x)$, $i = 1,\cdots m$, be m given nonnegative and bounded functions. Let $u_i \geqq 0$ be the solutions of the Cauchy problems:*

$$\begin{aligned} (u_i)_t &= \Delta u_i + f(u_i), \quad t > 0,\ x \in \mathbb{R}^N, \\ u_i(\cdot,0) &= u_{i,0} \end{aligned}$$

and let $u \geqq 0$ be the solution of

$$\begin{aligned} u_t &= \Delta u + f(u), \quad t > 0,\ x \in \mathbb{R}^N, \\ 0 \leqq u(\cdot,0) &\leqq u_{1,0} + \cdots + u_{m,0}. \end{aligned}$$

Then $u(x,t) \leqq u_1(x,t) + \cdots + u_m(x,t)$ for all $t \geqq 0$ and for all $x \in \mathbb{R}^N$.

Property (7) follows then immediately from Lemma 2.1 because U_n satisfies

$$U_n(x,t) \leqq \sum_{i=1}^{p-1} \varphi_{c_i}(x\cdot\nu_i + c_i t + h_i) + \xi(t+h_p)$$

for each $t \geqq -n$ and $x \in \mathbb{R}^N$.

From (6), it follows that $u(x,t) > 0$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$. On the other hand, $u(0,t) \to 0$ as $t \to -\infty$ because of (7). Therefore, the strong maximum principle implies that $u < 1$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$. The function u is then a solution of (1) such that $0 < u < 1$.

Proof of (8). Let (ν, c) be in $S^{N-1} \times [c^*, +\infty[$. Assume, say, that $c\nu\cdot\nu_j < c_j$ for all $1 \leqq j \leqq p-1$. From (7),

$$0 \leqq u(-ct\nu + x, t) \leqq \sum_{i=1}^{p-1} \varphi_{c_i}((c_i - c\nu\cdot\nu_i)t + x\cdot\nu_i + h_i) + \xi(t+h_p).$$

Therefore, $u(-ct\nu + x, t) \to 0$ locally in x as $t \to -\infty$. From standard parabolic estimates, the convergence also takes place in $C^2_{\rm loc}(\mathbb{R}^N_x)$. The other two cases ($c\nu\cdot\nu_i = c_i$ for some i, $c\nu\cdot\nu_j < c_j$ for all $j \neq i$; and $c\nu\cdot\nu_i > c_i$ for some i) can be treated similarly. □

Proof of (9). It is similar to (8). □

Proof of (10). From (6)–(7),

$$\begin{aligned} \xi(t+h_p)e^{-f'(0)t} &\leqq u(x,t)e^{-f'(0)t} \\ &\leqq \sum_{i=1}^{p-1} e^{-f'(0)t}\varphi_{c_i}(c_i t + x\cdot\nu_i + h_i) + \xi(t+h_p)e^{-f'(0)t}. \end{aligned}$$

Observe that $\xi(t+h_p)e^{-f'(0)t} \to e^{f'(0)h_p}$ as $t \to -\infty$, since $\xi(s) \sim e^{f'(0)s}$ as $s \to -\infty$. On the other hand, because of (4), (5), we have as $t \to -\infty$

$$\varphi_{c_i}(x \cdot \nu_i + c_i t + h_i) = \begin{cases} O(|t|e^{\lambda^* c^* t}) & \text{locally in } x \text{ if } 1 \leqq i \leqq k \\ O(e^{\lambda_{c_i} c_i t}) & \text{locally in } x \text{ if } k+1 \leqq i \leqq p-1. \end{cases}$$

Since $\lambda_c c = \lambda_c^2 + f'(0)F'(0)$ for all $c \geqq c^*$, it is found that

$$\sum_{i=1}^{p-1} e^{-f'(0)t}\varphi_{c_i}(c_i t + x \cdot \nu_i + h_i) \to 0 \text{ locally in } x \text{ as } t \to -\infty.$$

As a consequence, $u(x,t)e^{-f'(0)t} \to e^{f'(0)h_p}$ locally in x as $t \to -\infty$. Since u is a positive and bounded solution of (1), the standard parabolic estimates and Harnack inequality (see, e.g., FRIEDMAN [13], GRUBER [14], MOSER [27]) yield the existence of a constant C such that $|\nabla u(x,t)|, |u_{x_i x_j}(x,t)|, |u_{x_i x_j x_k}(x,t)| \leqq Cu(x,t+1)$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$. Hence, we conclude that $u(x,t)e^{f'(0)t} \to e^{f'(0)h_p}$ in $C^2_{\text{loc}}(\mathbb{R}^N_x)$ as $t \to -\infty$.

Take now $z \in \mathbb{R}^N$ such that $0 < |z| < c^* = 2\sqrt{f'(0)}$. We have

$$\begin{aligned} 0 &\leqq u(-zt + x, t)e^{-\frac{1}{4}(c^{*2}-|z|^2)t} \\ &\leqq \sum_{i=1}^{p-1} \varphi_{c_i}((c_i - z\cdot\nu_i)t + x\cdot\nu_i + h_i)e^{-\frac{1}{4}(c^{*2}-|z|^2)t} + \xi(t+h_p)e^{-\frac{1}{4}(c^{*2}-|z|^2)t}. \end{aligned}$$

Since $c^* = 2\sqrt{f'(0)}$, $|z| > 0$ and $\xi(s) \sim e^{f'(0)s}$ as $s \to -\infty$, it follows that $\xi(t+h_p)e^{-\frac{1}{4}(c^{*2}-|z|^2)t}$ approaches 0 as $t \to -\infty$, uniformly in x.

Consider the case where there exists i_0 such that $z = 2\lambda_{c_{i_0}}\nu_{i_0}$. Notice that there exists at most one such i_0 since $c_i \neq c_j$, i.e., $\lambda_{c_i} \neq \lambda_{c_j}$, as soon as $\nu_i = \nu_j$. For each $i \leqq k$, we have $\lambda_{c_i} = \lambda^* = \frac{c^*}{2}$. Since $|z| < c^*$, we find that $k+1 \leqq i_0 \leqq p-1$. Furthermore, for each $i \in \{1,\cdots,k\}$, $c_i = c^* > z\cdot\nu_i$ and

$$\varphi_{c_i}((c_i - z\cdot\nu_i)t + x\cdot\nu_i + h_i)e^{-\frac{1}{4}(c^{*2}-|z|^2)t} = O\big(|t|e^{(\lambda^*(c^*-z\cdot\nu_i)-f'(0)+\frac{1}{4}|z|^2)t}\big)$$

locally in x as $t \to -\infty$. Since $\lambda^*(c^* - z\cdot\nu_i) - f'(0) + \frac{1}{4}|z|^2 = \lambda^{*2} - \lambda^* z \cdot \nu_i + \frac{1}{4}|z|^2 = \frac{1}{4}|z - 2\lambda^*\nu_i|^2 > 2$, it follows that $\varphi_{c_i}((c_i - z\cdot\nu_i)t + x\cdot\nu_i + h_i)e^{-\frac{1}{4}(c^{*2}-|z|^2)t} \to 0$ locally in x as $t \to -\infty$. For each $i \in \{k+1,\cdots,p-1\}$ such that $i \neq i_0$, the latter also holds similarly. On the other hand, since $c_{i_0} > c^* > z\cdot\nu_{i_0}$, it is found that

$$\begin{aligned} &\varphi_{c_{i_0}}((c_{i_0} - z\cdot\nu_{i_0})t + x\cdot\nu_{i_0} + h_{i_0})e^{-\frac{1}{4}(c^{*2}-|z|^2)t} \\ &\quad\sim e^{\lambda_{c_{i_0}}(x\cdot\nu_{i_0}+h_{i_0})}e^{\frac{1}{4}|z-2\lambda_{c_{i_0}}\nu_{i_0}|^2 t} \\ &\quad\sim e^{\lambda_{c_{i_0}}(x\cdot\nu_{i_0}+h_{i_0})} = e^{\frac{1}{2}z\cdot x + \frac{1}{2}|z|h_{i_0}} \end{aligned}$$

locally in x as $t \to -\infty$. On the other hand, (6) implies that

$$\varphi_{c_{i_0}}((c_{i_0} - z \cdot \nu_{i_0})t + x \cdot \nu_{i_0} + h_{i_0})e^{-\frac{1}{4}(c^{*2}-|z|^2)t} \leqq u(-zt + x, t)e^{-\frac{1}{4}(c^{*2}-|z|^2)t}.$$

Eventually, we conclude that

$$u(-zt + x, t)e^{-\frac{1}{4}(c^{*2}-|z|^2)t} \to e^{\frac{1}{2}z\cdot x + \frac{1}{2}|z|h_{i_0}} \tag{23}$$

as $t \to -\infty$, locally in x, and also, as usual, in $C^2_{\mathrm{loc}}(\mathbb{R}^N_x)$.

Consider now the case where $z \neq 2\lambda_{c_i}\nu_i$ for all $i = 1, \cdots, p-1$. With the same arguments as above, it is found that

$$u(-zt + x, t)e^{-\frac{1}{4}(c^{*2}-|z|^2)t} \to 0 \text{ in } C^2_{\mathrm{loc}}(\mathbb{R}^N_x) \text{ as } t \to -\infty. \tag{24}$$

Notice here that, from (23) and (24), it easily follows that, for any sequence $t_n \to -\infty$ and for any z such that $0 < |z| < c^*$,

$$\begin{aligned} &u(-zt_n + x, t_n + t)\, e^{-\frac{1}{4}(c^{*2}-|z|^2)t_n} \to e^{(f'(0)+\frac{1}{4}|z|^2)t+\frac{1}{2}|z|h_i} e^{\frac{1}{2}z\cdot x} \\ &\qquad\qquad \text{if } \exists\, i,\ c_i < +\infty,\ 2\lambda_{c_i}\nu_i = z, \\ &u(-zt_n + x, t_n + t)\, e^{-\frac{1}{4}(c^{*2}-|z|^2)t_n} \to 0 \text{ otherwise,} \end{aligned} \tag{25}$$

in $C^1_{\mathrm{loc}}(\mathbb{R}_t)$ and $C^2_{\mathrm{loc}}(\mathbb{R}^N_x)$.

Let us now prove the last formula in (10). Take $\nu \in S^{N-1}$. If there exists i such that $(\nu, c^*) = (\nu_i, c_i)$ $(1 \leqq i \leqq k)$, then, for all $j \in \{1, \cdots, k\}\backslash\{i\}$, $c^*\nu \cdot \nu_j < c^*$ since $\nu_j \neq \nu_i$. Moreover, for each $j \geqq k+1$, $c^*\nu \cdot \nu_j \leqq c^* < c_j$. Therefore, (8) gives

$$\begin{aligned} &u(-c^*t\nu + x, t) \to \varphi_{c^*}(x \cdot \nu_i + h_i) \text{ in } C^2_{\mathrm{loc}}(\mathbb{R}^N_x) \text{ as } t \to -\infty \\ &\qquad\qquad \text{if } \exists\, i,\ (\nu, c^*) = (\nu_i, c_i). \end{aligned}$$

Otherwise, if $(\nu, c^*) \neq (\nu_i, c_i)$ for all i, then, for all $j \in \{1, \cdots, k\}$, $c^*\nu \cdot \nu_j < c^* = c_j$, and, for all $j \geqq k+1$, $c^*\nu \cdot \nu_j \leqq c^* < c_j$. Finally, the asymptotic limit

$$u(-c^*t\nu + x, t) \to 0 \text{ in } C^2_{\mathrm{loc}}(\mathbb{R}^N_x) \text{as } t \to -\infty$$

follows from (8).

Let us now check that the set of the so-built entire solutions u of (1) contains the planar travelling waves, the solutions that only depend on time and the solutions constructed in [16].

Indeed, if $(\nu, c) \in S^{N-1} \times [c^*, +\infty[$ and $h \in \mathbb{R}$, just take $p = 1$ and $(\nu_1, c_1, h_1) = (\nu, c, h)$; the function $u(x, t)$ is then equal to the planar travelling front $\varphi_c(x \cdot \nu + ct + h)$.

If $h \in \mathbb{R}$, take $p = 1$ and $(\nu_1, c_1, h_1) = (\nu_0, +\infty, h)$ for some arbitrary vector $\nu_0 \in S^{N-1}$; the function $u(x, t)$ is then equal to the function $\xi(t + h)$.

In dimension $N = 1$, under the notation of Theorem 1.1 in [16], if $c, c' \in (c^*, +\infty)$, $h, h' \in \mathbb{R}$ and $K > 0$, take $p = 3$ and $(\nu_1, c_1, h_1) = (-1, c', h')$, $(\nu_2, c_2, h_2) = (1, c, h)$ and $(\nu_3, c_3, h_3) = (\nu_0, +\infty, \frac{\ln K}{f'(0)})$ for some arbitrary $\nu_0 \in \{\pm 1\}$; by definition, the function $u(x, t)$ is then equal to the solution $u_{c,c',h,h',K}(x, t)$

constructed in Theorem 1.1 in [16] (other properties of the function u are also stated in [16]). Similarly, the entire solutions constructed in Theorems 1.3, 1.4, 1.5 in [16] can easily be obtained from the mixing of two travelling fronts or from the mixing of a travelling front with a solution depending only on time.

That completes the proof of Theorem 1.1. □

3. Construction of the infinite-dimensional manifold of entire solutions (proof of Theorem 1.2)

Let μ be a nonnegative and nonzero Radon measure on the set X and assume that the restriction μ^* of μ on the sphere $S^{N-1} \times \{c^*\}$ can be written as:

$$\mu^* = \sum_{1 \leqq i \leqq k} m_i \delta_{(\nu_i, c^*)},$$

where $k \in \mathbb{N}$ and $\nu_i \in S^{N-1}$, $0 < m_i < +\infty$ for each $i = 1, \cdots, k$. Let us moreover assume that $\nu_i \neq \nu_j$ if $i \neq j$. Let us define $\tilde{\mu}$ as the restriction of μ on $S^{N-1} \times (c^*, +\infty)$ and $\hat{\mu}$ as the restriction of μ on $\hat{X} := S^{N-1} \times (c^*, +\infty) \cup \{\infty\} = X \setminus \{(\nu, c^*),\ \nu \in S^{N-1}\}$. Let $\hat{M}$ be the set defined by

$$\hat{M} = \int_{\hat{X}} d\hat{\mu} = \mu(X) - \sum_{1 \leqq i \leqq k} m_i, \quad 0 \leqq \hat{M} < +\infty.$$

Given μ, we want to define an entire solution of (1) which should come from the mixing of a integrable sum, weighted by the measure μ, of planar travelling waves of the type $\varphi_c(x \cdot \nu + ct)$. The construction is divided into several steps: we first define a sequence of Cauchy problems starting at times $-n$ (Section 3.1), we find lower and upper bounds independent of n (Section 3.2), we pass to the limit $n \to +\infty$ (Section 3.3), we show in Section 3.5 that the limit function u_μ satisfies the asymptotic behavior (11), (12) as $t \to -\infty$ (property (i) in Theorem 1.2). We then prove the monotonicity of u_μ with respect to t and we study under what condition the function u_μ goes to 1 as $t \to +\infty$ uniformly in x (Section 3.4). Section 3.6 is devoted to the proof of property (iii) in Theorem 1.2. We prove in Section 3.7 that the functions u_μ are continuous with respect to μ on the set $\hat{\mathcal{M}}$. In Section 3.8, we deal with the case of a measure μ which is absolutely continuous with respect to the Lebesgue measure $d\nu \times dc$ (property (iv) of Theorem 1.2). In Section 3.9, we prove that the set of the functions u_μ contains the solutions described in Theorem 1.1, which are obtained from the mixing of a finite number of travelling waves.

3.1. Definition of a sequence of Cauchy problems

Let us first state the following lemma:

Lemma 3.1. (a) *If $\hat{M} > 0$, then, for each $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, the function*

$$\begin{aligned} \hat{X} &\to (0,1), \\ (\nu, c) \neq \infty &\mapsto \varphi_c(x \cdot \nu + ct + c \ln \hat{M}), \\ \infty &\mapsto \xi(t + \ln \hat{M}), \end{aligned}$$

is measurable with respect to $\hat{\mu}$. (The reason why we add the extra term $c \ln \hat{M}$ and $\ln \hat{M}$ will become clear later.)
(b) *Similarly, if $\hat{M} > 0$, the function*

$$\begin{aligned} \hat{X} &\to (0, +\infty), \\ (\nu, c) \neq \infty &\mapsto e^{\lambda_c (x \cdot \nu + ct + c \ln \hat{M})}, \\ \infty &\mapsto e^{f'(0)(t + \ln \hat{M})}, \end{aligned}$$

where $\lambda_c = \frac{c - \sqrt{c^2 - c^{*2}}}{2}$, is measurable with respect to the measure $\hat{\mu}$.

Note that in the definition of the map in (b), we have $e^{\lambda_{c_n}(x \cdot \nu_n + c_n t + c_n \ln \hat{M})} \to e^{f'(0)(t + \ln \hat{M})}$ for any sequence $c_n \to +\infty$ and $\nu_n \in S^{N-1}$, because $\lambda_c \to 0$ and $\lambda_c c \to f'(0)$ as $c \to +\infty$.

Proof of Lemma 3.1. (b) Because of the definition of $\hat{X}$ and $\hat{\mu}$, it is sufficient to show that the function $(\nu, c) \mapsto \lambda_c(x \cdot \nu + ct + c \ln \hat{M})$ is continuous on $S^{N-1} \times (c^*, +\infty)$. Since λ_c is continuous with respect to c, the conclusion follows.

(a) From what precedes, and since each function $s \mapsto \varphi_c(s)$ is continuous, we only have to prove that the functions $s \mapsto \varphi_{c_n}(s)$ converge locally to the function $s \mapsto \varphi_c(s)$ as soon as $c_n \to c \in (c^*, +\infty)$. But the latter follows from Proposition 5.5 in the paper by MALLORDY & ROQUEJOFFRE [23] (see also [16], Section 2). □

In the case $\hat{M} > 0$, let us now define, for each $n \in \mathbb{N}$, the solution $u_n(x,t)$ of the following Cauchy problem,

$$\begin{aligned} (u_n)_t &= \Delta u_n + f(u_n), \quad x \in \mathbb{R}^N, \ t > -n, \\ u_n(x, -n) &= \max\Bigg(\max_{1 \leqq i \leqq k} (\varphi_{c^*}(x \cdot \nu_i - c^* n + c^* \ln m_i)), \\ &\qquad \int_{S^{N-1} \times (c^*, +\infty)} \varphi_c(x \cdot \nu - cn + c \ln \hat{M}) \frac{1}{\hat{M}} \tilde{\mu}(d\nu \times dc) \\ &\qquad + \xi(-n + \ln \hat{M}) \, \frac{\mu(\infty)}{\hat{M}} \Bigg) \qquad (26) \\ &= \max\Bigg(\max_{1 \leqq i \leqq k} (\varphi_{c^*}(x \cdot \nu_i - c^* n + c^* \ln m_i)), \\ &\qquad \int_{\hat{X}} \varphi_c(x \cdot \nu - cn + c \ln \hat{M}) \frac{1}{\hat{M}} d\hat{\mu} \Bigg), \end{aligned}$$

by setting $\varphi_c(x \cdot \nu + ct + c \ln \hat{M}) := \xi(t + \ln \hat{M})$ if $(c, \nu) = \infty$.

In the case $\hat{M} = 0$, we simply take

$$u_n(x, -n) = \max_{1 \leqq i \leqq k} (\varphi_{c^*}(x \cdot \nu_i - c^* n + c^* \ln m_i)).$$

In the case $\hat{M} = 0$, the function $u_n(-n, x)$ is well defined, continuous with respect to x and satisfies $0 \leqq u_n(x, -n) \leqq 1$. These properties carry over in the case $\hat{M} > 0$ from Lemma 3.1 and from Lebesgue's dominated convergence theorem. As a consequence, in each case $\hat{M} > 0$ or $\hat{M} = 0$, the above Cauchy problem is itself well defined and the maximum principle yields

$$\forall\, t \geqq -n,\ \forall\, x \in \mathbb{R}^N, \quad 0 \leqq u_n(x, t) \leqq 1.$$

Remark 3.2. Before going further, let us consider the case $\mu = M_0 \delta_{\nu_0, c_0}$ where, say, $c_0 > c^*$, $M_0 > 0$ and $\delta_{(\nu_0, c_0)}$ is the Dirac distribution at the point (ν_0, c_0), and let us explain the role played by the total mass M_0. In this case, $u_n(x, t) = \varphi_{c_0}(x \cdot \nu_0 + c_0 t + c_0 \ln M_0)$ and $\ln M_0$ can be viewed as a shift in time for the travelling wave $\varphi_c(x \cdot \nu_0 + c_0 t)$.

In the general case, given a measure μ on X, each function u_n can be thought of as a superposition of travelling waves $\varphi_c(x \cdot \nu + ct)$ (with finite or infinite speeds), with some weights given by the density of the measure μ at the point (ν, c).

3.2. Lower and upper bounds

We first claim that, for all $t \geqq -n$ and for all $x \in \mathbb{R}^N$,

$$u_n(x, t) \geqq \max\Bigg(\max_{1 \leqq i \leqq k} \varphi_{c^*}(x \cdot \nu_i + c^* t + c^* \ln m_i), \\ \int_{\hat{X}} \varphi_c(x \cdot \nu + ct + c \ln \hat{M}) \frac{1}{\hat{M}} d\hat{\mu} \Bigg) \tag{27}$$

under the convention that the integral with respect to $\hat{\mu}$ drops out as soon as $\hat{M} = 0$, and that $\varphi_c(x \cdot \nu + ct + c \ln \hat{M}) := \xi(t + \ln \hat{M})$ if $(\nu, c) = \infty$.

Proof of (27). Let us first observe that $u_n(x, -n) \geqq \varphi_{c^*}(x \cdot \nu_i - c^* n + c^* \ln m_i)$ for each $i = 1, \cdots, k$. Since the function $\varphi_{c^*}(x \cdot \nu_i + c^* t + c^* \ln m_i)$ is an entire solution of (1), the maximum principle gives $u_n(x, t) \geqq \varphi_{c^*}(x \cdot \nu_i + c^* t + c^* \ln m_i)$ for all $t \geqq -n$ and for all $x \in \mathbb{R}^N$. That provides (27) in the case $\hat{M} = 0$.

In the case $\hat{M} > 0$, let $v(x, t)$ be the function defined by

$$v(x, t) := \int_{\hat{X}} \varphi_c(x \cdot \nu + ct + c \ln \hat{M}) \frac{1}{\hat{M}} d\hat{\mu}.$$

From standard parabolic estimates and since the function f is smooth, there exists a constant C_0 such that, if $0 \leqq u(t, x) \leqq 1$ is an entire solution of (1), then $|u_t|$, $|u_{x_i}|$, $|\Delta u| \leqq C_0$ globally in $(x, t) \in \mathbb{R}^N \times \mathbb{R}$. Any travelling wave $\varphi_c(x \cdot \nu + ct)$ is an entire solution of (1), whence $|c\varphi'(s)|$, $|\varphi'(s)|$, $|\varphi''(s)| \leqq C_0$ for all $c \geqq c^*$ and $s \in \mathbb{R}$. As far as the function $\xi(t)$ is concerned, we also have $|\xi'(t)| \leqq C_0$ for all $t \in \mathbb{R}$. As a consequence of Lebesgue's dominated convergence theorem, the function $v(t,x)$ is of class C^1 with respect to t and of class C^2 with respect to x and it satisfies

$$\begin{aligned} v_t - \Delta v &= \int_{\hat{X}} f\left(\varphi_c(x\cdot\nu + ct + c\ln\hat{M})\right)\frac{1}{\hat{M}}\,d\hat{\mu} \\ &\leqq f\left(\int_{\hat{X}} \varphi_c(x\cdot\nu + ct + c\ln\hat{M})\frac{1}{\hat{M}}\,d\hat{\mu}\right) \end{aligned}$$

since f is concave on $[0,1]$. The claim (27) follows then from the maximum principle. □

The inequality (27) provides a lower bound independent of n for the functions u_n. We shall now get upper bounds for the functions u_n. To this end, let us first state an auxiliary lemma:

Lemma 3.3. (a) *For each* $c > c^*$, $\varphi_c(s) \sim e^{\lambda_c s}$ *as* $s \to -\infty$ *from* (4). *Furthermore,* $\varphi_c(s) \leqq e^{\lambda_c s}$ *for all* $s \in \mathbb{R}$ *and the function* $v(s) = e^{\lambda_c s}$ *solves the linear equation* $v'' - cv' + f'(0)v = 0$ *in* $\mathbb{R}$.
(b) *Also,* $\xi(s) \leqq e^{f'(0)s}$ *for all* $s \in \mathbb{R}$.

Proof. Let us start with the proof of (a). It is rather standard but we give it for the sake of completeness. Choose $c > c^*$. Owing to the definition of λ_c in (3), the function $v(s) = e^{\lambda_c s}$ satisfies $v'' - cv' + f'(0)v = 0$. For each $t \in \mathbb{R}$, define $v^t(s) = v(s+t) = e^{\lambda_c s + \lambda_c t}$. Since φ_c is bounded and satisfies (4), it follows that there exists a real t_0 such that, for all $t \geqq t_0$, $v^t \geqq \varphi_c$ in $\mathbb{R}$. Let us now define $\tau = \inf\{t \in \mathbb{R},\ v^t \geqq \varphi_c \text{ in } \mathbb{R}\}$. From (4), we get $\tau \geqq 0$ and by continuity, we have $v^\tau(s) \geqq \varphi_c(s)$ for all $s \in \mathbb{R}$.

Assume now that $\tau > 0$ and consider a sequence $t^n \overset{<}{\to} \tau$ as $n \to +\infty$. There exists then a sequence of points $s_n \in \mathbb{R}$ such that $v^{t_n}(s_n) < \varphi_c(s_n)$. Since φ_c is bounded, the sequence (s_n) is bounded from above. Up to extraction of some subsequence, two cases may occur: $s_n \to s_\infty \in \mathbb{R}$ or $s_n \to -\infty$ as $n \to +\infty$. Assume first that $s_n \to s_\infty \in \mathbb{R}$ as $n \to +\infty$. It follows that $v^\tau(s_\infty) = \varphi_c(s_\infty)$. Define $z = v^\tau - \varphi_c$. This function z is nonnegative and vanishes at the point s_∞. Furthermore, the function φ_c satisfies $\varphi_c'' - c\varphi_c' + f'(0)\varphi_c \geqq \varphi_c'' - c\varphi_c' + f(\varphi_c) = 0$ since $f(u) \leqq f'(0)u$ for all $u \in [0,1]$. As a consequence, $z'' - cz' + f'(0)z \leqq 0$. The strong maximum principle then yields $z \equiv 0$. This is impossible because φ_c is bounded, unlike v. We deduce then that $s_n \to -\infty$ as $n \to +\infty$. Now, $\varphi_c(s_n) \sim e^{\lambda_c s_n}$ as $s_n \to -\infty$ whereas $\varphi_c(s_n) \geqq v^\tau(s_n) = e^{\lambda_c(s_n+\tau)}$. This is ruled out because $\tau > 0$. Eventually, we conclude that $\tau = 0$, which is the desired result.

Because $f(s) \leqq f'(0)s$ and $\xi(s) \sim e^{f'(0)s}$ as $s \to -\infty$, the assertion (b) is also straightforward. □

Let us now turn to the main upper bound for the functions u_n.

Lemma 3.4. *For all* $(x,t) \in \mathbb{R}^N \times \mathbb{R}$,

$$\limsup_{n\to+\infty} u_n(x,t) \leqq \sum_{1\leqq i\leqq k} \varphi_{c^*}\left(x\cdot\nu_i + c^*t + c^*\ln m_i\right) + \int_{\hat{X}} e^{\lambda_c(x\cdot\nu+ct+c\ln\hat{M})}\,\frac{1}{\hat{M}}\,d\hat{\mu} \tag{28}$$

under the convention that the second term disappears if $\hat{M}=0$, *and*

$$e^{\lambda_c(x\cdot\nu+ct+c\ln\hat{M})} = e^{f'(0)(t+\ln\hat{M})} \quad \textit{if } (\nu,c)=\infty.$$

Proof. Because of its definition, the function $u_n(x,-n)$ satisfies

$$\forall\, x\in\mathbb{R}^N,\quad 0\leqq u_n(x,-n)\leqq u_{1,0}(x)+\cdots+u_{k+2}(x),$$

where

$$\begin{aligned} u_{i,0}(x) &= \varphi_{c^*}(x\cdot\nu_i - c^*n + c^*\ln m_i) \quad \text{for } 1\leqq i\leqq k && \text{if } k>0,\\ u_{k+1,0}(x) &= \int_{S^{N-1}\times(c^*,+\infty)} \varphi_c(x\cdot\nu+ct+c\ln\hat{M})\frac{1}{\hat{M}}\,\tilde{\mu}(d\nu\times dc) && \text{if } \hat{M}>0,\\ u_{k+2,0}(x) &= \xi(-n+\ln\hat{M})\,\frac{\mu(\infty)}{\hat{M}} && \text{if } \hat{M}>0. \end{aligned} \tag{29}$$

For each $i=1,\cdots,k+2$, let $u_{i,n}(x,t)$ be the (nonnegative) solution of the Cauchy problem: $(u_{i,n})_t = \Delta u_{i,n} + f(u_{i,n})$, $t>-n$ and $u_{i,n}(x,-n)=u_{i,0}(x)$ (actually, $u_{k+2,n}(x,t)$ is only a function of t). From Lemma 2.1, it follows that

$$\forall\, t\geqq -n,\ \forall\, x\in\mathbb{R}^N,\quad 0\leqq u_n(x,t)\leqq u_{1,n}(x,t)+\cdots+u_{k+1,n}(x,t)+u_{k+2,n}(t).$$

If $1\leqq i\leqq k$, then $u_{i,n}(x,t)=\varphi_{c^*}(x\cdot\nu_i+c^*t+c^*\ln m_i)$.

Let us now find an upper bound for $u_{k+1,n}(x,t)$ (in the case $\hat{M}>0$). Choose any $(x_0,t_0)\in\mathbb{R}^N\times\mathbb{R}$. Let us first observe that the function $u_{k+1,0}(x)$ satisfies:

$$\begin{aligned} u_{k+1,0}(x) &= \int_{S^{N-1}\times(c^*,+\infty)} \varphi_c(x\cdot\nu - cn + c\ln\hat{M})\frac{1}{\hat{M}}\,\tilde{\mu}(d\nu\times dc)\\ &\leqq \int_{S^{N-1}\times(c^*,+\infty)} e^{\lambda_c(x\cdot\nu-cn+c\ln\hat{M})}\,\frac{1}{\hat{M}}\,\tilde{\mu}(d\nu\times dc) \ =: v_{n,0}(x) \end{aligned}$$

(from Lemmas 3.1 and 3.3). The $\tilde{\mu}$-measurability of the function $(\nu,c)\mapsto e^{\lambda_c(x\cdot\nu-cn+c\ln\hat{M})}$ on $S^{N-1}\times(c^*,+\infty)$ is guaranteed from Lemma 3.1 and, on the other hand, the integral

$$\int_{S^{N-1}\times(c^*,+\infty)} e^{\lambda_c(x\cdot\nu-cn+c\ln\hat{M})}\,\tilde{\mu}(d\nu\times dc)$$

converges because the functions $c\mapsto\lambda_c$ and $c\mapsto\lambda_c c=\lambda_c^2+f'(0)$ are globally bounded on $(c^*,+\infty)$ and because μ is finite.

Let us now consider the function

$$v(x,t) := \int_{S^{N-1}\times(c^*,+\infty)} e^{\lambda_c(x\cdot\nu+ct+c\ln\hat{M})}\ \frac{1}{\hat{M}}\ \tilde{\mu}(d\nu\times dc).$$

As for $v_{n,0}(x)$, this function $v(x,t)$ is well defined and $v(x,-n) = v_{n,0}(x)$. Furthermore, from Lebesgue's dominated convergence theorem, and because $\lambda_c c = \lambda_c^2 + f'(0)$, the function v solves the following Cauchy problem:

$$\begin{aligned} v_t &= \Delta v + f'(0)v,\\ v(x,-n) &= v_{n,0}(x). \end{aligned}$$

On the other hand, $f(s) \leqq f'(0)s$ for all $s \geqq 0$ (remember that f is extended by 0 outside the interval [0, 1]). The maximum principle then yields, for any $n \geqq |t_0|$,

$$u_{k+1,n}(x_0,t_0) \leqq v(x_0,t_0) = \int_{S^{N-1}\times(c^*,+\infty)} e^{\lambda_c(x_0\cdot\nu+ct_0+c\ln\hat{M})}\frac{1}{\hat{M}}\tilde{\mu}(d\nu\times dc).$$

Let us now find an upper bound for $u_{k+2,n}(t_0)$. This function solves the Cauchy problem $u'_{k+2,n}(t) = f(u_{k+2,n})$ and $u_{k+2,n}(-n) = \xi(-n+\ln\hat{M})\frac{\mu(\infty)}{\hat{M}}$. Since $f(s) \leqq f'(0)s$, we deduce that, for any $n \geqq |t_0|$,

$$u_{k+2,n}(t_0) \leqq \xi(-n+\ln\hat{M})\frac{\mu(\infty)}{\hat{M}}\ e^{f'(0)(t_0+n)}.$$

From Lemma 3.3(b), it follows then that

$$u_{k+2,n}(t_0) \leqq \frac{\mu(\infty)}{\hat{M}} e^{f'(0)(t_0+\ln\hat{M})}.$$

That completes the proof of Lemma 3.4. □

3.3. Passage to the limit $n \to +\infty$

From (27) and from the maximum principle, it follows that, for each $(x,t) \in \mathbb{R}^N\times\mathbb{R}$, the sequence $(u_n(x,t))_{n>|t|}$ is nondecreasing and satisfies $0 \leqq u_n(x,t) \leqq 1$. Hence, there exists a function $u_\mu(x,t)$ such that $u_n(x,t) \to u_\mu(x,t)$ for each $(x,t) \in \mathbb{R}^N\times\mathbb{R}$. Furthermore, from standard parabolic estimates, the functions $u_n(x,t)$ approach the function u_μ in the spaces $C^2_{\text{loc}}(\mathbb{R}^N_x)$ and $C^1_{\text{loc}}(\mathbb{R}_t)$. As a consequence, the function u_μ is an entire solution of (1), such that $0 \leqq u_\mu(x,t) \leqq 1$ for all $(x,t) \in \mathbb{R}^N\times\mathbb{R}$.

Moreover, from the lower and upper bounds (27) and (28), the function u_μ satisfies,

$$\begin{aligned} &\forall\,(x,t)\in\mathbb{R}^N\times\mathbb{R},\\ &\max\left(\max_{1\leqq i\leqq k}(\varphi_{c^*}(x\cdot\nu_i+c^*t+c^*\ln m_i)),\ \int_{\hat{X}}\varphi_c(x\cdot\nu+ct+c\ln\hat{M})\frac{1}{\hat{M}}d\hat{\mu}\right)\\ &\leqq\ u_\mu(x,t) \leqq \sum_{1\leqq i\leqq k}\varphi_{c^*}\left(x\cdot\nu_i+c^*t+c^*\ln m_i\right)+\int_{\hat{X}} e^{\lambda_c(x\cdot\nu+ct+c\ln\hat{M})}\frac{1}{\hat{M}}\,d\hat{\mu} \end{aligned} \tag{30}$$

under the convention that the integrals over $\hat{X}$ disappear as soon as $\hat{M} = 0$, and remembering that

$$\int_{\hat{X}} \varphi_c\big(x \cdot \nu + ct + c\ln \hat{M}\big)\frac{1}{\hat{M}}d\hat{\mu} = \int_{S^{N-1}\times(c^*,+\infty)} \varphi_c\big(x \cdot \nu + ct + c\ln \hat{M}\big) \times \frac{1}{\hat{M}}\tilde{\mu}(d\nu \times dc) + \xi\big(t + \ln \hat{M}\big)\frac{\mu(\infty)}{\hat{M}}$$

$$\int_{\hat{X}} e^{\lambda_c(x\cdot\nu+ct+c\ln\hat{M})}\frac{1}{\hat{M}}\,d\hat{\mu} = \int_{S^{N-1}\times(c^*,+\infty)} e^{\lambda_c(x\cdot\nu+ct+c\ln\hat{M})} \times \frac{1}{\hat{M}}\tilde{\mu}(d\nu \times dc) + e^{f'(0)(t+\ln\hat{M})}\frac{\mu(\infty)}{\hat{M}}.$$

From (30), it follows that $u_\mu(x,t) > 0$ for all (x,t). Furthermore, each of the two terms in the upper bound of (30) goes to 0 as $t \to -\infty$ for each given $x \in \mathbb{R}^N$ (the convergence of the second term

$$\int_{\hat{X}} e^{\lambda_c(x\cdot\nu+ct+c\ln\hat{M})}\frac{1}{\hat{M}}\hat{\mu}(d\nu \times dc)$$

as $t \to -\infty$ is a consequence of Lebesgue's dominated convergence theorem). Hence,

$$\forall\, x \in \mathbb{R}^N, \quad u_\mu(x,t) \to 0 \ \text{ as } t \to -\infty, \tag{31}$$

whence the function u_μ cannot be identically equal to 1. The strong maximum principle then yields $u_\mu(x,t) < 1$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$. Eventually, u_μ is an entire solution of (1) such that $0 < u < 1$.

Lastly, since f is of class C^2 on [0, 1] and from standard parabolic estimates, the functions $(u_\mu)_t$, ∇u_μ, $(u_\mu)_{x_ix_j}$, $(u_\mu)_{x_ix_jx_k}$ are globally bounded in $\mathbb{R}^N \times \mathbb{R}$.

3.4. Monotonicity in time and behavior of u_μ as $t \to +\infty$

Let us prove property (ii) in Theorem 1.2, saying that u_μ is increasing in time. Under the notation in (29), $u_n(x,-n) = \max(\max_{1\leqq i\leqq k} u_{i,0}(x), u_{k+1,0}(x) + u_{k+2,0}(x))$ for all $x \in \mathbb{R}^N$. Let us check that $\Delta u_n(x,-n) + f(u_n(x,-n)) \geqq 0$ in $\mathcal{D}'(\mathbb{R}^N)$. To do this, it is sufficient to show that $\Delta u_{i,0} + f(u_{i,0}) \geqq 0$ in $\mathbb{R}^N$ for each $i = 1, \cdots, k$ and $\Delta(u_{k+1,0} + u_{k+2,0}) + f(u_{k+1,0} + u_{k+2,0}) \geqq 0$ in $\mathbb{R}^N$.

First, we have, for each $i = 1, \cdots, k$ (provided $k > 0$),

$$\begin{aligned}\Delta u_{i,0} + f(u_{i,0}) &= \varphi''_{c^*}(x\cdot\nu_i - c^*n + c^*\ln m_i) \\ &\quad + f(\varphi_{c^*}(x\cdot\nu_i - c^*n + c^*\ln m_i)) \\ &= c^*\varphi'_{c^*}(x\cdot\nu_i - c^*n + c^*\ln m_i) > 0\end{aligned}$$

since $c^* > 0$ and $\varphi'_{c^*} > 0$. Next, with the same arguments as at the beginning of Section 3.2, the function $z(x) := u_{k+1,0}(x)+u_{k+2,0}(x)$ is of class C^2 and (provided $\hat{M} > 0$)

$$\begin{aligned}\Delta z + f(z) = & \int_{\hat{X}} \varphi''_c(x\cdot\nu - cn + c\ln\hat{M})\frac{1}{\hat{M}}d\hat{\mu} \\ & + f\left(\int_{\hat{X}} \varphi_c(x\cdot\nu - cn + c\ln\hat{M})\frac{1}{\hat{M}}\,d\hat{\mu}\right) \\ = & \int_{\hat{X}} c\varphi'_c(x\cdot\nu - cn + c\ln\hat{M})\frac{1}{\hat{M}}\,d\hat{\mu} \\ & - \int_{\hat{X}} f(\varphi_c(x\cdot\nu - cn + c\ln\hat{M}))\frac{1}{\hat{M}}\,d\hat{\mu} \\ & + f\left(\int_{\hat{X}} \varphi_c(x\cdot\nu - cn + c\ln\hat{M})\frac{1}{\hat{M}}\,d\hat{\mu}\right) \\ > & \ 0 \text{ in } \mathbb{R}^N\end{aligned}$$

since f is concave and $c\varphi'_c > 0$ for each $(\nu, c) \in \hat{X}$, under the convention that, for $(\nu, c) = \infty$, $c\varphi''_c(x\cdot\nu - cn + c\ln\hat{M}) = 0$ and $c\varphi'_c(x\cdot\nu - cn + c\ln\hat{M}) = f(\xi(-n+\ln\hat{M}))$ (> 0).

Therefore, $\Delta u_n(x,-n) + f(u_n(x,-n)) \geqq 0$ in $\mathcal{D}'(\mathbb{R}^N)$, whence the function $u_n(x,t)$ is nondecreasing with respect to t for all $x \in \mathbb{R}^N$ and $t > -n$. As a consequence, by passing to the limit $n \to +\infty$, the function $u_\mu(x,t)$ is nondecreasing with respect to t in $\mathbb{R}^N \times \mathbb{R}$. Since the nonnegative function $\partial_t u_\mu$ satisfies a linear parabolic equation, it follows from the strong maximum principle that either $\partial_t u_\mu \equiv 0$ or $\partial_t u_\mu > 0$ in $\mathbb{R}^N \times \mathbb{R}$. The first case is impossible since $0 < u_\mu(x,t) < 1$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$ and $u_\mu(x,t) \to 0$ as $t \to -\infty$ for each $x \in \mathbb{R}^N$, from (31). Eventually, we conclude that the function u_μ is increasing in time t.

Let us now study the behavior of u_μ when $t \to +\infty$ and prove the properties that are stated in Remark 1.3. Let us first consider the case where there exists a direction $\nu_0 \in S^{N-1}$ such that

$$\mu(\{c^* \leqq c < +\infty,\ \nu\cdot\nu_0 \geqq 0\} \cup \{\infty\}) = 0$$

and let us prove that $g(t) := \inf_{\mathbb{R}^N} u_\mu(\cdot,t) = 0$ for all $t \in \mathbb{R}$. Indeed, the above assumption and the upper bound in (30) yield, for all $\alpha \geqq 0$,

$$\begin{aligned}u_\mu(\alpha\nu_0, t) \leqq & \sum_{\substack{1\leqq i\leqq k\\ \nu_0\cdot\nu_i<0}} \varphi_{c^*}(\alpha\nu_0\cdot\nu_i + c^* t + c^*\ln m_i) \\ & + \int_{\{c^*<c<+\infty,\ \nu_0\cdot\nu<0\}} e^{\lambda_c(\alpha\nu_0\cdot\nu + ct + c\ln\hat{M})}\frac{1}{\hat{M}}\,d\tilde{\mu}.\end{aligned}$$

The limit $\alpha \to +\infty$ implies that $g(t) = 0$ for each time $t \in \mathbb{R}$.

Let us now consider the case where

$$\forall\,\nu_0 \in S^{N-1}, \exists\,\varepsilon > 0, \mu(\{c^* \leqq c < +\infty,\ \nu\cdot\nu_0 \geqq \varepsilon\} \cup \{\infty\}) > 0.$$

Suppose for contradiction that $g(t_0) = 0$ for some $t_0 \in \mathbb{R}$. From the lower bound in (30), we immediately get $\mu(\infty) = 0$. Furthermore, there exists a sequence of points $x_n = \alpha_n \nu_{0n}$ with $\alpha_n \geqq 0$ and $\nu_{0n} \in S^{N-1}$ such that $u(\alpha_n \nu_{0n}, t_0) \to 0$ as $n \to +\infty$. Up to extraction of some subsequence, we can assume that $\nu_{0n} \to \nu_\infty \in S^{N-1}$ as $n \to +\infty$. Since $\alpha_n \geqq 0$ and since each function φ_c is increasing, the lower bound in (30) yields

$$\max\Big(\max_{\substack{1 \leqq i \leqq k \\ \nu_{0n} \cdot \nu_i \geqq 0}} (\varphi_{c^*}(c^* t_0 + c^* \ln m_i)), \int_{\{\nu_{0n} \cdot \nu \geqq 0, c^* < c < +\infty\}} \varphi_c(ct_0 + c \ln \hat{M}) \frac{1}{\hat{M}} \, \tilde{\mu}(d\nu \times dc)\Big) \to 0$$

as $n \to +\infty$. Take any $\varepsilon > 0$. By passing to the limit $n \to +\infty$ in the above formula, it follows that $\{1 \leqq i \leqq k, \ \nu_\infty \cdot \nu_i \geqq \varepsilon\} = \emptyset$. Furthermore, since $\{\nu_{0n} \cdot \nu \geqq 0, c^* < c < +\infty\} \supset \{\nu_\infty \cdot \nu \geqq \varepsilon, c^* < c < +\infty\}$ for n large enough,

$$\int_{\{\nu_\infty \cdot \nu \geqq \varepsilon, \ c^* < c < +\infty\}} \varphi_c(ct_0 + c \ln \hat{M}) \frac{1}{\hat{M}} \, \tilde{\mu}(d\nu \times dc) = 0.$$

Hence, $\mu(\{\nu_\infty \cdot \nu \geqq \varepsilon, \ c^* < c < +\infty\}) = 0$. Eventually, we have $\mu(\{c^* \leqq c < +\infty, \nu \cdot \nu_\infty \geqq \varepsilon\} \cup \{\infty\}) = 0$ for all ε and we have then reached a contradiction. Therefore, $g(t) > 0$ for all time $t \in \mathbb{R}$.

Since $g(0) > 0$, the maximum principle implies that $u(x,t) \geqq \eta(t)$ for all $x \in \mathbb{R}^N$ and $t \geqq 0$, where $0 < \eta(t) < 1$ is the solution of the Cauchy problem $\eta' = f(\eta)$ with $\eta(0) = g(0)$. Since $\eta(t) \to 1$ as $t \to +\infty$, we conclude that $g(t) = \inf_{\mathbb{R}^N} u_\mu(\cdot, t) \to 1$ as $t \to +\infty$.

3.5. Asymptotic behavior of u_μ as $t \to -\infty$

In this section, we prove the formulas (11), (12) about the asymptotic behavior of the function u_μ as $t \to -\infty$.

Proof of (11). Assume that $k \geqq 1$ and choose $i_0 \in \{1, \cdots, k\}$. From (30), it follows that

$$\begin{aligned}\varphi_{c^*}(x \cdot \nu_{i_0} + c^* \ln m_{i_0}) &\leqq u_\mu(-c^* t \nu_{i_0} + x, t) \\ &\leqq \varphi_{c^*}(x \cdot \nu_{i_0} + c^* \ln m_{i_0}) + v(x,t) + w(x,t) + z(t),\end{aligned}$$

where

$$\begin{aligned} v(x,t) &= \sum_{i \neq i_0} \varphi_{c^*}(c^*(1 - \nu_{i_0} \cdot \nu_i)t + x \cdot \nu_i + c^* \ln m_i), \\ w(x,t) &= \int_{S^{N-1} \times (c^*, +\infty)} e^{\lambda_c(-c^* \nu_{i_0} \cdot \nu + c)t + \lambda_c x \cdot \nu + \lambda_c c \ln \hat{M}} \, \frac{1}{\hat{M}} \, \tilde{\mu}(d\nu \times dc), \\ z(t) &= \frac{\mu(\infty)}{\hat{M}} e^{f'(0)(t + \ln \hat{M})}. \end{aligned}$$

Since $\nu_{i_0}\cdot\nu_i < 1$ for each $i \neq i_0$, the function $v(x,t)$ goes to 0 locally in x as $t\to-\infty$. As far as the function w is concerned, we have $-c^*\nu_{i_0}\cdot\nu + c > 0$ for each $(\nu,c)\in S^{N-1}\times(c^*,+\infty)$. Furthermore, for each compact subset K of $\mathbb{R}^N$, there exists a constant $C(K)$ such that for all $x\in K$ and for all $(\nu,c)\in S^{N-1}\times(c^*,+\infty)$, we have $0 \leqq e^{\lambda_c x\cdot\nu+\lambda_c c\ln\hat M} \leqq C(K)$ (because λ_c and $\lambda_c c$ are bounded uniformly with respect to c). Hence, from Lebesgue's dominated convergence theorem, $w(x,t)\to 0$ as $t\to-\infty$, locally in x. Lastly, $z(t)\to 0$ as $t\to-\infty$, uniformly in x.

We finally get that $u_\mu(-c^*t\nu_{i_0}+x,t)\to\varphi_{c^*}(x\cdot\nu_{i_0}+c^*\ln m_{i_0})$ locally in x as $t\to-\infty$. Furthermore, this convergence also holds in the spaces $C^2_{\text{loc}}(\mathbb{R}^N_x)$ since the first, second and third derivatives of u_μ with respect to x are globally bounded.

If ν is such that $\nu\neq\nu_i$ for all $1\leqq i\leqq k$, then the same reasoning implies that $u_\mu(-c^*t\,\nu+x,t)\to 0$ as $t\to-\infty$ in $C^2_{\text{loc}}(\mathbb{R}^N_x)$. □

Proof of (12). Consider first the case $\hat M > 0$. Let us set

$$\alpha_N = \left(\int_{\mathbb{R}^N} e^{-\frac14|y|^2}dy\right)^{-1} = (4\pi)^{-N/2}.$$

Take a continuous function $\psi(z)$ with compact support, included in $B(0,c^*)$. Let $0\leqq a < c^*$ be such that the support of ψ is included in the open ball $B(0,a)$. Let t_n be a sequence such that $t_n\to-\infty$. We aim here to prove that

$$\begin{aligned} U_n(x,t) &:= \int_{B(0,c^*)} \alpha_N\sqrt{|t_n|}^N u_\mu(-zt_n+x,t_n+t)e^{-\frac14(c^{*2}-|z|^2)t_n}\,\psi(z)\,dz \\ &\underset{t_n\to-\infty}{\longrightarrow} \int_{B(0,c^*)} e^{(f'(0)+\frac14|z|^2)(t+\ln\hat M)+\frac12 x\cdot z}\,\psi(z)\,\frac{1}{\hat M}\Phi_*\hat\mu(dz) \end{aligned} \tag{32}$$

in $C^1_{\text{loc}}(\mathbb{R}_t)$ and $C^2_{\text{loc}}(\mathbb{R}^N_x)$, under the convention that the right-hand side is zero if $\hat M = 0$.

By additivity, it is sufficient to consider the case where ψ is nonnegative.

From standard parabolic regularity theory and since the function f is of class C^2, the function u_μ is at least of class C^2 with respect to t and of class C^3 with respect to x. As a consequence, the functions $U_n(x,t)$ are of class C^2 with respect to t and of class C^3 with respect to x. In order to show the above formula (32), it is enough to prove that the functions $U_n(x,t)$ converge pointwise to

$$\int_{B(0,c^*)} e^{(f'(0)+\frac14|z|^2)(t+\ln\hat M)+\frac12 x\cdot z}\psi(z)\frac{1}{\hat M}\Phi_*\hat\mu(dz)$$

as $t_n\to-\infty$ and that U_n and their second-order (or third-order) derivatives with respect to t (resp., x) are locally bounded.

First, from (30) and since ψ is nonnegative,

$$U_n(x,t) \geqq w_n'(x,t) \tag{33}$$

where

$$w_n'(x,t) = \int_{B(0,a)} \alpha_N \sqrt{|t_n|}^N \Big(\int_{\hat{X}} \varphi_c((c - z \cdot \nu)t_n + ct + x \cdot \nu + c \ln \hat{M}) \frac{1}{\hat{M}} d\hat{\mu} \Big) \times e^{-\frac{1}{4}(c^{*2} - |z|^2)t_n} \psi(z)\, dz.$$

Let us now prove that

$$w_n'(x,t) \to \int_{B(0,c^*)} e^{(f'(0) + \frac{1}{4}|z|^2)(t + \ln \hat{M}) + \frac{1}{2} x \cdot z}\, \psi(z)\, \frac{1}{\hat{M}}\, \Phi_* \hat{\mu}(dz)$$

as $t_n \to -\infty$, pointwise in (x,t). From Fubini's theorem, we have

$$\begin{aligned} w_n'(x,t) = &\int_{\hat{X}} \int_{B(0,a)} \alpha_N \sqrt{|t_n|}^N\, \varphi_c((c - z \cdot \nu)t_n + ct + x \cdot \nu + c \ln \hat{M}) \\ &\times\, e^{-\frac{1}{4}(c^{*2} - |z|^2)t_n}\, \psi(z)\, dz\, \frac{1}{\hat{M}} d\hat{\mu} \\ = &\int_{\hat{X}} \int_{B(0,a)} \alpha_N \sqrt{|t_n|}^N\, g(\nu, c, z, t_n, x, t) e^{\lambda_c((c - z \cdot \nu)t_n + ct + x \cdot \nu + c \ln \hat{M})} \\ &\times\, e^{-\frac{1}{4}(c^{*2} - |z|^2)t_n}\, \psi(z)\, dz\, \frac{1}{\hat{M}} d\hat{\mu}, \end{aligned}$$

where

$$0 \leqq g(\nu, c, z, t_n, x, t) = \frac{\varphi_c((c - z \cdot \nu)t_n + ct + x \cdot \nu + c \ln \hat{M})}{e^{\lambda_c((c - z \cdot \nu)t_n + ct + x \cdot \nu + c \ln \hat{M})}} \leqq 1$$

(the inequality $g \leqq 1$ follows from Lemma 3.3). Because of (3), we have

$$\lambda_c c - \lambda_c z \cdot \nu - \frac{c^{*2}}{4} + \frac{|z|^2}{4} = \lambda_c^2 - \lambda_c z \cdot \nu + \frac{|z|^2}{4} = \frac{1}{4}|2\lambda_c \nu - z|^2$$

(notice that these equalities are also true in the case $(\nu, c) = \infty$ with the convention that, in this case, $\lambda_c = 0$ and $\lambda_c c = f'(0)$). As a consequence, it follows that

$$w_n'(x,t) = \int_{\hat{X}} h(\nu, c, t_n, x, t) e^{\lambda_c c t + \lambda_c x \cdot \nu + \lambda_c c \ln \hat{M}} \frac{1}{\hat{M}}\, d\hat{\mu} \tag{34}$$

where

$$h(\nu, c, t_n, x, t) = \int_{B(0,a)} \alpha_N \sqrt{|t_n|}^N g(\nu, c, z, t_n, x, t) e^{\frac{1}{4}|2\lambda_c \nu - z|^2 t_n}\, \psi(z)\, dz.$$

For each compact subset K of $\mathbb{R}^N \times \mathbb{R}$, there exists a constant $C(K)$ such that $\forall\, (\nu, c) \in \hat{X},\ \forall\, (x,t) \in K$,

$$e^{\lambda_c c t + \lambda_c x \cdot \nu + \lambda_c c \ln \hat{M}}\, \frac{1}{\hat{M}}\, \leqq C(K).$$

Furthermore, after the change of variables $z = 2\lambda_c\nu + y|t_n|^{-1/2}$, we find that $\forall\,(x,t) \in \mathbb{R}^{N+1}$, $\forall\,(\nu,c) \in \hat{X}$, $\forall\,t_n < 0$,

$$|h(\nu,c,t_n,x,t)| \leqq \|\psi\|_\infty \int_{\mathbb{R}^N} \alpha_N e^{-\frac{1}{4}|y|^2} dy = \|\psi\|_\infty$$

because of the definition of α_N and because $|g|$ is bounded by 1. Putting together the above estimates into (34), it is found that $\forall\,(x,t) \in K$, $\forall\,(\nu,c) \in \hat{X}$, $\forall\,t_n < 0$,

$$|h(\nu,c,t_n,x,t)| e^{\lambda_c ct + \lambda_c x\cdot\nu + \lambda_c c \ln \hat{M}} \frac{1}{\hat{M}} \leqq \|\psi\|_\infty\, C(K).$$

Let us now prove that $\forall\,(x,t) \in \mathbb{R}^N \times \mathbb{R}$, $\forall\,(\nu,c) \in \hat{X}$,

$$h(\nu,c,t_n,x,t) \to \psi(2\lambda_c\nu) \text{ as } t_n \to -\infty. \tag{35}$$

Take $(x,t) \in \mathbb{R}^N \times \mathbb{R}$ and $(\nu,c) \in \hat{X}$. With the change of variables $z = 2\lambda_c\nu + y|t_n|^{-1/2}$ and from of the definition of α_N,

$$\begin{aligned} h(\nu,c,t_n,x,t) &- \psi(2\lambda_c\nu) \\ &= \int_{\sqrt{|t_n|}(B(0,a)-2\lambda_c\nu)} \alpha_N\, g(\nu,c,2\lambda_c\nu + y|t_n|^{-1/2}, t_n, x, t) e^{-\frac{1}{4}|y|^2} \\ &\quad \times\ \psi(2\lambda_c\nu + y|t_n|^{-1/2})\, dy - \int_{\mathbb{R}^N} \alpha_N\, e^{-\frac{1}{4}|y|^2}\, \psi(2\lambda_c\nu)\, dy \\ &= \int_{\mathbb{R}^N} \alpha_N\, k_{\nu,c,t_n,x,t}(y) e^{-\frac{1}{4}|y|^2} dy, \end{aligned}$$

where

$$\begin{aligned} k_{\nu,c,t_n,x,t}(y) = \Big(&\chi_{\sqrt{|t_n|}(B(0,a)-2\lambda_c\nu)}(y)\; g(\nu,c,2\lambda_c\nu + y|t_n|^{-1/2}, t_n, x, t) \\ &\times \psi(2\lambda_c\nu + y|t_n|^{1/2}) - \psi(2\lambda_c\nu)\Big) \end{aligned}$$

and where, for any subset A of $\mathbb{R}^N$, χ_A denotes the characteristic function of the set A. The function $y \mapsto k_{\nu,c,t_n,x,t}(y)$ is globally bounded by $2\|\psi\|_\infty$, independently of t_n (remember that $|g|$ is bounded by 1).

Two cases may now occur: $2\lambda_c\nu \notin B(0,a)$ or $2\lambda_c\nu \in B(0,a)$.

If $2\lambda_c\nu \notin B(0,a)$, then $\psi(2\lambda_c\nu) = 0$ and we immediately observe that $k_{\nu,c,t_n,x,t}(y) \to 0$ as $t_n \to -\infty$ for each $y \in \mathbb{R}^N$ since $\psi(2\lambda_c\nu + y|t_n|^{-1/2}) \to \psi(2\lambda_c\nu) = 0$ as $t_n \to -\infty$.

On the other hand, if $2\lambda_c\nu \in B(0,a)$, then $\chi_{\sqrt{|t_n|}(B(0,a)-2\lambda_c\nu)}(y) \to 1$ as $t_n \to -\infty$ for each $y \in \mathbb{R}^N$ (remember that $B(0,a)$ is open). Furthermore, for each $y \in \mathbb{R}^N$,

$$\begin{aligned} g(\nu,c,&2\lambda_c\nu + y|t_n|^{-1/2}, t_n, x, t) \\ &= \frac{\varphi_c\Big((c - (2\lambda_c\nu + y|t_n|^{-1/2})\cdot\nu)t_n + ct + x\cdot\nu + c\ln\hat{M}\Big)}{e^{\lambda_c\big((c-(2\lambda_c\nu + y|t_n|^{-1/2})\cdot\nu)t_n + ct + x\cdot\nu + c\ln\hat{M}\big)}} \to 1 \end{aligned}$$

as $t_n \to -\infty$ because of (4) and because $c - 2\lambda_c > 0$ (notice that the convergence $g(\nu, c, 2\lambda_c\nu + y|t_n|^{-1/2}, t_n, x, t) \to 1$ holds both in the case $(\nu, c) \neq \infty$ and in the case $(\nu, c) = \infty$). Eventually, we conclude that $k_{\nu,c,t_n,x,t}(y) \to 0$ as $t_n \to -\infty$ for each $y \in \mathbb{R}^N$. The claim (35) follows then from Lesbesgue's dominated convergence theorem.

As a consequence, in each case $2\lambda_c\nu \notin B(0, a)$ or $2\lambda_c\nu \in B(0, a)$, a second application of Lesbesgue's dominated convergence theorem yields

$$\begin{aligned} w_n'(x,t) \underset{t_n\to-\infty}{\longrightarrow} &\int_{\hat{X}} e^{\lambda_c ct+\lambda_c x\cdot\nu+\lambda_c c\ln\hat{M}}\psi(2\lambda_c\nu)\,\frac{1}{\hat{M}}\,d\hat{\mu} \\ &= \int_{\hat{X}} e^{(f'(0)+\lambda_c^2)t+\lambda_c x\cdot\nu+(f'(0)+\lambda_c^2)\ln\hat{M}}\psi(2\lambda_c\nu)\frac{1}{\hat{M}}\,d\hat{\mu} \\ &= \int_{B(0,c^*)} e^{(f'(0)+\frac{1}{4}|z|^2)(t+\ln\hat{M})+\frac{1}{2}x\cdot z}\,\psi(z)\frac{1}{\hat{M}}\,\Phi_*\hat{\mu}\,(dz) \end{aligned}$$

by definition of the map Φ. Therefore, remembering (33), it is found that

$$\begin{aligned} \liminf_{t_n\to-\infty} U_n(x,t) = &\liminf_{t_n\to-\infty}\int_{B(0,c^*)} \alpha_N\sqrt{|t_n|}^N\,u_\mu(-zt_n+x,t_n+t) \\ &\times e^{-\frac{1}{4}(c^{*2}-|z|^2)t_n}\psi(z)\,dz \\ &\geqq \int_{B(0,c^*)} e^{(f'(0)+\frac{1}{4}|z|^2)(t+\ln\hat{M})+\frac{1}{2}x\cdot z}\psi(z)\,\frac{1}{\hat{M}}\Phi_*\hat{\mu}\,(dz). \end{aligned}$$

Similarly, by using the upper bound in (30), we claim that

$$\begin{aligned} \limsup_{t_n\to-\infty}\int_{B(0,c^*)} &\alpha_N\sqrt{|t_n|}^N u_\mu(-zt_n+x,t_n+t)e^{-\frac{1}{4}(c^{*2}-|z|^2)t_n}\,\psi(z)\,dz \\ &\leqq \int_{B(0,c^*)} e^{(f'(0)+\frac{1}{4}|z|^2)(t+\ln\hat{M})+\frac{1}{2}x\cdot z}\psi(z)\,\frac{1}{\hat{M}}\Phi_*\hat{\mu}\,(dz). \end{aligned} \quad (36)$$

Indeed, we have $U_n(x,t) \leqq v_n''(x,t) + w_n''(x,t)$ with

$$\begin{aligned} v_n''(x,t) = &\int_{B(0,a)} \alpha_N\sqrt{|t_n|}^N \sum_{1\leqq i\leqq k}\big(\varphi_{c^*}((c^*-z\cdot\nu_i)t_n \\ &+ c^*t + x\cdot\nu_i + c^*\ln m_i)\big)e^{-\frac{1}{4}(c^{*2}-|z|^2)t_n}\,\psi(z)\,dz \\ w_n''(x,t) = &\int_{B(0,a)} \alpha_N\sqrt{|t_n|}^N\left(\int_{\hat{X}} e^{\lambda_c((c-z\cdot\nu)t_n+ct+x\cdot\nu+c\ln\hat{M})}\frac{1}{\hat{M}}\,d\hat{\mu}\right) \\ &\times\; e^{-\frac{1}{4}(c^{*2}-|z|^2)t_n}\,\psi(z)\,dz. \end{aligned}$$

Let us first prove that $v_n''(x,t) \to 0$ as $t_n \to -\infty$. Choose a compact subset K of $\mathbb{R}^N\times\mathbb{R}$. Because $c^* - z\cdot\nu_i \geqq c^* - a > 0$ for all $z \in B(0,a)$ and for all $1 \leqq i \leqq k$, and because $\varphi_{c^*}(s) \sim |s|e^{\lambda^* s}$ as $s \to -\infty$, it follows that there exists a constant $C = C(K)$ and a real number T such that, for all $(x,t) \in K$ and for all $t_n \leqq -T$, $\forall\, 1 \leqq i \leqq k$,

$$\varphi_{c^*}((c^*-z\cdot\nu_i)t_n + c^*t + x\cdot\nu_i + c^*\ln m_i) \leqq C(|t_n|+1)e^{\lambda^*(c^*-z\cdot\nu_i)t_n}.$$

Since $\lambda^*c^* = (\lambda^*)^2 + f'(0) = (\lambda^*)^2 + \frac{(c^*)^2}{4}$, we have

$$\begin{aligned}\lambda^*c^* - \lambda^* z\cdot\nu_i - \frac{c^{*2}}{4} + \frac{|z|^2}{4} &= (\lambda^*)^2 - \lambda^* z\cdot\nu_i + \frac{|z|^2}{4}\\ &= \tfrac{1}{4}\,|2\lambda^*\nu_i - z|^2 = \tfrac{1}{4}\,|c^*\nu_i - z|^2\\ &> \tfrac{1}{4}\,(c^*-a)^2 > 0\end{aligned}$$

for all $z\in B(0,a)$ and for all $1 \leqq i \leqq k$. Hence, even if it means changing the constant C, we get

$$\forall\,(x,t)\in K,\ \forall\,t_n \leqq -T,\quad |v_n''(x,t)| \leqq C\sqrt{|t_n|}^N(|t_n|+1)e^{\frac{1}{4}(c^*-a)^2 t_n}.$$

Hence, $v_n''(x,t)\to 0$ as $t_n\to-\infty$, uniformly for $(x,t)\in K$.

On the other hand, as for $w_n'(x,t)$, we have

$$w_n''(x,t) \to \int_{B(0,c^*)} e^{(f'(0)+\frac{1}{4}|z|^2)(t+\ln\hat{M})+\frac{1}{2}x\cdot z}\psi(z)\,\frac{1}{\hat{M}}\,\Phi_*\hat{\mu}(dz) \text{ as } t_n\to-\infty.$$

(Here, unlike the case of $w_n'(x,t)$, we do not have to use the function $g(\nu,c,z,t_n,x,t)$.) Hence, we get (36).

As a conclusion,

$$U_n(x,t) \to \int_{B(0,c^*)} e^{(f'(0)+\frac{1}{4}|z|^2)(t+\ln\hat{M})+\frac{1}{2}x\cdot z}\,\psi(z)\,\frac{1}{\hat{M}}\Phi_*\hat{\mu}(dz)$$

as $t_n\to-\infty$, for each $(x,t)\in\mathbb{R}^N\times\mathbb{R}$.

Furthermore, from the arguments above, the functions $U_n(x,t)$ are uniformly (with respect to t_n) bounded in each compact subset K of $\mathbb{R}^N\times\mathbb{R}$. On the other hand, since the function u_μ is a positive entire and globally bounded solution of (1), it follows from standard parabolic estimates and Harnack inequality that there exists a constant C such that, for all $(x,t)\in\mathbb{R}^N\times\mathbb{R}$, we have: $\|\nabla u_\mu(x,t)\|$, $|(u_\mu)_{x_ix_j}(x,t)|, |(u_\mu)_{x_ix_jx_k}(x,t)| \leqq Cu(x,t+1)$. As a consequence, the derivatives of the functions U_n (at least up to the second order in t and the third order in x) are locally bounded in (x,t), uniformly with respect to t_n. This implies that the convergence

$$\begin{aligned}U_n(x,t) &= \int_{B(0,c^*)} \alpha_N\sqrt{|t_n|}^N u_\mu(-zt_n+x,t_n+t)e^{-\frac{1}{4}(c^{*2}-|z|^2)t_n}\psi(z)\,dz\\ &\to \int_{B(0,c^*)} e^{(f'(0)+\frac{1}{4}|z|^2)(t+\ln\hat{M})+\frac{1}{2}x\cdot z}\psi(z)\,\frac{1}{\hat{M}}\,\Phi_*\hat{\mu}\,(dz)\end{aligned}$$

actually takes place in $C^1_{\rm loc}(\mathbb{R}_t)$ and $C^2_{\rm loc}(\mathbb{R}^N_x)$.

Consider now the case $\hat{M}=0$. Under the same notation as above, the term $w_n''(x,t)$ disappears and we have $0 \leqq U_n(x,t) \leqq v_n''(x,t)$, whence $U_n(x,t)\to 0$ in $\mathcal{T}$ as $n\to+\infty$.

This completes the proof of (32), which gives (12). □

From (11), (12), we deduce the following

Lemma 3.5. *The map $\mu \mapsto u_\mu$ is one-to-one.*

Proof. Consider two measures μ_1 and μ_2 in $\mathcal{M}$ and assume that $u_{\mu_1} = u_{\mu_2}$. From (11), it follows that the ν_i's and the m_i's are identical for μ_1 and μ_2, that is to say, that $\mu_1^* = \mu_2^*$.

Formula (12) especially implies that either $\hat{M}_1 = \hat{M}_2$, or both $\hat{M}_1$ and $\hat{M}_2$ are positive. In the first case, then $\hat{\mu}_1 = \hat{\mu}_2 = 0$ and, eventually, $\mu_1 = \mu_2$. Consider now the case where both $\hat{M}_1$ and $\hat{M}_2$ are positive. Formula (12) applied to $x = 0$ and $t = -\ln \hat{M}_1$ gives

$$\frac{1}{\hat{M}_1}\int_{B(0,c^*)} \psi(z)\Phi_*\hat{\mu}_1(dz) = \int_{B(0,c^*)} e^{(f'(0)+\frac{1}{4}|z|^2)(\ln \hat{M}_2 - \ln \hat{M}_1)}\ \psi(z)\ \frac{1}{\hat{M}_2}\ \Phi_*\hat{\mu}_2(dz)$$

for each function $\psi \in C_c(B(0,c^*))$. Take a sequence of functions $\psi_n \in C_c(B(0,c^*))$ such that $0 \leqq \psi_n \leqq 1$ and $\psi_n = 1$ in $B(0, c^* - 1/n)$, and pass to the limit $n \to +\infty$. It follows that

$$\frac{1}{\hat{M}_1}\ \Phi_*\hat{\mu}_1(B(0,c^*)) = \int_{B(0,c^*)} e^{(f'(0)+\frac{1}{4}|z|^2)(\ln \hat{M}_2 - \ln \hat{M}_1)} \frac{1}{\hat{M}_2} \Phi_*\hat{\mu}_2(dz).$$

By definition of $\hat{M}_1$ and of the map Φ, the left-hand side is equal to 1. Applying the mean value theorem to the right-hand side, gives $1 = e^{(f'(0)+\frac{1}{4}|z_0|^2)(\ln \hat{M}_2 - \ln \hat{M}_1)}$ for some z_0 such that $|z_0| \leqq c^*$. This yields $\hat{M}_1 = \hat{M}_2$. From (12), we conclude that $\Phi_*\hat{\mu}_1 = \Phi_*\hat{\mu}_2$ on $B(0,c^*)$, whence $\hat{\mu}_1 = \hat{\mu}_2$ on $\hat{X}$ from the definition of the map Φ. Eventually, we get $\mu_1 = \mu_2$. $\square$

Before ending this section, let us make more precise the behavior of $u_\mu(x,t)$ when $t \to -\infty$, locally in $x \in \mathbb{R}^N$. This corresponds to the case $z = 0$ in (12). We claim that

$$u_\mu(x, t_n + t)e^{-f'(0)t_n} \to \frac{\mu(\infty)}{\hat{M}} e^{f'(0)(t+\ln \hat{M})} \tag{37}$$

in the sense of $\mathcal{T}$ for each sequence $t_n \to -\infty$ (under the convention that the right-hand side is zero if $\hat{M} = 0$). Let us first consider the case $\hat{M} > 0$. The inequalities (30) yield

$$\begin{aligned}\xi(t_n + t + \ln \hat{M})\frac{\mu(\infty)}{\hat{M}} e^{-f'(0)t_n} &\leqq u_\mu(x, t_n + t)e^{-f'(0)t_n} \\ &\leqq v_n'''(x,t) + w_n'''(x,t) + \frac{\mu(\infty)}{\hat{M}} e^{f'(0)(t+\ln \hat{M})}\end{aligned} \tag{38}$$

where

$$v_n'''(x,t) = \sum_{1 \leqq i \leqq k} \varphi_{c^*}(x \cdot \nu_i + c^* t_n + c^* t + c^* \ln m_i) e^{-f'(0)t_n}$$

$$w_n'''(x,t) = \int_{S^{N-1}\times(c^*,+\infty)} e^{(\lambda_c c - f'(0))t_n + \lambda_c ct + \lambda_c x\cdot\nu + \lambda_c c \ln \hat{M}} \frac{1}{\hat{M}} \tilde{\mu}(d\nu \times dc).$$

Since $\xi(s) \sim e^{f'(0)s}$ as $s \to -\infty$, the left-hand side of (38) goes to $\frac{\mu(\infty)}{\hat{M}} e^{f'(0)(t+\ln \hat{M})}$ as $t_n \to -\infty$. Let us now investigate the term $v_n'''(x,t)$ of the right-hand side. Let K be a compact subset of $\mathbb{R}^N \times \mathbb{R}$. Since $\varphi_{c^*}(s) \sim |s| e^{\lambda^* s}$ as $s \to -\infty$, there exists a positive constant $C(K)$ and a real T such that

$$\forall (x,t) \in K,\ \forall t_n \leqq -T, \quad 0 \leqq v_n'''(x,t) \leqq C(K)(|t_n|+1) e^{(\lambda^* c^* - f'(0))t_n}.$$

Because $\lambda^* c^* - f'(0) = \lambda^{*2} = f'(0) > 0$, we find that $v_n'''(x,t) \to 0$ as $t_n \to -\infty$ locally in (x,t).

As far as the term $w_n'''(x,t)$ is concerned, since $\lambda_c c - f'(0) = \lambda_c^2 > 0$ for each $c \in (c^*, +\infty)$, we conclude from Lebesgue's dominated convergence theorem that $w_n'''(x,t) \to 0$ as $t_n \to -\infty$ locally in (x,t).

Eventually,

$$u_\mu(x, t_n + t) e^{-f'(0)t_n} \to \frac{\mu(\infty)}{\hat{M}} e^{f'(0)(t+\ln \hat{M})}$$

locally in (x,t) as $t_n \to -\infty$. On the other hand, since

$$\|\nabla u_\mu(x,t)\|, |(u_\mu)_{x_i x_j}(x,t)|, |(u_\mu)_{x_i x_j x_k}(x,t)| \leqq C u_\mu(x, t+1)$$

for some constant C and for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, the functions $(x,t) \mapsto u_\mu(x, t_n + t) e^{-f'(0)t_n}$ and their derivatives in t (or in x) up to the second order (resp., third order) are locally bounded in (x,t), uniformly with respect to t_n. We finally conclude that $u_\mu(x, t_n+t) e^{-f'(0)t_n} \to \frac{\mu(\infty)}{\hat{M}} e^{f'(0)(t+\ln \hat{M})}$ as $t_n \to -\infty$ in the sense of the topology $\mathcal{T}$.

If $\hat{M} = 0$, then $\mu(\infty) = 0$, the term $w_n'''(x,t)$ disappears and the convergence $u_\mu(x, t_n + t) e^{-f'(0)t_n} \to 0$ in $\mathcal{T}$ follows.

Remark 3.6. For each entire solution u of (1), $\max_{|x| \leqq c|t|} u(x,t) \to 0$ as $t \to -\infty$ for each $c \in [0, c^*[$ (see Lemma 4.1 in Section 4) and (12) gives the asymptotic behavior of the function $z \in B(0, c^*) \mapsto u_\mu(zt, t)$ as $t \to -\infty$, for each entire solution of (1) of the type u_μ with $\mu \in \mathcal{M}$. Similarly, we know that $\min_{|x| \leqq ct} u(x,t) \to 1$ as $t \to +\infty$ for each $c \in [0, c^*[$. We could try to define more precisely the behavior of the function $z \in B(0, c^*) \mapsto 1 - u_\mu(zt, t)$ when $t \to +\infty$. But that seems intricate because of the lack of a suitable upper bound of u_μ for large time.

3.6. Multiplication of μ by positive constants

The puropose of this section is to prove property (iii) in Theorem 1.2. Take a measure $\mu \in \mathcal{M}$ and write μ as

$$\mu = \sum_{1 \leqq i \leqq k} m_i \delta_{(\nu_i, c^*)} + \hat{\mu},$$

where k is a nonnegative integer and $m_i \geqq 0$.

Choose any positive real number α. The measure $\alpha\mu$ belongs to $\mathcal{M}$. By definition, $u_{\alpha\mu}(x,t) = \lim_{n\to+\infty} U_n(x,t)$ where U_n is the solution of the Cauchy problem $(U_n)_t = \Delta U_n + f(U_n)$, $t > -n$, $x \in \mathbb{R}^N$, with initial condition at time $t = -n$

$$\begin{aligned} U_n(x,-n) = {} & \max\Bigg(\max_{1\leqq i\leqq k}(\varphi_{c^*}(x\cdot\nu_i - c^*n + c^*\ln(\alpha m_i))), \\ & \int_{\hat{X}} \varphi_c(x\cdot\nu - cn + c\ln(\alpha\hat{M}))\,\frac{1}{\alpha\hat{M}}\,d(\alpha\hat{\mu})\Bigg) \\ = {} & u_{n-\ln\alpha}(x, -n+\ln\alpha), \end{aligned}$$

where $u_{n-\ln\alpha}$ is defined as in (26) by n replaced by $n - \ln\alpha$. By uniqueness of the above Cauchy problem, it follows that $U_n(x,t) = u_{n-\ln\alpha}(x, t+\ln\alpha)$ for any n and $t \geqq -n$, $x \in \mathbb{R}^N$.

As shown in Section 3.3, it is true that the sequence $(u_{n'}(x,t))_{n'}$ is nondecreasing for any nondecreasing sequence of positive numbers n', the n' being not necessarily integers. Therefore, $u_{n-\ln\alpha}(x, t+\ln\alpha) \to u_\mu(x, t+\ln\alpha)$ as $n \to +\infty$. Eventually, that yields $u_{\alpha\mu}(x,t) = u_\mu(x, t+\ln\alpha)$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, which is the desired result.

In addition, as a consequence of the general asymptotic properties (15) and (16) that are satisfied by any solution u of (1), it immediately follows that, for each measure $\mu \in \mathcal{M}$, $u_{\alpha\mu} \to 1$ in the sense of the topology $\mathcal{T}$, as $\alpha \to +\infty$, and $u_{\alpha\mu} \to 0$ as $\alpha \to 0^+$.

3.7. Continuity with respect to μ

Let μ^n be a sequence of $\hat{\mathcal{M}}$ such that μ^n converges to $\mu \in \hat{\mathcal{M}}$ in the sense that:

(a) $\int_{\hat{X}} f\,d\hat{\mu}^n \to \int_{\hat{X}} f\,d\hat{\mu}$ for any continuous function f on $\hat{X}$ such that $f \equiv 0$ on $S^{N-1} \times (c^*, c)$ for some $c > c^*$,
(b) $\hat{M}^n = \mu^n(\hat{X}) \to \hat{M} = \mu(\hat{X})$,
(c) $\mu^n(\infty) \to \mu(\infty)$ as $n \to +\infty$.

The functions $u_{\mu^n}(x,t)$ are entire solutions of (1). From standard parabolic estimates, they converge in the sense of the topology $\mathcal{T}$, up to extraction of some subsequence, to a solution $U(x,t)$ of (1). We then have to prove that $U = u_\mu$.

The formula (30) applied to u_{μ_n} yields

$$\begin{aligned}
&\int_{S^{N-1}\times(c^*,+\infty)} \varphi_c(x\cdot\nu+ct+c\ln\hat M^n)\frac{1}{\hat M^n}d\tilde\mu^n+\xi(t+\ln\hat M^n)\frac{\mu^n(\infty)}{\hat M^n}\\
&\leqq u_{\mu^n}(x,t) \qquad (39)\\
&\leqq \int_{S^{N-1}\times(c^*,+\infty)} e^{\lambda_c(x\cdot\nu+ct+c\ln\hat M^n)}\frac{1}{\hat M^n}\,d\tilde\mu^n+\frac{\mu^n(\infty)}{\hat M^n}e^{f'(0)(t+\ln\hat M^n)}
\end{aligned}$$

for all $(x,t)\in\mathbb{R}^N\times\mathbb{R}$. From assumptions (b) and (c), it immediately follows that

$$\xi(t+\ln\hat M^n)\frac{\mu^n(\infty)}{\hat M^n}\to\xi(t+\ln\hat M)\frac{\mu(\infty)}{\hat M}\text{ as } n\to+\infty.$$

Choose now any ε, $A>0$ such that $c^*+\varepsilon<A$ and let $\chi(c)$ be a continuous function defined on $\mathbb{R}$ and such that $0\leqq\chi\leqq 1$, $\chi(c)=1$ if $c^*+\varepsilon\leqq c\leqq A$ and $\chi(c)=0$ if $c\notin[c^*+\varepsilon/2,2A]$. We have

$$\begin{aligned}
&\int_{S^{N-1}\times(c^*,+\infty)} \varphi_c(x\cdot\nu+ct+c\ln\hat M^n)\frac{1}{\hat M^n}\,d\tilde\mu^n\\
&\qquad\geqq I_n:=\int_{S^{N-1}\times(c^*,+\infty)}\chi(c)\varphi_c(x\cdot\nu+ct+c\ln\hat M^n)\frac{1}{\hat M^n}\,d\tilde\mu^n.
\end{aligned}$$

The term I_n also reads $\mathrm{I}_n=\mathrm{II}_n+\mathrm{III}_n$ where

$$\begin{aligned}
\mathrm{II}_n&=\int_{S^{N-1}\times(c^*,+\infty)}\chi(c)\Big(\varphi_c(x\cdot\nu+ct+c\ln\hat M^n)\frac{1}{\hat M^n}\\
&\qquad-\varphi_c(x\cdot\nu+ct+c\ln\hat M)\frac{1}{\hat M}\Big)\,d\tilde\mu^n\\
\mathrm{III}_n&=\int_{S^{N-1}\times(c^*,+\infty)}\chi(c)\varphi_c(x\cdot\nu+ct+c\ln\hat M)\frac{1}{\hat M}\,d\tilde\mu^n.
\end{aligned}$$

From the assumption (a) and from the choice of χ,

$$\mathrm{III}_n\to\int_{S^{N-1}\times(c^*,+\infty)}\chi(c)\;\varphi_c(x\cdot\nu+ct+c\ln\hat M)\;\frac{1}{\hat M}\;d\tilde\mu\;\text{ as } n\to+\infty.$$

On the other hand,

$$|\mathrm{II}_n|\leqq\int_{S^{N-1}\times(c^*,+\infty)}\left(\left|\frac{1}{\hat M^n}-\frac{1}{\hat M}\right|+c\|\varphi_c'\|_\infty|\ln\hat M^n-\ln\hat M|\frac{1}{\hat M}\right)\chi(c)\,d\tilde\mu^n.$$

Since the functions $u_{\nu,c}(x,t)=\varphi_c(x\cdot\nu+ct)$ are bounded solutions of the parabolic equation (1), there exists a constant K, independent of (ν,c) such that $\|\partial_t u_{\nu,c}(x,t)\|\leqq K$ for all $(x,t)\in\mathbb{R}^{N+1}$. Therefore, $c\|\varphi_c'\|_\infty\leqq K$ for all $c\in(c^*,+\infty)$. Since the sequence $(\mu^n(X))$ is bounded, we finally conclude that $\mathrm{II}_n\to 0$ as $n\to+\infty$.

Thus,

$$\mathrm{I}_n \underset{n\to+\infty}{\to} \int_{S^{N-1}\times(c^*,+\infty)} \chi(c)\varphi_c(x\cdot\nu+ct+c\ln\hat{M})\frac{1}{\hat{M}}\,d\tilde{\mu} \geqq \int_{S^{N-1}\times(c^*+\varepsilon,A)} \varphi_c(x\cdot\nu+ct+c\ln\hat{M})\frac{1}{\hat{M}}\,d\tilde{\mu}.$$

Passing to the limits $\varepsilon\to 0$ and $A\to+\infty$ eventually implies, thanks to the monotone convergence theorem, that

$$\liminf_{n\to+\infty}\int_{S^{N-1}\times(c^*,+\infty)} \varphi_c(x\cdot\nu+ct+c\ln\hat{M}^n)\frac{1}{\hat{M}^n}d\tilde{\mu}^n \geqq \int_{S^{N-1}\times(c^*,+\infty)} \varphi_c(x\cdot\nu+ct+c\ln\hat{M})\frac{1}{\hat{M}}d\tilde{\mu}.$$

Similarly, we can prove that

$$\limsup_{n\to+\infty}\int_{S^{N-1}\times(c^*,+\infty)} e^{\lambda_c(x\cdot\nu+ct+c\ln\hat{M}^n)}\frac{1}{\hat{M}^n}\,d\tilde{\mu}^n \leqq \int_{S^{N-1}\times(c^*,+\infty)} e^{\lambda_c(x\cdot\nu+ct+c\ln\hat{M})}\frac{1}{\hat{M}}\,d\tilde{\mu}.$$

Putting all the above results into (39) leads to:

$$\begin{aligned}\int_{S^{N-1}\times(c^*,+\infty)} &\varphi_c(x\cdot\nu+ct+c\ln\hat{M})\frac{1}{\hat{M}}\,d\tilde{\mu}+\xi(t+\ln\hat{M})\frac{\mu(\infty)}{\hat{M}}\\ &\leqq U(x,t)\\ &\leqq \int_{S^{N-1}\times(c^*,+\infty)} e^{\lambda_c(x\cdot\nu+ct+c\ln\hat{M})}\frac{1}{\hat{M}}\,d\tilde{\mu}+e^{f'(0)(t+\ln\hat{M})}\frac{\mu(\infty)}{\hat{M}}\end{aligned}$$

for all $(x,t)\in\mathbb{R}^{N+1}$. In other words, for all $(x,t)\in\mathbb{R}^N\times\mathbb{R}$,

$$\int_{\hat{X}}\varphi_c(x\cdot\nu+ct+c\ln\hat{M})\frac{1}{\hat{M}}\,d\hat{\mu}\leqq U(x,t)\leqq\int_{\hat{X}} e^{\lambda_c(x\cdot\nu+ct+c\ln\hat{M})}\,\frac{1}{\hat{M}}\,d\hat{\mu}. \tag{40}$$

Remember that, by definition, the function u_μ is the pointwise limit of the functions $u_n(x,t)$, which are solutions of the Cauchy problems $\partial_t u_n=\Delta u_n+f(u_n)$, $t>-n$,

$$u_n(x,-n)=\int_{\hat{X}}\varphi_c(x\cdot\nu-cn+c\ln\hat{M})\frac{1}{\hat{M}}\,d\hat{\mu}.$$

From the maximum principle, it follows then that $u_n(x,t)\leqq U(x,t)$ for all $t\geqq -n$ and $x\in\mathbb{R}^N$.

Let v_n be the function defined by $v_n(x,t) = U(x,t) - u_n(x,t) \geqq 0$. The function v_n satisfies $\partial_t v_n = \Delta v_n + f(U) - f(u_n) \leqq \Delta v_n + f'(0)v_n$ for all $t > -n$, $x \in \mathbb{R}^N$. Fix a couple $(x,t) \in \mathbb{R}^{N+1}$. For $n > |t|$,

$$\begin{aligned} 0 \leqq v_n(x,t) &\leqq \frac{e^{f'(0)(t+n)}}{\sqrt{4\pi(t+n)}^N} \int_{\mathbb{R}^N} v_n(y,-n) e^{-\frac{|y-x|^2}{4(t+n)}}\, dy \\ &\leqq \frac{e^{f'(0)(t+n)}}{\sqrt{4\pi(t+n)}^N} \int_{\mathbb{R}^N} \Big(\int_{\hat{X}} (e^{\lambda_c(x\cdot\nu - cn + c\ln\hat{M})} \\ &\qquad - \varphi_c(x\cdot\nu - cn + c\ln\hat{M}))\, \frac{1}{\hat{M}}\, d\hat{\mu} \Big) \\ &\qquad \times e^{-\frac{|y-x|^2}{4(t+n)}}\, dy \end{aligned}$$

because of (40). Moreover, from Lemma 3.3, $e^{\lambda_c(x\cdot\nu - cn + c\ln\hat{M})} - \varphi_c(x\cdot\nu - cn + c\ln\hat{M}) \geqq 0$ for all $(\nu, c) \in \hat{X}$ (the case (ν, c) also works because of our conventions and because $\xi(s) \leqq e^{f'(0)s}$ for all $s \in \mathbb{R}$). We then get

$$0 \leqq v_n(x,t) \leqq \int_{\hat{X}} w_n(\nu, c)\, d\hat{\mu}, \tag{41}$$

where

$$\begin{aligned} w_n(\nu,c) = {} & \frac{e^{f'(0)(t+n)}}{\sqrt{4\pi(t+n)}^N} \int_{\mathbb{R}^N} \Big(e^{\lambda_c(y\cdot\nu - cn + c\ln\hat{M})} - \varphi_c(y\cdot\nu - cn + c\ln\hat{M}) \Big) \\ & \times \frac{1}{\hat{M}} e^{-\frac{|y-x|^2}{4(t+n)}}\, dy. \end{aligned}$$

On the one hand,

$$\begin{aligned} 0 \leqq w_n(\nu,c) &\leqq \phi_n(x,t) \\ &= \frac{e^{f'(0)(t+n)}}{\sqrt{4\pi(t+n)}^N} \int_{\mathbb{R}^N} e^{\lambda_c(y\cdot\nu - cn + c\ln\hat{M})}\, \frac{1}{\hat{M}}\, e^{-\frac{|y-x|^2}{4(t+n)}}\, dy. \end{aligned}$$

By definition, the function ϕ_n is a solution of the linear Cauchy problem $\partial_t \phi_n = \Delta\phi_n + f'(0)\phi_n$ for $t > -n$ and $\phi_n(x,-n) = \frac{1}{\hat{M}}\, e^{\lambda_c(x\cdot\nu - cn + c\ln\hat{M})}$. By uniqueness of this Cauchy problem and since $\lambda_c c = \lambda_c^2 + f'(0)$, we conclude that $\phi_n(x,t) = \frac{1}{\hat{M}} e^{\lambda_c(x\cdot\nu + ct + c\ln\hat{M})}$. Therefore, $0 \leqq w_n(\nu,c) \leqq \frac{1}{\hat{M}} e^{\lambda_c(x\cdot\nu + ct + c\ln\hat{M})}$ and this function $(\nu,c) \mapsto \frac{1}{\hat{M}}\, e^{\lambda_c(x\cdot\nu + ct + c\ln\hat{M})}$ is such that

$$\int_{\hat{X}} \frac{1}{\hat{M}} e^{\lambda_c(x\cdot\nu + ct + c\ln\hat{M})}\, d\hat{\mu} < +\infty.$$

Choose now any couple $(\nu, c) \in S^{N-1} \times (c^*, +\infty)$. By making the change of variables $y = x + 2\lambda_c(t+n)\nu + \sqrt{4(t+n)}z$ and by using (3), a straightforward calculation gives

$$w_n(\nu,c) = \frac{1}{\hat{M}}\, e^{\lambda_c(x\cdot\nu + ct + c\ln\hat{M})} \int_{\mathbb{R}^N} \pi^{-N/2} e^{-|z|^2} (1 - \eta_n(z))\, dz$$

where

$$\eta_n(z) = e^{-\lambda_c(x\cdot\nu+ct+c\ln\hat{M})+f'(0)(t+n)-\lambda_c^2(t+n)-\lambda_c\sqrt{4(t+n)}z\cdot\nu} \times \varphi_c\left(x\cdot\nu + 2\lambda_c(t+n) + \sqrt{4(t+n)}z\cdot\nu - cn + c\ln\hat{M}\right).$$

Lemma 3.3 implies that $0 \leqq \eta_n(z) \leqq 1$ and $\eta_n(z) \to 1$ for all $z \in \mathbb{R}^N$ as $n \to +\infty$. Therefore, Lebesgue's dominated convergence theorem implies that $w_n(\nu, c) \to 0$ as $n \to +\infty$.

Similarly, we can prove that $w_n(\infty) \to 0$ as $n \to +\infty$. Eventually, another application of Lebesgue's dominated convergence theorem in (41) leads to $v_n(x,t) \to 0$ as $n \to +\infty$.

As a conclusion, $U(x,t) - u_n(x,t) \to 0$ as $n \to +\infty$, whence $U(x,t) = u_\mu(x,t)$. Since the couple $(x,t) \in \mathbb{R}^{N+1}$ is arbitrary, we conclude that $U = u_\mu$. Lastly, since the limit u_μ is uniquely determined by the sequence (μ^n) and does not depend on its subsequences, it follows that the whole sequence (u_{μ^n}) converges to u_μ in the sense of the topology $\mathcal{T}$ as $n \to +\infty$.

3.8. Case where $\tilde{\mu}$ is absolutely continuous with respect to $d\nu \times dc$

This section is devoted to the proof of the non-convergence property (14) in the case of a measure $\mu \in \mathcal{M}$ such that $\mu^* = 0$ and $\tilde{\mu}$ is absolutely continuous with respect to the Lebesgue measure $d\mu \times dc$.

The formula (14) is actually a consequence of more general results that we state below. Consider a measure $\mu \in \mathcal{M}$ such that $\mu^* = 0$ and

$$\mu(\{(\nu, c) \in S^{N-1} \times (c^*, +\infty), c_0\nu_0 \cdot \nu = c\}) = 0$$

for some $c_0 \geqq c^*$ and $\nu_0 \in S^{N-1}$. Note that the set $E = \{(\nu, c) \in S^{N-1} \times (c^*, +\infty),\ c_0\nu_0 \cdot \nu = c\}$ can also be written as $E = S(c_0\nu_0/2, c_0/2) \setminus \overline{B(0, c^*)}$ where $S(c_0\nu_0/2, c_0/2)$ is the sphere centered at the point $c_0\nu_0/2$ with radius $c_0/2$. Then we claim that

$$\forall h \in \mathbb{R}, \quad u_\mu(-c_0t\ \nu_0 + x, t) \not\to \varphi_{c_0}(x \cdot \nu_0 + h) \text{ as } t \to \pm\infty. \tag{42}$$

Postponing the proof, we see that property (14) immediately follows from (42). Indeed, if a measure $\mu \in \mathcal{M}$ is such that $\mu^* = 0$ and $\tilde{\mu} \ll d\nu \times dc$, then $\mu(E) = 0$ for all (c_0, ν_0).

Let us now turn to the

Proof of (42). Choose a measure $\mu \in \mathcal{M}$ such that $\mu^* = 0$ and such that $\mu(\{(\nu, c) \in S^{N-1} \times (c^*, +\infty), c_0\nu_0 \cdot \nu = c\}) = 0$ for some $c_0 \geqq c^*$ and $\nu_0 \in S^{N-1}$.

Let us first study the limit $t \to -\infty$. Assume that there exists a real number $h_0 \in \mathbb{R}$ such that

$$u_\mu(-c_0t\ \nu_0 + x, t) \to \varphi_{c_0}(x \cdot \nu_0 + h_0) \text{ as } t \to -\infty \tag{43}$$

for each $x \in \mathbb{R}^N$ (this implies that the convergence actually takes place in $C^2_{\text{loc}}(\mathbb{R}^N_x)$).

Let us first consider the case where $\tilde{\mu}(S^{N-1}\times(c^*,+\infty))=0$ (which implies that $\hat{M}=\mu(\infty)>0$, since $\mu^*=0$). From (30), we have

$$u_\mu(-c_0 t\ \nu_0+x,t)\leqq \frac{\mu(\infty)}{\hat{M}}\ e^{f'(0)(t+\ln\hat{M})}.$$

Passing to the limit $t\to-\infty$ leads to $\varphi_{c_0}(x\cdot\nu_0+h_0)\leqq 0$ for all $x\in\mathbb{R}^N$. That is clearly impossible.

We now have to consider the case where $\tilde{\mu}(S^{N-1}\times(c^*,+\infty))>0$ (that implies in particular that $\hat{M}>0$). Let F be the set

$$F=\{(\nu,c)\in S^{N-1}\times(c^*,+\infty),\ c<c_0\nu_0\cdot\nu\}.$$

The set F can also be written as $F=B(c_0\nu_0/2,c_0/2)\setminus\overline{B(0,c^*)}$ where $B(c_0\nu_0/2,c_0/2)$ is the open ball centered at the point $c_0\nu_0/2$ with radius $c_0/2$. Suppose that $\mu(F)>0$. Take now any point $x\in\mathbb{R}^N$. From the lower bound of (30), it follows that

$$u_\mu(-c_0 t\nu_0+x,t)\geqq\int_F\varphi_c\left((c-c_0\nu_0\cdot\nu)t+x\cdot\nu+c\ln\hat{M}\right)\frac{1}{\hat{M}}\,d\tilde{\mu}.$$

For any couple (ν,c) in F, we have $c-c_0\nu_0\cdot\nu<0$, whence $\varphi_c((c-c_0\nu_0\cdot\nu)t+x\cdot\nu+c\ln\hat{M})\to 1$ as $t\to-\infty$. Hence, from Lebesgue's dominated convergence theorem, the right-hand side of the previous inequality goes to

$$\beta:=\int_F\frac{1}{\hat{M}}\,d\tilde{\mu}=\frac{\mu(F)}{\hat{M}}>0$$

as $t\to-\infty$. Therefore, $\varphi_{c_0}(x\cdot\nu_0+h_0)\geqq\beta>0$ for each $x\in\mathbb{R}^N$, where β is independent of x. This is impossible. We deduce then that

$$\mu(F)=0.$$

From the upper bound of (30), and since $\mu(E)=\mu(F)=0$, it follows that

$$u_\mu(-c_0 t\ \nu_0+x,t)\leqq w(x,t)+z(t), \tag{44}$$

where

$$w(x,t)=\int_G e^{\lambda_c(c-c_0\nu_0\cdot\nu)t+\lambda_c x\cdot\nu+\lambda_c c\ln\hat{M}}\,\frac{1}{\hat{M}}\,d\tilde{\mu}$$
$$z(t)=\frac{\mu(\infty)}{\hat{M}}e^{f'(0)(t+\ln\hat{M})}$$

and

$$G=\{(\nu,c)\in S^{N-1}\times(c^*,+\infty),\ c>c_0\nu_0\cdot\nu\}.$$

Choose any $x\in\mathbb{R}^N$. For each $(\nu,c)\in G$, we have $c-c_0\nu_0\cdot\nu>0$. Furthermore, $0\leqq\lambda_c\leqq c^*/2$ and $0\leqq\lambda_c c=\lambda_c^2+f'(0)\leqq 2f'(0)$. Hence, for $t\leqq 0$,

$$e^{\lambda_c(c-c_0\nu_0\cdot\nu)t+\lambda_c x\cdot\nu+\lambda_c c\ln\hat{M}}\leqq e^{c^*|x|/2+2f'(0)|\ln\hat{M}|}$$

and

$$e^{\lambda_c(c-c_0\nu_0\cdot\nu)t+\lambda_c x\cdot\nu+\lambda_c c\ln\hat{M}} \to 0 \quad \text{as } t\to-\infty.$$

We conclude from Lebesgue's dominated convergence theorem that $w(x,t)\to 0$ as $t\to-\infty$ for each $x\in\mathbb{R}^N$. The passage to the limit $t\to-\infty$ in (44) leads to $\varphi_{c_0}(x\cdot\nu_0+h_0)\leqq 0$ for all $x\in\mathbb{R}^N$.

Eventually, the assumption (43) is impossible and therefore we have the formula (42) when $t\to-\infty$.

Let us now turn to the proof of (42) for the limit $t\to+\infty$. We just outline it because it is very similar to the previous case $t\to-\infty$. Assume then that

$$u_\mu(-c_0t\nu_0+x,t)\to\varphi_{c_0}(x\cdot\nu_0+h_0) \ \text{ as } t\to+\infty$$

for some $h_0\in\mathbb{R}$. From (30),

$$\begin{aligned}
&\max\Bigg(\int_{S^{N-1}\times(c^*,+\infty)}\varphi_c(x\cdot\nu+(c-c_0\nu_0\cdot\nu)t+c\ln\hat{M})\\
&\qquad\times\frac{1}{\hat{M}}\,d\tilde{\mu},\ \xi(t+\ln\hat{M})\frac{\mu(\infty)}{\hat{M}}\Bigg)\\
&\leqq u_\mu(-c_0\nu_0t+x,t)\\
&\leqq \int_{S^{N-1}\times(c^*,+\infty)} e^{\lambda_c(x\cdot\nu+(c-c_0\nu_0\cdot\nu)t+c\ln\hat{M})}\ \frac{1}{\hat{M}}\,d\tilde{\mu}\\
&\qquad+\ \frac{\mu(\infty)}{\hat{M}}\ e^{f'(0)(t+\ln\hat{M})}.
\end{aligned}$$

Assume first that $\mu(\infty)>0$. Then, $\hat{M}>0$, and, passing to the limit $t\to+\infty$ yields $\varphi_{c_0}(x\cdot\nu_0+h_0)\geqq\frac{\mu(\infty)}{\hat{M}}$ (>0) for all $x\in\mathbb{R}^N$. That is impossible. Hence, $\mu(\infty)=0$.

Second, as was shown above, $\mu(G)=0$, otherwise $\varphi_{c_0}(x\cdot\nu_0+h_0)\geqq\beta':=\frac{\mu(G)}{\hat{M}}>0$ for all $x\in\mathbb{R}^N$.

Third, it follows then that

$$u_\mu(x-c_0\nu_0t,t)\leqq\int_F e^{\lambda_c(x\cdot\nu+(c-c_0\nu_0\cdot\nu)t+c\ln\hat{M})}\frac{1}{\hat{M}}\,d\tilde{\mu}.$$

The limit $t\to+\infty$ yields $\varphi_{c_0}(x\cdot\nu_0+h_0)\leqq 0$ for all $x\in\mathbb{R}^N$, which is clearly impossible. Hence, the claim (42) also holds when $t\to+\infty$. □

Let us now prove an additional property that also shows that when $\tilde{\mu}$ is absolutely continuous with respect to the Lebesgue measure $d\nu\times dc$, then u_μ does not behave, along the rays zt with $|z|<c^*$ and $t\to-\infty$, in the same way as a solution obtained from the mixing of a finite number of travelling waves. More precisely, if $\mu\in\mathcal{M}$ is such that $\tilde{\mu}$ is absolutely continuous with respect to $d\nu\times dc$, then

$$\forall z\in\mathbb{R}^N,\ 0<|z|<c^*,\quad u_\mu(-zt,t)=o(e^{\frac{1}{4}(c^{*2}-|z|^2)t}) \ \text{ as } t\to-\infty. \qquad (45)$$

Note that, from (10), for each function u in Theorem 1.1, there exists $z \in B(0, c^*)\backslash\{0\}$ such that $u(-zt, t) \neq o(e^{\frac{1}{4}((c^*)^2-|z|^2)t})$ as $t \to -\infty$.

Let $\mu \in \mathcal{M}$ be such that $\mu^* = 0$ and $d\tilde{\mu} = g(\nu, c)d\nu \times dc$ form some L^1 function g on $S^{N-1} \times (c^*, +\infty)$. Choose $z \in B(0, c^*)\backslash\{0\}$. From (30), it follows that

$$u_\mu(-zt, t)e^{-\frac{1}{4}((c^*)^2-|z|^2)t} \leqq v(t) + w(t) + z(t),$$

where

$$v(t) = \sum_{1\leqq i\leqq k} \varphi_{c^*}((c^* - z\cdot\nu_i)t + c^*\ln m_i),$$

$$w(t) = \int_{S^{N-1}\times(c^*,+\infty)} e^{[\lambda_c(-z\cdot\nu+c)-\frac{1}{4}(c^*)^2+\frac{1}{4}|z|^2]t+\lambda_c c\ln\hat{M}}\,\frac{1}{\hat{M}}\,g(\nu, c)\,d\nu\times dc,$$

$$z(t) = \frac{\mu(\infty)}{\hat{M}}\,e^{\frac{1}{4}|z|^2t+f'(0)\ln\hat{M}}.$$

As was shown in the proof of (10), $v(t) \to 0$ as $t \to -\infty$, since $|z| < c^*$. On the other hand, the term $z(t)$ clearly goes to 0 as $t \to -\infty$. Finally, let us observe that, because of (3),

$$\begin{aligned}\lambda_c(-z\cdot\nu + c) - \tfrac{1}{4}(c^*)^2 + \tfrac{1}{4}|z|^2 &= \lambda_c^2 - \lambda_c z\cdot\nu + \tfrac{1}{4}|z|^2\\ &= \tfrac{1}{4}|2\lambda_c\nu - z|^2 \geqq 0.\end{aligned}$$

Furthermore, the Lebesgue measure of the set $\{(\nu, c) \in S^{N-1}\times(c^*, +\infty),\ 2\lambda_c\nu = z\}$ (which is a single point) is equal to 0. Since the function $\frac{1}{\hat{M}}\,e^{\lambda_c c\ln\hat{M}}$ is uniformly bounded, Lebesgue's dominated convergence theorem implies that $w(t) \to 0$ as $t \to -\infty$. That completes the proof of (45).

3.9. The set $\{u_\mu\}$ contains the solutions obtained from the mixing of a finite number of travelling waves

This section is devoted to proving that the entire solutions of (1) that are obtained from the mixing of a finite number of travelling waves (see Theorem 1.1) are actually of the type u_μ. In other words, the set of the entire solutions of the type u_μ contains the solutions obtained from the mixing of a finite number of travelling waves.

In order to do this, let p be a positive integer $p \geqq 1$ and, for each $i = 1, \cdots, p$, choose $(\nu_i, c_i, h_i) \in S^{N-1}\times[c^*, +\infty]\times\mathbb{R}$. Assume that $c_i \neq c_j$ if $\nu_i = \nu_j$ and assume that there is at most one index i such that $c_i = +\infty$. We want to prove that the entire solution $u(x, t)$ of (1) constructed in Theorem 1.1 is of the type u_μ for some $\mu \in \mathcal{M}$.

As in Section 2, let us consider the case where $k := \#\{i,\ c_i = c^*\} \geqq 1$ and $\#\{i,\ c_i = +\infty\} = 1$ (the other cases being easier). Up to a renumbering, we can assume that

$$c_1 = \cdots = c_k = c^* \leqq c_{k+1} \leqq \cdots \leqq c_{p-1} < +\infty = c_p.$$

The function $u(x,t)$ is the limit of the solutions $U_n(x,t)$ of the Cauchy problems

$$(U_n)_t = \Delta U_n + f(U_n), \quad x \in \mathbb{R}^N,\ t > -n,$$

where $U_n(x,-n)$ is a maximum of travelling waves (with finite or infinite speeds):

$$U_n(x,-n) = \max\left(\max_{1\leqq i\leqq p-1}(\varphi_{c_i}(x\cdot\nu_i - c_i n + h_i)), \xi(-n+h_p)\right).$$

Notice that $0 \leqq U_n(x,-n) \leqq 1$.

Let us now consider the following measure $\mu \in \mathcal{M}$, which is the sum of a finite number of Dirac distributions:

$$\mu = \sum_{i=1}^{k} e^{h_i/c^*}\delta_{(\nu_i,c^*)} + \sum_{i=k+1}^{p-1}\alpha_i\delta_{(\nu_i,c_i)} + \alpha_p\delta_\infty,$$

where the α_i are defined as follows: first, elementary arguments give the existence of a unique positive real number $\hat{M}$ such that

$$\sum_{i=k+1}^{p-1} e^{\lambda_{c_i}(h_i - c_i\ln\hat{M})} + e^{f'(0)(h_p-\ln\hat{M})} = 1;$$

if we then set $\alpha_i = \hat{M}e^{\lambda_{c_i}(h_i-c_i\ln\hat{M})} > 0$ for each $i = k+1,\cdots,p-1$ and $\alpha_p = \hat{M}e^{f'(0)(h_p-\ln\hat{M})}$, we have, by definition, $\Sigma_{i=k+1}^{p}\alpha_i = \hat{M}$.

The function $u_\mu(x,t)$ is the limit of the solutions $u_n(x,t)$ of the Cauchy problems

$$(u_n)_t = \Delta u_n + f(u_n), \quad x \in \mathbb{R}^N,\ t > -n$$

where, owing to the definition given in (26), $u_n(x,-n)$ is the maximum of some travelling waves with the minimal speed c^* and of an average of travelling waves with speeds greater than c^*:

$$\begin{aligned} u_n(x,-n) = \max\Bigg(&\max_{1\leqq i\leqq k}(\varphi_{c^*}(x\cdot\nu_i - c^*n + h_i)),\\ &\sum_{i=k+1}^{p-1}\varphi_{c_i}(x\cdot\nu_i - c_i n + c_i\ln\hat{M})e^{\lambda_{c_i}(h_i-c_i\ln\hat{M})}\\ &+\xi(-n+\ln\hat{M})e^{f'(0)(h_p-\ln\hat{M})}\Bigg).\end{aligned}$$

We have $0 \leqq u_n(x,-n) \leqq 1$ for all $x \in \mathbb{R}^N$.

The key point consists in proving that, by considering these above two sequences of Cauchy problems with different initial data, we actually get the same function at the limit. This is done in the following

Lemma 3.7. *For all* $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, $u(x,t) = u_\mu(x,t)$.

Postponing the proof of this lemma, we see that the manifold of the solutions of (1) of the type u_μ contains all the solutions u constructed in Theorem 1.1. From Theorem 1.1, it follows that the manifold $\{u_\mu\}$ then contains the finite-dimensional manifold of the planar travelling waves, the manifold $\{t \mapsto \xi(t+h),\ h \in \mathbb{R}\}$ and the finite-dimensional manifolds of the planar solutions that have been constructed in [16].

Before doing the proof of Lemma 3.7, we state an auxiliary result. In what follows, we call $(S(t))_{t>0}$ the semi-group generated by the Laplace operator in $\mathbb{R}^N$. In particular, for each bounded measurable function g on $\mathbb{R}^N$ and for each $t > 0$ and $x \in \mathbb{R}^N$,

$$(S(t)\cdot g)(x) = \frac{1}{\sqrt{4\pi t}^N}\int_{\mathbb{R}^N} g(y)\, e^{-\frac{|y-x|^2}{4t}}\, dy.$$

Lemma 3.8. (a) *For each $\gamma > c^*$ and $(x,t) \in \mathbb{R}^N \times \mathbb{R}$,*

$$z_n(x,t) := e^{f'(0)n}\left(S(t+n)\cdot \mathbf{1}_{|\cdot|\geqq \gamma n}\right)(x) \ \to 0$$

as $n \to +\infty$ (with $t+n > 0$), where $\mathbf{1}_{|\cdot|\geqq\gamma n}(y) = 1$ if $|y| \geqq \gamma n$ and 0 otherwise.

(b) *For each $\gamma > c^*$, $\tau < 0$ and $x \in \mathbb{R}^N$, the integral*

$$h_\gamma(x,\tau) := \int_{-\infty}^{\tau} e^{f'(0)(\tau-s)}(S(\tau-s)\cdot \mathbf{1}_{|\cdot|\geqq\gamma|s|})(x)\, ds$$

converges.

Proof. (a) For $n > |t|$,

$$0 \leqq z_n(x,t) \leqq \frac{e^{f'(0)n}}{\sqrt{4\pi(t+n)}^N}\int_{|y|\geqq\gamma n} e^{-\frac{|y-x|^2}{4(t+n)}}\, dy.$$

The change of variables $y = \gamma n z + x$ leads to

$$0 \leqq z_n(x,t) \leqq \frac{\gamma^N n^N}{\sqrt{4\pi(t+n)}^N}\int_{|z+\frac{x}{\gamma n}|\geqq 1} e^{(f'(0)-\frac{\gamma^2 n|z|^2}{4(t+n)})n} dz.$$

Since $\gamma > c^* = 2\sqrt{f'(0)}$, there exists $\eta > 0$ and $n_0 \in \mathbb{N}$ such that, if $n \geqq n_0$ and $|z + \frac{x}{\gamma n}| \geqq 1$, then

$$f'(0) - \frac{\gamma^2 n|z|^2}{4(t+n)} \leqq -\eta - \eta|z|^2.$$

Therefore, for $n \geqq n_0$, it follows that

$$\begin{aligned} 0 \leqq z_n(x,t) &\leqq \frac{\gamma^N n^N}{\sqrt{4\pi}^N\ (t+n)^{N/2}} e^{-\eta n}\int_{|z+\frac{x}{\gamma n}|\geqq 1} e^{-\eta n|z|^2} dz \\ &\leqq \frac{\gamma^N n^N}{\sqrt{4\pi}^N\ (\eta n(t+n))^{N/2}} e^{-\eta n}\int_{\mathbb{R}^N} e^{-|y|^2} dy \end{aligned}$$

after the change of variables $\sqrt{\eta n} z = y$. Therefore, $z_n(x,t) \to 0$ as $n \to +\infty$.

(b) Take $\tau < 0$ and $x \in \mathbb{R}^N$. Since $0 \leqq (S(\tau - s) \cdot \mathbf{1}_{|\cdot| \geqq \gamma |s|})(x) \leqq 1$ for all $s < \tau$, we have only to prove that the integral I,

$$0 \leqq \mathrm{I} := \int_{-\infty}^{\tau-1} e^{f'(0)(\tau-s)} \frac{1}{\sqrt{4\pi(\tau-s)}^N} \int_{|y| \geqq \gamma|s|} e^{-\frac{|y-x|^2}{4(\tau-s)}}\, dy\, ds,$$

converges. With the changes of variables $y = |s|z$ (possible because $s \leqq \tau < 0$) and $t = \tau - s$, it is found that

$$\begin{aligned}
\mathrm{I} &= \int_1^{\infty} (4\pi t)^{-N/2} (t-\tau)^N \int_{|z| \geqq \gamma} e^{f'(0)t - \frac{|(t-\tau)z - x|^2}{4t}}\, dz\, dt \\
&= \int_1^{\infty} (4\pi t)^{-N/2} (t-\tau)^N \int_{|z| \geqq \gamma} e^{(f'(0) - \frac14 |z|^2)t + \frac12 z\cdot x + \frac12 \tau |z|^2 - \frac{|x+\tau z|^2}{4t}}\, dz\, dt.
\end{aligned}$$

In the above integral, $e^{-\frac{|x+\tau z|^2}{4t}} \leqq 1$. Furthermore, since $c^* = 2\sqrt{f'(0)}$ and $\gamma > c^*$, there exists $\delta > 0$ such that $f'(0) - \frac14 |z|^2 \leqq -\delta$ as soon as $|z| \geqq \gamma$. Hence,

$$0 \leqq \mathrm{I} \leqq \left(\int_1^{\infty} (4\pi t)^{-N/2} (t-\tau)^N e^{-\delta t} dt \right) \times \left(\int_{\mathbb{R}^N} e^{\frac12 z\cdot x + \frac12 \tau |z|^2} dz \right).$$

The integral in t converges because $\delta > 0$. So does the integral in z, because $\tau < 0$. That completes the proof of Lemma 3.8(b). □

Note that since $0 \leqq (S(\tau - s) \cdot \mathbf{1}_{|\cdot| \geqq \gamma |s|})(x) \leqq 1$ for all $\tau \in \mathbb{R}$, $s < \tau$ and $x \in \mathbb{R}^N$, it follows that the integral $h_\gamma(x, \tau)$ converges for all $(x, \tau) \in \mathbb{R}^N \times \mathbb{R}$. Let us now turn to the

Proof of Lemma 3.7. Remember the definitions of the sequences of the functions u_n and U_n at the beginning of this subsection. Since $0 \leqq u_n \leqq U_n \leqq 1$ and $f'(s) \leqq f'(0)$ (> 0) on $[0, 1]$, it follows that, for each n, the function $w_n = |u_n - U_n|$ satisfies

$$(w_n)_t \leqq \Delta w_n + f'(0) w_n, \quad t > -n,\ x \in \mathbb{R}^N.$$

Therefore,

$$0 \leqq w_n(x, t) \leqq e^{f'(0)(t+n)} (S(t+n) \cdot w_n(\cdot, -n))(x).$$

Choose a couple $(x, t) \in \mathbb{R}^N \times \mathbb{R}$. Let ε be an arbitrary positive real number and let γ be such that $c^* < \gamma < c_{k+1}$ ($\leqq c_i$ for all $i \geqq k+1$). From Lemma 3.8(a) and since $|w_n| \leqq 2$, we have

$$e^{f'(0)(t+n)} (S(t+n) \cdot (w_n(\cdot, -n) \mathbf{1}_{|\cdot| \geqq \gamma n}))(x) \to 0 \text{ as } n \to +\infty.$$

Therefore,

$$\limsup_{n\to+\infty} w_n(x, t) \leqq \limsup_{n\to+\infty} e^{f'(0)(t+n)} (S(t+n) \cdot (w_n(\cdot, -n) \mathbf{1}_{|\cdot| < \gamma n}))(x).$$

Let us now find an upper bound for the function $w_n(y,-n)$ for $|y|<\gamma n$. Owing to the definitions of $u_n(\cdot,-n)$ and $U_n(\cdot,-n)$, we have (for all $y\in\mathbb{R}^N$),

$$\begin{aligned}0\leqq w(y,-n)\leqq\ &\Bigg|\max\left(\max_{k+1\leqq i\leqq p-1}(\varphi_{c_i}(y\cdot\nu_i-c_in+h_i)),\ \xi(-n+h_p)\right)\\&-\Bigg(\sum_{i=k+1}^{p-1}\varphi_{c_i}(y\cdot\nu_i-c_in+c_i\ln\hat M)e^{\lambda_{c_i}(h_i-c_i\ln\hat M)}\\&+\xi(-n+\ln\hat M)e^{f'(0)(h_p-\ln\hat M)}\Bigg)\Bigg|.\end{aligned}$$

Since $\gamma<c_i$ for each $i\geqq k+1$, $y\cdot\nu_i-c_in\to-\infty$ as $n\to+\infty$, uniformly for $|y|<\gamma n$. From (2) and (4), we have, for n large enough and for all $|y|<\gamma n$,

$$\begin{aligned}\left|\varphi_{c_i}(y\cdot\nu_i-c_in+h_i)-e^{\lambda_{c_i}(y\cdot\nu_i-c_in+h_i)}\right|&\leqq\varepsilon\, e^{\lambda_{c_i}(y\cdot\nu_i-c_in)},\\ \left|\varphi_{c_i}(y\cdot\nu_i-c_in+c_i\ln\hat M)e^{\lambda_{c_i}(h_i-c_i\ln\hat M)}-e^{\lambda_{c_i}(y\cdot\nu_i-c_in+h_i)}\right|&\leqq\varepsilon\, e^{\lambda_{c_i}(y\cdot\nu_i-c_in)},\\ \left|\xi(-n+h_p)-e^{-f'(0)n+f'(0)h_p}\right|&\leqq\varepsilon\, e^{-f'(0)n},\\ \left|\xi(-n+\ln\hat M)e^{f'(0)(h_p-\ln\hat M)}-e^{-f'(0)n+f'(0)h_p}\right|&\leqq\varepsilon\, e^{-f'(0)n},\end{aligned}$$

where the first two inequalities hold for $i=k+1,\cdots,p-1$. In what follows, we set $\lambda_{c_p}(y\cdot\nu_p+c_pt+h_p):=f'(0)t+f'(0)h_p$ and $\lambda_{c_p}(y\cdot\nu_p+c_pt):=f'(0)t$ for all $t\in\mathbb{R}$. Thus,

$$\begin{aligned}0\leqq\ &\limsup_{n\to+\infty}w_n(x,t)\\ \leqq\ &\limsup_{n\to+\infty}e^{f'(0)(t+n)}\int_{|y|<\gamma n}\frac{1}{\sqrt{4\pi(t+n)}^N}|\mathrm{I}_n(y)|\,e^{-\frac{|y-x|^2}{4(t+n)}}\,dy\\&+2\varepsilon\limsup_{n\to+\infty}\sum_{i=k+1}^{p}z_n^i(x,t),\end{aligned}\tag{46}$$

where

$$\forall\,|y|<\gamma n,\quad \mathrm{I}_n(y)=\max_{k+1\leqq i\leqq p}(e^{\lambda_{c_i}(y\cdot\nu_i-c_in+h_i)})-\sum_{i=k+1}^{p}e^{\lambda_{c_i}(y\cdot\nu_i-c_in+h_i)}$$

and

$$z_n^i(x,t)=e^{f'(0)(t+n)}\int_{|y|<\gamma n}\frac{1}{\sqrt{4\pi(t+n)}^N}e^{\lambda_{c_i}(y\cdot\nu_i-c_in)-\frac{|y-x|^2}{4(t+n)}}\,dy.$$

Let us first estimate the terms $z_n^i(x,t)$. We have

$$z_n^i(x,t)\leqq\phi_n^i(x,t)=\frac{e^{f'(0)(t+n)}}{\sqrt{4\pi(t+n)}^N}\int_{\mathbb{R}^N}e^{\lambda_{c_i}(y\cdot\nu_i-c_in)}e^{-\frac{|y-x|^2}{4(t+n)}}\,dy.$$

As already observed in Section 3.7, and since $\lambda_{c_i}^2 - c_i\lambda_{c_i} + f'(0) = 0$, the right-hand side of the above inequality is equal to

$$\phi_n^i(x,t) = e^{\lambda_{c_i}(x\cdot\nu_i + c_i t)}$$

(in both cases $k+1 \leqq i \leqq p-1$, i.e., $c_i < \infty$, and $i = p$, i.e., $c_i = +\infty$).

Let us find an upper bound for $\mathrm{I}_n(y)$ for all $|y| \leqq \gamma n$. For each $i = k+1, \cdots, p$, let Ω_n^i be the set

$$\Omega_n^i = \left\{|y| < \gamma n,\, e^{\lambda_{c_i}(y\cdot\nu_i - c_i n + h_i)} = \max_{k+1\leqq j\leqq p} e^{\lambda_{c_j}(y\cdot\nu_j - c_j n + h_j)}\right\},$$

and, for each $i = k+1, \cdots, p$ and $j \neq i$, let us define

$$\begin{aligned} A_n^{ij} &= \{y \in \Omega_n^i,\ \ln\varepsilon \leqq \lambda_{c_j}(y\cdot\nu_j - c_j n + h_j) - \lambda_{c_i}(y\cdot\nu_i - c_i n + h_i) \leqq 0\},\\ B_n^{ij} &= \{y \in \Omega_n^i, \lambda_{c_j}(y\cdot\nu_j - c_j n + h_j) - \lambda_{c_i}(y\cdot\nu_i - c_i n + h_i) < \ln\varepsilon\} \end{aligned}$$

(with ε small enough so that $\ln\varepsilon < 0$). Due to the definition of the sets Ω_n^i, we have

$$\{|y| < \gamma n\} = \bigcup_{k+1\leqq i\leqq p}\ \bigcap_{j\neq i} (A_n^{ij} \cup B_n^{ij}).$$

As a consequence, $\forall\, |y| < \gamma n$,

$$|\mathrm{I}_n(y)| \leqq \sum_{i=k+1}^{p}\sum_{j\neq i}\left(\mathbf{1}_{\{y\in A_n^{ij}\}}\, e^{\lambda_{c_j}(y\cdot\nu_j - c_j n + h_j)} + \mathbf{1}_{\{y\in B_n^{ij}\}}\, e^{\lambda_{c_i}(y\cdot\nu_i - c_i n + h_i) + \ln\varepsilon}\right)$$

and

$$\begin{aligned} &\limsup_{n\to+\infty} \int_{|y|<\gamma n} \frac{e^{f'(0)(t+n)}}{\sqrt{4\pi(t+n)}^N}\, |\mathrm{I}_n(y)| e^{-\frac{|y-x|^2}{4(t+n)}}\, dy \\ &\qquad\qquad \leqq \limsup_{n\to+\infty} \sum_{i=k+1}^{p}\sum_{j\neq i} (a_n^{ij}(x,t) + b_n^{ij}(x,t)) \end{aligned}$$

where

$$\begin{aligned} 0 \leqq a_n^{ij}(x,t) &= \int_{y\in A_n^{ij}} \frac{e^{f'(0)(t+n)}}{\sqrt{4\pi(t+n)}^N} e^{\lambda_j(y\cdot\nu_j - c_j n + h_j) - \frac{|y-x|^2}{4(t+n)}}\, dy,\\ 0 \leqq b_n^{ij}(x,t) &= \int_{y\in B_n^{ij}} \frac{e^{f'(0)(t+n)}}{\sqrt{4\pi(t+n)}^N} \varepsilon e^{\lambda_{c_i}(y\cdot\nu_i - c_i n + h_i) - \frac{|y-x|^2}{4(t+n)}}\, dy. \end{aligned}$$

The change of variables $y = x + 2(t+n)\lambda_{c_j}\nu_j + \sqrt{4(t+n)}\,\zeta$ in $a_n^{ij}(x,t)$ leads, after a straightforward calculation, to

$$a_n^{ij}(x,t) = e^{\lambda_{c_j}(\nu_j\cdot x + c_j t + h_j)} \int_{\{(x+2(t+n)\lambda_{c_j}\nu_j + \sqrt{4(t+n)}\ \zeta)\ \in A_n^{ij}\}} \pi^{-N/2} e^{-|\zeta|^2}\, d\zeta.$$

But it is found that

$$\Big[(x+2(t+n)\lambda_{c_j}\nu_j+\sqrt{4(t+n)}\,\zeta)\ \in A_n^{ij}\Big]$$
$$\Rightarrow\Big[\alpha_n+\frac{\ln\varepsilon}{\sqrt{4(t+n)}}\leqq(\lambda_{c_j}\nu_j-\lambda_{c_i}\nu_i)\cdot\zeta\leqq\alpha_n\Big]$$

where $\alpha_n=(\lambda_{c_j}c_j-\lambda_{c_i}c_i)n+\lambda_{c_i}h_i-\lambda_{c_j}h_j-(\lambda_{c_j}\nu_j-\lambda_{c_i}\nu_i)\cdot(x+2(t+n)\lambda_{c_j}\nu_j)$. By assumption, $(c_i,\nu_i)\neq(c_j,\nu_j)$ as soon as $i\neq j$. Therefore, for each $i\neq j$, the vector $\lambda_{c_j}\nu_j-\lambda_{c_i}\nu_i$ is not zero. Set

$$e_1=\frac{\lambda_{c_j}\nu_j-\lambda_{c_i}\nu_i}{|\lambda_{c_j}\nu_j-\lambda_{c_i}\nu_i|}$$

and complete e_1 into an orthonormal basis $(e_1,e_2,\cdots,e_N)$. By making the change of variables $z_l=e_l\cdot\zeta$ $(l=1,\cdots,N)$, we get

$$\begin{aligned}0&\leqq a_n^{ij}(x,t)\\&\leqq e^{\lambda_{c_j}(\nu_j\cdot x+c_jt+h_j)}\pi^{-N/2}\int_{\{\alpha_n+\frac{\ln\varepsilon}{\sqrt{4(t+n)}}\leqq|\lambda_{c_j}\nu_j-\lambda_{c_i}\nu_i|z_1\leqq\alpha_n\}}e^{-z_1^2}dz_1\\&\quad\times\int_{\mathbb{R}^{N-1}}e^{-(z_2^2+\cdots+z_N^2)}dz_2\cdots dz_N.\end{aligned}$$

Eventually, $a_n^{ij}(x,t)\to0$ as $n\to+\infty$.

On the other hand,

$$\begin{aligned}0\leqq b_n^{ij}(x,t)&\leqq\varepsilon\int_{\mathbb{R}^N}\frac{e^{f'(0)(t+n)}}{\sqrt{4\pi(t+n)}^N}e^{\lambda_{c_i}(y\cdot\nu_i-c_in+h_i)-\frac{|y-x|^2}{4(t+n)}}\,dy\\&=\varepsilon\,e^{\lambda_{c_i}(x\cdot\nu_i+c_it+h_i)},\end{aligned}$$

as already observed in Section 3.7.

Putting together all the previous estimates leads to

$$0\leqq\limsup_{n\to+\infty}w_n(x,t)\leqq\varepsilon\sum_{i=k+1}^{p}(p-k+1)e^{\lambda_{c_i}(x\cdot\nu_i+c_it+h_i)}.$$

Since $\varepsilon>0$ was arbitrary, it follows that $w_n(x,t)\to0$ as $n\to+\infty$. In other words, $u(x,t)=u_\mu(x,t)$ and the proof of Lemma 3.7 is done. □

For each $(\nu,c,h)\in S^{N-1}\times[c^*,+\infty]\times\mathbb{R}$, let us set

$$\begin{aligned}\phi_{(\nu,c,h)}&=\varphi_c(x\cdot\nu+ct+h)&&\text{if } c<+\infty,\\\phi_{(\nu,c,h)}&=\xi(t+h)&&\text{if } c=+\infty,\end{aligned}$$

and let us call $\mathcal{TW}$ the set of such functions $\phi_{(\nu,c,h)}$, namely, the set of all planar travelling waves for (1), with finite speed $(c<+\infty)$ or infinite speed $(c=+\infty)$.

We can define a law from $\mathcal{TW}$ to the set $\mathcal{E}$ of all entire solutions of (1) as follows:

Definition 3.9. For any integer $p \geqq 1$ and any p-uple $(\nu_i, c_i, h_i) \in (S^{N-1} \times [c^*, +\infty] \times \mathbb{R})^p$, we denote by $\oplus_{i=1}^{p} \phi_{(\nu_i, c_i, h_i)}(x, t)$ the function defined by

$$\bigoplus_{i=1}^{p} \phi_{(\nu_i, c_i, h_i)}(x, t) := \lim_{n \to +\infty} U_n(x, t),$$

where U_n is the solution of the Cauchy problem

$$\begin{aligned} (U_n)_t &= \Delta U_n + f(U_n), \quad t > -n, \; x \in \mathbb{R}^N \\ U_n(x, -n) &= \max_{1 \leqq i \leqq p} \phi_{(\nu_i, c_i, h_i)}(x, -n). \end{aligned}$$

As it was shown in Section 2, the function $\oplus_{i=1}^{p} \phi_{(\nu_i, c_i, h_i)}(x, t)$ is well defined and it belongs to $\mathcal{E}$.

The law $\oplus$ is commutative and associative. Furthermore, each function $\oplus_{i=1}^{p} \phi_{(\nu_i, c_i, h_i)}(x, t)$ is a solution of (1) of the type described in Theorem 1.1. Indeed, given a p-uple (ν_i, c_i, h_i), there exists a subset $I \subset \{1, \cdots, p\}$ such that $(\nu_i, c_i) \neq (\nu_j, c_j)$ for $i \neq j$, $i, j \in I$, and such that, for all $k \in \{1, \cdots, p\}$, there exists $i \in I$ such that $(\nu_k, c_k) = (\nu_i, c_i)$ and $h_k \leqq h_i$. Then, we immediately have

$$U_n(x, -n) = \max_{i \in I} \phi_{(\nu_i, c_i, h_i)}(x, -n).$$

Therefore, by definition, the function $\oplus_{i=1}^{p} \phi_{(\nu_i, c_i, h_i)}(x, t)$ is an entire solution of the type described in Theorem 1.1.

Conversely, each solution u constructed as in Theorem 1.1 is of the type $\oplus_{i=1}^{m} \phi_{(\nu_i, c_i, h_i)}(x, t)$ for some m-uple $(\nu_i, c_i, h_i)_{1 \leqq i \leqq m}$.

Finally, we formulate the following

Conjecture 3.10. *The set $\mathcal{E}$ of all entire solutions u of (1), such that $0 \leqq u \leqq 1$, is the closure, in the sense of the topology $\mathcal{T}$ of all the solutions of the type $\oplus_{i=1}^{p} \phi_{(\nu_i, c_i, h_i)}(x, t)$, when p varies in $\mathbb{N}^*$ and $(\nu_i, c_i, h_i) \in S^{N-1} \times [c^*, +\infty] \times \mathbb{R}$.*

4. Partial uniqueness results

Our goal in this section is to prove Theorem 1.4 and 1.5. First of all, we need a preliminary lemma, whose result has already been mentionned in Section 1.

4.1. A preliminary lemma

Lemma 4.1. *For any solution $u(x, t)$ of (1), we have:* $\forall\, 0 \leqq c < c^*$,

$$\lim_{t \to -\infty} \max_{|x| \leqq c|t|} u(x, t) = 0$$

$$\lim_{t \to +\infty} \min_{|x| \leqq ct} u(x, t) = 1$$

Proof. Let $u(x,t)$ be a solution of (1). Since u is positive, there exists a function $\rho(x)$ which is positive in the open ball of radius 1 and center $0 \in \mathbb{R}^N$, which vanishes outside this open ball and which is such that $\rho(x) \leqq u(x,0)$ in $\mathbb{R}^N$.

Let $v(x,t)$ be the solution of the Cauchy problem

$$\begin{aligned} v_t &= \Delta v + f(v), \quad x \in \mathbb{R}^N,\ t > 0, \\ v(x,0) &= \rho(x). \end{aligned} \tag{47}$$

The maximum principle implies then that $v(x,t) \leqq u(x,t)$ for all $x \in \mathbb{R}^N$ and $t \geqq 0$. Since $\liminf_{u\to 0} u^{-(1+2/N)} f(u) > 0$, the results of ARONSON & WEINBERGER (see [2]) imply that, for all $0 \leqq c < c^*$, we have $\lim_{t\to+\infty} \min_{|x|\leqq ct} v(x,t) = 1$. The same assertion then holds well for u.

Fix now a speed $c \in [0, c^*[$ and assume that $\limsup_{t\to-\infty} \max_{|x|\leqq c|t|} u(x,t) > 0$. There exist then a real $\varepsilon > 0$ and two sequences $x_n \in \mathbb{R}^N$ and $t_n \to -\infty$ such that $|x_n| \leqq c|t_n|$ and $u(x_n,t_n) \geqq \varepsilon$. By the standard parabolic estimates, $\nabla_x u(x,t)$ is uniformly bounded in $\mathbb{R}^N \times \mathbb{R}$. Hence, there exists a real $r > 0$ such that $u(x,t_n) \geqq \varepsilon/2$ for any x such that $|x - x_n| \leqq r$. Let $\rho(x)$ be a continuous nonnegative function such that $0 < \rho(x) \leqq \varepsilon/2$ if $|x| < r$ and $\rho(x) = 0$ if $|x| \geqq r$. Let v be the solution of the Cauchy problem (47). On the one hand, the maximum principle implies that $v(x,t) \leqq u(x+x_n, t+t_n)$ for all $x \in \mathbb{R}^N$ and $t \geqq 0$. In particular, $v(-x_n,-t_n) \leqq u(0,0) < 1$. On the other hand, since $-t_n \to +\infty$, $|x_n| \leqq c|t_n|$ and $c < c^*$, the above result of Aronson and Weinberger yields that $v(-x_n,-t_n) \to 1$ as $-t_n \to +\infty$. This eventually leads to a contradiction and Lemma 4.1 is proved. □

4.2. Partial uniqueness (*proof of Theorem* 1.4)

This section is devoted to the proof of Theorem 1.4. Before entering into the proof, we first state a few general lemmas.

The following lemma states that an entire solution U of (1) can be approximated by a suitable sequence of solutions of Cauchy problems.

Lemma 4.2. *Let $U(x,t)$ be an entire solution of* (1) *and let $\gamma > c^*$. For each $n \in \mathbb{N}$, let $U_n(x,t)$ be the solution of the Cauchy problem*

$$\begin{aligned} (U_n)_t &= \Delta U_n + f(U_n), \quad x \in \mathbb{R}^N,\ t > -n, \\ U_n(x,-n) &= \begin{cases} U(x,-n) & \text{if } |x| < \gamma n, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Then $U_n(x,t) \overset{\leqq}{\longrightarrow} U(x,t)$ as $n \to +\infty$.

Proof. From the maximum principle, $0 \leqq U_n(x,t) \leqq U(x,t) \leqq 1$ for each $n \in \mathbb{N}$ and for all $x \in \mathbb{R}^N$, $t \geqq -n$.

The nonnegative function $v_n(x,t) = U(x,t) - U_n(x,t)$ satisfies

$$\begin{aligned}\partial_t v_n &= \Delta v_n + f(U) - f(U_n)\\ &\leqq \Delta v_n + f'(0)v_n\end{aligned}$$

because $f'(s) \leqq f'(0)$ for all $s \in [0,1]$.

Choose now any $(x,t) \in \mathbb{R}^N \times \mathbb{R}$. For any $n > |t|$,

$$\begin{aligned}0 \leqq v_n(x,t) &\leqq \frac{e^{f'(0)(t+n)}}{\sqrt{4\pi(t+n)}^N}\int_{R^N} v_n(y,-n)e^{-\frac{|y-x|^2}{4(t+n)}}dy\\ &= \frac{e^{f'(0)(t+n)}}{\sqrt{4\pi(t+n)}^N}\int_{|y|>\gamma n} e^{-\frac{|y-x|^2}{4(t+n)}}\,dy\end{aligned}$$

by definition of $U_n(\cdot,-n)$. In other words,

$$0 \leqq v_n(x,t) \leqq e^{f'(0)(t+n)}\left(S(t+n)\cdot \mathbf{1}_{|\cdot|>\gamma n}\right)(x).$$

From Lemma 3.8(a), it follows that $v_n(x,t) \to 0$ as $n \to +\infty$, that is to say, $U_n(x,t) \to U(x,t)$. □

The following lemma states that if an entire solution of (1) converges to 0 in a cone $\{|x| \leqq c|t|\}$ when $t \to -\infty$, then it has exponential decay in strict subcones.

Lemma 4.3. *Let $U(x,t)$ be an entire solution of (1) and assume that $\max_{|x|\leqq c|t|} U(x,t) \to 0$ as $t \to -\infty$, for some $c > 0$. Then, for each $\gamma \in [0, c[$, there exists $\alpha_0 > 0$ such that*

$$\forall\, \alpha \in [0,\alpha_0], \qquad \max_{|x|\leqq \gamma|t|} U(x,t) = o(e^{\alpha t}) \;\; \textit{as } t \to -\infty.$$

Proof. Let c and γ be as in Lemma 4.3. Take $\alpha > 0$ (to be chosen later) and assume that the conclusion does not hold, namely, that there exists $\delta > 0$ and a sequence $t'_n \to -\infty$ such that $U(x_n, t'_n) \geqq \delta e^{\alpha t'_n}$ for some $|x_n| \leqq \gamma|t'_n|$.

Since U is a positive entire solution of (1), the Harnack inequality yields the existence of a positive constant C_0 such that

$$U(x, t'_n + 1) \geqq C_0 \delta e^{\alpha t'_n} \;\text{ for all } x \text{such that } |x - x_n| \leqq 1.$$

Therefore, even if it means changing δ, we have, by setting $t_n = t'_n + 1$,

$$U(x, t_n) \geqq \delta e^{\alpha t_n} \;\text{ for all } x \text{ such that } |x - x_n| \leqq 1.$$

Let us fix $\eta > 0$ such that $\eta < \min(f'(0), \frac{1}{2}(c-\gamma)^2)$ and $\mu > 0$ such that $f(u) \geqq (f'(0) - \eta)u$ for all $u \in [0,\mu]$. There exists then a real number $T < 0$ such that

$$\forall\, t \leqq T,\; \forall\, |x| \leqq c|t|, \quad 0 \leqq U(x,t) \leqq \mu.$$

Let v be the function defined by

$$v(x,t) = U(x,t)e^{-(f'(0)-\eta)t}.$$

It satisfies $0 \leqq v(x,t) \leqq e^{-(f'(0)-\eta)t}$ and

$$v_t - \Delta v \geqq \begin{cases} 0 & \text{if } t \leqq T,\ |x| \leqq c|t|, \\ -(f'(0)-\eta)v & \text{if } t \leqq T,\ |x| \geqq c|t|. \end{cases}$$

On the other hand, for n large enough such that $t_n < T$,

$$v(x,t_n) \geqq \begin{cases} \delta e^{(\alpha - f'(0)+\eta)t_n} & \text{if } |x-x_n| \leqq 1, \\ 0 & \text{if } |x-x_n| \geqq 1. \end{cases}$$

The maximum principle gives

$$v(x_n, T) \geqq \mathrm{I}_n + \mathrm{II}_n,$$

where

$$\mathrm{I}_n = \frac{\delta e^{(\alpha - f'(0)+\eta)t_n}}{\sqrt{4\pi(T-t_n)}^N} \int_{|y|\leqq 1} e^{-\frac{|y|^2}{4(T-t_n)}}\, dy,$$

$$\mathrm{II}_n = -(f'(0)-\eta) \int_{t_n}^{T} \frac{1}{\sqrt{4\pi(T-s)}^N} \int_{|y|\geqq \gamma|s|} v(y,s) e^{-\frac{|y-x_n|^2}{4(T-s)}}\, dy\, ds.$$

When $n \to +\infty$,

$$\mathrm{I}_n \sim C_1 |t_n|^{-N/2} e^{(\alpha - f'(0)+\eta)t_n},$$

where $C_1 = \delta(4\pi)^{-N/2}|B(0,1)| > 0$ and $|B(0,1)|$ is the Lebesgue measure of the unit ball.

Let us now find an upper bound for $|\mathrm{II}_n|$. Remember first that $0 \leqq v(y,s) \leqq e^{-(f'(0)-\eta)s}$ for all (y,s). Make the change of variables $y = x_n + z|s|$ (possible because $s \leqq \tau < 0$). If $|y| \geqq c|s|$, then $|z| \geqq \max(0, c - \frac{|x_n|}{|s|})$. Therefore,

$$|\mathrm{II}_n| \leqq C_1' \int_{t_n}^{T} \frac{|s|^N}{\sqrt{T-s}^N} \int_{|z|\geqq \max(0, c - \frac{|x_n|}{|s|})} e^{-(f'(0)-\eta)s} e^{-\frac{s^2|z|^2}{4(T-s)}}\, dz\, ds,$$

where $C_1' = (f'(0)-\eta)(4\pi)^{-N/2}$. After a straightforward calculation, the change of variables $t = T - s$ leads to

$$\begin{aligned} |\mathrm{II}_n| \leqq\ & C_1' \int_0^{T-t_n} \frac{(t-T)^N}{t^{N/2}} \\ & \times \int_{|z|\geqq \max(0, c - \frac{|x_n|}{t-T})} e^{\frac{T}{2}|z|^2 - \frac{T^2}{4t}|z|^2 + (f'(0)-\eta-\frac{1}{4}|z|^2)t}\, dz\, dt \\ \leqq\ & C_1'' |t_n|^N\, \mathrm{III}_n, \end{aligned}$$

where $C_1'' = C_1' e^{-(f'(0)-\eta)T}$ and

$$\mathrm{III}_n = \int_0^{T-t_n} t^{-N/2} \int_{|z|\geqq \max(0, c - \frac{|x_n|}{t-T})} e^{\frac{T}{2}|z|^2 - \frac{T^2}{4t}|z|^2 + (f'(0)-\eta-\frac{1}{4}|z|^2)t}\, dz\, dt.$$

Since $\gamma < c$ and $\eta > 0$, it is possible to fix a real number β such that

$$\frac{\gamma}{c} < \beta < 1$$

$$f'(0) - \eta - \frac{1}{4}\left(c - \frac{\gamma}{\beta}\right)^2 < f'(0) - \frac{\eta}{2} - \frac{1}{4}(c-\gamma)^2.$$

From now on, n is taken large enough so that $1 < T - \beta t_n$. Let us divide III_n into three parts:

$$\mathrm{III}_1 = \int_0^1 \cdots, \quad \mathrm{III}_2 = \int_1^{T-\beta t_n} \cdots \text{ and } \mathrm{III}_3 = \int_{T-\beta t_n}^{T-t_n} \cdots.$$

Since $T < 0$ and $\eta < f'(0)$, the term III_1 can be bounded by

$$\mathrm{III}_1 \leqq \int_0^1 t^{-N/2} \int_{\mathbb{R}^N} e^{-\frac{T^2}{4t}|z|^2} e^{f'(0)-\eta}\, dz\, dt = e^{f'(0)-\eta}(2|T|^{-1})^N \int_{\mathbb{R}^N} e^{-|y|^2} dy.$$

Therefore, III_1 is bounded independently of n.

When $t \geqq 1$, we have $t^{-N/2} \leqq 1$. The second term III_2 can then be bounded by

$$\begin{aligned}\mathrm{III}_2 &\leqq \int_1^{T-\beta t_n} \int_{\mathbb{R}^N} e^{\frac{T}{2}|z|^2} e^{(f'(0)-\eta)t}\, dz\, dt \\ &= (2|T|^{-1})^{N/2} \left(\int_{\mathbb{R}^N} e^{-|y|^2} dy\right) (f'(0)-\eta)^{-1} \\ &\quad \times \left(e^{(f'(0)-\eta)(T-\beta t_n)} - e^{f'(0)-\eta}\right) \\ &= O(e^{\beta(f'(0)-\eta)|t_n|}) \quad \text{as } n \to +\infty.\end{aligned}$$

Let us now estimate the third term

$$\mathrm{III}_3 = \int_{T-\beta t_n}^{T-t_n} t^{-N/2} \int_{|z| \geqq \max(0, c - \frac{|x_n|}{t-T})} e^{\frac{T}{2}|z|^2 - \frac{T^2}{4t}|z|^2 + (f'(0)-\eta-\frac{1}{4}|z|^2)t}\, dz\, dt.$$

Remember that $|x_n| \leqq \gamma |t_n'| = \gamma |t_n - 1|$. Therefore, since $\beta > \frac{\gamma}{c}$, we have, for all t such that $T - \beta t_n \leqq t \leqq T - t_n$,

$$c - \frac{|x_n|}{t-T} \geqq c - \frac{\gamma}{\beta}\frac{|t_n - 1|}{|t_n|} > 0$$

for n large enough. Hence, by dropping the term $e^{-\frac{T^2}{4t}|z|^2} \leqq 1$ in III_3, we get, for n large enough,

$$\mathrm{III}_3 \leqq \int_{T-\beta t_n}^{T-t_n} e^{(f'(0)-\eta-\frac{1}{4}(c-\frac{\gamma}{\beta}\frac{|t_n-1|}{|t_n|})^2)t} dt \times \int_{\mathbb{R}^N} e^{\frac{T}{2}|z|^2} dz.$$

From our choice of β,

$$f'(0) - \eta - \frac{1}{4}(c - \frac{\gamma}{\beta}\frac{|t_n - 1|}{|t_n|})^2 < f'(0) - \frac{\eta}{2} - \frac{1}{4}(c-\gamma)^2$$

for n large enough. As a consequence,

$$\mathrm{III}_3 \leqq C \int_{T-\beta t_n}^{T-t_n} e^{(f'(0)-\frac{\eta}{2}-\frac{1}{4}(c-\gamma)^2)t}\, dt$$

for some constant $C = C(T)$. Whatever the sign of $f'(0) - \frac{\eta}{2} - \frac{1}{4}(c-\gamma)^2$ may be, it is easily found that

$$\mathrm{III}_3 = O(|t_n| e^{(f'(0)-\frac{\eta}{2}-\frac{1}{4}(c-\gamma)^2)^+ |t_n|}) \text{ as } n \to +\infty.$$

Eventually, we obtain

$$|\mathrm{II}_n| = O\left(|t_n|^{N+1}(e^{\beta(f'(0)-\eta)|t_n|} + e^{(f'(0)-\frac{\eta}{2}-\frac{1}{4}(c-\gamma)^2)^+|t_n|})\right) \text{ as } n \to +\infty.$$

On the other hand, we had

$$\mathrm{I}_n \sim C_1 |t_n|^{-N/2} e^{(f'(0)-\eta-\alpha)|t_n|} \text{ as } n \to +\infty.$$

Since $\beta < 1$ and $\eta < \min(f'(0), \frac{1}{2}(c-\gamma)^2)$, it is possible to fix $\alpha_0 > 0$ such that $\forall\, \alpha \in [0, \alpha_0]$,

$$0 < \beta(f'(0) - \eta) < f'(0) - \eta - \alpha$$
$$(f'(0) - \frac{\eta}{2} - \frac{1}{4}(c-\gamma)^2)^+ < f'(0) - \eta - \alpha.$$

Take now $\alpha \in [0, \alpha_0]$. It follows that $|\mathrm{II}_n| = o(\mathrm{I}_n)$ as $n \to +\infty$. Therefore,

$$v(x_n, T) \geqq \frac{C_1}{2} |t_n|^{-N/2} e^{(f'(0)-\eta-\alpha)|t_n|}$$

for n large enough. Since $f'(0) - \eta - \alpha > 0$, we conclude that

$$U(x_n, T) = v(x_n, T) e^{(f'(0)-\eta)T} \to +\infty \text{ as } n \to +\infty.$$

This is impossible because $U \leqq 1$.

Finally, it follows that, if $\alpha \in [0, \alpha_0]$, then

$$\max_{|x| \leqq \gamma |t|} U(x, t) = o(e^{\alpha t}) \text{ as } t \to -\infty.$$

The proof of Lemma 4.3 is complete. □

Let us now turn to the

Proof of Theorem 1.4. Let u be an entire solution of (1) such that there exists $\varepsilon > 0$ such that

$$\max_{|x| \leqq (c^*+\varepsilon)|t|} u(x, t) \to 0 \text{ as } t \to -\infty.$$

For each $n \in \mathbb{N}$, let u_n and v_n be the solutions of the following Cauchy problems:

$$(u_n)_t = \Delta u_n + f(u_n), \quad x \in \mathbb{R}^N,\ t > -n$$
$$u_n(x, -n) = \begin{cases} u(x, -n) & \text{if } |x| < (c^* + \varepsilon/2)n \\ 0 & \text{otherwise} \end{cases}$$

and

$$\begin{aligned}(v_n)_t &= \Delta v_n + f'(0)v_n, \quad x \in \mathbb{R}^N,\ t > -n\\ v_n(x,-n) &= u_n(x,-n).\end{aligned}$$

Since $c^* + \varepsilon/2 > c^*$, we know from Lemma 4.2 that $u_n(x,t) \overset{\leqq}{\to} u(x,t)$ as $n \to +\infty$. We are now going to compare u_n with the function v_n, which is a solution of a linear (more tractable) parabolic equation.

From the maximum principle, it immediately follows that $0 \leqq u_n \leqq 1$ and $v_n \geqq 0$. Furthermore, since $f(s) \leqq f'(0)s$ for all $s \in [0,1]$, we get

$$u_n(x,t) \leqq v_n(x,t) \text{ for all } x \in \mathbb{R}^N,\ t \geqq -n.$$

Let us now find an upper bound for v_n. Since f is of class C^2 and $f'(0) > 0$, there exist two positive real numbers η and κ such that f is increasing in $[0,\eta]$ and $f(s) \geqq f'(0)s - \kappa s^2$ for all $s \in [0,\eta]$. Since $c^* + \varepsilon/2 < c^* + \varepsilon$, Lemma 4.3 provides the existence of a real number $\alpha \in (0, f'(0))$ and a, say, negative time T, such that

$$0 \leqq u(x,t) \leqq e^{\alpha t} \leqq \eta \text{ for all } t \leqq T \text{ and } |x| \leqq (c^* + \varepsilon/2)|t|. \tag{48}$$

Lemma 4.4. *There exists a constant $C_2 = C_2(f,\alpha,\kappa,T)$ such that, for each $t \leqq T$ and $x \in \mathbb{R}^N$, we have*

$$\forall n > |t|, \quad u_n(x,t) \leqq v_n(x,t) \leqq u_n(x,t)e^{\frac{\kappa}{\alpha}e^{\alpha t}} + C_2 h_{c^*+\varepsilon/2}(x,t)$$

under the notation of Lemma 3.8(b).

Proof. First of all, we have already observed that $u_n \leqq v_n$.

Let us now prove the upper bound for v_n. Remember that $0 \leqq u_n(x,t) \leqq u(x,t)$ from the maximum principle. From (48) and from our choice of η and κ, $\forall t \leqq T,\ |x| \leqq (c^* + \varepsilon/2)|t|,\ n > |t|$,

$$\begin{aligned}f(u(x,t)) &\geqq f(u_n(x,t))\\ &\geqq f'(0)u_n(x,t) - \kappa\, u_n(x,t)^2\\ &\geqq f'(0)u_n(x,t) - \kappa\, e^{\alpha t}\, u_n(x,t).\end{aligned}$$

Set

$$U_n(x,t) = u_n(x,t)e^{\frac{\kappa}{\alpha}e^{\alpha t}} \ (\geqq u_n(x,t)) \text{ and } w_n(x,t) = U_n(x,t) - v_n(x,t).$$

Take $n \geqq |T|$. The function w_n satisfies

$$\begin{aligned}(w_n)_t - \Delta w_n - f'(0)w_n &= (f(u_n) - f'(0)u_n + \kappa e^{\alpha t}u_n)e^{\frac{\kappa}{\alpha}e^{\alpha t}}\\ &\geqq \begin{cases}0 & \text{for all } -n \leqq t \leqq T,\ |x| \leqq (c^* + \varepsilon/2)|t|\\ -C_2 & \text{for all } -n \leqq t \leqq T,\ |x| \geqq (c^* + \varepsilon/2)|t|,\end{cases}\end{aligned}$$

where

$$C_2 = (\|f\|_\infty + f'(0) + \kappa e^{\alpha T}) e^{\frac{\kappa}{\alpha} e^{\alpha T}}.$$

On the other hand, $w_n(x, -n) = u_n(x, -n)(e^{\frac{\kappa}{\alpha} e^{-\alpha n}} - 1) \geqq 0$. From the maximum principle, it follows that, for all $-n < t \leqq T$ and for all $x \in \mathbb{R}^N$,

$$w_n(x, t) \geqq -C_2 \int_{-n}^{t} e^{f'(0)(t-s)} (S(t-s) \cdot \mathbf{1}_{|\cdot| \geqq (c^* + \varepsilon/2)|s|})(x)\, ds.$$

Since $c^* + \varepsilon/2 > c^*$, Lemma 3.8(b) implies that, for all $-n < t \leqq T$ and $x \in \mathbb{R}^N$,

$$\begin{aligned} w_n(x, t) &\geqq -C_2 h_{c^* + \varepsilon/2}(x, t) \\ &= -C_2 \int_{-\infty}^{t} e^{f'(0)(t-s)} (S(t-s) \cdot \mathbf{1}_{|\cdot| \geqq (c^* + \varepsilon/2)|s|})(x)\, ds. \end{aligned}$$

By definition of w_n, it follows that

$$\forall\ -n < t \leqq T,\ \forall x \in \mathbb{R}^N, \quad v_n(x, t) \leqq u_n(x, t) e^{\frac{\kappa}{\alpha} e^{\alpha t}} + C_2 h_{c^* + \varepsilon/2}(x, t)$$

and the proof of Lemma 4.4 is complete. □

Lemma 4.5. *Up to extraction of some subsequence, the functions v_n locally converge in $\mathbb{R}^N \times (-\infty, T)$ to a positive function v, which is a C^∞ solution of*

$$\partial_t v = \Delta v + f'(0) v, \quad x \in \mathbb{R}^N,\ t < T.$$

Furthermore, under the notation of Lemma 4.4,

$$\forall t < T,\ \forall x \in \mathbb{R}^N, \quad u(x, t) \leqq v(x, t) \leqq v(x, t) e^{\frac{\kappa}{\alpha} e^{\alpha t}} + C_2 h_{c^* + \varepsilon/2}(x, t). \quad (49)$$

Proof. From Lemma 4.4, we have

$$u_n(0, T) \leqq v_n(0, T) \leqq u_n(0, T) e^{\frac{\kappa}{\alpha} e^{\alpha T}} + C_2 h_{c^* + \varepsilon/2}(0, T).$$

Lemma 4.2 implies that $u_n(0, T) \to u(0, T)$ as $n \to +\infty$. Therefore, the sequence $(v_n(0, T))_n$ is bounded. On the other hand, each function $v_n(x, t)$ is positive for $t > -n$ and for all $x \in \mathbb{R}^N$, from the strong maximum principle. We finally get from the Harnack inequality that the sequence of functions $(v_n(x, t))_n$ is locally bounded in $\mathbb{R}^N \times (-\infty, T)$. From standard parabolic estimates, it is also bounded in each $C^k(K)$ for each compact subset $K \subset \mathbb{R}^N \times (-\infty, T)$. Up to extraction of some subsequence, the functions $v_n(x, t)$ locally converge to a nonnegative C^∞ function $v(x, t)$, which is a solution of

$$\partial_t v = \Delta v + f'(0) v, \quad x \in \mathbb{R}^N,\ t < T.$$

The estimates (49) follow from Lemmas 4.2 and 4.4. Furthermore, from (49), we deduce that v is not identically equal to 0. Hence, $v(x, t) > 0$ for all $(x, t) \in \mathbb{R}^N \times (-\infty, T)$, from the strong maximum principle. □

Since the functions v_n are solutions of the linear heat equation, it turns out that we can find an explicit formula for the limit function v:

Lemma 4.6. *Up to extraction of some subsequence, the functions $v_n(x,t)$ actually converge for each $(x,t) \in \mathbb{R}^N \times \mathbb{R}$ to a C^∞ function $v(x,t)$ solving $v_t = \Delta v + f'(0)v$, and there exists a nonzero and nonnegative Radon measure ρ on the open ball $B = B(0, c^* + \varepsilon/2)$ such that*

$$\forall\,(x,t) \in \mathbb{R}^N \times \mathbb{R}, \quad v(x,t) = e^{f'(0)t} \int_B e^{\frac{1}{2} z\cdot x + \frac{t}{4}|z|^2} \rho(dz). \tag{50}$$

Furthermore, there is a real number $\beta \in (0, c^*)$ such that the support of ρ belongs to $\overline{B(0,\beta)}$.

Proof. By definition of the functions v_n, we have, for n large enough,

$$\begin{aligned} v_n(0, T-1) &= \frac{e^{f'(0)(T-1+n)}}{\sqrt{4\pi(T-1+n)}^N} \int_{|y|<(c^*+\varepsilon/2)n} u(y,-n) e^{-\frac{|y|^2}{4(T-1+n)}}\, dy \\ &= e^{f'(0)(T-1)} \left(\frac{n}{T-1+n}\right)^{N/2} \\ &\quad \times \int_{|z|<c^*+\varepsilon/2} (4\pi)^{-N/2} n^{N/2} u(nz,-n) e^{(f'(0)-\frac{1}{4}|z|^2)n} \\ &\quad \times e^{\frac{(T-1)n}{4(T-1+n)}|z|^2} dz. \end{aligned}$$

Since $v_n(0,T-1)$ converges (to $v(0,T-1)$) as $n \to +\infty$ and since the positive functions $e^{\frac{(T-1)n}{4(T-1+n)}|z|^2}$ are uniformly bounded away from 0 in B as $n \to +\infty$, it follows that the positive functions

$$f_n(z) := (4\pi)^{-N/2} n^{N/2}\, u(nz,-n)\, e^{(f'(0)-\frac{1}{4}|z|^2)n}$$

are bounded in $L^1(B)$. Up to extraction of some subsequence, there exists then a nonnegative Radon measure ρ on B such that

$$f_n(z)dz \ \rightharpoonup\ \rho(dz) \text{ in } (C_c(B(0, c^* + \varepsilon/2)))' \ \text{ as } n \to +\infty.$$

Remember that $\alpha \in (0, f'(0))$ has been chosen so that (48) is satisfied. Set

$$\beta = 2\sqrt{f'(0) - \alpha} \in (0, c^*).$$

Take any continuous function ψ whose support is compactly included in $\{z,\ \beta < |z| < c^* + \varepsilon/2\}$. In particular, there exists a real number $\delta > 0$ such that $\operatorname{supp}\psi \subset \{z,\ \beta + \delta \leqq |z| < c^* + \varepsilon/2\}$. By definition of ρ, we have

$$\int_B f_n(z)\psi(z)dz \to \int_B \psi(z)\rho(dz) \text{as } n \to +\infty.$$

Let us prove that this limit is equal to 0. By definition,

$$\begin{aligned} &\left|\int_B f_n(z)\psi(z)dz\right| \\ &\quad = \left|\int_{\beta+\delta \leqq |z| < c^*+\varepsilon/2} (4\pi)^{-N/2} n^{N/2} u(nz,-n) e^{(f'(0)-\frac{1}{4}|z|^2)n} \psi(z)\, dz\right| \\ &\quad \leqq \int_{\beta+\delta \leqq |z| < c^*+\varepsilon/2} (4\pi)^{-N/2} n^{N/2} u(nz,-n) e^{(f'(0)-\frac{1}{4}(\beta+\delta)^2)n} |\psi(z)|\, dz. \end{aligned}$$

Since α satisfies (48), $0 \leqq u(nz,-n) \leqq e^{-\alpha n}$ in B for n large enough. Due to our choice of β, $f'(0) - \frac{1}{4}(\beta+\delta)^2 - \alpha < 0$. Therefore,

$$\int_B f_n(z)\psi(z)dz \to 0 \text{ as } n \to +\infty.$$

As a consequence, $\int_B \psi(z)\rho(dz) = 0$ for any continuous function ψ whose support is compact and included in $\{z,\ \beta < |z| < c^* + \varepsilon/2\}$. In other words, the support of ρ is included in $\overline{B(0,\beta)}$.

Note that the above arguments also imply that

$$\forall\, \beta' \in (\beta, c^* + \varepsilon/2), \quad \int_{z,\ \beta'<|z|<c^*+\varepsilon/2} f_n(z)dz \ \to 0 \text{ as } n \to +\infty. \tag{51}$$

Choose now any couple $(x,t) \in \mathbb{R}^N \times \mathbb{R}$. For all $n > |t|$, it is found that

$$\begin{aligned} v_n(x,t) &= e^{f'(0)(t+n)} \frac{1}{\sqrt{4\pi(t+n)}^N} \int_{B(0,(c^*+\varepsilon/2)n)} u(y,-n)\, e^{-\frac{|y-x|^2}{4(t+n)}}\, dy \\ &= \left(\frac{n}{t+n}\right)^{N/2} e^{f'(0)t} I_n, \end{aligned}$$

where

$$I_n = \int_{B(0,c^*+\varepsilon/2)} f_n(z)\, e^{\frac{1}{2}z\cdot x + \frac{t}{4}|z|^2 - \frac{|tz+x|^2}{4(t+n)}}\, dz.$$

Let $\chi(z)$ be a fixed smooth function such that $0 \leqq \chi \leqq 1$, $\chi = 1$ in $\overline{B(0,c^*)}$ and $\chi = 0$ outside $B(0, c^* + \varepsilon/4)$. Let ε' be an arbitrary positive real number. For n large enough, $e^{-\frac{|tz+x|^2}{4(t+n)}} \leqq 1 + \varepsilon'$ for all $z \in B(0, c^* + \varepsilon/2)$, whence

$$I_n \leqq (1+\varepsilon')(A_1 + A_2),$$

where

$$\begin{aligned} A_1 &= \int_{B(0,c^*+\varepsilon/2)} \chi(z)\ f_n(z) e^{\frac{1}{2}z\cdot x + \frac{t}{4}|z|^2}\, dz, \\ A_2 &= \int_{B(0,c^*+\varepsilon/2)} (1-\chi(z))\ f_n(z) e^{\frac{1}{2}z\cdot x + \frac{t}{4}|z|^2} dz. \end{aligned}$$

Since χ is a continuous function whose support is compactly included in $B(0, c^* + \varepsilon/2)$, and due to the definition of the measure ρ, we have

$$A_1 \to \int_{B(0,c^*+\varepsilon/2)} \chi(z)\, e^{\frac{1}{2}z\cdot x + \frac{t}{4}|z|^2} \rho(dz) \ \text{ as } n \to +\infty.$$

Furthermore, since the support of ρ is included in $\overline{B(0,\beta)}$ with $\beta < c^*$ and $\chi = 1$ on $\overline{B(0,c^*)}$, it follows that

$$A_1 \to \int_{B(0,c^*+\varepsilon/2)} e^{\frac{1}{2}z\cdot x + \frac{t}{4}|z|^2} \rho(dz) \ \text{ as } n \to +\infty.$$

On the other hand, since $\chi = 1$ on $\overline{B(0, c^*)}$ and $0 \leqq \chi \leqq 1$ on $B(0, c^* + \varepsilon/2)$,

$$|A_2| \leqq C(t, x) \int_{z,\ c^* \leqq |z| \leqq c^* + \varepsilon/2} f_n(z)\, dz$$

for some constant $C(t, x) \in \mathbb{R}$. From (51), we get $A_2 \to 0$ as $n \to +\infty$.

Therefore,

$$\limsup_{n \to +\infty} \mathrm{I}_n \leqq (1 + \varepsilon') \int_{B(0, c^* + \varepsilon/2)} e^{\frac{1}{2} z \cdot x + \frac{t}{4} |z|^2} \rho\,(dz).$$

Similarly, we can show that

$$\liminf_{n \to +\infty} \mathrm{I}_n \geqq (1 - \varepsilon') \int_{B(0, c^* + \varepsilon/2)} e^{\frac{1}{2} z \cdot x + \frac{t}{4} |z|^2} \rho(dz).$$

Since ε' is arbitrary, we get

$$\mathrm{I}_n \to \int_{B(0, c^* + \varepsilon/2)} e^{\frac{1}{2} z \cdot x + \frac{t}{4} |z|^2} \rho(dz) \text{ as } n \to +\infty.$$

Since $v_n(x, t) = (\frac{n}{t+n})^{N/2} e^{f'(0)t} I_n$, it follows that $v_n(x, t)$ converges to a function $v(x, t)$, for each $(x, t) \in \mathbb{R}^N \times \mathbb{R}$, and that the function v is given by the formula (50).

Lastly, it follows from

$$\begin{aligned} e^{f'(0)(T-1)} \rho(B) \leqq v(0, T - 1) &= e^{f'(0)(T-1)} \int_B e^{\frac{1}{4}|z|^2}\ \rho(dz) \\ &\leqq e^{f'(0)(T-1) + \frac{1}{4}(c^* + \varepsilon/2)^2}\ \rho(B) \end{aligned}$$

and $0 < v(0, T - 1) < +\infty$ that $\rho(B) = \rho(B(0, c^* + \varepsilon/2)) = \rho(B(0, c^*)) \in (0, +\infty)$. From the formula (50), it follows then that the function v is actually a positive and locally bounded C^∞ solution of $v_t = \Delta v + f'(0)v$ in $\mathbb{R}^N \times \mathbb{R}$. □

So far, we have proved the existence of a nonnegative finite Radon measure ρ on $B(0, c^* + \varepsilon/2)$, the support of which is included in $\overline{B(0, \beta)}$ for some $\beta < c^*$. For the sake of simplicity, we also call ρ the restriction of the measure ρ to the ball $B(0, c^*)$.

Since ρ is nonnegative and nonzero on $B(0, c^*)$, elementary arguments provide the existence of a unique positive real number $\hat{M} > 0$ such that

$$\int_{B(0, c^*)} e^{-(f'(0) + \frac{1}{4}|z|^2) \ln \hat{M}} \rho(dz) = 1. \tag{52}$$

Let us now call μ the unique nonzero, nonnegative and finite Radon measure on $\hat{X} = S^{N-1} \times (c^*, +\infty) \cup \{\infty\}$ such that

$$\Phi_* \mu(dz) = \hat{M} e^{-(f'(0) + \frac{1}{4}|z|^2) \ln \hat{M}} \rho(dz). \tag{53}$$

By definition of $\hat{M}$, $\int_{B(0,c^*)} \Phi_*\hat{\mu}(dz) = \hat{M}$, that is to say, $\hat{\mu}(\hat{X}) = \hat{M}$. By extending μ by 0 on $S^{N-1} \times \{c^*\}$, we get $\mu \in \mathcal{M}$. Furthermore, due to the definition of the map Φ, the support of μ is included in $S^{N-1} \times [c_0, +\infty[\cup \{\infty\}$ where $c_0 > c^*$ is such that $\beta = 2\lambda_{c_0}$.

The remaining part of this section consists in proving that $u = u_\mu$.

In order to do this, we first prove the following

Lemma 4.7. *For each* $\theta \in [0, c_0[$,

$$\max_{|x| \leqq \theta |t|} u_\mu(x,t) \to 0 \ \text{ as } t \to -\infty.$$

Proof. Choose $\theta \in [0, c_0[$. From the upper bound in (30), it follows that

$$\begin{aligned} u_\mu(x,t) &\leqq \int_{\hat{X}} e^{\lambda_c(x\cdot\nu + ct + c\ln\hat{M})} \frac{1}{\hat{M}}\, d\mu \\ &= \int_{\{\nu \in S^{N-1},\ c \geqq c_0\}} e^{\lambda_c(x\cdot\nu + ct + c\ln\hat{M})}\ \frac{1}{\hat{M}}\, d\mu + \frac{\mu(\infty)}{\hat{M}}\ e^{f'(0)(t+\ln\hat{M})}. \end{aligned}$$

For each $t \leqq 0$ and $|x| \leqq \theta|t|$, we have

$$\forall \nu \in S^{N-1},\ \forall c \geqq c_0, \quad \lambda_c(x\cdot\nu + ct) \leqq \lambda_c(\theta|t| - c_0|t|) = \lambda_c(\theta - c_0)|t|.$$

On the other hand, $0 \leqq \lambda_c c \leqq 2f'(0)$ for all $c \geqq c^*$. Therefore, for $t \leqq 0$,

$$\begin{aligned} &\frac{1}{\hat{M}} \int_{\{\nu\in S^{N-1},\ c\geqq c_0\}} e^{\lambda_c(x\cdot\nu + ct + c\ln\hat{M})}\, d\mu \\ &\qquad \leqq \frac{1}{\hat{M}} e^{2f'(0)|\ln\hat{M}|} \int_{\{\nu\in S^{N-1},\ c\geqq c_0\}} e^{\lambda_c(\theta - c_0)|t|}\, d\mu \to 0 \end{aligned}$$

as $t \to -\infty$, from Lesbesgue's dominated convergence theorem. Eventually, the conclusion of Lemma 4.6 follows. □

Remark 4.8. By slightly modifying the proof of the above Lemma 4.7, we get the following more general result: if $m \in \mathcal{M}$ is such that $m(S^{N-1} \times [c^*, \overline{c}]) = 0$ for some $\overline{c} \geqq c^*$, then $\max_{|x|\leqq\overline{c}|t|} u_m(x,t) \to 0$ as $t \to -\infty$. Indeed,

$$u_m(x,t) \leqq \int_{\{\nu\in S^{N-1}, c>\overline{c}\}} e^{\lambda_c(x\cdot\nu + ct + c\ln\hat{M})} \frac{1}{\hat{M}}\, d\mu + \frac{\mu(\infty)}{\hat{M}} e^{f'(0)(t+\ln\hat{M})}.$$

(Note that, for the measure m, $\hat{M} > 0$ because $\mu^* = 0$.) Take any $\eta > 0$ and let $\delta > 0$ be such that $m(S^{N-1} \times (\overline{c}, \overline{c}+\delta)) \leqq \eta$. For each $|x| \leqq \overline{c}|t|$, $t \leqq 0$, $\nu \in S^{N-1}$ and $c > \overline{c}$, we have $x\cdot\nu + ct \leqq -\overline{c}t + ct = (c - \overline{c})t \leqq 0$. Therefore,

$$\begin{aligned} \max_{|x|\leqq\overline{c}|t|} u_m(x,t) &\leqq \frac{e^{2f'(0)|\ln\hat{M}|}}{\hat{M}} \eta \\ &\quad + \max_{|x|\leqq\overline{c}|t|} \int_{\{\nu\in S^{N-1},\ c\geqq\overline{c}+\delta\}} e^{\lambda_c(x\cdot\nu + ct + c\ln\hat{M})} \frac{d\mu}{\hat{M}} + \frac{\mu(\infty)}{\hat{M}} e^{f'(0)(t+\ln\hat{M})}. \end{aligned}$$

As was shown in the course of Lemma 4.7, the second and third terms of the right-hand side converge to 0 as $t \to -\infty$. Since $\eta > 0$ is arbitrary, we conclude that $\max_{|x| \leqq \overline{c}|t|} u_m(x,t) \to 0$ as $t \to -\infty$.

Let us now turn to the proof of

Lemma 4.9. *The function u is equal to the function u_μ.*

Proof. Let us first choose a real number $\tilde{\gamma}$ such that

$$c^* < \tilde{\gamma} < \min\,(c_0, c^* + \varepsilon/2).$$

Let $\tilde{u}_n$, $\tilde{v}_n$, $\tilde{U}_n$ and $\tilde{V}_n$ be the solutions of the following Cauchy problems:

$$\begin{aligned} (\tilde{u}_n)_t &= \Delta \tilde{u}_n + f(\tilde{u}_n), \qquad x \in \mathbb{R}^N,\ t > -n \\ (\tilde{v}_n)_t &= \Delta \tilde{v}_n + f'(0)\tilde{v}_n, \qquad x \in \mathbb{R}^N,\ t > -n \\ \tilde{u}_n(x,-n) = \tilde{v}_n(x,-n) &= \begin{cases} u(x,-n) & \text{if } |x| < \tilde{\gamma} n \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

$$\begin{aligned} (\tilde{U}_n)_t &= \Delta \tilde{U}_n + f(\tilde{U}_n), \qquad x \in \mathbb{R}^N,\ t > -n \\ (\tilde{V}_n)_t &= \Delta \tilde{V}_n + f'(0)\tilde{V}_n, \qquad x \in \mathbb{R}^N,\ t > -n \\ \tilde{U}_n(x,-n) = \tilde{U}_n(x,-n) &= \begin{cases} u_\mu(x,-n) & \text{if } |x| < \tilde{\gamma} n \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Since $\tilde{\gamma} > c^*$, Lemma 4.2 yields

$$\forall\,(x,t) \in \mathbb{R}^N \times \mathbb{R}, \quad \tilde{u}_n(x,t) \to u(x,t), \quad \tilde{U}_n(x,t) \to u_\mu(x,t) \text{ as } n \to +\infty.$$

On the other hand,

$$\tilde{v}_n(x,t) = \left(\frac{n}{t+n}\right)^{N/2} e^{f'(0)t} \int_{B(0,\tilde{\gamma})} f_n(z)\, e^{\frac{1}{2} z\cdot x + \frac{t}{4}|z|^2 - \frac{|tz+x|^2}{4(t+n)}}\, dz,$$

where we recall that

$$f_n(z) = (4\pi)^{-N/2} n^{N/2}\, u(nz,-n)\, e^{(f'(0) - \frac{|z|^2}{4})n}.$$

As in the proof of Lemma 4.6 and since $\tilde{\gamma} > c^* > \beta$, we get

$$\tilde{v}_n(x,t) \to \tilde{v}(x,t) := e^{f'(0)t} \int_{B(0,\tilde{\gamma})} e^{\frac{1}{2} z\cdot x + \frac{t}{4}|z|^2} \rho(dz) = v(x,t). \qquad (54)$$

Similarly,

$$\tilde{V}_n(x,t) = \left(\frac{n}{t+n}\right)^{N/2} e^{f'(0)t} \int_{B(0,\tilde{\gamma})} F_n(z) e^{\frac{1}{2} z\cdot x + \frac{t}{4}|z|^2 - \frac{|tz+x|^2}{4(t+n)}}\, dz,$$

where

$$F_n(z) = (4\pi)^{-N/2} n^{N/2}\, u_\mu(nz,-n)\, e^{(f'(0) - \frac{|z|^2}{4})n}.$$

Furthermore, since the function u_μ is such that

$$\max_{|x|\leqq c|t|} u_\mu(x,t) \to 0 \text{ as } t \to -\infty$$

for some $c > c^*$ (take for instance $c = \frac{c^*+c_0}{2}$ and apply Lemma 4.7), it also follows, as in Lemma 4.6, that there exists a finite nonnegative Radon measure $\tilde\rho$ on $B(0,\tilde\gamma)$, whose support is included in $\overline{B(0,\tilde\beta)}$ for some $\tilde\beta < c^*$, and such that

$$F_n(z)dz \rightharpoonup \tilde\rho(dz) \text{ in } (C_c(B(0,\tilde\gamma)))' \text{ as } n \to +\infty$$

(up to extraction of some subsequence), and

$$\tilde V_n(x,t) \to V(x,t) := e^{f'(0)t}\int_{B(0,\tilde\gamma)} e^{\frac12 z\cdot x+\frac t4|z|^2}\tilde\rho(dz) \text{ as } n \to +\infty.$$

From the asymptotic behavior (12), which is satisfied by the function u_μ, we finds that

$$F_n(z)dz \rightharpoonup \frac{1}{\hat M}e^{(f'(0)+\frac14|z|^2)\ln \hat M}\Phi_*\hat\mu(dz) \text{ in } (C_c(B(0,c^*)))'.$$

Eventually, from the definition of $\hat\mu$ in (53), it follows that $\tilde\rho = \rho$ on $B(0,c^*)$, whence

$$\forall (x,t)\in\mathbb{R}^N\times\mathbb{R},\quad \tilde V(x,t) = v(x,t). \tag{55}$$

Since $\tilde\gamma < \min(c^*+\varepsilon/2, c_0)$, Lemma 4.3 yields the existence of a real number $\tilde\alpha\in(0,f'(0))$ and of a, say, negative time $\tilde T$ such that

$$\forall t\leqq \tilde T,\ \forall x\in\mathbb{R}^N,\quad 0\leqq u(x,t),\ u_\mu(x,t)\leqq e^{\tilde\alpha t}\leqq \eta,$$

where $\eta > 0$ is such that f is increasing in $[0,\eta]$ and $f(s)\geqq f'(0)s-\kappa s^2$ on $[0,\eta]$, with $\kappa > 0$. With the same proof as for Lemma 4.4, and by using (54) and (55), we finally finds that $\forall t\leqq\tilde T,\ \forall x\in\mathbb{R}^N$,

$$\begin{aligned} u(x,t)\leqq v(x,t)\leqq u(x,t)e^{\frac{\kappa}{\tilde\alpha}e^{\tilde\alpha t}}+C_3h_{\tilde\gamma}(x,t),\\ u_\mu(x,t)\leqq v(x,t)\leqq u_\mu(x,t)e^{\frac{\kappa}{\tilde\alpha}e^{\tilde\alpha t}}+C_3h_{\tilde\gamma}(x,t),\end{aligned} \tag{56}$$

where $C_3 = (\|f\|_\infty + f'(0)+\kappa e^{\tilde\alpha\tilde T})e^{\frac{\kappa}{\tilde\alpha}e^{\tilde\alpha\tilde T}}$.

Define $w = u-u_\mu$. Since $f'(s)\leqq f'(0)$ for all $s\in[0,1]$, the function $|w|$ satisfies

$$\partial_t w\leqq \Delta w + f'(0)|w|,\quad t\in\mathbb{R},\ x\in\mathbb{R}^N.$$

For each n large enough, it easily follows from (56) that

$$\begin{aligned}|w(x,-n)| &= |u(x,-n)-u_\mu(x,-n)|\\ &\leqq (u(x,-n)+u_\mu(x,-n))2\frac{\kappa}{\tilde\alpha}e^{-\tilde\alpha n}+C_3h_{\tilde\gamma}(x,-n).\end{aligned}$$

Choose any $x\in\mathbb{R}^N$ and, say, $t\leqq 0$. The maximum principle yields

$$|w(x,t)|\leqq \mathrm{I}_n+\mathrm{II}_n,$$

where

$$\mathrm{I}_n = e^{f'(0)(t+n)}(S(t+n)\cdot(2\frac{\kappa}{\tilde\alpha}e^{-\tilde\alpha n}(u(\cdot,-n)+u_\mu(\cdot,-n))))(x)$$
$$\mathrm{II}_n = e^{f'(0)(t+n)}(S(t+n)\cdot(C_3 h_{\tilde\gamma}(\cdot,-n)))(x)$$

for n large enough.

Let us first estimate the term I_n. By definition of $\tilde v_n$ and $\tilde V_n$,

$$\mathrm{I}_n = \left(e^{f'(0)(t+n)}\int_{|y|\geqq\tilde\gamma n}\frac{1}{\sqrt{4\pi(t+n)}^N}e^{-\frac{|y-x|^2}{4(t+n)}}(u(y,-n)+u_\mu(y,-n))\,dy\right)$$
$$\times\, 2\frac{\kappa}{\tilde\alpha}e^{-\tilde\alpha n} + 2\frac{\kappa}{\tilde\alpha}e^{-\tilde\alpha n}\big(\tilde v_n(x,t)+\tilde V_n(x,t)\big).$$

Since $\tilde\gamma > c^*$, Lemma 4.2 yields

$$e^{f'(0)(t+n)}\int_{|y|\geqq\tilde\gamma n}\frac{1}{\sqrt{4\pi(t+n)}^N}e^{-\frac{|y-x|^2}{4(t+n)}}(u(y,-n)+u_\mu(y,-n))\,dy$$
$$\to 0 \ \text{ as } n\to+\infty.$$

Since $\tilde v_n(x,t)$ and $\tilde V_n(x,t)$ are bounded, we finally conclude that $\mathrm{I}_n \to 0$ as $n\to +\infty$.

On the other hand, because of the definition of $h_{\tilde\gamma}$, the term II_n is equal to

$$\mathrm{II}_n = C_3 e^{f'(0)(t+n)}$$
$$\times\left(S(t+n)\cdot\left(\int_{-\infty}^{-n}e^{f'(0)(-n-s)}(S(-n-s)\cdot\mathbf{1}_{|\cdot|\geqq\tilde\gamma|s|})(y)\,ds\right)\right)(x)$$
$$= C_3\int_{-\infty}^{-n}e^{f'(0)(t-s)}(S(t-s)\cdot\mathbf{1}_{|\cdot|\geqq\tilde\gamma|s|})(x)\,ds.$$

Since

$$\int_{-\infty}^{t}e^{f'(0)(t-s)}(S(t-s)\cdot\mathbf{1}_{|\cdot|\geqq\tilde\gamma|s|})(x)\,ds = h_{\tilde\gamma}(x,t)$$

converges (because of Lemma 3.8(b)), Lebesgue's dominated convergence theorem implies that $\mathrm{II}_n \to 0$ as $n\to+\infty$.

As a consequence, $|w|(x,t)=0$ for each $x\in\mathbb{R}^N$ and $t\leqq 0$. The maximum principle for $|w|$ yields $w(x,t)=0$ for each couple $(x,t)\in\mathbb{R}^N\times\mathbb{R}$. In other words, $u=u_\mu$ and the proof of Lemma 4.9 is complete. □

In order to complete the proof of Theorem 1.4, we have only to show the following

Lemma 4.10. *The support of μ is included in $S^{N-1}\times[c^*+\varepsilon,+\infty[\,\cup\,\{\infty\}$.*

Proof. We already know that supp $\mu \subset S^{N-1}\times[c_0,+\infty[\cup\{\infty\}$ for some $c_0 > c^*$. Set $\hat{M} = \mu(\hat{X})$. From the definition of $\hat{M}$ in (52), we have $\hat{M} > 0$.

Choose any couple $(\overline{\nu},\overline{c}) \in S^{N-1}\times(c^*, c^*+\varepsilon)$ and let $B_{(\overline{\nu},\overline{c})} \subset S^{N-1}\times(c^*,+\infty)$ be an open neighborhood of $(\overline{\nu},\overline{c})$ such that

$$\forall\,(\nu,c)\in B_{(\overline{\nu},\overline{c})},\quad (c^*+\varepsilon)(\overline{\nu}\cdot\nu)-c \geqq \frac{1}{2}(c^*+\varepsilon-\overline{c}) =: \delta > 0. \tag{57}$$

From the lower bound in (30) applied to the point $(x,t) = ((c^*+\varepsilon)n\overline{\nu}, -n)$, it is found that

$$\int_{B_{(\overline{\nu},\overline{c})}} \varphi_c((c^*+\varepsilon)n\overline{\nu}\cdot\nu - cn + c\ln\hat{M})\frac{1}{\hat{M}}d\hat{\mu} \leqq u((c^*+\varepsilon)n\overline{\nu}, -n).$$

Because of (57) and because of Lebesgue's dominated convergence theorem, the left-hand side in the previous inequality approaches $\frac{1}{\hat{M}}\hat{\mu}(B_{(\overline{\nu},\overline{c})})$ as $n\to+\infty$. On the other hand, the hypothesis made on u implies that the right-hand side approaches 0. As a consequence,

$$\mu(B_{(\overline{\nu},\overline{c})}) = 0.$$

Since $S^{N-1}\times(c^*,c^*+\varepsilon)$ can be covered by a countable sets of the type $B_{(\overline{\nu},\overline{c})}$, it follows that

$$\mu(S^{N-1}\times(c^*,c^*+\varepsilon)) = 0. \qquad \square$$

The proof of Theorem 1.4 is now complete. □

Remark 4.11. Note that, under the assumption of Theorem 1.4, μ is not necessarily concentrated on $S^{N-1}\times(c^*+\varepsilon,+\infty)\cup\{\infty\}$, that is to say, $\mu(S^{N-1}\times\{c^*+\varepsilon\})$ may not be 0.

Indeed, for any $c_0 > c^*$, we can prove that the measure $\mu = d\nu\times\delta_{c_0}$, which is concentrated on $S^{N-1}\times\{c_0\}$, gives rise to a function u_μ satisfying $\max_{|x|\leqq c_0|t|} u(x,t)\to 0$ as $t\to-\infty$. The measure μ being radially symmetric, each function $u_n(x,-n)$ defined as in (26) is radially symmetric with respect to the origin, and, eventually, the function u_μ is itself radially symmetric with respect to the origin (see more details in Section 5.2). Therefore,

$$\begin{aligned}\max_{|x|\leqq c_0|t|} u(x,t) &= \max_{0\leqq r\leqq c_0} u(r|t|,0,\cdots,0,t)\\ &\leqq \frac{e^{\lambda_{c_0}c_0\ln\hat{M}}}{\hat{M}}\int_{S^{N-1}} e^{\lambda_{c_0}(-r\nu_1+c_0)t}\,d\nu\end{aligned}$$

by definition of μ and from (30). The function

$$g(r) := \int_{S^{N-1}} e^{\lambda_{c_0}(-r\nu_1+c_0)t}d\nu$$

is such that

$$\begin{aligned} g'(r) &= -\lambda_{c_0} \int_{S^{N-1}} \nu_1 e^{\lambda_{c_0}(-r\nu_1+c_0)t}\, d\nu \\ &= -\lambda_{c_0} \int_{S^{N-1}\cap\{\nu_1 \geqq 0\}} \nu_1 (e^{\lambda_{c_0}(-r\nu_1+c_0)t} - e^{\lambda_{c_0}(r\nu_1+c_0)t})\, d\nu \\ &\geqq 0. \end{aligned}$$

Therefore,

$$\max_{|x|\leqq c_0|t|} u(x,t) \leqq \frac{e^{\lambda_{c_0} c_0 \ln \hat{M}}}{\hat{M}} \int_{S^{N-1}} e^{\lambda_{c_0}(-c_0\nu_1+c_0)t}\, d\nu \ \to 0$$

as $t \to -\infty$ (from Lebesgue's dominated convergence theorem).

4.3. Uniqueness in the class of the solutions bounded away from 1 *(Proof of Theorem 1.5)*

This section is devoted to the proof of Theorem 1.5. Let $u(x,t)$ be an entire solution of (1) and assume that there exists a time t_0 such that $\sup\ u(\cdot, t_0) < 1$. Our goal is to prove that $u(x,t)$ depends only on t.

Let us first prove the following

Lemma 4.12. *Set $M(t) = \sup u(\cdot, t)$. Then $M(t) \to 0$ as $t \to -\infty$.*

Proof. Assume otherwise. There exist then a real $\varepsilon > 0$ and two sequences $t_n \to -\infty$ and $x_n \in \mathbb{R}^N$ such that $u(x_n, t_n) \geqq \varepsilon$. By standard parabolic estimates, $\nabla_x u(x,t)$ is uniformly bounded in $\mathbb{R}^N \times \mathbb{R}$. Hence, there exists a real $r > 0$ such that $u(x, t_n) \geqq \varepsilon/2$ if $|x - x_n| \leqq r$.

Let $\rho(x)$ now be a continuous nonnegative function such that $0 < \rho(x) \leqq \varepsilon/2$ if $|x| < r$ and $\rho(x) = 0$ otherwise. From the results of Aronson & Weinberger [2], the function $v(x,t)$ solving the Cauchy problem

$$v_t = \Delta v + f(v),\ \ t > 0, \qquad v(x,0) = \rho(x),$$

goes to 1 as $t \to +\infty$, uniformly in any compact subset of $\mathbb{R}^N$.

From the maximum principle, it follows that

$$\forall\, t \geqq t_n, \quad v(0, t - t_n) \leqq u(x_n, t).$$

Take $t = t_0$ and pass to the limit $t_n \to -\infty$ in this inequality. The left-hand side goes to 1 whereas $u(x_n, t_0) \leqq \sup u(\cdot, t_0) < 1$ by hypothesis. This is impossible. □

Let us now turn to the

Proof of Theorem 1.5. Take u as above (there exists $t_0 \in \mathbb{R}$ such that $\sup\ u(\cdot, t_0) < 1$). From Lemma 4.12 and Theorem 1.4, there exists a measure $\mu \in \mathcal{M}$ such that $u = u_\mu$. Furthermore, from Lemma 4.10, μ is concentrated on $S^{N-1} \times [c, +\infty) \cup \{\infty\}$

for each $c > c^*$. Therefore, $\mu = \mu(\infty)\delta_\infty$. As a consequence, the functions u_n defined in (26) do not depend on x. Neither does u_μ. In other words, $u = u_\mu$ only depends on time t. Actually, if $\mu = \mu(\infty)\delta_\infty$, then $\hat{M} = \mu(\infty)$ and the formula (37) implies that $u_\mu(t) \sim e^{f'(0)\ln\mu(\infty)}e^{f'(0)t}$ as $t \to -\infty$. Therefore, it eventually follows that the set of such solutions u_μ, where $\mu = \mu(\infty)\delta_\infty$ and $\mu(\infty)$ describes $(0, +\infty)$, is equal to the one-dimensional family of solutions $\{t \mapsto \xi(t+h),\ h \in \mathbb{R}\}$.

As a consequence, if a solution u_μ of (1) is such that μ is not concentrated on $\{\infty\}$, then u cannot depend on t only, whence $\sup_{x\in\mathbb{R}^N} u_\mu(x,t) = 1$ for all $t \in \mathbb{R}$. That completes the proof of Theorem 1.5. □

5. Nonplanar travelling waves and radial solutions

In this section, we apply the general results stated in Theorems 1.2 and 1.4, and we deal with special solutions of (1), namely, travelling waves and radial solutions.

5.1. Nonplanar travelling waves

This subsection is devoted to the

Proof of Theorem 1.7. (i) Let $u(x,t)$ be a travelling wave for (1), satisfying (17) for some $(\nu_0, c_0) \in S^{N-1} \times [0, +\infty[$ and let v be defined by (18).

(i-a) Assume that $c_0 < c^*$. From (18), $v(0) = u(-c_0 t\nu_0, t)$ for all $t \in \mathbb{R}$. Since $0 \leqq c_0 < c^*$, Lemma 4.1 yields $\lim_{t\to+\infty} u(-c_0 t\nu_0, t) \to 1$, whence $v(0) = 1$. This is impossible since $0 < v(y) < 1$ for all y.

Before proving the monotonicity properties satisfied by each travelling wave for (1) in a cone of directions (Theorem 1.7 (i-b), let us state the following

Lemma 5.1. *Let $u(x,t)$ be an entire solution of* (1) *such that the fields u_t/u and $\nabla_x u/u$ are globally bounded. Then, for each vector $\rho \in \mathbb{R}^N$ such that $|\rho| = \sqrt{\rho\cdot\rho} < c^* = 2\sqrt{f'(0)}$, we have $u_t + \rho\cdot\nabla_x u > 0$ in $\mathbb{R}^N \times \mathbb{R}$.*

Proof. To this end, it is enough to prove that $\partial_t u(x,t) + \rho\cdot\nabla_x u(x,t) \geqq 0$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$. Indeed, suppose the latter is true. The function $U = \partial_t u + \rho\cdot\nabla_x u$ satisfies the linear parabolic equation

$$\partial_t U = \Delta U + f'(u)U.$$

From the strong parabolic maximum principle, U is then either identically equal to 0 or $U(x,t) > 0$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$. The first case would imply that the function $w(t) = u(\rho t, t)$ is constant, but, since $|\rho| < c^*$, that would be in contradiction with Lemma 4.1. Hence, $U = \partial_t u + \rho\cdot\nabla_x u > 0$ and the conclusion of Lemma 5.1 will follow.

Let us now denote by $v(x,t)$ the function

$$v(x,t) = \frac{\partial_t u(x,t) + \rho\cdot\nabla_x u(x,t)}{u(x,t)}.$$

By assumption, this function v is globally bounded and we then only have to prove that $\inf_{\mathbb{R}^N\times\mathbb{R}} v \geqq 0$.

Suppose for contradiction that $\inf_{\mathbb{R}^N\times\mathbb{R}} v = -\varepsilon < 0$. There exists a sequence $(x_n, t_n) \in \mathbb{R}^N \times \mathbb{R}$ such that $v(x_n, t_n) \to -\varepsilon$ as $n \to +\infty$. Up to extraction of some subsequence, two and only two cases may occur:

Case 1. $u(x_n, t_n) \to \alpha \in (0, 1]$ as $n \to +\infty$,

Case 2. $u(x_n, t_n) \to 0$ as $n \to +\infty$.

Let us first deal with Case 1. After a straightforward calculation, it is found that the function v satisfies

$$v_t = \Delta v + 2\frac{\nabla_x u}{u} \cdot \nabla_x v + \left(f'(u) - \frac{f(u)}{u}\right) v \ \text{ in } \mathbb{R}^N \times \mathbb{R}.$$

Let us set

$$u_n(x,t) = u(x + x_n, t + t_n) \text{ and } \ v_n(x,t) = v(x + x_n, t + t_n).$$

From standard parabolic estimates, the functions u_n converge in $C^1_{\text{loc}}(\mathbb{R}_t)$ and $C^2_{\text{loc}}(\mathbb{R}^N_x)$ to a function u_∞ (up to extraction of some subsequence). The function u_∞ is such that $0 \leqq u_\infty \leqq 1$ and it solves

$$\partial_t u_\infty = \Delta u_\infty + f(u_\infty) \quad \text{in } \mathbb{R}^N \times \mathbb{R}.$$

Furthermore, since $u(x_n, t_n) \to \alpha \in (0, 1]$ as $n \to +\infty$, we have $u_\infty(0, 0) = \alpha > 0$. Therefore, the function $u_\infty(x, t)$ is positive everywhere (because of the strong parabolic maximum principle) and the globally bounded sequences of functions $\nabla_x u_n/u_n$, $f'(u_n)$ and $f(u_n)/u_n$ converge to the globally bounded functions $\nabla_x u_\infty/u_\infty$, $f'(u_\infty)$ and $f(u_\infty)/u_\infty$, respectively.

Similarly, the globally bounded functions v_n converge locally in the sense of the topology $\mathcal{T}$ (up to extraction of some subsequence) to a globally bounded function v_∞, which is equal to

$$v_\infty = \frac{\partial_t u_\infty + \rho \cdot \nabla_x u_\infty}{u_\infty}.$$

The function v_∞ is such that $v_\infty(x, t) \geqq -\varepsilon$ for all $(x, t) \in \mathbb{R}^N \times \mathbb{R}$ and $v_\infty(0, 0) = -\varepsilon$. Furthermore, v_∞ satisfies

$$\partial_t v_\infty = \Delta v_\infty + 2\frac{\nabla_x u_\infty}{u_\infty} \cdot \nabla_x v_\infty + \left(f'(u_\infty) - \frac{f(u_\infty)}{u_\infty}\right) v_\infty \text{ in } \mathbb{R}^N \times \mathbb{R}.$$

The point $(0, 0)$ is a global minimum for the function v_∞ and $v_\infty(0, 0) = -\varepsilon < 0$. On the other hand, $u_\infty(0, 0) = \alpha \in (0, 1]$ and $f'(\alpha) - f(\alpha)/\alpha \leqq 0$ since the function f is concave on $[0, 1]$ and $f(0) = 0$. From the strong parabolic maximum principle for the function v_∞, it follows then that $v_\infty \equiv -\varepsilon$ in $\mathbb{R}^N \times \mathbb{R}^-$. In other words,

$$\frac{\partial_t u_\infty + \rho \cdot \nabla_x u_\infty}{u_\infty} \equiv -\varepsilon < 0$$

in $\mathbb{R}^N \times \mathbb{R}^-$. Since u_∞ is positive,

$$\partial_t u_\infty + \rho \cdot \nabla_x u_\infty < 0 \text{ in } \mathbb{R}^N \times \mathbb{R}^-. \tag{58}$$

But, since u_∞ is a solution of $\partial_t u_\infty = \Delta u_\infty + f(u_\infty)$ such that $u_\infty \leqq 1$, we have either $u_\infty \equiv 1$ or $u_\infty < 1$. The case $u_\infty \equiv 1$ is in contradiction with (58). The case $u_\infty < 1$ means that the function u_∞ is a solution of (1), such that $0 < u_\infty < 1$. Since $|\rho| < c^*$, Lemma 4.1 implies in particular that $w(t) = u_\infty(\rho t, t) \to 0$ as $t \to -\infty$. But this positive function w is decreasing for $t \leqq 0$ by (58). We have then reached a contradiction. As a conclusion, Case 1 is ruled out.

Let us now deal with Case 2. Up to extraction of some subsequence,

$$u(x_n, t_n) \to 0 \ \text{ as } n \to +\infty.$$

Let us set

$$w_n(x,t) = \frac{u(x + \rho t + x_n, t + t_n)}{u(x_n, t_n)}\, e^{\frac{1}{2}\rho\cdot x}, \quad (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

Since the fields u_t/u and $\nabla_x u/u$ are globally bounded, there exists a constant C such that $w_n(x,t) \leqq e^{C(|t|+|x|)}$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$ and all n. In particular, the sequence (w_n) is locally bounded and the functions $(x,t) \mapsto u(x + x_n, t + t_n)$ approach 0 locally in $\mathbb{R}^N \times \mathbb{R}$. On the other hand, each function w_n satisfies

$$(w_n)_t = \Delta w_n + \left(\frac{f(u(x + \rho t + x_n, t + t_n))}{u(x + \rho t + x_n, t + t_n)} - \frac{1}{4}|\rho|^2 \right) w_n, \quad (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

From standard parabolic estimates, the functions w_n converge locally in the sense of the topology $\mathcal{T}$ (up to extraction of some subsequence), to a nonnegative and locally bounded function w_∞. The function w_∞ solves

$$\partial_t w_\infty = \Delta w_\infty + (f'(0) - \frac{1}{4}|\rho|^2)\, w_\infty \ \text{ in } \mathbb{R}^N \times \mathbb{R} \tag{59}$$

and it satisfies

$$\forall t \in \mathbb{R},\ \forall x \in \mathbb{R}^N, \quad w_\infty(x,t) \leqq e^{C(|t|+|x|)}. \tag{60}$$

Due to the definition of w_n and to the choice of (x_n, t_n), we have

$$\partial_t w_n(0,0) = \frac{\partial_t u(x_n, t_n) + \rho \cdot \nabla_x u(x_n, t_n)}{u(x_n, t_n)} = v(x_n, t_n) \to -\varepsilon \ \text{ as } n \to +\infty.$$

Hence,

$$\partial_t w_\infty(0,0) = -\varepsilon. \tag{61}$$

Choose now any point $(x,t) \in \mathbb{R}^N \times \mathbb{R}$. Because of (59) and (60), $w_\infty(x,t)$ can be written as

$$w_\infty(x,t) = e^{(f'(0) - \frac{1}{4}|\rho|^2)(t+k)} \int_{\mathbb{R}^N} p(x - y, t + k) w_\infty(y, -k)\, dy$$

for all $k > |t|$, where $p(z,\tau) = (4\pi\tau)^{-N/2} e^{-\frac{|z|^2}{4\tau}}$ for any $\tau > 0$ and $z \in \mathbb{R}^N$. As a consequence,

$$\partial_t w_\infty(x,t) = e^{(f'(0)-\frac{1}{4}|\rho|^2)(t+k)} \int_{\mathbb{R}^N} \partial_t p(x-y, t+k)\, w_\infty(y,-k)\, dy \\ + (f'(0) - \frac{1}{4}|\rho|^2)\, w_\infty(x,t).$$

Notice that $\partial_\tau p(z,\tau) \geqq -\frac{N}{2\tau} p(z,\tau)$ for all $\tau > 0$ and $z \in \mathbb{R}^N$. Since w_∞ is nonnegative, it follows that

$$\partial_t w_\infty(x,t) \geqq \left(f'(0) - \frac{1}{4}|\rho|^2 - \frac{N}{2(t+k)} \right) w_\infty(x,t).$$

Passing to the limit $k \to +\infty$ in the above formula leads to

$$\partial_t w_\infty(x,t) \geqq (f'(0) - \frac{1}{4}|\rho|^2)\, w_\infty(x,t).$$

Since $|\rho| < c^* = 2\sqrt{f'(0)}$ and $w_\infty \geqq 0$, we get $\partial_t w_\infty(x,t) \geqq 0$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$. That is in contradiction with (61). Therefore, Case 2 is ruled out too and the proof of Lemma 5.1 is complete. □

Let us now return to the proof of Theorem 1.7.

(i-b) Let $\nu \in S^{N-1}$ be such that $\nu \cdot \nu_0 > \cos(\arcsin(\frac{c^*}{c_0}))$. Let ρ be the vector defined by $\rho = c_0(\nu_0 \cdot \nu)\nu - c_0\nu_0$. We have

$$|\rho|^2 = c_0^2 - c_0^2(\nu_0 \cdot \nu)^2 < c_0^2 - c_0^2 \cos^2(\arcsin(\frac{c^*}{c_0})) = (c^*)^2.$$

Let us now check that the function u satisfies the assumption of Lemma 5.1, that is to say that u_t/u and $\nabla_x u/u$ are globally bounded. Indeed, since u is written as $u(x,t) = v(x + c_0 t \nu_0)$, we have $u_t/u = c_0 \partial_{\nu_0} v/v$ and $\nabla_x u/u = \nabla v/v$. Therefore, we only have to check that $\nabla v/v$ is bounded. But since v is a positive solution of $\Delta v - c_0 \partial_{\nu_0} v + f(v) = 0$ in $\mathbb{R}^N$, Schauder interior estimates imply that $|\nabla v(y)| \leqq C_1 \max_{|z-y| \leqq 1} v(z)$ and Harnack-type inequalities [14] imply that $\max_{|z-y|\leqq 1} v(z) \leqq C_2 \min_{|z-y|\leqq 1} v(z) \leqq C_2 v(y)$ for some constants C_1 and C_2 independent of y. Therefore, $|\nabla v(y)| \leqq C_1 C_2 v(y)$ for all $y \in \mathbb{R}^N$, which was the desired result.

As a consequence, Lemma 5.1 can be applied and yields $\partial_t u + \rho \cdot \nabla_x u > 0$ in $\mathbb{R}^N \times \mathbb{R}$. Due to the definition of v, it follows that $c_0 \nu_0 \cdot \nabla v + \rho \cdot \nabla v > 0$ in $\mathbb{R}^N$, i.e., $c_0(\nu_0 \cdot \nu)\nu \cdot \nabla v > 0$. Since $\nu_0 \cdot \nu > 0$ and $c_0 > 0$, we get $\nu \cdot \nabla v > 0$ in $\mathbb{R}^N$. Let ν be as above and choose $a \in \mathbb{R}^N$. We have $v(a + c_0(\nu \cdot \nu_0)s\nu) = u(a + c_0(\nu \cdot \nu_0)s\nu - c_0 s\nu_0, s)$. From the calculation above,

$$\limsup_{s \to +\infty} \frac{|a + c_0(\nu \cdot \nu_0)s\nu - c_0 s \nu_0|}{|s|} < c^*.$$

From Lemma 4.1, $\lim_{s\to-\infty} v(a + c_0(\nu \cdot \nu_0)s\nu) = 0$ and $\lim_{s\to+\infty} v(a + c_0 (\nu \cdot \nu_0)s\nu) = 1$. Lastly, since $c_0(\nu \cdot \nu_0) > 0$, the conclusion in (i-b) follows.

(i-c) Suppose that $c_0 = c^*$. From (i-b) and by continuity, the function v is then nondecreasing in any direction ν such that $\nu \cdot \nu_0 \geqq 0$. It is then both nondecreasing and nonincreasing in any direction ν such that $\nu \cdot \nu_0 = 0$. Therefore, v is planar and can be written as $v(y) = w(\nu_0 \cdot y)$. The function w satisfies $0 < w < 1$ on $\mathbb{R}$ and $w'' - c^* w' + f(w) = 0$ in $\mathbb{R}$ with $w(-\infty) = 0$, $w(+\infty) = 1$ (from (i-b)). As a consequence, $w(s) = \varphi_{c^*}(s + h)$ for some $h \in \mathbb{R}$. In other words, $u(x,t) = \varphi_{c^*}(x \cdot \nu_0 + c^* t + h)$ is a planar travelling wave propagating with the speed c^*.

(ii-a) From Theorem 1.2, the only thing we have to prove is that, when

$$\mu = \sum_{i=1}^{k} m_i \delta_{(\nu_i, c^*)} + \hat{\mu} \ \in \mathcal{M}$$

is concentrated on $S_{(\nu_0, c_0)}$ for some (ν_0, c_0), then u_μ is a travelling wave for (1) satisfying (17) and the function v_μ defined by (18) is the smallest solution of (19) such that (20) holds.

Let μ be as above. Since μ is concentrated on $S_{(\nu_0, c_0)}$, we have $\mu(\infty) = 0$. By definition, $u_\mu(x,t)$ is the limit of $u_n(x,t)$ where u_n is the solution of the Cauchy problem

$$\begin{aligned} (u_n)_t &= \Delta u_n + f(u_n), \quad t > -n,\ x \in \mathbb{R}^N \\ u_n(x,-n) &= \max\Bigg(\max_{1 \leqq i \leqq k} (\varphi_{c^*}(x \cdot \nu_i - c^* n + c^* \ln m_i)), \\ &\qquad \int_{S^{N-1} \times (c^*, +\infty)} \varphi_c(x \cdot \nu - cn + c \ln \hat{M}) \frac{1}{\hat{M}} d\tilde{\mu} \Bigg). \end{aligned} \tag{62}$$

Choose any $(x,t) \in \mathbb{R}^N \times \mathbb{R}$ and $\tau \in \mathbb{R}$. We shall prove that $u_\mu(x, t+\tau) = u_\mu(x + c_0 \tau \nu_0, t)$. The proof is quite similar to that given in Section 3.6 to prove property (iii) of Theorem 1.2. Observe that $u_\mu(x, t+\tau) = \lim_{n \to +\infty} u_n(x, t+\tau)$ and that $u_n(x, t+\tau)$ can be written as $u_n(x, t+\tau) = U_n(x,t)$ where U_n is the solution of the Cauchy problem

$$\begin{aligned} (U_n)_t &= \Delta U_n + f(U_n), \quad t > -n - \tau,\ x \in \mathbb{R}^N, \\ U_n(x, -n-\tau) &= u_n(x, -n). \end{aligned}$$

Since $c_0 \nu_0 \cdot \nu = c$ for each $(\nu, c) \in S_{(\nu_0, c_0)}$ and μ is concentrated on $S_{(\nu_0, c_0)}$, the function $U_n(x, -n-\tau)$ can be rewritten as

$$\begin{aligned} &U_n(x, -n-\tau) \\ &\quad = \max\Bigg(\max_{1 \leqq i \leqq k} (\varphi_{c^*}((x + c_0 \nu_0 \tau) \cdot \nu_i - c^*(n+\tau) + c^* \ln m_i)), \\ &\qquad\quad \int_{S^{N-1} \times (c^*, +\infty)} \varphi_c((x + c_0 \nu_0 \tau) \cdot \nu - c(n+\tau) + c \ln \hat{M}) \frac{1}{\hat{M}} d\tilde{\mu} \Bigg). \end{aligned}$$

In other words, $U_n(x, -n-\tau) = u_{n+\tau}(x + c_0 \nu_0 \tau, -n-\tau)$, where $u_{n+\tau}$ is defined as in (62) by replacing n with $n+\tau$. By uniqueness of the Cauchy problem, it follows that $U_n(x,t) = u_{n+\tau}(x + c_0\nu_0\tau, t)$ for each n. On the other hand, as already observed in Section 3, the functions $u_n(x,t)$ are nondecreasing with respect to $n \geqq 0$ (n may not necessarily be an integer). As a consequence, $u_{n+\tau}(x + c_0\nu_0\tau, t) \to u_\mu(x + c_0\nu_0\tau, t)$ as $n \to +\infty$. Remember now that $u_\mu(x, t+\tau) = \lim_{n\to+\infty} U_n(x,t)$ by definition of U_n. Eventually, (17) follows.

From the first inequality in (30) and using the definition of $v_\mu(y) = u_\mu(y,0)$, we immediately get (20). On the other hand, let $w(y)$ be a solution of (19) such that w satisfies (20) (with w instead of v). The function $U(x,t) = w(x + c_0 t\nu_0)$ is a solution of (1) such that

$$\begin{aligned} U(x,-n) &= w(x - c_0 n\nu_0) \\ &\geqq \max\Bigg(\max_{1\leqq i\leqq k} (\varphi_{c^*}((x - c_0 n\nu_0)\cdot\nu_i + c^*\ln m_i)), \\ &\qquad \int_{S^{N-1}\times(c^*,+\infty)} \varphi_c((x - c_0 n\nu_0)\cdot\nu + c\ln\hat{M})\frac{1}{\hat{M}}d\tilde{\mu}\Bigg) \\ &= u_n(x,-n) \end{aligned}$$

since μ is concentrated on $S_{(\nu_0,c_0)}$ and $c_0\nu_0\cdot\nu = c$ for each $(\nu,c) \in S_{(\nu_0,c_0)}$. Therefore, $U(x,t) \geqq u_n(x,t)$ for each n and passing to the limit $n \to +\infty$ leads to $U(x,t) \geqq u_\mu(x,t)$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$. In particular, $w(y) = U(y,0) \geqq u_\mu(y,0) = v_\mu(y)$, which gives the desired result.

(ii-b) Take $c_0 > c^*$ and $\nu_0 \in S^{N-1}$ and define $\mathcal{M}_{(\nu_0,c_0)}$ as the set

$$\mathcal{M}_{(\nu_0,c_0)} = \{\mu \in \mathcal{M},\ \mu \text{ is concentrated on } S_{(\nu_0,c_0)}\}.$$

The application $\mu \mapsto v_\mu(\cdot)$ $(= u_\mu(\cdot,0))$ is one-to-one on $\mathcal{M}_{(\nu_0,c_0)} \cap \hat{\mathcal{M}}$. Indeed, if $v_{\mu_1} = v_{\mu_2}$, then it is immediately found that $u_{\mu_1} = u_{\mu_2}$, whence $\mu_1 = \mu_2$ from Theorem 1.2. Furthermore, if μ^n $(\in \mathcal{M}_{(\nu_0,c_0)}) \rightharpoonup \mu$ $(\in \mathcal{M}_{(\nu_0,c_0)})$ in the sense described in Section 1.1, then $u_{\mu^n} \to u_\mu$ in the sense of $\mathcal{T}$, whence $v_{\mu^n} \to v_\mu$ in $C^2_{\text{loc}}(\mathbb{R}^N)$. Therefore, in dimension $N \geqq 2$, there exists an infinite-dimensional manifold of solutions v of (19) such that $0 < v < 1$.

(ii-c) Let u be an entire solution of (1) of the type u_μ and assume that u is a travelling wave satisfying (17). We have to prove that the measure μ is concentrated on $S_{(\nu_0,c_0)}$. Let v_μ be the function defined as in (18) by $u_\mu(x,t) = v_\mu(x + c_0 t\nu_0)$. From the lower bound in (30), it follows that

$$\begin{aligned} v_\mu(y) &= u_\mu(y - c_0 t\nu_0, t) \\ &\geqq \max\Bigg(\max_{1\leqq i\leqq k} (\varphi_{c^*}((c^* - c_0\nu_0\cdot\nu_i)t + y\cdot\nu_i + c^*\ln m_i)), \\ &\qquad \int_{\hat{X}} \varphi_c((c - c_0\nu_0\cdot\nu)t + y\cdot\nu + c\ln\hat{M})\frac{1}{\hat{M}}\,d\hat{\mu}\Bigg) \\ &\geqq \max\Bigg(\max_{1\leqq i\leqq k,\ c_0\nu_0\cdot\nu_i > c^*} (\varphi_{c^*}((c^* - c_0\nu_0\cdot\nu_i)t + y\cdot\nu_i + c^*\ln m_i)), \\ &\qquad \int_{\hat{X}\cap\{c_0\nu_0\cdot\nu > c\}} \varphi_c((c - c_0\nu_0\cdot\nu)t + y\cdot\nu + c\ln\hat{M})\frac{1}{\hat{M}}\,d\hat{\mu}\Bigg). \end{aligned}$$

If there exists an integer $i \in \{1, \cdots, k\}$ such that $c_0\nu_0 \cdot \nu_i > c^*$, then the right hand side of the above inequality goes to 1, for each $y \in \mathbb{R}^N$, as t goes to $-\infty$. That would imply that v_μ is identically equal to 1, which is impossible. Similarly, if $\beta := \mu(\hat{X} \cap \{c_0\nu_0 \cdot \nu > c\})$ is positive, then $\hat{M}$ is itself positive and, passing to the limit $t \to -\infty$ in the above inequality leads to, through Lebesgue's dominated convergence theorem, $v_\mu(y) \geqq \beta\frac{1}{\hat{M}}$ for all $y \in \mathbb{R}^N$. Therefore, $u_\mu(x,t) \geqq \beta\frac{1}{\hat{M}}$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$. Since $\beta\frac{1}{\hat{M}}$ is a positive real number, that contradicts property (16). Eventually, the measure of the set $X \cap \{c_0\nu_0 \cdot \nu > c\}$ is zero.

Similarly, by studying the limit as $t \to +\infty$, it follows that $\mu(X \cap \{c_0\nu_0 \cdot \nu > c\}) = 0$. As a consequence, the measure μ is concentrated on the set $S_{(\nu_0,c_0)}$.

(iii-a), (iii-b) Property (iii-a) immediately follows from (16) and from the definition of v in (18). Property (iii-b) follows from Theorem 1.4 and from property (ii-c) in Theorem 1.7.

That completes the proof of Theorem 1.7. □

5.2. Radial solutions

This subsection is devoted to the

Proof of Theorem 1.8. (i-a). Take any couple $(\mu, a) \in \mathcal{M}_R \times \mathbb{R}^N$ and define $u_{\mu,a}(x,t) = u_\mu(x-a,t)$. Proving that $u_{\mu,a}$ is radially symmetric with respect to a, it is equivalent to proving that u_μ is radially symmetric with respect to the origin. By definition, we have $u_\mu(x,t) = \lim_{n\to+\infty} u_n(x,t)$, where u_n is the solution of the Cauchy problem (26) with initial condition

$$u_n(x,-n) = \int_{\hat{X}} \varphi_c(x\cdot\nu - cn + c\ln\hat{M})\frac{1}{\hat{M}}\,d\hat{\mu}$$

(remember that $\mu^* = 0$ for $\mu \in \mathcal{M}_R$, whence $\hat{M} = \mu(X) > 0$). For any rotation $\rho \in SO(N)$, we have $u_n(\rho(x), -n) = u_n(x,-n)$, because μ is itself rotationally invariant. By uniqueness of the Cauchy problem, it follows that $u_n(\rho(x),t) = u_n(x,t)$ for all $t \geqq -n$ and $x \in \mathbb{R}^N$. The passage to the limit $n \to +\infty$ leads to $u_\mu(\rho(x),t) = u_\mu(x,t)$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$. In other words, the function u_μ is radially symmetric with respect to the origin, that is to say, the function $u_{\mu,a}$ is radially symmetric with respect to the point a.

The function v defined by $u_{\mu,a}(x,t) = v(|x-a|,t)$ clearly satisfies (21). Furthermore, if μ is not concentrated on the single point $\{\infty\}$, then $\sup_{x\in\mathbb{R}^N} u_{\mu,a}(x,t) = 1$ for all $t \in \mathbb{R}$. Since $u_{\mu,a}(x,t) < 1$ for all x and t, we conclude that $v(r,t) \to 1$ as $r \to +\infty$, for all $t \in \mathbb{R}$.

Consider now a sequence $(\mu^n, a^n) \in \mathcal{M}_R \times \mathbb{R}^N$ such that $\mu^n \rightharpoonup \mu \in \mathcal{M}_R$ (in the sense of Section 1.1) and $a^n \to a \in \mathbb{R}^N$. From Theorem 1.2, the functions u_{μ^n} converge to the function u_μ in the sense of the topology $\mathcal{T}$. Since these functions (u_{μ^n}) are locally bounded, say, up to their first-order (or second-order) derivatives in t (resp., x), we conclude that the functions u_{μ^n,a^n} converge to the function $u_{\mu,a}$ in $\mathcal{T}$.

Lastly, choose two measures μ_1 and μ_2 in $\mathcal{M}_R$, such that μ_1 and μ_2 are not concentrated on $\{\infty\}$. Let a_1 and a_2 be two points in $\mathbb{R}^N$. Suppose that $u_{\mu_1,a_1} = u_{\mu_2,a_2}$. We have

$$u_{\mu_1,a_1}(a_1, 0) = u_{\mu_2,a_2}(a_1, 0) = u_{\mu_2,a_2}(2a_2 - a_1, 0)$$

since u_{μ_2,a_2} is radially symmetric with respect to the point a_2. Similarly, it is found that

$$u_{\mu_2,a_2}(2a_2 - a_1, 0) = u_{\mu_1,a_1}(2a_2 - a_1, 0) = u_{\mu_1,a_1}(3a_1 - 2a_2, 0)$$

since u_{μ_1,a_1} is radially symmetric with respect to the point a_1. Going one step further, we get $u_{\mu_1,a_1}(3a_1-2a_2, 0) = u_{\mu_2,a_2}(3a_1-2a_2, 0) = u_{\mu_2,a_2}(4a_2-3a_1, 0)$. By induction, it is then found that

$$u_{\mu_1,a_1}(a_1, 0) = u_2(2k(a_2 - a_1) + a_1, 0)$$

for each integer $k \in \mathbb{N}$. Since μ_2 is not concentrated on the single point $\{\infty\}$, we have $u_{\mu_2,a_2}(x, 0) \to 1$ as $|x| \to +\infty$. On the other hand, $u_{\mu_1,a_1}(a_1, 0) < 1$. Therefore, by passing to the limit $k \to +\infty$, it follows that $a_2 - a_1 = 0$. As a consequence, since we had assumed that $u_{\mu_1,a_1} = u_{\mu_2,a_2}$, we get $u_{\mu_1} = u_{\mu_2}$ and Lemma 3.5 yields $\mu_1 = \mu_2$. Hence, $(\mu_1, a_1) = (\mu_2, a_2)$. In other words, the map $(\mu, a) \to u_{\mu,a}$ is one-to-one if μ is in the set of measures $\mu \in \mathcal{M}_R$ which are not concentrated on the single point $\{\infty\}$.

(i-b) Fix $a = 0$. The map $\mu \in \mathcal{M}_R \mapsto v_\mu$ such that $v_\mu(|x|, t) = u_\mu(x, t)$ ranges in the set of solutions $v(r, t)$ of (21). Furthermore, with the same arguments which were used in the proof of (i-a), it follows that this map is one-to-one on the set of measures μ which are not concentrated on $\{\infty\}$. On the other hand, this map is continuous in the sense that if $\mu^n \rightharpoonup \mu$, then $v_{\mu^n} \to v_\mu$ in C^1_{loc} with respect to t and in C^2_{loc} with respect to r.

(ii) Property (2) immediately follows from (16) and from Theorem 1.4.

That completes the proof of Theorem 1.8. □

Acknowledgements. We are grateful to the referee for the valuable comments. FRANÇOIS HAMEL is grateful to R. FEFFERMAN and N. NADIRASHVILI for the invitation to the Department of Mathematics of the University of Chicago, where part of this work was done. NIKOLAÏ NADIRASHVILI was partly supported by NSF grant.

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(*Accepted October 30, 2000*)
Published online March 21, 2001 –