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Optimal Joint Spare Stocking and Block Replacement Policy (Cost Modelling of Spare Stocking and Block Replacement)

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When a group of identical units are put in service simultaneously, a popular replacement policy is the block replacement policy (BRP). Under the BRP, all units are replaced by new ones at intervals and an individual unit is replaced at failure during the interval. Since many units are replaced during the block replacement interval, adequate spare stocking becomes an important factor for performing BRP. In this paper, an expected cost model is formulated for the joint spare stocking and block replacement policy using the renewal process.

Keywords: Block replacement; Pooled process; Renewal process; Spare stock

1. Introduction

Most classical maintenance policies consider only repair and replacement. Spares for replacing failed units are assumed to be available whenever needed. In practice, a maintenance schedule is influenced by the spares stock level. If the spares stock is maintained at a low level, maintenance activities will be affected. On the other hand, if the stock level is too high, excessive carrying costs will be incurred. Falkner [1], Mine and Kawai [2], and Park and Park [3] have studied the joint maintenance-stock problem. In their studies, ordering/stocking policies are based mainly on failure replacement, age replacement, or minimal repair–replacement policies.

Barlow and Proschan [4] and Jardine [5] have discussed block replacement policy (BRP) for a multi-unit system. Under the BRP, all units are replaced by new ones at block replacement intervals and any individual unit is replaced at failure during the interval. Therefore, the stocking of adequate spares becomes a very important factor for performing BRP.

Acharya et al. [6] and Sivazlian and Danusaputro [7] considered joint spares stocking and the BRP problem. Acharya et al. [6] assumed that the demand for spares for replacing failed units follows a normal distribution law. Sivazlian and Danusaputro [7] assumed that the number of spares is enough to cover the failures during the block replacement intervals. In many situations, those are unnatural assumptions. In this paper, we assume that an unlimited supply of spares follows a renewal process. The expected long-run cost model for the joint spare stocking and BRP is derived in terms of the renewal process.

2. Expected Cost Under the Joint Spare Stocking and BRP

Assume that there are N units in an operating fleet at time 0 and the initial spares stock level is S. Whenever a unit fails, the failed unit is replaced by a new one from the spares stock. The operating fleet is replaced when the time interval T elapses or the first failure occurs after zero stock level ($S = 0$), whichever comes first. At block replacement, the stock level is restored to S, and then a new cycle begins. Thus, prior to the block replacement, as the interval T approaches, the replacement stock is ordered to ensure a level of S immediately following block replacement. A typical behaviour of the stock level under the policy is shown in Fig.1 where S is the initial spares stock level. We assume that the procurement lead time is negligible.

2.1. Expected Length of a Renewal Cycle

For the joint problem (spare stocking and BRP), all failures are immediately replaced by new ones from the spare stock. Let $n_i(t)$ number of failure replacements of unit i the interval $[0,t)$ for $i = 1,2,\ldots,N$. Assume that an unlimited supply of spares follows a renewal process [8]. Since there are N units

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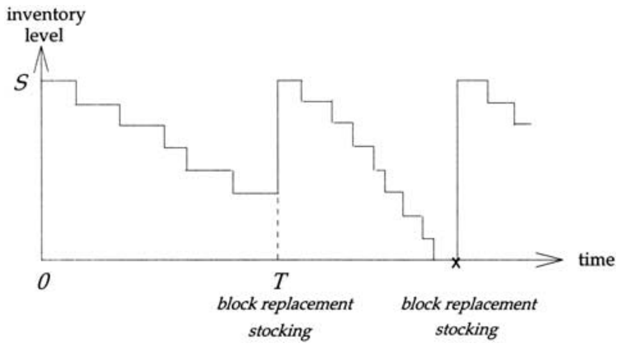

Fig. 1. A typical behaviour of the stock level under the proposed joint spare stocking and BRP. x on the horizontal axis represents the first failure after zero inventory, S is the initial spares stock level and T is the elapsed time.

in the operating fleet, the total number of failure replacements during $[0,t]$ is

$$n_p(t) \equiv n_1(t) + n_2(t) + \ldots + n_N(t) = \sum_{i=1}^{N} n_i(t)$$

In renewal theory, $\{n_p(t),\ t \geq 0\}$ is called a pooled process, or a superposition of N statistically identical renewal processes $\{n_i(t),\ t \geq 0\}$ for $i = 1,2,\ldots,N$ [8].

Now, let

$$p(n,t) \equiv Pr\{n_p(t) = n\} \tag{1}$$

and W_n be the time to the nth renewal in the pooled process $\{n_p(t),\ t \geq 0\}$.

For the joint problem, one cycle terminates at elapsed time T or the first failure after zero stock. Therefore, the expected length of a cycle can be represented as

$$E[cycle] = E[cycle|cycle = T]Pr\{cycle = t\} + E[cycle|cycle < T]Pr\{cycle < T\} \tag{2}$$

The event $\{cycle = T\}$ occurs if and only if the total number of failure replacements is less than or equal to the initial stock level S in the interval $[0,T]$. Therefore, the first term in Eq. (2) becomes

$$E[cycle|cycle = T]Pr\{cycle = T\} = T\,Pr\{n_p(T) \leq S\} = T\sum_{n=0}^{S} Pr\{n_p(T) = n\} = T\sum_{n=0}^{S} p(n,T) \tag{3}$$

The event $\{cycle<T\}$ happens if and only if the total number of failure replacements exceeds the stock level S before T. $\{W_{S+1} \leq t\}$ if and only if $\{n_p(t) \geq S + 1\}$ (see [8]). Therefore, the second term in Eq. (2) is reduced to

$$E[cycle|cycle < T]Pr\{cycle < T\} = \int_0^T t\mathrm{d}\,Pr\{W_{S+1} \leq t\} = \int_0^T t\mathrm{d}\,Pr\{n_p(t) \geq S + 1\} = \int_0^T t\mathrm{d}\left[\sum_{n=S+1}^{\infty} Pr\{n_p(t) = n\}\right] = \sum_{n=S+1}^{\infty}\int_0^T t\mathrm{d}p(n,t) = \sum_{n=S+1}^{\infty}\left[Tp(n,T) - \int_0^T p(n,t)\mathrm{d}t\right] \tag{4}$$

The last equality follows from the integration by parts. By combining (3) and (4) with the relation of

$$\sum_{n=0}^{\infty} p(n,t) = \sum_{n=0}^{S} p(n,t) + \sum_{n=S+1}^{\infty} p(n,t) = 1 \tag{5}$$

the expected cycle length is

$$E[cycle] = T\sum_{n=0}^{S} p(n,T) + T\sum_{n=S+1}^{\infty} p(n,T) - \sum_{n=S+1}^{\infty}\int_0^T p(n,t)\mathrm{d}t = T\left(\sum_{n=0}^{S} p(n,T) + \sum_{n=S+1}^{\infty} p(n,T)\right) - \sum\int_0^T p(n,t)\mathrm{d}t = T - \sum_{n=S+1}^{\infty}\int_0^T p(n,t)\mathrm{d}t = T - \int_0^T\left[1 - \sum_{n=0}^{S} p(n,t)\right]\mathrm{d}t = \sum_{n=0}^{S}\int_0^T p(n,t)\mathrm{d}t \tag{6}$$

2.2. Expected Total Cost During a Renewal Cycle

The total cost per cycle is made up of four parts: failure replacement cost, block replacement cost, ordering cost, and inventory holding cost. The number of failure replacements during a cycle depends on whether the cycle terminates at T or at the first failure after zero inventories. The expected number of failure replacements is

$$E[FR] = E[Fr|cycle = T]Pr\{cycle = T\} + E[FR|cycle < T]Pr\{cycle < T\} = \sum_{n=0}^{S} nPr\{n_p(T) = n\} + S\,Pr\{n_p(T) \geq S + 1\} = \sum_{n=0}^{S} np(n,T) + S\sum_{n=S+1}^{\infty} p(n,T) = \sum_{n=0}^{S} np(n,T) + S\left[1 - \sum_{n=0}^{S} p(n,T)\right] = S - \sum_{n=0}^{S}(S - n)p(n,T) \tag{7}$$

Hence, the expected failure replacement cost during a cycle is

$$c_f\left[S - \sum_{n=0}^{S}(S - n)p(n,T)\right] \tag{8}$$

where c_f is the failure replacement cost per unit.

The holding inventory time also depends on the cycle termination point.

1. If $T < W_1$, a cycle terminates at T with no failure replacement. Then, the carrying inventory time is ST.
2. If $W_1 \le T < W_2$, only one failure occurs during a cycle time T. Then, the inventory holding time is $W_1 + (S-1)T$.
3. In the same way, $W_{S-1} \le T < W_S$ implies a cycle time T with $S-1$ failure replacements, resulting in stock holding time $W_1 + W_2 + \dots + W_{S-1} + T$.
4. Finally, If $T \ge W_S$, the inventory holding time becomes $W_1 + W_2 + \dots + W_S$.

From these results, the inventory holding time for a cycle, I, can be expressed as follows:

$$I = \sum_{n=1}^{S} Min\{T,W_n\} \tag{9}$$

From (9),

$$\begin{aligned}
E[Min\{T,W_n\}] &= T\,Pr\{W_n > T\} + \int_0^T t\mathrm{d}\,Pr\{W_n \le t\} \\
&= T\,Pr\{n_p(T) < n\} + \int_0^T t\mathrm{d}\,Pr\{n_p(t) \ge n\} \\
&= T\sum_{i=0}^{n-1} p(i,T) + \sum_{i=n}^{\infty} \int_0^T t\mathrm{d}p(i,t) \\
&= T\sum_{i=0}^{n-1} p(i,T) + \sum_{i=n}^{\infty} \left[Tp(i,T) - \int_0^T p(i,t)\mathrm{d}t \right] \\
&= T - \int_0^T \left[1 - \sum_{i=0}^{n-1} p(i,t) \right] \mathrm{d}t \\
&= \sum_{i=0}^{n-1} \int_0^T p(i,t)\mathrm{d}t
\end{aligned} \tag{10}$$

Therefore, the expected inventory holding cost for a cycle becomes

$$\begin{aligned}
c_h E[I] &= c_h \sum_{n=1}^{S} \sum_{i=0}^{n-1} \int_0^T p(i,t)\mathrm{d}t \\
&= c_h \sum_{n=0}^{S-1} (S-n) \int_0^T p(n,t)\mathrm{d}t
\end{aligned} \tag{11}$$

where c_h is the inventory holding cost per unit time per unit.

Then, the expected total cost is

$$C_N(S,T) = \frac{c_O + Nc_b + c_f\left[S - \sum_{n=0}^{S} (S-n)p(n,T) \right] + c_h \sum_{n=0}^{S-1} (S-n) \int_0^T p(n,t)\mathrm{d}t}{\sum_{n=0}^{S} \int_0^T p(n,t)\mathrm{d}t} \tag{12}$$

where,

N = number of units to operate (all units are identical)

c_o = ordering cost per order

c_b = block replacement per unit

c_f = failure replacement cost per unit

c_h = inventory holding cost per unit

3. Evaluating $p(n,t)$ Using Recursive Relational Algorithm

We seek an optimum replacement time T^* and an optimal initial spares stock level S^* which minimise $C_N(S,T)$ in Eq. (12). Therefore, $p(n,t)$ should be evaluated first to find the optimal solution. The following method is derived from Buzen's computational algorithm in closed queueing networks [9].

Let

$$\begin{aligned}
f_i(n_i,t) &\equiv Pr\{n_i(t) = n_i\} \\
&= F_{n_i}(t) - F_{n_i+1}(t)
\end{aligned} \tag{13}$$

and define an ordered set

$$R(n,N) \equiv \left\{ \mathbf{n} \equiv (n_1,n_2,\dots,n_N) \middle| \sum_{i=1}^{N} n_i = n \right\}$$

Then, the probability mass function $p(n,t)$ can be expressed as

$$\begin{aligned}
p(n,t) &= Pr\{n_p(t) = n\} \\
&= Pr\left[\sum_{i=1}^{N} n_i(t) = n \right] \\
&= \sum_{\substack{n_1+\dots n_N = n \\ n_i \ge 0}} Pr\{n_1(t) = n_1, n_2(t) = n_2, \dots, n_N(t) = n_N\} \\
&= \sum_{n \in R(n,N)} \prod_{i=1}^{N} Pr\{n_i(t) = n_i\} \\
&= \sum_{n \in R(n,N)} \prod_{i=1}^{N} f_i(n_i,t)
\end{aligned} \tag{14}$$

since $n_i(t)$ for $i=1,\dots, N$ are independent.

Defining an auxiliary function

$$q_m(n,t) = \sum_{n \in R(n,m)} \sum_{i=1}^{m} f_1(n_i,t) \tag{15}$$

From Eqs (14) and (15), we have $p(n,t) = q_N(n,t)$. Then,

$$\begin{aligned}
q_m(n,t) &= \sum_{n \in R(n,m)} \prod_{i=1}^{w} f_i(n_i,t) \\
&= \sum_{j=0}^{n} \left\{ \sum_{\substack{n \in R(n,m) \\ n_m = j}} \prod_{i=1}^{m} f_i(n_i,t) \right\} \\
&= \sum_{j=0}^{n} f_m(j,t) \left\{ \sum_{(n \in R(n-j,m-1)} \prod_{i=1}^{m-1} f_i(n_i,t) \right\} \\
&= \sum_{j=0}^{n} f_m(j,t) q_{m-1}(n-j,t)
\end{aligned} \tag{16}$$

Therefore, $f_i(n_i,t) \equiv Pr\{n_i(t) = n_i\}$ can be obtained numerically (by a numerical inversion program after taking a Laplace transform [10]), starting with the initial condition

$$q_0(n,t) = \begin{cases} 1 \text{ if } n = 0 \\ 0 \text{ otherwise} \end{cases}$$

to the recursive relation (16), $q_m(n,t)$ can be evaluated recursively for sequential values of $m = 1, 2, \ldots$. The algorithm stops when m reaches N with the value of $p(n,t) = q_N(n,t)$.

4. Conclusion

In this paper, we formulated a new expected long-run cost model for the joint spare stocking and BRP. Although the cost model was obtained in a compact form, the complexity of the pooled demand process made it difficult to analyse the obtained model. However, optimising the spares stock level and the block replacement interval time are determined after evaluating $p(n,t)$ using the method in Section 3. The suggested method to evaluate $p(n,t)$ is complicated and difficult to use in practice. For further study, we must find a better method for evaluating $p(n,t)$, and then present an optimal procedure for the spares stock level and the block replacement interval using the cost model.

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