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Neoclassical life-cycle consumption: a textbook example★

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Summary. We present a simple neoclassical life-cycle model in continuous time, in which the effects of endogenous labor supply, uncertain lifetime, and family composition on consumption and income profiles are jointly analyzed. Due to a parsimonious specification, analytical solutions for consumption growth are available for constant intertemporal elasticity of substitution preferences. Without relying on borrowing constraints, the model can generate a hump in the consumption profile, and a comovement of consumption and income during working life.

Keywords and Phrases: Life-cycle consumption profiles, Consumption-income correlation.

JEL Classification Number: D91.

1 Introduction

The basic Modigliani and Brumberg (1954) life-cycle model implies a constant growth rate of consumption over time. In a continuous time life-cycle model, consumption growth can be easily computed as

$$\frac{\dot{c}(t)}{c(t)} = \eta(t)\,(r - \rho)\,,$$

where r and ρ are interest rate and rate of time preference, respectively, and $\eta(t)$ is the elasticity of intertemporal substitution. Most empirical studies, however, have

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found strongly correlated, hump–shaped consumption and income paths over the life-cycle. Consumption tracking income has been recognized as a robust empirical fact since the analysis of Thurow (1969) and Ghez and Becker (1975, chapter 2), who demonstrated that income and consumption expenditures both peak around age 50. Carroll and Summers (1991) draw income and consumption profiles for a number of educational and occupational subgroups of the population, and take the strong correlation between consumption and expected income growth as an indication for liquidity or borrowing constraints, which prevent agents from achieving the efficient consumption path. Carroll (1997) attributes the "consumption/income parallel" to buffer–stock savings of impatient agents in the presence of income uncertainty. Gourinchas and Parker (1999), who estimate a structural model of optimal life-cycle consumption under realistic income uncertainty, find that precautionary savings by young household and life-cycle savings of the middle-aged account for the correlation between consumption and income.

The aim of this paper is to present a simple neoclassical life-cycle model which is able to replicate the most important empirical regularities over the life-cycle without relying on borrowing constraints and income uncertainty. The framework blends elastic labor supply, uncertain lifetime, and a demographic structure into a single model, for which elegant analytical solutions can be obtained. It can therefore be useful as a (textbook) benchmark model, and as a building block for larger macroeconomic models.

The paper pulls together results from many previous studies of life-cycle consumption, following the seminal contributions of Yaari (1965) and Heckman (1974). The importance of lifetime uncertainty on optimal intertemporal consumption and saving decisions was first pointed out by Yaari (1965). Heckman (1974) has shown that nonseparability of consumption and leisure can explain the comovement of consumption and earnings. The consumption path depends on the wage rate if the latter changes systematically over the life-cycle, as leisure can be substituted for consumption. Heckman (1976) extends his earlier model by allowing human capital (and consequently the wage rate) to evolve endogenously. Demographic variables have played a somewhat smaller role in the formal analysis of consumption and income, although the relevance of family composition for life-cycle choices has already been observed by the fathers of the life-cycle model, as in Modigliani and Ando (1957) and Modigliani and Brumberg (1954). Attanasio and Browning (1995) argue that the apparent "excess sensitivity" of consumption to income changes disappears when one controls for age, demographic variables and labor supply patterns of both spouses.

The paper is organized as follows. The general setup and basic assumptions are outlined in Section 2. Section 3 presents the life-cycle model with a time–dependent discount factor, followed by two applications: first, the impact of lifetime uncertainty with and without annuities, and second, the inclusion of a stylized demographic structure. Section 4 explores the consequences of endogenous labor supply in view of a distinctive age–wage pattern. A modification which allows for the endogenous accumulation of human capital is also discussed. The results are summarized in Section 5.

2 Model setup and assumptions

To keep the model tractable, a number of simplifying assumptions are imposed. Time is continuous, and all time–dependent variables are assumed continuous and differentiable. There is a fixed and known maximum age $T_{\max}$, to which people can live. Households without bequest or gift motives maximize their expected lifetime utility. Income opportunities are non–stochastic, and retirement is voluntary, induced by the age–wage profile.

Our economy is equiped with a single asset which—under certainty—yields a constant real interest rate r.[1] There is no wedge between borrowing and lending rates, and agents can lend and borrow freely at an interest rate $R(t)$, which may depend the relevant market structure and on their age, but not on the size of their actual asset holdings. Apart from the requirement that agents are not allowed to die in debt at the very last period $T_{\max}$, there are no constraints on asset holdings $a(t)$. Note that under lifetime uncertainty, lending without security to a person with a mortality risk is equivalent to providing insurance, as already argued in Yaari (1965). We therefore assume that agents face sufficiently low income during the later periods of their lives, preventing them from going into debt when very old.

Preferences are time separable. The instantaneous utility function $U[\cdot]$ depends on consumption $c(t)$ and leisure $l(t)$ and satisfies the usual concavity requirement. To get closed–form solutions for endogenous labor supply, the utility function is specialized to the constant intertemporal elasticity of substitution case,

$$U[c,l] = \begin{cases} \frac{(c^{\theta} l^{1-\theta})^{1-\sigma}}{1-\sigma} & \text{for} \quad \sigma \neq 1, \\ \theta \ln c + (1-\theta) \ln l & \text{for} \quad \sigma = 1. \end{cases} \tag{1}$$

$\frac{1}{\sigma}$ is the elasticity of intertemporal substitution and θ describes the trade–off between leisure and consumption.

3 Age–dependent discount factors

Let the discount factor $\Phi(t)$ be a continuous function of age, but independent of the control variables. The optimization problem of an agent facing an exogenous income stream $m(t)$ can thus be expressed as maximizing lifetime utility

[1] The real interest rate r is usually assumed to be greater than the pure rate of time preference ρ. Hurd (1989) points out that most estimates for the rate of time preference are upward biased as mortality risks are not accounted for. If lifetime uncertainty is included, the rate of time preference is estimated to be very low or even negative. In addition, Davies (1981) observes that non–credit–constraint households consume at an increasing rate, which is only consistent with an interest rate greater than the rate of time preference.

$$\mathscr{U}(0) = \int_0^{T_{\max}} U[c(t)]\Phi(t)\,dt. \tag{2}$$

subject to the constraints

$$\begin{aligned} \dot{a}(t) &= a(t)R(t) + m(t) - c(t) \\ a(0) &= a_0 \\ a(T_{\max}) &\geq 0 \end{aligned} \tag{3}$$

Equations (2) and (3) form a standard problem in optimal control:

Proposition 1. The optimal consumption path resulting from the optimization problem as stated in equations (2) and (3), can be expressed as

$$\frac{\dot{c}(t)}{c(t)} = \eta(t)\left\{R(t) + \frac{\dot{\Phi}(t)}{\Phi(t)}\right\} \tag{4}$$

where $\eta(t) \equiv -\frac{U_c}{cU_{cc}} > 0$ is the elasticity of intertemporal substitution.

Proof. See Appendix A.

Note that apart from concavity, no assumptions were made about the form of the utility function. It is straightforward to prove that a potential non–separability between consumption and leisure (as presented in the next section) does not change the result, as long as the discount factor does not depend on the control variables.

A constant rate of time preference ρ forms a special case with $\Phi(t) = \exp(-\rho t)$. We get $\frac{\dot{\Phi}(t)}{\Phi(t)} = -\rho$, and the resulting consumption trajectory represents the standard life-cycle model with $\frac{\dot{c}(t)}{c(t)} = \eta(t)\,(r - \rho)$.

3.1 Application I: Uncertain lifetime and annuity markets

We now consider individuals whose lifetime $\tilde{T}$ is a random variable with survival distribution $\Psi(t)$ and support $[0, T_{\max}]$, $0 < T_{\max} < \infty$. The hazard rate of death—or mortality rate—$h(t)$ is defined as[2]

$$h(t) \equiv -\frac{d}{dt}\ln\Psi(t) = -\frac{\dot{\Psi}(t)}{\Psi(t)} \tag{5}$$

Lifetime uncertainty can easily be accounted for by adjusting the relevant discount factor. If the pure rate of time preference is ρ, the inclusion of the survival probability $\Psi(t)$ yields an implicit discount factor $\Phi = \{\exp(-\rho t)\Psi(t)\}$, which decreases over the life-cycle. According to Proposition 1, the growth rate of consumption is

[2] Due to a finite maximum lifetime, the density function $\frac{-d\Psi(t)}{dt}$ might have a mass point at $T_{\max}$, and $h(T_{\max}) = \infty$. Typically hazard rates are very low until about age 60 whereafter they exhibit an exponential increase.

$$\frac{\dot{c}(t)}{c(t)} = \eta(t)\left\{R(t) - \rho - h(t)\right\}. \tag{6}$$

If annuities are actuarially fair, consumers will hold their entire wealth in the form of annuities. The applicable rate of return then equals the regular interest rate r plus the instantaneous probability of death or hazard rate $h(t)$, $R(t) = r + h(t)$. This leads again to a monotonic consumption path as in the basic life-cycle model,

$$\frac{\dot{c}(t)}{c(t)} = \eta(t)\left\{r - \rho\right\}. \tag{7}$$

In a world with perfect annuity markets, consumption with uncertain lifetime grows at the same rate as under certainty, a result first derived by Barro and Friedman (1977). The implicitly higher rate of time preference is exactly offset by a corresponding discount in the budget constraint (equation (22), Appendix A) for future income and consumption.

Although private annuity markets do exist, participation is generally limited, as shown by Friedman and Warshawsky (1990). If individuals are restricted from writing debt contracts with payoffs contingent on survival, all agents face the same interest rate, and therefore $R(t) = r$, irrespective of age.[3] The growth rate of consumption is then

$$\frac{\dot{c}(t)}{c(t)} = \eta(t)\left\{r - \rho - h(t)\right\}. \tag{8}$$

Thus, the consumption profile depends not only on the difference between interest rate and rate of time preference, but also on the mortality hazard $h(t)$. Consumption reaches its lifetime maximum when $h(t) = r - \rho$, and declines thereafter. Lifetime uncertainty alone can partly account for a decrease in consumption for the elderly. However, consumption in this simple setup peaks much later in life than both consumption observed in the data and a typical income pattern.

3.2 Application II: A model with a family structure

As many authors have pointed out, lifetime consumption also depends on the size of the family.[4] For ease of exposition, we follow an approach mainly used in development economics which maximizes utility in consumption per "adult equivalent", a term coined by Modigliani and Ando (1957) and recently used by Attanasio (1995). Consumption per family member is therefore not prespecified, but follows from an optimizing strategy of the household head.

Let the number of adult equivalents $n(t)$ be a continuous and differentiable function. If the relevant consumption for utility maximization is consumption per

[3] A number of imperfect annuity contracts can be easily integrated into our framework. The offered rate of return can, for example, be a fraction γ of the perfect insurance, $R[r, h(t)] = \max\{r, \gamma(r + h(t))\}$, or the insurance company can charge a fee f, $R[r, h(t)] = \max\{r, r + h(t) - \mathsf{f}\}$.

[4] See Browning (1992) for an excellent review of determinants of consumption in models with a family structure.

"adult equivalent", the dependency of the utility on the size of the family can be captured by the following utility function

$$U[c(t)] = \frac{\left(\frac{c(t)}{n(t)}\right)^{1-\sigma}}{1-\sigma} = n(t)^{-(1-\sigma)} \frac{c(t)^{1-\sigma}}{1-\sigma} \tag{9}$$

The expression $n(t)^{(\sigma-1)}$ acts like an additional time–dependent discount factor. A time–varying family size changes the marginal utility of consumption, and adds an additional *additive* term to the growth rate of consumption,

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\sigma}\left\{(R(t) - \rho - h(t)) + (\sigma - 1)\frac{\dot{n}(t)}{n(t)}\right\}. \tag{10}$$

In fact, any other household characteristics that yields a multiplicatively separable utility specification can be dealt with in a similar way.

4 Endogenous labor supply

If consumption goods and leisure are substitutes (i.e. $0 < \theta < 1$ in the CIES case), agents can be expected to adjust their labor supply according to the changing price of labor, working more when wages are high and enjoying more leisure and consuming less when wages are low. In order to arrive at a well defined wage determination, it is postulated that the wage rate is proportional to an age–dependent labor productivity $e(t)$. An individual who supplies $(1 - l(t))$ units of labor thus gets a labor income of $m(t) = (1 - l(t))e(t)w$. We retain longevity uncertainty from the previous section, and assume closed annuity markets ($R(t) \equiv r$).

As usual, individuals maximize their lifetime utility

$$\mathscr{U}(0) = \int_0^{T_{\max}} U[c(t), l(t)] \exp(-\rho t)\, \Psi(t)\, dt, \tag{11}$$

subject to the constraints

$$\begin{aligned} \dot{a}(t) &= a(t)r + we(t)(1 - l(t)) - c(t) && (12)\\ l(t) &\leq 1 && (13)\\ a(0) &= a_0 \\ a(T_{\max}) &\geq 0 \end{aligned}$$

Again, equations (11) and (12)–(13) form a standard problem in optimal control, which can be solved accordingly.

Proposition 2. The optimal consumption trajectory resulting from optimization problem (11)–(13) is

$$\frac{\dot{c}(t)}{c(t)} = \begin{cases} \frac{1}{\sigma}\left((r - \rho - h(t)) + (\sigma - 1)(1 - \theta)\frac{\dot{e}(t)}{e(t)}\right) & \text{if} \quad l(t) < 1 \\ \frac{1}{\theta(\sigma-1)+1}\left(r - \rho - h(t)\right) & \text{if} \quad l(t) = 1 \end{cases} \tag{14}$$

Proof. See Appendix B.

In addition to being dependent on the mortality hazard, consumption growth is now also a linear function of productivity growth as long as the agent choses to work. Note that the growth rate of consumption can be discontinuous at the transition age between work and retirement. For standard parametrization of the utility function, the constraint $l(t) < 1$ usually does not bind for ages under 65 (see also Figure 1).

The factor $(\sigma - 1)(1 - \theta)/\sigma$ being less than one for $\sigma > 1$, implies that the consumption profile is flatter than the productivity profile. Assuming continuity of both labor productivity and its first derivative, it is clear from the mean value theorem that the growth rate of consumption will be zero at an age that lies in between the time of maximal labor productivity[5] ($\frac{\dot{e}}{e} = 0$) and the peak in consumption due to uncertain lifetime alone ($r - \rho = h(t)$). Labor income attains its maximum even before productivity peaks. Using the linear relationship between consumption and leisure (see Appendix B), $l(t) = c(t)\frac{1-\theta}{\theta e(t) w}$, labor income $m(t)$ can be written as

$$m(t) = (1 - l(t))e(t)w = e(t)w - c(t)\frac{1 - \theta}{\theta}.$$

Assuming $\dot{m}(0) > 0$, the mean value theorem states that the time derivative of income has a root between $t = 0$ and the time where $\dot{e}(t) = 0$.

In the data, however, total income and consumption peak at about the same age. Two factors mitigate the discrepancy: First, from basic life-cycle considerations one can deduce that total income peaks later than labor income, as agents save during the more productive periods in life. Second, if family size is accounted for, as in the previous section, consumption evolves as

$$\frac{\dot{c}(t)}{c(t)} = \begin{cases} \frac{1}{\sigma}\left((r - \rho - h(t)) + (\sigma - 1)(1 - \theta)\frac{\dot{e}(t)}{e(t)} + (\sigma - 1)\theta\frac{\dot{n}(t)}{n(t)}\right) & \text{if} \quad l(t) < 1 \\ \frac{1}{\theta(\sigma - 1) + 1}\left((r - \rho - h(t)) + (\sigma - 1)\theta\frac{\dot{n}(t)}{n(t)}\right) & \text{if} \quad l(t) = 1 \end{cases} \tag{15}$$

For $\sigma > 1$ the profile has a more pronounced hump, and it is not possible to rank the peaks in consumption and labor income in time anymore. As an illustration, Figure 1 draws optimal trajectories for endogenous labor supply with and without a family structure.

4.1 Extension: Endogenous human capital and the value of leisure

Labor productivity $e(t)$ can also be viewed as human capital resulting from an optimal investment strategy. Higher human capital, moreover, might increase the utility from a given amount of leisure. Following Heckman (1976), our model is supplemented by a dynamic human capital accumulation equation.

[5] The "typical productivity pattern" $e(t)$ can be described as follows: Labor productivity typically grows until it reaches a maximum level at the age of ± 50. The sharpest decline is assumed to be around retirement age, whereafter there is a slowdown in productivity loss.

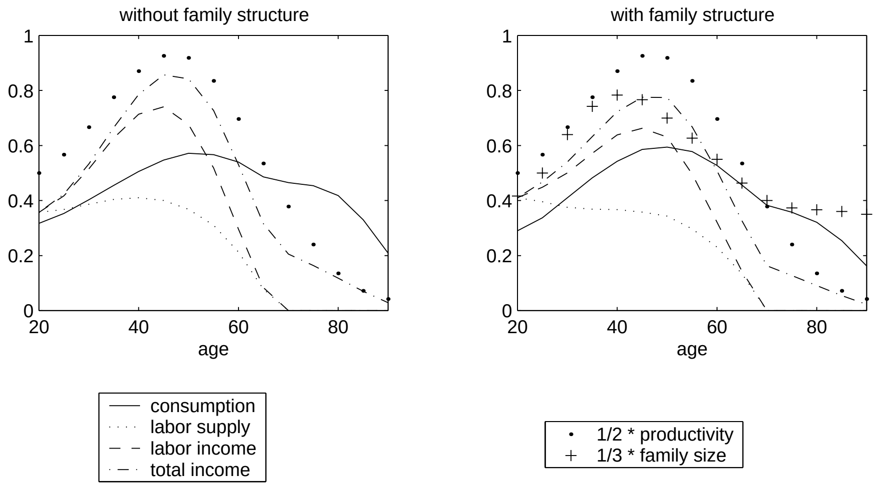

Figure 1. Optimal consumption, labor supply and income profiles over the life-cycle for models without and with a family structure (parameters used for simulation: $\sigma = 4$, $\theta = .33$, $\rho = 0$, $r = 0.03$, artificial productivity and family profiles, Swiss mortality data)

$$\dot{e}(t) = F[e(t), i(t)] - \xi e(t). \tag{16}$$

where the "production function" of human capital $F[\cdot, \cdot]$ is strictly concave and it is required that $F[e(t), 0] \equiv 0$. $i(t)$ denotes the fraction of time devoted to learning, and ξ is the depreciation rate of human capital.[6] Taking into account the proposed human capital accumulation formation, the new budget constraint becomes

$$\dot{a}(t) = a(t)r + we(t)(1 - l(t) - i(t)) - c(t) \tag{17}$$

and the time constraints are

$$0 \leq i(t) + l(t) \leq 1, \qquad i(t)l(t) \geq 0. \tag{18}$$

In order to explore the role of human capital for the valuation of leisure in explaining the optimal consumption trajectory more carefully, let the utility function be of the following form

$$U = U[c(t), e(t)^{\alpha} l(t)]. \tag{19}$$

The parameter α describes the extent to which productivity yields non–market benefits, and $e(t)^{\alpha} l(t)$ is productivity adjusted leisure. In a first polar case, human capital does not enter the utility function in terms of adjusted leisure ($\alpha = 0$). In Heckman (1976), on the other hand, human capital's non–market benefits are captured by formulating preferences concerning leisure in efficiency units $e(t)l(t)$ ($\alpha = 1$).[7]

[6] As an additional simplification and unlike Heckman (1976), no consumption goods are needed to acquire human capital. This latter assumption does not influence the analysis.

[7] Heckman (1976) views the productivity adjusted leisure as key in his analysis, but acknowledges the inability of his model to explain non–monotonous consumption profiles.

Although the model is too complicated to be solved analytically even for simple forms of the functions $U[\cdot,\cdot]$ and $F[\cdot,\cdot]$, Proposition 3 shows that the problem is separable.

Proposition 3. Given the path of human capital $e(t)$, the optimal path of consumption $c(t, e(t))$ conditional on $e(t)$ can be computed. The resulting rate of consumption growth is

$$\frac{\dot{c}(t,e(t))}{c(t,e(t))} \tag{20}$$
$$= \begin{cases} \frac{1}{\sigma}\left((r-\rho-h(t)) + (1-\alpha)(\sigma-1)(1-\theta)\frac{\dot{e}(t)}{e(t)}\right) & \text{if } l(t)+i(t)<1 \\ \frac{1}{\theta(\sigma-1)+1}\left(r-\rho-h(t)\right) & \text{if } l(t)+i(t)=1 \end{cases}$$

With $c(t, e(t))$ and the remaining first order conditions, we get a solution for $i(t)$, which in turn can be used to compute $e(t)$.

Proof. See Appendix C.

Although we have not explored the human capital accumulation process *per se*, it can be concluded that changes in productivity affect consumption in an analogous way, independent of whether productivity is exogenous or evolves endogenously by an optimal human capital accumulation process. Consumption trajectories, however, hinge on the extent to which human capital yields non–market benefits. The more productivity increases the valuation of leisure, the flatter the consumption profile and the later in life consumption peaks.

5 Summary

This paper has presented several important determinants of the consumption profile and its correlation with income which can be derived from a very simple neoclassical model without borrowing constraints, income uncertainty or myopia. The basic life-cycle model has been supplemented by uncertain lifetime, endogenous labor supply with age–dependent wage–pattern, and changing family composition. The considered CIES–specification of preferences allows for analytical solutions of the problem.

The main findings can be summarized by the following equation

$$\frac{\dot{c}(t)}{c(t)} = \gamma_1\left\{r-\rho-h(t)\right\} + \gamma_2\frac{\dot{e}(t)}{e(t)} + \gamma_3\frac{\dot{n}(t)}{n(t)},$$

where the coefficients γ_i depend on preference parameters as well as on assumptions about longevity uncertainty and the valuation of leisure in the utility function. They also depend on whether the individual choses to work or not. Note that the different influences act *additively* on the growth rate of consumption. The inclusion of any additional factor does not change the coefficient of the previous ones.

Lifetime uncertainty leads to a reduction in consumption expenditures as agents age. The availability of annuities lessens the decline to a certain extent, with the extreme outcome of fair annuities leading to essentially the same consumption path as under certainty. If consumption and leisure in the utility function are nonseparable, there is indeed a relationship between consumption changes and changes in (labor) income. The two profiles however do not fully coincide: The peak in labor income (but not necessarily total income) precedes the peak in consumption, and the consumption profile is flatter than the income profile. Changing family composition reinforces the effects of endogenous labor, as family size often moves with productivity and thus with labor income. Consumption profiles become more pronouncedly hump–shaped and the correlation with income increases.

The paper has shown that consumption tracking income cannot *per se* be taken as evidence against the hypothesis of optimal allocation of consumption over the life-cycle. Nevertheless, the analysis does not preclude alternative explanations for the observed patterns of consumption and labor supply.

Appendix

A: Proof of Proposition 1

The Hamiltonian to the optimization problem stated in (2) and (3) is given by

$$\mathcal{H} = U[c(t)]\Phi(t) + \lambda(t)\{a(t)R(t) + m(t) - c(t)\}$$

where $\lambda(t)$ is the multiplier for the state equation (3). Let U_c denote the partial derivative of the utility function with respect to c. The necessary first order conditions are then given by

$$\frac{\partial \mathcal{H}}{\partial c(t)} = \Phi(t)U_c[c(t)] - \lambda(t) = 0 \tag{21}$$

$$\dot{\lambda}(t) = -\frac{\partial \mathcal{H}}{\partial a(t)} = -\lambda(t)R(t) \tag{22}$$

$$\dot{a}(t) = \frac{\partial \mathcal{H}}{\partial \lambda(t)} = a(t)R(t) + m(t) - c(t)$$

To solve the problem, rearrange (21) to get an expression for the marginal utility of consumption

$$U_c = \frac{\lambda(t)}{\Psi(t)}.$$

Taking the natural logarithm on both sides, and computing time derivatives yields

$$\dot{c}(t)\frac{U_{cc}}{U_c} = \frac{\dot{\lambda}(t)}{\lambda(t)} - \frac{\dot{\Phi}(t)}{\Phi(t)}.$$

Using the costate equation (22) and solving for $\dot{c}$ yields

$$\dot{c}(t) = \frac{U_c}{U_{cc}}\left\{-R(t) + \frac{\dot{\Phi}(t)}{\Phi(t)}\right\}. \tag{23}$$

B: Proof of Proposition 2

Abstracting from corner solutions (i.e. $l(t) < 1$), the Hamiltonian is given by

$$\mathscr{H} = U[c(t), l(t)] \exp(-\rho t)\Psi(t) + \lambda(t)\left\{a(t)r + we(t)(1 - l(t)) - c(t)\right\},$$

where $\lambda(t)$ is the multiplier for the state equation (12). The necessary first order conditions are

$$\frac{\partial \mathscr{H}}{\partial c(t)} = \Psi(t)\exp(-\rho t)U_c[c(t), l(t)] - \lambda(t) = 0 \tag{24}$$

$$\frac{\partial \mathscr{H}}{\partial l(t)} = \Psi(t)\exp(-\rho t)U_l[c(t), l(t)] - \lambda(t)e(t)w = 0 \tag{25}$$

$$\dot{\lambda}(t) = -\frac{\partial \mathscr{H}}{\partial a(t)} = -\lambda(t)r \tag{26}$$

$$\dot{a}(t) = \frac{\partial \mathscr{H}}{\partial \lambda(t)} = a(t)r + we(t)(1 - l(t)) - c(t)$$

Equations (24) and (25) can be rewritten for the CIES utility as

$$\theta\left[c(t)^{\theta} l(t)^{1-\theta}\right]^{-\sigma} c(t)^{\theta-1} l(t)^{1-\theta} = \lambda(t)\exp(\rho t)\Psi(t)^{-1} \tag{27}$$

$$(1-\theta)\left[c(t)^{\theta} l(t)^{1-\theta}\right]^{-\sigma} c(t)^{\theta} l(t)^{-\theta} = \lambda(t)\exp(\rho t)\Psi(t)^{-1}we(t) \tag{28}$$

Dividing (28) by (27) one arrives at a relationship between consumption and leisure

$$\frac{c(t)}{l(t)} = \frac{\theta}{1-\theta}e(t)w,$$

which can be substituted into (27) to eliminate leisure,

$$\theta c(t)^{-\sigma}\left[\frac{\theta}{1-\theta}e(t)w\right]^{-(1-\sigma)(1-\theta)} = \lambda(t)\exp(\rho t)\Psi(t)^{-1}.$$

Taking the natural logarithm on both sides yields

$$-\sigma\log(c(t)) - (1-\sigma)(1-\theta)\log(e(t)) + \log\left(\theta\left[\frac{\theta}{1-\theta}w\right]^{-(1-\sigma)(1-\theta)}\right) = \log(\lambda(t)) + \rho t - \log(\Psi(t))$$

Taking the time derivative, and using equations (5) and (26), one gets an expression for the growth rate of consumption for $l(t) < 1$,

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\sigma}\left\{(r - \rho - h(t)) + (\sigma - 1)(1-\theta)\frac{\dot{e}(t)}{e(t)}\right\}.$$

In an analogous way, the growth rate of consumption can be derived for periods where $l(t) = 1$. We get

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta(\sigma-1)+1}\Big(r - \rho - h(t)\Big). \tag{29}$$

C: Proof of Proposition 3

Assuming interior solutions ($l(t) + i(t) < 1$), the Hamiltonian can be written as

$$\begin{aligned}\mathscr{H} &= U[c(t), e(t)^{\alpha} l(t)] \exp(-\rho t)\Psi(t)\\ &\quad +\lambda(t)\{a(t)r + we(t)(1 - l(t) - i(t)) - c(t)\}\\ &\quad +\mu(t)\{F[e(t), i(t)] - \xi e(t)\}\end{aligned}$$

where $\lambda(t)$ and $\mu(t)$ are the multipliers for the state equations in asset holdings (17) and human capital (16). Let L denote productivity adjusted leisure, $L(t) \equiv e(t)^{\alpha} l(t)$. The necessary first order conditions are then given by

$$\frac{\partial \mathscr{H}}{\partial c(t)} = \Psi(t) \exp(-\rho t) U_c[c(t), e(t)^{\alpha} l(t)] - \lambda(t) = 0 \tag{30}$$

$$\frac{\partial \mathscr{H}}{\partial l(t)} = \Psi(t) \exp(-\rho t) e(t)^{\alpha} U_L[c(t), e(t)^{\alpha} l(t)] - \lambda(t) e(t) w = 0 \tag{31}$$

$$\frac{\partial \mathscr{H}}{\partial i(t)} = -\lambda(t) w e(t) + \mu(t) F_i[e(t), i(t)] = 0 \tag{32}$$

$$\dot{\lambda}(t) = -\frac{\partial \mathscr{H}}{\partial a(t)} = -\lambda(t) r \tag{33}$$

$$\dot{\mu}(t) = -\frac{\partial \mathscr{H}}{\partial e(t)} = \mu(t)\{\xi - F_e(e(t), i(t))\} - \lambda(t)(1 - l(t) - i(t)) w \tag{34}$$

$$\dot{a}(t) = \frac{\partial \mathscr{H}}{\partial \lambda(t)} = a(t) r + w e(t)(1 - l(t) - i(t)) - c(t) \tag{35}$$

$$\dot{e}(t) = \frac{\partial \mathscr{H}}{\partial \mu(t)} = F[e(t), i(t)] - \xi e(t) \tag{36}$$

An inspection of the first order conditions above reveals that the problem is separable. Conditional on $e(t)$, equations (30), (31), and (33) can be used to get an expression for consumption growth $\frac{\dot{c}(t,e(t))}{c(t,e(t))}$. Its derivation is completely analogous to the derivation in Appendix B.

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