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# On the Construction of Pseudorandom Permutations: Luby–Rackoff Revisited*

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**Abstract.** Luby and Rackoff [26] showed a method for constructing a pseudorandom permutation from a pseudorandom function. The method is based on composing four (or three for weakened security) so-called Feistel permutations, each of which requires the evaluation of a pseudorandom function. We reduce somewhat the complexity of the construction and simplify its proof of security by showing that two Feistel permutations are sufficient together with initial and final pairwise independent permutations. The revised construction and proof provide a framework in which similar constructions may be brought up and their security can be easily proved. We demonstrate this by presenting some additional adjustments of the construction that achieve the following:

- Reduce the success probability of the adversary.
- Provide a construction of pseudorandom permutations with *large* input-length using pseudorandom functions with *small* input-length.

**Key words.** Pseudorandomness, Block ciphers, Modes of operation.

## 1. Introduction

Pseudorandom permutations, which were introduced by Luby and Rackoff [26], formalize the well-established cryptographic notion of block ciphers. Block ciphers are private-key encryption schemes such that the encryption of every plaintext-block is a single ciphertext-block *of the same length*. Therefore we can think of the private key as determining a permutation on strings of the length of the block. A highly influential example of a block cipher is the Data Encryption Standard (DES) [32].

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The advantage of block ciphers (compared with using pseudorandom functions for private-key encryption) is that the plaintext and ciphertext are of the same length. This property saves on memory and prevents wasting communication bandwidth. Furthermore, it enables the easy incorporation of the encryption scheme into existing protocols or hardware components.

Luby and Rackoff defined the security of pseudorandom permutations in analogy to the different attacks considered in the context of block ciphers:

- Pseudorandom permutations can be interpreted as block ciphers that are secure against an adaptive *chosen plaintext attack*. Informally, this means that an (efficient) adversary, with access to the encryptions of messages of its choice, cannot tell apart those encryptions from the values of a truly random permutation.
- Strong pseudorandom permutations can be interpreted as block ciphers that are secure against an adaptive *chosen plaintext and ciphertext attack*. Here, the adversary has the additional power to ask for the decryption of ciphertexts of its choice.

Pseudorandom permutations are closely related (both in definition and in their construction) to the earlier concept of pseudorandom functions which was defined by Goldreich et al. [17]. These are efficiently sampled and computable functions that are indistinguishable from random functions under all (efficient) black-box attacks (see Section 2 for a formal definition). Pseudorandom functions play a major role in private-key cryptography and have many additional applications (for some of these applications, see [10], [16], and [25]).

Luby and Rackoff [26] provided a construction of strong pseudorandom permutations (**LR-Construction**) which was motivated by the structure of DES. The basic building block is the so-called Feistel permutation[1] based on a pseudorandom function defined by the key. Their construction consists of four rounds of Feistel permutations (or three rounds, for pseudorandom permutations), each round involves an application of a (different) pseudorandom function (see Fig. 1(a) for an illustration). The LR-Construction's main source of attraction is, most probably, its elegance.

Goldreich et al. [17] showed a construction of pseudorandom functions from pseudorandom generators [9], [50]. Thus, the construction of pseudorandom permutations reduces to the construction of pseudorandom generators. Recently a different construction of pseudorandom functions was introduced by Naor and Reingold [31]; this is a parallel construction based on a new primitive called a pseudorandom *synthesizer* that in particular can be constructed from any trapdoor permutation. This implies a parallel construction of pseudorandom permutations. Nevertheless, all known constructions of pseudorandom functions involve nontrivial (though of course polynomial-time) computation, so it makes sense to attempt to minimize the number of invocations of pseudorandom functions.

Alongside cryptographic pseudorandomness the last two decades saw the development of the notion of limited independence in various settings and formulations [3], [4],

[1] A Feistel permutation for a function $f\colon \{0,1\}^n \mapsto \{0,1\}^n$ is a permutation on $\{0,1\}^{2n}$ defined by $\mathbf{D}_f(L,R) \stackrel{\text{def}}{=} (R, L \oplus f(R))$, where $|L| = |R| = n$. Each of the 16 rounds of DES involves a Feistel permutation of a function determined by the 56 key bits.

[12], [13], [24], [30], [49]. For a family of functions $\mathcal{F}$ to have some sort of (limited) independence means that if we consider the value of a function $f$, chosen uniformly at random from $\mathcal{F}$, at each point as a random variable (in the probability space defined by choosing $f$), then these random variables possess the promised independence property. Thus, a family of permutations on $\{0,1\}^n$ is pairwise independent if for all $x \neq y$ the values of $f(x)$ and $f(y)$ are uniformly distributed over strings $(a,b) \in \{0,1\}^{2n}$ such that $a \neq b$. Functions of limited independence are typically much simpler to construct and easier to compute than (cryptographic) pseudorandom functions.

### 1.1. *New Results and Organization*

The goal of this paper is to provide a better understanding of the LR-Construction and as a result improve the construction in several respects. Our main observation is that the different rounds of the LR-Construction serve significantly different roles. We show that the first and last rounds can be replaced by pair-wise independent permutations and use this in order to:

1. Simplify the proof of security of the construction (especially in the case of strong pseudorandom permutations) and provide a framework for proving the security of similar constructions.
2. Derive generalizations of the construction that are of practical and theoretical interest. The proof of security for each one of the constructions is practically "free of charge" given the framework.
3. Achieve an improvement in the computational complexity of the pseudorandom permutations—two applications of a pseudorandom function on $n$ bits suffice for computing the value of a pseudorandom permutation on $2n$ bits at a given point (versus four applications in the original LR-Construction). This implies that the reduction is "optimal."

As discussed in Section 5.2, the new construction is in fact a generalization of the original LR-Construction. Thus, the proof of security (Theorem 3.2) also applies to the original construction. The following is a brief and informal description of the paper's main results and organization:

**Section 2** reviews the notation and the definitions regarding pseudorandomness and $k$-wise independence.

**Section 3** presents the main construction and proves its security: pairwise independent permutations can replace the first and fourth rounds of the LR-Construction (see Fig. 1(b) for an illustration).

**Section 4** highlights the high-level structure of the proof of security which provides a framework that enables us to relax and generalize the main construction.

**Section 5** shows how the main construction can be relaxed by:

- **5.1.** Using a single pseudorandom function (instead of two).
- **5.2.** Using weaker and more efficient permutations (or functions) instead of the pairwise independent permutations.

**Section 6** provides a simple generalization of the main construction: using $t$ rounds of (generalized) Feistel permutations (instead of two) the success probability of the distinguisher is reduced from approximately $m^2/2^{\ell/2}$ to approximately $t/2 \cdot$

$m^2/2^{(1-1/t)\ell}$, where the permutation is on $\ell$ bits and the distinguisher makes at most $m$ queries (see Fig. 3 for an illustration).

**Section 7** provides a second generalization of the main construction. Instead of applying Feistel permutations on the entire outputs of the first and second rounds, Feistel permutations can be separately applied on each one of their subblocks. This is a construction of a strong pseudorandom permutation on *many* blocks using pseudorandom functions on a *single* block (see Fig. 4 for an illustration).

**Section 8** analyzes the different constructions of the paper as constructions of $k$-wise $\delta$-dependent permutations.

**Section 9** suggests directions for further research.

### 1.2. *Related Work*

The LR-Construction inspired a considerable amount of research. We try to refer to the more relevant (to this paper) part of these directions.

Several alternative proofs of the LR-Construction were presented over the years. Maurer [28] gives a proof of the three-round construction. His proof concentrates on the nonadaptive case, i.e., when the distinguisher has to specify all its queries in advance. A point worth noticing is that indistinguishability under nonadaptive attacks does not necessarily imply indistinguishability under adaptive attacks. For example, a random involution (an involution is a permutation which is the inverse of itself) and a random permutation are indistinguishable under nonadaptive attacks and can be distinguished using a very simple adaptive attack.[2] A different approach toward the proof was described by Patarin [34] (this is the only published proof, we are aware of, for the LR-Construction of *strong* pseudorandom permutations; another proof was given by Koren [22]).

Other papers consider the security of possible variants of the construction. A significant portion of this research deals with the construction of pseudorandom permutations and strong pseudorandom permutations from a *single* pseudorandom function. This line of work is described in Section 5.1.

Lucks [27] shows that a hash function can replace the pseudorandom function in the first round of the three-round LR-Construction. His proof is based on [28] and is motivated by his suggestion of using the LR-Construction when the input is divided into two *unequal* parts. Lucks left open the question of the construction of strong pseudorandom permutations.

Somewhat different questions were considered by Even and Mansour [14] and by Kilian and Rogaway [21]. Loosely speaking, the former construct several pseudorandom permutations from a single one, while the latter show how to make exhaustive key-search attacks more difficult. The construction itself amounts, in both cases, to XORing the input of the pseudorandom permutation with a random key and XORing the output of the permutation with a second random key.

The background and related work concerning other relevant issues are discussed in the appropriate sections: definitions and constructions of efficient hash functions in Section 5.2, reducing the distinguishing probability in Section 6, and the construction of

[2] An even more striking example is obtained by comparing a random permutation $P$ that satisfies $P(P(0)) = 0$ with a truly random permutation.

pseudorandom permutations (or functions) with large input-length from pseudorandom permutations (or functions) with small input-length in Section 7.

## 2. Preliminaries

In this section the concepts of pseudorandom functions and pseudorandom permutations are briefly reviewed. A more thorough and formal treatment can be found in [15] and [25].

### 2.1. *Notation*

- $I_n$ denotes the set of all $n$-bit strings, $\{0,1\}^n$.
- $F_n$ denotes the set of all $I_n \mapsto I_n$ functions and $P_n$ denotes the set of all such permutations ($P_n \subset F_n$).
- Let $x$ and $y$ be two bit strings of equal length, then $x \oplus y$ denotes their bit-by-bit exclusive-or.
- For any $f, g \in F_n$, denote their composition by $f \circ g$ (i.e., $f \circ g(x) = f(g(x))$).
- For $x \in I_{2n}$, denote the first (left) $n$ bits of $x$ by $x_{|_L}$ and the last (right) $n$ bits of $x$ by $x_{|_R}$ .

**Definition 2.1 (Feistel Permutations).** [3] For any function $f \in F_n$, let $\mathbf{D}_f \in P_{2n}$ be the permutation defined by $\mathbf{D}_f(L, R) \stackrel{\text{def}}{=} (R, L \oplus f(R))$, where $|L| = |R| = n$.

Notice that Feistel permutations are as easy to invert as they are to compute (since the inverse permutation satisfies $\mathbf{D}_f^{-1}(L, R) = (R \oplus f(L), L)$; that is, $\mathbf{D}_f^{-1}(L, R) \equiv \rho \circ \mathbf{D}_f \circ \rho$ for $\rho(L, R) \stackrel{\text{def}}{=} (R, L)$). Therefore, the LR-Construction (and its different variants which are introduced in Sections 6 and 7) are easy to invert.

### 2.2. *Pseudorandomness*

Pseudorandomness is fundamental to cryptography and, indeed, essential in order to perform such tasks as encryption, authentication, and identification. Loosely speaking, pseudorandom distributions cannot be efficiently distinguished from the truly random distributions (usually, random here means uniform). However, the pseudorandom distributions have substantially smaller entropy than the truly random distributions and are able to be sampled efficiently.

#### 2.2.1. *Overview of Pseudorandom Primitives*

In the case of **pseudorandom (bit) generators**, which were introduced by Blum and Micali [9] and Yao [50], the pseudorandom distribution is of bit-sequences. The distribution is efficiently sampled using a, relatively small, truly random bit-sequence (the seed). Hastad et al. [20] showed how to construct a pseudorandom generator from any

[3] **D** stands for DES-like, another common term for these permutations.

one-way function (informally, a function is one-way if it is easy to compute its value but hard to invert it).

**Pseudorandom function ensembles (PFE)**, which were introduced by Goldreich et al. [17], are distributions of functions. These distributions are indistinguishable from the uniform distribution under all (polynomially bounded) black-box attacks (i.e., the distinguisher can only access the function by specifying inputs and getting the value of the function on these inputs). Goldreich et al. provided a construction of such functions based on the existence of pseudorandom generators.

Luby and Rackoff [26] define **pseudorandom permutation ensembles (PPE)** to be distributions of permutations that are indistinguishable from the uniform distribution to an efficient observer (that, again, has access to the value of the permutation at points of its choice). In addition, they consider a stronger notion of pseudorandomness which they call *super pseudorandom permutation generators*. Here the distinguisher can also access the inverse permutation at points of its choice. Following [15] we use the term **strong pseudorandom permutation ensembles (SPPE)** instead.

Luby and Rackoff provided a simple construction of PPE and SPPE (*LR-Construction*) which is the focus of this work. Their construction is based on a basic compound of the structure of DES [32], namely, the compositions of several Feistel permutations. Their design of the PPE (resp. SPPE) is $\mathbf{D}_{f_3} \circ \mathbf{D}_{f_2} \circ \mathbf{D}_{f_1}$ (resp. $\mathbf{D}_{f_4} \circ \mathbf{D}_{f_3} \circ \mathbf{D}_{f_2} \circ \mathbf{D}_{f_1}$) where all $f_i$'s are independent pseudorandom functions and $\mathbf{D}_{f_i}$ is as in Definition 2.1 (see Fig. 1(a) for an illustration).

### 2.2.2. *Definitions*

A *function ensemble* is a sequence $H = \{H_n\}_{n\in\mathbb{N}}$ such that $H_n$ is a distribution over $F_n$, $H$ is the *uniform function ensemble* if $H_n$ is uniformly distributed over $F_n$. A *permutation ensemble* is a sequence $H = \{H_n\}_{n\in\mathbb{N}}$ such that $H_n$ is a distribution over $P_n$, $H$ is the *uniform permutation ensemble* if $H_n$ is uniformly distributed over $P_n$.

A function ensemble (or a permutation ensemble), $H = \{H_n\}_{n\in\mathbb{N}}$, is *efficiently computable* if the distribution $H_n$ can be sampled efficiently and the functions in $H_n$ can be computed efficiently. That is, there exist probabilistic polynomial-time Turing machines, $I$ and $V$, and a mapping from strings to functions, $\varphi$, such that $\varphi(I(1^n))$ and $H_n$ are identically distributed and $V(i, x) = (\varphi(i))(x)$ (so, in fact, $H_n \equiv V(I(1^n), \cdot)$).

We would like to consider efficiently computable function (or permutation) ensembles that cannot be efficiently distinguished from the uniform ensemble. In our setting, the distinguisher is an oracle machine that can make queries to a length-preserving function (or functions) and outputs a single bit. We assume that on input $1^n$ the oracle machine makes only $n$-bit long queries, $n$ also serves as the security parameter. An oracle machine has an interpretation both under the uniform complexity model and under the nonuniform model. In the former it is interpreted as a Turing machine with a special oracle-tape (in this case efficient means probabilistic polynomial time) and in the latter as a circuit-family with special oracle-gates (in this case efficient means polynomial size). The discussion of this paper is independent of the chosen interpretation.

Let $M$ be an oracle machine, let $f$ be a function in $F_n$, and let $H_n$ be a distribution over $F_n$. Denote by $M^f(1^n)$ the distribution of $M$'s output when its queries are answered by $f$ and denote by $M^{H_n}(1^n)$ the distribution $M^f(1^n)$, where $f$ is distributed according

to $H_n$. We would also like to consider oracle machines with access both to a permutation and to its inverse. Let $M$ be such a machine, let $f$ be a permutation in $P_n$, and let $H_n$ be a distribution over $P_n$. Denote by $M^{f,f^{-1}}(1^n)$ the distribution of $M$'s output when its queries are answered by $f$ and $f^{-1}$ and denote by $M^{H_n,H_n^{-1}}(1^n)$ the distribution $M^{f,f^{-1}}(1^n)$, where $f$ is distributed according to $H_n$.

**Definition 2.2** (Advantage). Let $M$ be an oracle machine and let $H = \{H_n\}_{n\in\mathbb{N}}$ and $\tilde{H} = \{\tilde{H}_n\}_{n\in\mathbb{N}}$ be two function (or permutation) ensembles. We call the function

$$\left|\Pr[M^{H_n}(1^n) = 1] - \Pr[M^{\tilde{H}_n}(1^n) = 1]\right|$$

the *advantage* $M$ achieves in distinguishing between $H$ and $\tilde{H}$.

Let $M$ be an oracle machine and let $H = \{H_n\}_{n\in\mathbb{N}}$ and $\tilde{H} = \{\tilde{H}_n\}_{n\in\mathbb{N}}$ be two permutation ensembles. We call the function

$$\left|\Pr[M^{H_n,H_n^{-1}}(1^n) = 1] - \Pr[M^{\tilde{H}_n,\tilde{H}_n^{-1}}(1^n) = 1]\right|$$

the advantage $M$ achieves in distinguishing between $\langle H, H^{-1}\rangle$ and $\langle \tilde{H}, \tilde{H}^{-1}\rangle$.

**Definition 2.3** ($\varepsilon$-Distinguish). We say that $M$ $\varepsilon$-distinguishes between $H$ and $\tilde{H}$ (resp. $\langle H, H^{-1}\rangle$ and $\langle \tilde{H}, \tilde{H}^{-1}\rangle$) for $\varepsilon = \varepsilon(n)$ if for infinitely many $n$'s the advantage $M$ achieves in distinguishing between $H$ and $\tilde{H}$ (resp. $\langle H, H^{-1}\rangle$ and $\langle \tilde{H}, \tilde{H}^{-1}\rangle$) is at least $\varepsilon(n)$.

**Definition 2.4** (Negligible Functions). A function $h\colon \mathbb{N} \mapsto \mathbb{N}$ is *negligible* if, for every constant $c > 0$ and all sufficiently large $n$'s,

$$h(n) < \frac{1}{n^c}.$$

**Definition 2.5** (PFE). Let $H = \{H_n\}_{n\in\mathbb{N}}$ be an efficiently computable *function* ensemble and let $R = \{R_n\}_{n\in\mathbb{N}}$ be the uniform function ensemble. $H$ is a *pseudorandom function ensemble* if, for every efficient oracle machine $M$, the advantage $M$ has in distinguishing between $H$ and $R$ is negligible.

**Definition 2.6** (PPE). Let $H = \{H_n\}_{n\in\mathbb{N}}$ be an efficiently computable *permutation* ensemble and let $R = \{R_n\}_{n\in\mathbb{N}}$ be the uniform permutation ensemble. $H$ is a *pseudorandom permutation ensemble* if, for every efficient oracle machine $M$, the advantage $M$ has in distinguishing between $H$ and $R$ is negligible.

**Definition 2.7** (SPPE). Let $H = \{H_n\}_{n\in\mathbb{N}}$ be an efficiently computable permutation ensemble and let $R = \{R_n\}_{n\in\mathbb{N}}$ be the uniform permutation ensemble. $H$ is a *strong pseudorandom permutation ensemble* if, for every efficient oracle machine $M$, the advantage $M$ has in distinguishing between $\langle H, H^{-1}\rangle$ and $\langle R, R^{-1}\rangle$ is negligible.

*Remark* 2.1. We use the phrase "$f$ is a pseudorandom function" as an abbreviation for "$f$ is distributed according to a pseudorandom function ensemble" and similarly for "$f$ is a pseudorandom permutation" and "$f$ is a strong pseudorandom permutation."

### 2.3. *k-Wise Independent Functions and Permutations*

The notions of $k$-wise independent functions and $k$-wise "almost" independent functions [3], [4], [12], [13], [24], [30], [49] (under several different formulations) play a major role in contemporary computer science. These are distributions of functions such that their value on any given $k$ inputs is uniformly or "almost" uniformly distributed. Several constructions of such functions and a large variety of applications were suggested over the years.

We briefly review the definitions of $k$-wise independence (and $k$-wise $\delta$-dependence). The definitions of pairwise independence (and pairwise $\delta$-dependence) can be derived by taking $k = 2$.

**Definition 2.8.** Let $D_1$ and $D_2$ be two distributions defined over $\Omega$, the variation distance between $D_1$ and $D_2$ is

$$\|D_1 - D_2\| = \frac{1}{2} \sum_{\omega \in \Omega} |D_1(\omega) - D_2(\omega)| .$$

**Definition 2.9.** Let $A$ and $B$ be two sets, $0 \leq \delta \leq 1$, let $k$ be an integer ($2 \leq k \leq |A|$), and let $F$ be a distribution of $A \mapsto B$ functions. Let $x_1, x_2, \ldots, x_k$ be $k$ different members of $A$, and consider the following two distributions:

1. $\langle f(x_1), f(x_2), \ldots, f(x_k) \rangle$ where $f$ is distributed according to $F$.
2. The uniform distribution over $B^k$.

$F$ is $k$-wise independent if for all $x_1, x_2, \ldots, x_k$ the two distributions are identical. $F$ is $k$-wise $\delta$-dependent if for all $x_1, x_2, \ldots, x_k$ the two distributions are of variation distance at most $\delta$.

These definitions are naturally extended to permutations:

**Definition 2.10.** Let $A$ be a set, $0 \leq \delta \leq 1$, let $k$ be an integer ($2 \leq k \leq |A|$), and let $F$ be a distribution of permutations over $A$. Let $x_1, x_2, \ldots, x_k$ be $k$ different members of $A$, and consider the following two distributions:

1. $\langle f(x_1), f(x_2), \ldots, f(x_k) \rangle$ where $f$ is distributed according to $F$.
2. The uniform distribution over sequences of $k$ *different* elements of $A$.

$F$ is $k$-wise independent if for all $x_1, x_2, \ldots, x_k$ the two distributions are identical. $F$ is $k$-wise $\delta$-dependent if for all $x_1, x_2, \ldots, x_k$ the two distributions are of variation distance at most $\delta$.

The connection of this paper to $k$-wise independence is bidirectional as described in the following two paragraphs.

As shown in Section 3, pairwise independent permutations can replace the first and fourth rounds of the LR-Construction. Let $A$ be a finite field, then the permutation $f_{a,b}(x) \stackrel{\text{def}}{=} a \cdot x + b$, where $a \neq 0, b \in A$ are uniformly distributed, is pairwise independent. Thus, there are pairwise independent permutations over $I_n$ (the permutations $f_{a,b}$ with operations over $GF(2^n)$). In Section 5.2 it is shown that we can use even more efficient functions and permutations in our construction. In particular, we consider the concept of $\varepsilon$-AXU$_2$ functions.

In contrast with the case of pairwise independent permutations, we are not aware of any "good" constructions of $k$-wise $\delta$-dependent permutations for general $k$ and $\delta$. The different variants of the LR-Construction offer a partial solution to this problem ("partial" because of the bounded values of $\delta$ that can be achieved). For example, using $k$-wise $\delta'$-dependent functions on $n$ bits instead of pseudorandom functions in the original LR-Construction yields a $k$-wise $\delta$-dependent permutation on $2n$ bits (for $\delta = O(k^2/2^n + \delta')$). In Section 8 we analyze the different constructions of this paper as constructions of $k$-wise $\delta$-dependent permutations.

## 3. Construction of PPE and SPPE

### 3.1. *Intuition*

As mentioned in the Introduction, a principle observation of this paper is that the different rounds of the LR-Construction serve significantly different roles. To illustrate this point, consider two rounds of the construction. Namely, $E = \mathbf{D}_{f_2} \circ \mathbf{D}_{f_1}$, where $f_1, f_2 \in F_n$ are two independently chosen pseudorandom functions. It is not hard to verify that $E$ is computationally indistinguishable from a random permutation to any efficient algorithm that has access to pairs $\{\langle x_i, E(x_i)\rangle\}_{i=1}^m$, where the sequence $\{x_i\}_{i=1}^m$ is uniformly distributed. The intuition is as follows: Note that it is enough to prove the pseudorandomness of $E$ when $f_1$ and $f_2$ are *truly random functions* (instead of pseudorandom). Let $(L_i^0, R_i^0) = x_i$ and $(L_i^2, R_i^2) = E(x_i)$, by the definition of $E$ we get that $L_i^2 = L_i^0 \oplus f_1(R_i^0)$ and $R_i^2 = R_i^0 \oplus f_2(L_i^2)$. Since the sequence $\{x_i\}_{i=1}^m$ is uniformly distributed, we have that with good probability (better than $(1 - m^2/2^{n+1})$) $R_i^0 \neq R_j^0$ for all $i \neq j$. Conditioned on this event, the sequence $\{L_i^2\}_{i=1}^m$ is uniformly distributed and independent of the sequence $\{x_i\}_{i=1}^m$ (since $f_1$ is random). We now have that with good probability $L_i^2 \neq L_j^2$ for all $i \neq j$. Conditioned on this event, the sequence $\{R_i^2\}_{i=1}^m$ is uniformly distributed and independent of both $\{L_i^2\}_{i=1}^m$ and $\{x_i\}_{i=1}^m$. Notice that this argument still works if the sequence $\{x_i\}_{i=1}^m$ is only pairwise independent.

Nevertheless, as Luby and Rackoff showed, $E$ can be easily distinguished from a random permutation by an algorithm that gets to see the value of $E$ or $E^{-1}$ on inputs of its choice. The reason is that for any values $L_1$, $L_2$, and $R$ such that $L_1 \neq L_2$ we have that $E(L_1, R)_{|_{\mathrm{L}}} \oplus E(L_2, R)_{|_{\mathrm{L}}} = L_1 \oplus L_2$. In contrast, for a truly random permutation, the probability of this event is $2^{-n}$. This is the reason that the LR-Construction includes three or four rounds.

If we think of the second and third rounds of the LR-Construction as the permutation $E$, then the discussion above implies that the role of the first and fourth rounds is to prevent the distinguisher from directly choosing the inputs of $E$ and $E^{-1}$. We show

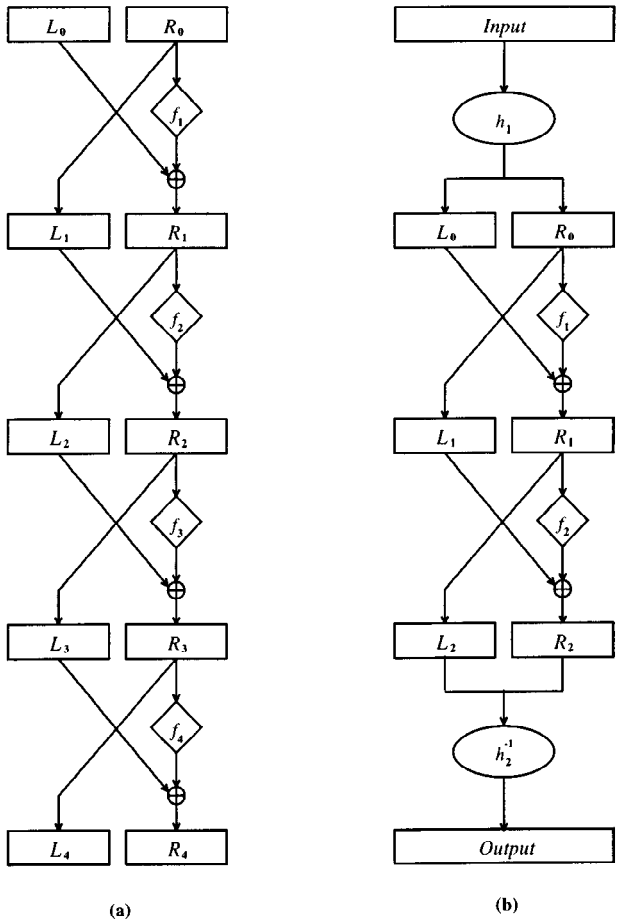


**Fig. 1.** Constructions of SPPE: (a) the original LR-Construction and (b) the revised construction. In (a) and (b): $\forall i \geq 1$, $L_i = R_{i-1}$ and $R_i = L_{i-1} \oplus f_i(R_{i-1})$. In (b): $\langle L_0, R_0 \rangle = h_1(Input)$ and $Output = h_2^{-1}(\langle L_2, R_2 \rangle)$.

that this goal can also be achieved with "combinatorial" constructions (e.g., pairwise independent permutations) rather than "cryptographic" (i.e., pseudorandom functions). In particular, the LR-Construction remains secure when the first and fourth rounds are replaced with pairwise independent permutations (see Fig. 1 for an illustration).

### 3.2. *Construction and Main Result*

**Definition 3.1.** For any $f_1, f_2 \in F_n$ and $h_1, h_2 \in P_{2n}$, define

$$W(h_1, f_1, f_2) \stackrel{\text{def}}{=} \mathbf{D}_{f_2} \circ \mathbf{D}_{f_1} \circ h_1$$

and

$$S(h_1, f_1, f_2, h_2) \stackrel{\text{def}}{=} h_2^{-1} \circ \mathbf{D}_{f_2} \circ \mathbf{D}_{f_1} \circ h_1.$$

**Theorem 3.1.** *Let $h_1, h_2 \in P_{2n}$ be pairwise independent permutations* (*similarly to Remark* 2.1 *this is an abbreviation for "distributed according to a pairwise independent permutation ensemble"*) *and let $f_1, f_2 \in F_n$ be pseudorandom functions; $h_1, h_2, f_1$, and $f_2$ are independently chosen. Then $W = W(h_1, f_1, f_2)$ is a pseudorandom permutation and $S = S(h_1, f_1, f_2, h_2)$ is a strong pseudorandom permutation* ($W$ *and* $S$ *as in Definition* 3.1).

*Furthermore, assume that no efficient oracle machine that makes at most $m = m(n)$ queries, $\varepsilon$-distinguishes between the pseudorandom functions and random functions for $\varepsilon = \varepsilon(n)$* (*see Definition* 2.3). *Then no efficient oracle machine that makes at most $m$ queries to $W$* (*resp. $S$ and $S^{-1}$*) *$\varepsilon'$-distinguishes $W$* (*resp. $S$*) *from a random permutation for $\varepsilon' = 2\varepsilon + m^2/2^n + m^2/2^{2n}$.*

*Remark* 3.1. The conditions of Theorem 3.1 are meant to simplify the exposition of the theorem and of its proof. These conditions can be relaxed, as discussed in Section 5. The main points are the following:

1. A single pseudorandom function $f$ can replace both $f_1$ and $f_2$.
2. $h_1$ and $h_2$ may obey weaker requirements than pairwise independence. For example, it is enough that, for every $x \neq y$,

$$\Pr[h_1(x)_{|_R} = h_1(y)_{|_R}] \leq 2^{-n} \quad \text{and} \quad \Pr[h_2(x)_{|_L} = h_2(y)_{|_L}] \leq 2^{-n}.$$

### 3.3. *Proof of Security*

We now prove the security of the SPPE construction; the proof of security for the PPE construction is very similar (and, in fact, a bit simpler). As with the original LR-Construction, the main task is to prove that the permutations are pseudorandom when $f_1$ and $f_2$ are truly random (instead of pseudorandom).

**Theorem 3.2.** *Let $h_1, h_2 \in P_{2n}$ be pairwise independent permutations and let $f_1, f_2 \in F_n$ be random functions. Define $S = S(h_1, f_1, f_2, h_2)$* (*as in Definition* 3.1) *and let $R \in P_{2n}$ be a random permutation. Then, for any oracle machine $M$* (*not necessarily an efficient one*) *that makes at most $m$ queries,*

$$\left|\Pr[M^{S,S^{-1}}(1^{2n}) = 1] - \Pr[M^{R,R^{-1}}(1^{2n}) = 1]\right| \leq \frac{m^2}{2^n} + \frac{m^2}{2^{2n}}.$$

Theorem 3.1 follows easily from Theorem 3.2 (see a proof-sketch in what follows). In order to prove Theorem 3.2, we introduce additional notation.

Let $G$ denote the permutation that is accessible to the machine $M$ ($G$ is either $S$ or $R$). There are two types of queries $M$ can make: either $(+, x)$ which denotes the query "what is $G(x)$?" or $(-, y)$ which denotes the query "what is $G^{-1}(y)$?" For the $i$th query $M$ makes, define the query–answer pair $\langle x_i, y_i \rangle \in I_{2n} \times I_{2n}$, where either $M$'s query was $(+, x_i)$ and the answer it got was $y_i$ or $M$'s query was $(-, y_i)$ and the answer it got was $x_i$. We assume that $M$ makes exactly $m$ queries and refer to the sequence $\{\langle x_1, y_1 \rangle, \ldots, \langle x_m, y_m \rangle\}$ of all these pairs as the *transcript* (of $M$'s computation).

Notice that no limitations were imposed on the computational power of $M$. Therefore, $M$ can be assumed to be deterministic (we can always fix the random tape that maximizes

the advantage $M$ achieves). This assumption implies that for every $1 \leq i \leq m$ the $i$th query of $M$ is fully determined by the first $i-1$ query–answer pairs. Thus, for every $i$ it can be determined from the transcript whether the $i$th query was $(+, x_i)$ or $(-, y_i)$. We also get that $M$'s output is a (deterministic) function of its transcript. Denote by $C_M[\{\langle x_1, y_1\rangle, ..., \langle x_{i-1}, y_{i-1}\rangle\}]$ the $i$th query of $M$ as a function of the previous query–answer pairs and denote by $C_M[\{\langle x_1, y_1\rangle, ..., \langle x_m, y_m\rangle\}]$ the output of $M$ as a function of its transcript.

**Definition 3.2.** Let $\sigma$ be a sequence $\{\langle x_1, y_1\rangle, ..., \langle x_m, y_m\rangle\}$, where for $1 \leq i \leq m$ we have that $\langle x_i, y_i\rangle \in I_{2n} \times I_{2n}$. Then $\sigma$ is a *possible $M$-transcript* if, for every $1 \leq i \leq m$,

$$C_M[\{\langle x_1, y_1\rangle, ..., \langle x_{i-1}, y_{i-1}\rangle\}] \in \{(+, x_i), (-, y_i)\}.$$

We consider yet another distribution on the answers to $M$'s queries (which, in turn, induces another distribution on the possible $M$-transcripts). Consider a random process $\tilde{R}$ that on the $i$th query of $M$ answers as follows:

1. If $M$'s query is $(+, x)$ and for some $1 \leq j < i$ the $j$th query–answer pair is $\langle x, y\rangle$, then $\tilde{R}$'s answer is $y$ (for an arbitrary such query–answer pair, $\langle x, y\rangle$).
2. If $M$'s query is $(-, y)$ and for some $1 \leq j < i$ the $j$th query–answer pair is $\langle x, y\rangle$, then $\tilde{R}$'s answer is $x$ (for an arbitrary such query–answer pair, $\langle x, y\rangle$).
3. If neither 1 nor 2 holds, then $\tilde{R}$'s answer is a uniformly chosen $2n$-bit string.

It is possible that $\tilde{R}$ provides answers that are not consistent with *any* permutation:

**Definition 3.3.** Let $\sigma = \{\langle x_1, y_1\rangle, ..., \langle x_m, y_m\rangle\}$ be any possible $M$-transcript. $\sigma$ is *inconsistent* if for some $1 \leq j < i \leq m$ the corresponding query–answer pairs satisfy $x_i = x_j$ and $y_i \neq y_j$ or $y_i = y_j$ and $x_i \neq x_j$. Otherwise, $\sigma$ is *consistent*.

We first show (in Proposition 3.3) that the advantage $M$ might have in distinguishing between the process $\tilde{R}$ and the random permutation $R$ is small. The reason is that as long as $\tilde{R}$ answers consistently (which happens with good probability) it "behaves" exactly as a random permutation. In order to formalize this, we consider the different distributions on the transcript of $M$ (induced by the different distributions on the answers it gets).

**Definition 3.4.** Let $T_S$, $T_R$, and $T_{\tilde{R}}$ be the random variables such that $T_S$ is the transcript of $M$ when its queries are answered by $S$, $T_R$ is the transcript of $M$ when its queries are answered by $R$, and $T_{\tilde{R}}$ is the transcript of $M$ when its queries are answered by $\tilde{R}$. Notice that by these definitions (and by our assumptions) $M^{S,S^{-1}}(1^{2n}) = C_M(T_S)$ (are the same random variables) and $M^{R,R^{-1}}(1^{2n}) = C_M(T_R)$.

**Proposition 3.3.**

$$\left| \Pr_{\tilde{R}}[C_M(T_{\tilde{R}}) = 1] - \Pr_{R}[C_M(T_R) = 1] \right| \leq \frac{m^2}{2^{2n+1}}.$$

**Proof.** For any possible and consistent $M$-transcript $\sigma$ we have that

$$\Pr_R[T_R = \sigma] = \frac{2^{2n}!}{(2^{2n} - m)!} = \Pr_{\tilde{R}}[T_{\tilde{R}} = \sigma \mid T_{\tilde{R}} \text{ is consistent}].$$

Therefore, the distribution of $T_{\tilde{R}}$ *conditioned* on $T_{\tilde{R}}$ being consistent is exactly the distribution of $T_R$. Furthermore, the probability that $T_{\tilde{R}}$ is inconsistent is small: $T_{\tilde{R}}$ is inconsistent if for some $1 \le j < i \le m$ the corresponding query–answer pairs satisfy $x_i = x_j$ and $y_i \ne y_j$ or $y_i = y_j$ and $x_i \ne x_j$. For a given $i$ and $j$ this event happens with probability at most $2^{-2n}$. Hence,

$$\Pr_{\tilde{R}}[T_{\tilde{R}} \text{ is inconsistent}] \le \binom{m}{2} \cdot 2^{-2n} < \frac{m^2}{2^{2n+1}}.$$

The proposition follows:

$$\begin{aligned}
&\left|\Pr_{\tilde{R}}[C_M(T_{\tilde{R}}) = 1] - \Pr_R[C_M(T_R) = 1]\right| \\
&\quad \le \left|\Pr_{\tilde{R}}[C_M(T_{\tilde{R}}) = 1 \mid T_{\tilde{R}} \text{ is consistent}] - \Pr_R[C_M(T_R) = 1]\right| \\
&\qquad \cdot \Pr_{\tilde{R}}[T_{\tilde{R}} \text{ is consistent}] \\
&\qquad + \left|\Pr_{\tilde{R}}[C_M(T_{\tilde{R}}) = 1 \mid T_{\tilde{R}} \text{ is inconsistent}] - \Pr_R[C_M(T_R) = 1]\right| \\
&\qquad \cdot \Pr_{\tilde{R}}[T_{\tilde{R}} \text{ is inconsistent}] \\
&\quad \le \Pr_{\tilde{R}}[T_{\tilde{R}} \text{ is inconsistent}] \\
&\quad < \frac{m^2}{2^{2n+1}}.
\end{aligned}$$

□

It remains to bound the advantage $M$ might have in distinguishing between $T_{\tilde{R}}$ and $T_S$. The intuition is that for every possible and consistent $M$-transcript $\sigma$ unless some "bad" and "rare" event on the choice of $h_1$ and $h_2$ (as in the definition of $S$) happens, the probability that $T_S = \sigma$ is exactly the same as the probability that $T_{\tilde{R}} = \sigma$. We now formally define this event (Definition 3.5) and bound its probability (Proposition 3.4).

**Convention 3.1.** For any possible $M$-transcript $\sigma = \{\langle x_1, y_1\rangle, ..., \langle x_m, y_m\rangle\}$ we can assume hereafter that if $\sigma$ is consistent, then for $i \ne j$ both $x_i \ne x_j$ and $y_i \ne y_j$ (this means that $M$ never asks a query if its answer is determined by a previous query–answer pair).

**Definition 3.5.** For every specific choice of pairwise independent permutations $h_1, h_2 \in P_{2n}$ (in the definition of $S$) define $\mathrm{BAD}(h_1, h_2)$ to be the set of all possible and consistent $M$-transcripts, $\sigma = \{\langle x_1, y_1\rangle, \ldots, \langle x_m, y_m\rangle\}$, satisfying:

$$\exists 1 \le i < j \le m \quad \text{such that} \quad h_1(x_i)_{|\mathrm{R}} = h_1(x_j)_{|\mathrm{R}} \quad \textit{or} \quad h_2(y_i)_{|\mathrm{L}} = h_2(y_j)_{|\mathrm{L}}.$$

**Proposition 3.4.** *Let $h_1, h_2 \in P_{2n}$ be pairwise independent permutations, then for any possible and consistent $M$-transcript $\sigma = \{\langle x_1, y_1\rangle, ..., \langle x_m, y_m\rangle\}$ we have that*

$$\Pr_{h_1,h_2}[\sigma \in \mathrm{BAD}(h_1, h_2)] < \frac{m^2}{2^n}.$$

**Proof.** By definition, $\sigma \in \mathrm{BAD}(h_1, h_2)$ if there exist $1 \le i < j \le m$ such that either $h_1(x_i)_{|\mathrm{R}} = h_1(x_j)_{|\mathrm{R}}$ or $h_2(y_i)_{|\mathrm{L}} = h_2(y_j)_{|\mathrm{L}}$. For any given $i$ and $j$ both $\Pr_{h_1}[h_1(x_i)_{|\mathrm{R}} = h_1(x_j)_{|\mathrm{R}}]$ and $\Pr_{h_2}[h_2(y_i)_{|\mathrm{L}} = h_2(y_j)_{|\mathrm{L}}]$ are smaller than $2^{-n}$ (since $h_1$ and $h_2$ are pairwise independent). Therefore,

$$\Pr_{h_1,h_2}[\sigma \in \mathrm{BAD}(h_1, h_2)] < \binom{m}{2} \cdot 2 \cdot 2^{-n} < \frac{m^2}{2^n}. \qquad \square$$

The key lemma for proving Theorem 3.2 is:

**Lemma 3.5.** *Let $\sigma = \{\langle x_1, y_1\rangle, ..., \langle x_m, y_m\rangle\}$ be any possible and consistent $M$-transcript, then*

$$\Pr_S[T_S = \sigma \mid \sigma \notin \mathrm{BAD}(h_1, h_2)] = \Pr_{\tilde{R}}[T_{\tilde{R}} = \sigma].$$

**Proof.** Since $\sigma$ is a possible $M$-transcript we have that, for all $1 \le i \le m$,

$$C_M[\{\langle x_1, y_1\rangle, ..., \langle x_{i-1}, y_{i-1}\rangle\}] \in \{(+, x_i), (-, y_i)\}.$$

Therefore, $T_{\tilde{R}} = \sigma$ iff, for all $1 \le i \le m$, the $i$th answer $\tilde{R}$ gives is $y_i$ in the case that $C_M[\{\langle x_1, y_1\rangle, ..., \langle x_{i-1}, y_{i-1}\rangle\}] = (+, x_i)$ and otherwise its $i$th answer is $x_i$. Assume that $\tilde{R}$ answered "correctly" (i.e., $y_i$ or $x_i$ as above) for each one of the first $i-1$ queries. Then by Convention 3.1 and the definition of $\tilde{R}$ its $i$th answer is an independent and uniform $2n$-bit string. Therefore,

$$\Pr_{\tilde{R}}[T_{\tilde{R}} = \sigma] = 2^{-2nm}.$$

Since $\sigma$ is a possible $M$-transcript we have that $T_S = \sigma$ iff, for all $1 \le i \le m$, $y_i = S(x_i)$. Consider any specific choice of permutations $h_1$ and $h_2$ (for which $S = S(h_1, f_1, f_2, h_2)$) such that $\sigma \notin \mathrm{BAD}(h_1, h_2)$. Let $(L_i^0, R_i^0) = h_1(x_i)$ and $(L_i^2, R_i^2) = h_2(y_i)$. By the definition of $S$, we get that

$$y_i = S(x_i) \iff f_1(R_i^0) = L_i^0 \oplus L_i^2 \quad \text{and} \quad f_2(L_i^2) = R_i^0 \oplus R_i^2.$$

For every $1 \le i < j \le m$ both $R_i^0 \ne R_j^0$ and $L_i^2 \ne L_j^2$ (otherwise $\sigma \in \mathrm{BAD}(h_1, h_2)$). Therefore, since $f_1$ and $f_2$ are random, we have that for every choice of $h_1$ and $h_2$ such that $\sigma \notin \mathrm{BAD}(h_1, h_2)$ the probability that $T_S = \sigma$ is exactly $2^{-2nm}$. We can conclude:

$$\Pr_S[T_S = \sigma \mid \sigma \notin \mathrm{BAD}(h_1, h_2)] = 2^{-2nm},$$

which complete the proof of the lemma. $\square$

**Proof of Theorem 3.2.** Let $\Gamma$ be the set of all possible and consistent $M$-transcripts $\sigma$ such that $M(\sigma) = 1$.

$$\begin{aligned}
&\left|\Pr_S[C_M(T_S) = 1] - \Pr_{\tilde{R}}[C_M(T_{\tilde{R}}) = 1]\right| \\
&\quad \le \left|\sum_{\sigma\in\Gamma}\left(\Pr_S[T_S = \sigma] - \Pr_{\tilde{R}}[T_{\tilde{R}} = \sigma]\right)\right| + \Pr_{\tilde{R}}[T_{\tilde{R}} \text{ is inconsistent}] \\
&\quad \le \sum_{\sigma\in\Gamma}\left|\Pr_S[T_S = \sigma | \sigma \notin \mathrm{BAD}(h_1,h_2)] - \Pr_{\tilde{R}}[T_{\tilde{R}} = \sigma]\right| \\
&\qquad \cdot \Pr_{h_1,h_2}[\sigma \notin \mathrm{BAD}(h_1,h_2)] \qquad (1) \\
&\qquad + \left|\sum_{\sigma\in\Gamma}\left(\Pr_S[T_S = \sigma | \sigma \in \mathrm{BAD}(h_1,h_2)] - \Pr_{\tilde{R}}[T_{\tilde{R}} = \sigma]\right)\right. \\
&\qquad \left. \cdot \Pr_{h_1,h_2}[\sigma \in \mathrm{BAD}(h_1,h_2)]\right| \qquad (2) \\
&\qquad + \Pr_{\tilde{R}}[T_{\tilde{R}} \text{ is inconsistent}]. \qquad (3)
\end{aligned}$$

We already showed in the proof of Proposition 3.3 that the value of (3) is smaller than $m^2/2^{2n+1}$, by Lemma 3.5 we get that the value of (1) is zero. Therefore, it remains to bound the value of (2): Assume without loss of generality that

$$\begin{aligned}
&\sum_{\sigma\in\Gamma}\Pr_S[T_S = \sigma \mid \sigma \in \mathrm{BAD}(h_1,h_2)] \cdot \Pr_{h_1,h_2}[\sigma \in \mathrm{BAD}(h_1,h_2)] \\
&\qquad \le \sum_{\sigma\in\Gamma}\Pr_{\tilde{R}}[T_{\tilde{R}} = \sigma] \cdot \Pr_{h_1,h_2}[\sigma \in \mathrm{BAD}(h_1,h_2)],
\end{aligned}$$

then using Proposition 3.4 we get that

$$\begin{aligned}
&\left|\sum_{\sigma\in\Gamma}\left(\Pr_S[T_S = \sigma \mid \sigma \in \mathrm{BAD}(h_1,h_2)] - \Pr_{\tilde{R}}[T_{\tilde{R}} = \sigma]\right) \cdot \Pr_{h_1,h_2}[\sigma \in \mathrm{BAD}(h_1,h_2)]\right| \\
&\quad \le \sum_{\sigma\in\Gamma}\Pr_{\tilde{R}}[T_{\tilde{R}} = \sigma] \cdot \Pr_{h_1,h_2}[\sigma \in \mathrm{BAD}(h_1,h_2)] \\
&\quad \le \max_{\sigma\in\Gamma}\Pr_{h_1,h_2}[\sigma \in \mathrm{BAD}(h_1,h_2)] \\
&\quad < \frac{m^2}{2^n}.
\end{aligned}$$

Thus, we can conclude that

$$\left|\Pr_S[C_M(T_S) = 1] - \Pr_{\tilde{R}}[C_M(T_{\tilde{R}}) = 1]\right| < \frac{m^2}{2^n} + \frac{m^2}{2^{2n+1}}.$$

Using Proposition 3.3 we complete the proof:

$$\begin{aligned}
&\left|\Pr_S[M^{S,S^{-1}}(1^{2n})=1]-\Pr_R[M^{R,R^{-1}}(1^{2n})=1]\right| \\
&\quad=\left|\Pr_S[C_M(T_S)=1]-\Pr_R[C_M(T_R)=1]\right| \\
&\quad\le\left|\Pr_S[C_M(T_S)=1]-\Pr_{\tilde{R}}[C_M(T_{\tilde{R}})=1]\right|+\left|\Pr_{\tilde{R}}[C_M(T_{\tilde{R}})=1]-\Pr_R[C_M(T_R)=1]\right| \\
&\quad<\frac{m^2}{2^n}+\frac{m^2}{2^{2n}}.
\end{aligned}$$

□

Given Theorem 3.2, the proof of Theorem 3.1 is essentially the same as the corresponding proof of the original LR-Construction (the proof of Theorem 1 of [26], given their main lemma). The proof idea is the following: Define three distributions:

- $S_1 = S(h_1,\ f_1,\ f_2, h_2)$, where $h_1, h_2 \in P_{2n}$ are pairwise independent and $f_1,\ f_2 \in F_n$ are pseudorandom functions.
- $S_2 = S(h_1, g_1,\ f_2, h_2)$, where $h_1, h_2 \in P_{2n}$ are pairwise independent, $f_2 \in F_n$ is a pseudorandom function, and $g_1 \in F_n$ is a random function.
- $S_3 = S(h_1, g_1, g_2, h_2)$, where $h_1, h_2 \in P_{2n}$ are pairwise independent and $g_1, g_2 \in F_n$ are random functions.

It is enough to show that, for every oracle machine, for all but a finite number of $n$'s:

1. $|\Pr[M^{S_1,S_1^{-1}}(1^{2n})=1]-\Pr[M^{S_2,S_2^{-1}}(1^{2n})=1]| \le \varepsilon(n)$.
2. $|\Pr[M^{S_2,S_2^{-1}}(1^{2n})=1]-\Pr[M^{S_3,S_3^{-1}}(1^{2n})=1]| \le \varepsilon(n)$.

If 1 or 2 do not hold, then we can construct an efficient oracle machine $M'$ that $\varepsilon$-distinguishes the pseudorandom functions from the random functions in contradiction to the assumption. Assume, for example, that, for infinitely many $n$'s,

$$\left|\Pr[M^{S_1,S_1^{-1}}(1^{2n})=1]-\Pr[M^{S_2,S_2^{-1}}(1^{2n})=1]\right| > \varepsilon(n).$$

The oracle machine $M'$ on input $1^n$ and with access to a function $O \in F_n$ first samples pairwise independent permutations, $h_1, h_2 \in P_{2n}$, and a pseudorandom function $f_2 \in F_n$. $M'$ then invokes $M$ with input $1^{2n}$ and answers its queries with the values of $S$ and $S^{-1}$, for $S = S(h_1, O,\ f_2, h_2)$. When $M$ halts so does $M'$ and it outputs whatever was the output of $M$. Notice that if $O$ is a pseudorandom function, then the distribution of $S$ is $S_1$ whereas if $O$ is a truly random function, then the distribution of $S$ is $S_2$. This is the reason that $M'$ distinguishes a pseudorandom function from a random one with advantage greater than $\varepsilon(n)$. Similar hybrid arguments apply to all other constructions of this paper.

## 4. The Framework

As we shall see in Sections 5–7, the construction of Section 3 can be relaxed and generalized in several ways. The different pseudorandom permutations obtained share a

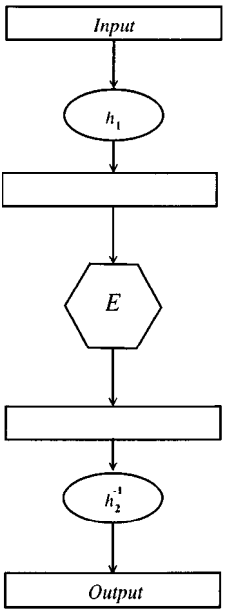


**Fig. 2.** The high-level structure of the different constructions of SPPE.

similar structure and almost identical proof of security. In this section we examine the proof of Theorem 3.2 in a more abstract manner. Our goal is to establish a framework for proving (almost) all the constructions of this paper and to suggest a way for designing and proving additional constructions.

Our framework deals with constructions of a pseudorandom permutation $S$ on $\ell$ bits which is the composition of three permutations: $S \equiv h_2^{-1} \circ E \circ h_1$ (see Fig. 2 for an illustration). In general, $h_1$ and $h_2^{-1}$ are "lightweight" and $E$ is where most of the work is done. $E$ is constructed from pseudorandom functions and for the purpose of the analysis we assume (as in Theorem 3.2) that these functions are truly random. In Section 3, for example, $\ell = 2n$, $h_1$ and $h_2$ are chosen as pairwise independent permutations, and $E \equiv \mathbf{D}_{f_2} \circ \mathbf{D}_{f_1}$ for random $f_1, f_2 \in F_n$.

The framework starts with $E$ which may be easily distinguished from a truly random permutation and transforms it via $h_1$ and $h_2$ into a pseudorandom permutation. The property $E$ should have is that for almost every sequence, $\{\langle x_1, y_1\rangle, \ldots, \langle x_m, y_m\rangle\}$, the probability that, $\forall i,\ y_i = E(x_i)$ is "close" to what we have for a truly random permutation:

**Definition 4.1.** A sequence, $\{\langle x_1, y_1\rangle, \ldots, \langle x_m, y_m\rangle\}$, is $E$-Good if $\Pr_E[\forall i,\ y_i = E(x_i)] = 2^{-\ell \cdot m}$.

We assume that apart from some "rare" sequences all others are $E$-Good. Loosely speaking, the role of $h_1$ and $h_2$ is to ensure that under any (adaptive chosen plaintext and ciphertext) attack on $S$ the inputs and outputs of $E$ form an $E$-Good sequence with very high probability.

For the exact properties needed from the distributions on $h_1$, $h_2$, and $E$, we try to follow the statement and proof of Theorem 3.2. The goal is to show that $S$ is indistinguishable from a truly random permutation $R$ on $\ell$ bits. Specifically, that for some small $\varepsilon$ (whose choice will be explained hereafter), for any oracle machine $M$ (not necessarily an efficient one) that makes at most $m$ queries:

$$\left|\Pr[M^{S,S^{-1}}(1^\ell) = 1] - \Pr[M^{R,R^{-1}}(1^\ell) = 1]\right| \leq \varepsilon + \frac{m^2}{2^\ell}.$$

Let the notions of a query–answer pair, a transcript, the function $C_M$, a possible $M$-transcript, the random process $\tilde{R}$, a consistent transcript, and the different random variables $T_S$, $T_R$, and $T_{\tilde{R}}$ be as in the proof of Theorem 3.2. Proposition 3.3 (saying that the distance between $T_R$ and $T_{\tilde{R}}$ is bounded by the probability that $T_{\tilde{R}}$ is inconsistent and that this probability is bounded by $m^2/2^{\ell+1}$) still holds. The heart of applying the framework is in specifying the "bad" $M$-transcripts for given $h_1$ and $h_2$. This set $\mathrm{BAD}_E(h_1, h_2)$ replaces $\mathrm{BAD}(h_1, h_2)$ in Definition 3.5 and in the rest of the proof. It contains possible and consistent $M$-transcripts and should have the property that any $\{\langle x_1, y_1\rangle, \ldots, \langle x_m, y_m\rangle\}$ *not in* $\mathrm{BAD}_E(h_1, h_2)$ satisfies that $\{\langle h_1(x_1), h_2(y_1)\rangle, \ldots, \langle h_1(x_m), h_2(y_m)\rangle\}$ is $E$-Good. Note that Definition 3.5 is indeed a special case of the above and also that, by this property,

$$\Pr_S[T_S = \sigma \mid \sigma \notin \mathrm{BAD}_E(h_1, h_2)] = 2^{-\ell \cdot m}.$$

This implies that Lemma 3.5 where $\mathrm{BAD}(h_1, h_2)$ is replaced with $\mathrm{BAD}_E(h_1, h_2)$ is true:

**Lemma 4.1.** *Let $\sigma = \{\langle x_1, y_1\rangle, \ldots, \langle x_m, y_m\rangle\}$ be any possible and consistent $M$-transcript, then*

$$\Pr_S[T_S = \sigma \mid \sigma \notin \mathrm{BAD}_E(h_1, h_2)] = \Pr_{\tilde{R}}[T_{\tilde{R}} = \sigma].$$

For $\mathrm{BAD}_E(h_1, h_2)$ to be useful we must have that

$$\Pr_{h_1,h_2}[\sigma \in \mathrm{BAD}_E(h_1, h_2)] \leq \varepsilon \tag{4}$$

and this substitutes Proposition 3.4. This is the only place in the proof where we use the definition of $\varepsilon$ and the definition of the distributions of $h_1$ and $h_2$. As will be demonstrated in Sections 5.2 and 7.1, there is actually a tradeoff between reducing the requirements from $h_1$ and $h_2$ and having a somewhat larger value of $\varepsilon$. Applying (4) and Lemma 4.1 as in the proof of Theorem 3.2 we conclude:

**Theorem 4.2.** *Let $h_1$, $h_2$, $E$ be distributed over permutations in $P_\ell$, let $S \equiv h_2^{-1} \circ E \circ h_1$, and let $R \in P_\ell$ be a random permutation. Suppose that* $\mathrm{BAD}_E(h_1, h_2)$ *is as above and $\varepsilon$ satisfies* (4)*. Then, for any oracle machine $M$ (not necessarily an efficient one) that makes at most $m$ queries,*

$$\left|\Pr[M^{S,S^{-1}}(1^\ell) = 1] - \Pr[M^{R,R^{-1}}(1^\ell) = 1]\right| \leq \varepsilon + \frac{m^2}{2^\ell}.$$

To summarize, the major point in proving the security of the different constructions is to define the set $\mathrm{BAD}_E(h_1, h_2)$ such that for any possible and consistent $M$-transcript, $\sigma$, both $\Pr_S[T_S = \sigma \mid \sigma \notin \mathrm{BAD}_E(h_1, h_2)] = 2^{-\ell \cdot m}$ and $\Pr_{h_1,h_2}[\sigma \in \mathrm{BAD}_E(h_1, h_2)] \leq \varepsilon$ (for the specific $\varepsilon$ in the claim we are proving). This suggests that the critical step for designing a pseudorandom permutation, using the framework described in this section, is to come up with a permutation $E$ such that the set of $E$-Good sequences is "large enough" and "nice enough." Note that to meet this end different or more general definitions of an $E$-Good sequence can be used with only minor changes to the proof (as is the case for the permutation $\hat{S}$ in Section 7).

## 5. Relaxing the Construction

### 5.1. *PPE and SPPE with a Single Pseudorandom Function*

Since Luby and Rackoff introduced their construction a considerable amount of research [33]–[36], [38], [42]–[44], [46], [51] has been focused on the following question: Can we obtain a similar construction of PPE or SPPE such that every permutation will be constructed from a *single* pseudorandom function?

Apparently, this line of research originated in the work of Schnorr [46]. Schnorr considered the LR-Construction, where the functions used are truly random, as a pseudorandom generator that is secure if not too many bits are accessible. The security of Schnorr's generator does not depend on any unproven assumption. This notion of local-randomness is further treated in [28] and [29]. Since the key of a random function is huge it makes sense to minimize the number of functions and, indeed, Schnorr suggested $\mathbf{D}_f \circ \mathbf{D}_f \circ \mathbf{D}_f$ as pseudorandom (the suggested permutation was later shown to be distinguishable from random [42]).

Following is an informal description of some of these results. Let $f \in F_n$ be a random function, then:

- For all $i, j, k \geq 1$ the permutation $\mathbf{D}_{f^i} \circ \mathbf{D}_{f^j} \circ \mathbf{D}_{f^k}$ is not pseudorandom [51].
- For all $i, j, k, \ell \geq 1$ the permutation $\mathbf{D}_{f^i} \circ \mathbf{D}_{f^j} \circ \mathbf{D}_{f^k} \circ \mathbf{D}_{f^\ell}$ is not strongly pseudorandom [43].
- $\mathbf{D}_{f^2} \circ \mathbf{D}_f \circ \mathbf{D}_f \circ \mathbf{D}_f$ is pseudorandom [38].
- $\mathbf{D}_f \circ \mathbf{D}_I \circ \mathbf{D}_{f^2} \circ \mathbf{D}_f \circ \mathbf{D}_I \circ \mathbf{D}_{f^2}$ is strongly pseudorandom, where $I \in F_n$ is the identity function [44].
- $\mathbf{D}_{f \circ \xi \circ f} \circ \mathbf{D}_f \circ \mathbf{D}_f$ is pseudorandom and $\mathbf{D}_{f \circ \xi \circ f} \circ \mathbf{D}_f \circ \mathbf{D}_f \circ \mathbf{D}_f$ is strongly pseudorandom, where $\xi$ is, for example, a rotation of one bit [36].

A critique which has often been voiced is that using only one pseudorandom function does not seem too significant: A pseudorandom function on $n + 2$ bits can replace four pseudorandom functions on $n$ bits or, alternatively, a small key can be used to pseudorandomly generate a larger key. It should also be noticed that the new constructions require additional invocations of the pseudorandom functions which imply an increase in the computation time. Furthermore, these results involve detailed and nontrivial proofs (to a point where some papers claim to find inaccuracies in others).

The adjustment of the LR-Construction we suggest in Section 3 can easily be converted into a construction of PPE and SPPE from a single pseudorandom function. Simply re-

place both (pseudorandom) functions $f_1$ and $f_2$ with a single (pseudorandom) function $f$. This solution does not suffer from the drawbacks of the previous ones. The construction and the proof remain as simple as before and the pseudorandom function is only invoked twice at each computation of the permutation. The additional key-length for the pairwise independent functions ($h_1$ and $h_2$) is not substantial (especially compared with the length of a truly random function). Consider, for example, the construction of SPPE when we use a truly random function $f$:

**Theorem 5.1.** *Let $h_1, h_2 \in P_{2n}$ be pairwise independent permutations and let $f \in F_n$ be a random function. Define $S = S(h_1, f, f, h_2)$ (as in Definition* 3.1*) and let $R \in P_{2n}$ be a random permutation. Then, for any oracle machine $M$ (not necessarily an efficient one) that makes at most $m$ queries,*

$$\left|\Pr[M^{S,S^{-1}}(1^{2n}) = 1] - \Pr[M^{R,R^{-1}}(1^{2n}) = 1]\right| \leq \frac{2m^2}{2^n} + \frac{m^2}{2^{2n}}.$$

The proof follows the framework described in Section 4. The set $\mathrm{BAD}(h_1, h_2)$ (Definition 3.5) is replaced with the set $\mathrm{BAD}_1(h_1, h_2)$ defined to be:

> The set of all possible and consistent $M$-transcripts,
>
> $$\sigma = \{\langle x_1, y_1\rangle, \ldots, \langle x_m, y_m\rangle\},$$
>
> satisfying that there exist $1 \leq i < j \leq m$ such that either $h_1(x_i)_{|\mathrm{R}} = h_1(x_j)_{|\mathrm{R}}$ or $h_2(y_i)_{|\mathrm{L}} = h_2(y_j)_{|\mathrm{L}}$ (as before), *or* there exist $1 \leq i, j \leq m$ such that $h_1(x_i)_{|\mathrm{R}} = h_2(y_j)_{|\mathrm{L}}$.

In order to apply Theorem 4.2, it is enough to note that by this definition we get that for any possible and consistent $M$-transcripts, $\sigma$, both $\Pr_S[T_S = \sigma \mid \sigma \notin \mathrm{BAD}_1(h_1, h_2)] = 2^{-2nm}$ (hence, it is a proper definition according to the framework) and $\Pr_{h_1,h_2}[\sigma \in \mathrm{BAD}_1(h_1, h_2)] < 2m^2/2^n$.

### 5.2. *Relaxing the Pairwise Independence Requirement*

The construction of Section 3 may be interpreted in the following way: given the task of constructing *efficient* pseudorandom permutations it is enough to concentrate on the efficient construction of pseudorandom *functions*. The assumption that supports such a claim is that the computation of pseudorandom functions is much more expensive than the computation of pairwise independent permutations. Therefore, computing the value of the pseudorandom permutation (that is constructed in Section 3) on any input of $2n$ bits is essentially equivalent to two invocations of a pseudorandom function with $n$-bit inputs. In this section we show that we can use even weaker permutations instead of the pairwise independent ones—resulting in an even more efficient construction of pseudorandom permutations.

As mentioned in Section 4, the only place in Section 3 we use the fact that $h_1$ and $h_2$ are pairwise independent permutations is in the proof of Proposition 3.4. In fact, the exact requirement on $h_1$ and $h_2$ we use is that, for every $x \neq y$,

$$\Pr_{h_1}[h_1(x)_{|\mathrm{R}} = h_1(y)_{|\mathrm{R}}] \leq 2^{-n} \quad \text{and} \quad \Pr_{h_2}[h_2(x)_{|\mathrm{L}} = h_2(y)_{|\mathrm{L}}] \leq 2^{-n}.$$

Furthermore, we can replace $2^{-n}$ with any $\varepsilon \geq 2^{-n}$ and still get a construction of pseudorandom permutations (with somewhat larger distinguishing probability). Consider, for example, the revised statement of Theorem 3.2:

**Theorem 5.2.** *Let $H^1$ and $H^2$ be distributions of permutations in $P_{2n}$ such that, for every pair of $2n$-bit strings $x \neq y$,*

$$\Pr_{h_1 \in H^1}[h_1(x)_{|\mathrm{R}} = h_1(y)_{|\mathrm{R}}] \leq \varepsilon \quad \textit{and} \quad \Pr_{h_2 \in H^2}[h_2(x)_{|\mathrm{L}} = h_2(y)_{|\mathrm{L}}] \leq \varepsilon.$$

*Let $h_1$ be distributed according to $H^1$, let $h_2$ be distributed according to $H^2$, and let $f_1, f_2 \in F_n$ be random functions. Define $S = S(h_1, f_1, f_2, h_2)$ (as in Definition 3.1) and let $R \in P_{2n}$ be a random permutation. Then, for any oracle machine $M$ (not necessarily an efficient one) that makes at most $m$ queries,*

$$\left|\Pr[M^{S,S^{-1}}(1^{2n}) = 1] - \Pr[M^{R,R^{-1}}(1^{2n}) = 1]\right| < m^2 \cdot \varepsilon + \frac{m^2}{2^{2n}}.$$

The proof follows the framework described in Section 4. This time the definition of $\mathrm{BAD}(h_1, h_2)$ stays unchanged and, in order to apply Theorem 4.2, we only need to note that, for any possible and consistent $M$-transcript $\sigma$, $\Pr_{h_1,h_2}[\sigma \in \mathrm{BAD}(h_1, h_2)] < m^2 \cdot \varepsilon$.

The conditions on $H^1$ and $H^2$ in Theorem 5.2 are somewhat nonstandard (since the requirements are on half the bits of the output). Nevertheless, these conditions are satisfied by more traditional requirements on function families. In particular, the concept of $\varepsilon$-$\mathrm{AXU}_2$ functions can be used:

**Definition 5.1.** A distribution on $I_n \mapsto I_n$ functions (or permutations), $H$, is $\varepsilon$-$\mathrm{AXU}_2$ if, for every $x \neq y$ and every $z$ ($x, y, z \in I_n$),

$$\Pr_{h \in H}[h(x) \oplus h(y) = z] \leq \varepsilon.$$

This concept was originally defined by Carter and Wegman [12]; we use the terminology of Rogaway [40].

It is easy to verify that the conditions on $H^1$ and $H^2$ in Theorem 5.2 are satisfied if both $H^1$ and $H^2$ are $((2^n - 1)^{-1} \cdot \varepsilon)$-$\mathrm{AXU}_2$. Such a distribution of permutations over $I_{2n}$, for $\varepsilon = (2^n + 1)^{-1}$, is $h_a(x) \stackrel{\mathrm{def}}{=} a \cdot x$ where $a$ is uniform in $I_{2n} \backslash \{0\}$ and the multiplication is in $GF(2^{2n})$.

Another way to construct $H^1$ and $H^2$ is by *using Feistel permutations* with $\varepsilon$-$\mathrm{AXU}_2$ *functions*. Let $H$ be a distribution of $\varepsilon$-$\mathrm{AXU}_2$ functions on $n$-bit strings, then we can define $H^1$ to be $\{\mathbf{D}_h\}_{h \in H}$ and $H^2$ to be $\{\mathbf{D}_h^{-1}\}_{h \in H}$. The reason is that for every two *different* $2n$-bit strings $x = (L^1, R^1)$ and $y = (L^2, R^2)$ and every function $h \in F_n$ we have by definition that

$$\mathbf{D}_h(x)_{|_R} = \mathbf{D}_h(y)_{|_R} \quad \Longleftrightarrow \quad h(R^1) \oplus h(R^2) = L^1 \oplus L^2.$$

If $R^1 = R^2$, then $L^1 \neq L^2$ and therefore $\mathbf{D}_h(x)_{|\mathrm{R}} \neq \mathbf{D}_h(y)_{|\mathrm{R}}$ otherwise, by the definition

of $\varepsilon$-AXU$_2$ functions:

$$\Pr_{h\in H}[\mathbf{D}_h(x)_{|\mathrm{R}} = \mathbf{D}_h(y)_{|\mathrm{R}}] = \Pr_{h\in H}[h(R^1)\oplus h(R^2) = L^1 \oplus L^2] \le \varepsilon.$$

Thus, $H^1$ satisfies its requirement and similarly for $H^2$.

By using Feistel permutations to construct $H^1$ and $H^2$ we get the original LR-Construction as a special case (since a random function is in particular $2^{-n}$-AXU$_2$). Thus, the proof of security in Section 3 also holds for the original LR-Construction. The idea of using $\varepsilon$-AXU$_2$ functions instead of pseudorandom functions for the first round of the LR-Construction was previously suggested by Lucks [27].

Another advantage of this approach is that it allows us to use many efficient constructions of function families. An example of efficient $2^{-n}$-AXU$_2$ functions are Vazirani's "shift" family [48]. A key of such a function is a uniformly chosen string $a \in I_{2n-1}$ and the $j$th bit of $f_a(x)$ $(1 \le j \le n)$ is defined to be $\sum_{i=1}^{n} x_i a_{j+i-1} \bmod 2$.

A substantial amount of research [12], [19], [23], [40], [47], [49] deals with the construction of *efficient* hash functions. This line of work contains constructions that obey weaker definitions on function families than pairwise independence and in particular contains constructions of $\varepsilon$-AXU$_2$ functions. Unfortunately, these functions were designed to be especially efficient when their output is substantially smaller than their input (since they were mainly brought up in the context of authentication) which is not true in our case (but is relevant in Section 7). An additional objective is to reduce the size of the family of hash functions (e.g., [18] and [23]). In our setting the purpose of this is to reduce the key-length of the pseudorandom permutations.

## 6. Reducing the Distinguishing Probability

There are various circumstances where it is desirable to have a pseudorandom permutation on relatively *few* bits (say 128). This is especially true when we want to minimize the size of the hardware circuit that implements the permutation or the communication bandwidth with the (hardware or software) component that computes the permutation.

Let $F$ be a pseudorandom permutation on $\ell$ bits (note that $n = \ell/2$ in Section 3) constructed from truly random functions (on $\ell/2$ bits) using the LR-Construction. As shown by Patarin [35], $F$ can be distinguished (with constant probability) from a random permutation using $O(2^{\ell/4})$ queries (which means that the analysis of the LR-Construction, where the distinguishing probability for $m$ queries is $O(m^2/2^{\ell/2})$, is tight). Therefore, the LR-Construction on $\ell$ bits can only be used if $2^{\ell/4}$ is large enough to bound the number of queries in the attack on the block cipher.

In this section a simple generalization of the construction of Section 3 is presented. Using this construction, the adversary's probability of distinguishing between the pseudorandom and random permutations can be reduced to roughly $t/2 \cdot m^2/2^{(1-1/t)\ell}$ for every integer $2 \le t \le \ell$ (for $t = 2$ we get the original construction). To achieve this security $t+2$ permutations are composed. The initial and final are pairwise independent permutations, the rest are (generalized) Feistel permutations defined by $I_{(1-1/t)\ell} \mapsto I_{\ell/t}$ random (or pseudorandom) functions (see Fig. 3 for an illustration).

Patarin [37] shows that if we take six rounds of the LR-Construction (instead of three or four), then the resulting permutation cannot be distinguished from a random permutation

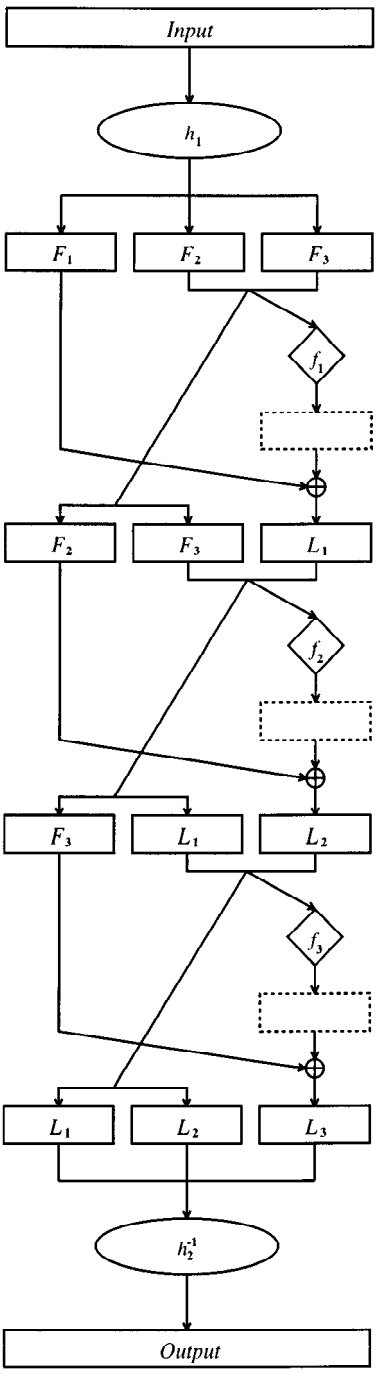


**Fig. 3.** Construction of strong pseudorandom permutations with reduced distinguishing probability using $t + 2$ rounds (here $t = 3$). Recall, $f_i\colon I_{(1-1/t)\ell} \mapsto I_{\ell/t}$ (here $f_i\colon I_{2\ell/3} \mapsto I_{\ell/3}$).

with advantage better than $5m^3/2^\ell$ (improving [35]). This means that distinguishing the six-round construction from a truly random permutation (with constant probability) requires at least $\Omega(2^{\ell/3})$ queries. The bound we achieve in this section ($\Omega(2^{(1-1/t)\cdot\ell/2})$) is better (for any $t \geq 4$). Note that our construction uses pseudorandom functions with larger input-length, which might be a disadvantage for some applications.

In order to describe our generalized constructions we first extend Feistel permutations to deal with the case where the underlying functions have arbitrary input and output lengths (instead of length-preserving functions as in Definition 2.1). We note that using such "unbalanced" Feistel permutations was previously suggested in [5], [27], and [45].

**Definition 6.1** (Generalized Feistel Permutations). For any two positive integers, $s$ and $\ell'$, and any function $f\colon I_{\ell'} \mapsto I_s$, let $\ell = \ell' + s$ and let $\mathbf{D}_f \in P_\ell$ be the permutation defined by $\mathbf{D}_f(L, R) \stackrel{\text{def}}{=} (R, L \oplus f(R))$, where $|L| = s$ and $|R| = \ell'$.

We can now define the revised construction and consider its security. These are simple generalizations of the construction in Section 3 and of its proof of security.

**Definition 6.2** ($(t+2)$-Round Construction). For any integers $2 \leq t \leq \ell$, let $s$ and $r$ be integers such that $\ell = s \cdot t + r$ (where $r < t$). For any $h_1, h_2 \in P_\ell$,

$$f_1, f_2, \ldots, f_r\colon I_{\ell-s-1} \mapsto I_{s+1},$$

and $f_{r+1}, \ldots, f_t\colon I_{\ell-s} \mapsto I_s$ define

$$W(h_1, f_1, f_2 \ldots, f_t) \stackrel{\text{def}}{=} \mathbf{D}_{f_t} \circ \mathbf{D}_{f_{t-1}} \circ \cdots \circ \mathbf{D}_{f_1} \circ h_1$$

and

$$S(h_1, f_1, f_2, \ldots, f_t, h_2) \stackrel{\text{def}}{=} h_2^{-1} \circ \mathbf{D}_{f_t} \circ \mathbf{D}_{f_{t-1}} \circ \cdots \circ \mathbf{D}_{f_1} \circ h_1.$$

(We get the construction of Definition 3.1 by choosing $t = 2$, $s = \ell/2$, and $r = 0$.)

**Theorem 6.1.** *Let $W$ and $S$ be as in Definition* 6.2*, where $h_1$ and $h_2$ are pairwise independent permutations and $f_1, f_2, \ldots, f_t$ are pseudorandom functions ($t$ is allowed to be a function of $\ell$); $h_1, h_2$, and $f_1, f_2, \ldots, f_t$ are independently chosen. Then $W$ is a pseudorandom permutation and $S$ is a strong pseudorandom permutation.*

*Furthermore, assume that no efficient oracle machine that makes at most $m = m(\ell)$ queries, $\varepsilon$-distinguishes between the pseudorandom functions and random functions for $\varepsilon = \varepsilon(n)$. Then no efficient oracle machine that makes at most $m$ queries to $W$ (resp. $S$ and $S^{-1}$) $\varepsilon'$-distinguishes $W$ (resp. $S$) from a random permutation for $\varepsilon' = t \cdot \varepsilon + t/2 \cdot m^2/2^{\ell-\lceil \ell/t \rceil} + m^2/2^\ell$.*

In case the middle functions are truly random this reduces to:

**Theorem 6.2.** *Let $S$ be as in Definition* 6.2*, where $h_1$ and $h_2$ are pairwise independent permutations and $f_1, f_2, \ldots, f_t$ are random functions and let $R \in P_\ell$ be a random*

*permutation. Then, for any oracle machine $M$ (not necessarily an efficient one) that makes at most $m$ queries,*

$$\left|\Pr[M^{S,S^{-1}}(1^\ell)=1]-\Pr[M^{R,R^{-1}}(1^\ell)=1]\right| \le \frac{t}{2}\cdot\frac{m^2}{2^{\ell-\lceil \ell/t\rceil}}+\frac{m^2}{2^\ell}.$$

The proof of Theorem 6.2 follows the framework described in Section 4. Assume for simplicity that $\ell = s\cdot t$, the set $\mathrm{BAD}(h_1,h_2)$ (Definition 3.5) is replaced with the set $\mathrm{BAD}_2(h_1,h_2)$ defined to be:

> The set of all possible and consistent $M$-transcripts, $\sigma = \{\langle x_1,y_1\rangle,\ldots,\langle x_m,y_m\rangle\}$, satisfying that there exist $1\le i<j\le m$ and $1\le k\le t$ such that
>
> $$(F_i^{k+1},\ldots,F_i^t,L_i^1,\ldots,L_i^{k-1}) = (F_j^{k+1},\ldots,F_j^t,L_j^1,\ldots,L_j^{k-1}),$$
>
> where $(F_i^1,F_i^2,\ldots,F_i^t)=h_1(x_i)$ and $(L_i^1,L_i^2,\ldots,L_i^t)=h_2(y_i)$ ($|F_i^1| = |F_i^2| = \cdots = |F_i^t| = |L_i^1| = |L_i^2| = \cdots = |L_i^t| = s$).

This guarantees that for any possible and consistent $M$-transcript $\sigma$ we have that $\Pr_S[T_S=\sigma \mid \sigma\notin \mathrm{BAD}_2(h_1,h_2)] = 2^{-\ell m}$ (and, hence, it is a proper definition according to the framework). The reason is that, under the notation above,

$$\forall i,\ y_i = S(x_i) \Longleftrightarrow \forall 1\le i\le m,\ \forall 1\le k\le t,\ \ f_k(F_i^{k+1},\ldots,F_i^t,L_i^1,\ldots,L_i^{k-1}) = F_i^k\oplus L_i^k.$$

Therefore, given any specific choice of $h_1$ and $h_2$ (in the definition of $S$) such that $\sigma\notin \mathrm{BAD}_2(h_1,h_2)$ the event $T_S=\sigma$ is composed of $m\cdot t$ independent events, each of which has probability $2^{-s}$ of happening. In order to apply Theorem 4.2, it remains to note that for any such $\sigma$ we have that

$$\Pr_{h_1,h_2}[\sigma\in \mathrm{BAD}_2(h_1,h_2)] < t\cdot\binom{m}{2}\cdot 2^{-(\ell-\lceil \ell/t\rceil)} < \frac{t}{2}\cdot\frac{m^2}{2^{\ell-\lceil \ell/t\rceil}}.$$

*Remark* 6.1. The construction of this section achieves a substantial improvement in security over the construction in Section 3 even for a small constant $t>2$ (that is, with a few additional applications of the pseudorandom functions). Nevertheless, it might be useful for some applications to take a larger value of $t$. Choosing $t=\ell$ reduces the advantage the distinguisher may achieve to roughly $(\ell\cdot m^2)/2^\ell$.

## 7. SPPE on Many Blocks Using PFE or PPE on a Single Block

Consider the application of pseudorandom permutations to encryption, i.e., using $f(M)$ in order to encrypt a message $M$, where $f$ is a pseudorandom permutation. Assume also that we want to use DES for this purpose. We now have the following problem: while DES works on fixed and relatively small length strings, we need a permutation on $|M|$-bit long strings, where the length of the message, $|M|$, may be large and may vary between different messages.

This problem is not restricted to the usage of DES (though the fact that DES was designed for hardware implementation contributes to it). Usually, a direct construction

of pseudorandom permutations or pseudorandom functions (if we want to employ the LR-Construction) with large input-length is expensive. Therefore, we would like a way to construct pseudorandom permutations (or functions) on *many blocks* from pseudorandom permutations (or functions) on *a single block.*

Several such constructions were suggested in the context of DES (see, e.g., [10] for the different modes of operation for DES). The simplest, known as the electronic codebook mode (ECB mode), is to divide the input into subblocks and to apply the pseudorandom permutation on each subblock separately. This solution suffers from the obvious drawback that every subblock of output solely depends on a single subblock of input (and, in particular, the permutation on the complete input is not pseudorandom). This may leak information about the message being encrypted (see further discussion in Section 7.2).

In this section we consider a generalization of the construction of Section 3 that uses pseudorandom functions (or permutations) on a *single* block to construct strong pseudorandom permutations on *many* blocks. The idea is as follows: apply a pairwise independent permutation on the entire input, divide the value you get into subblocks, and apply two rounds of Feistel permutations (or one round of a pseudorandom permutation) on each subblock separately, finally, apply a second pairwise independent permutation on the entire value you get (see Fig. 4 for an illustration).

This solution resembles the ECB mode, it is almost as simple and it is highly suitable for parallel implementation. Contrary to the ECB mode, this construction does give a pseudorandom permutation on *the entire message* (though the security parameter is still relative to the length of a subblock).

For simplicity, we only describe the construction using truly random functions (or a truly random permutation). The analysis of the construction when pseudorandom functions are used follows easily. In addition, we restrict our attention to the construction of *strong* pseudorandom permutations.

**Definition 7.1.** For any two integers $b$ and $s$, for any function $g \in F_s$ let $g^{\times b} \in F_{b \cdot s}$ be the function defined by

$$g^{\times b}(x_1, x_2, \ldots, x_b) \stackrel{\text{def}}{=} (g(x_1), g(x_2), \ldots, g(x_b)).$$

For any $f_1, f_2 \in F_n$ and $h_1, h_2 \in P_{2nb}$, define

$$S(h_1, f_1, f_2, h_2) \stackrel{\text{def}}{=} h_2^{-1} \circ \mathbf{D}_{f_2}^{\times b} \circ \mathbf{D}_{f_1}^{\times b} \circ h_1.$$

For any $p \in P_{2n}$ and $h_1, h_2 \in P_{2nb}$, define

$$\hat{S}(h_1, p, h_2) \stackrel{\text{def}}{=} h_2^{-1} \circ p^{\times b} \circ h_1.$$

**Theorem 7.1.** *Let $h_1, h_2 \in P_{2nb}$ be pairwise independent permutations, let $f_1, f_2 \in F_n$ be random functions, and let $p \in P_{2n}$ be a random permutation. Define $S = S(h_1, f_1, f_2, h_2)$ and $\hat{S} = \hat{S}(h_1, p, h_2)$ (as in Definition* 7.1*) and let $R \in P_{2nb}$ be a random permutation. Then, for any oracle machine $M$ (not necessarily an efficient one) that makes at most $m$ queries,*

$$\left| \Pr[M^{S, S^{-1}}(1^{2nb}) = 1] - \Pr[M^{R, R^{-1}}(1^{2nb}) = 1] \right| \leq \frac{m^2 \cdot b^2}{2^n} + \frac{m^2}{2^{2nb}}$$

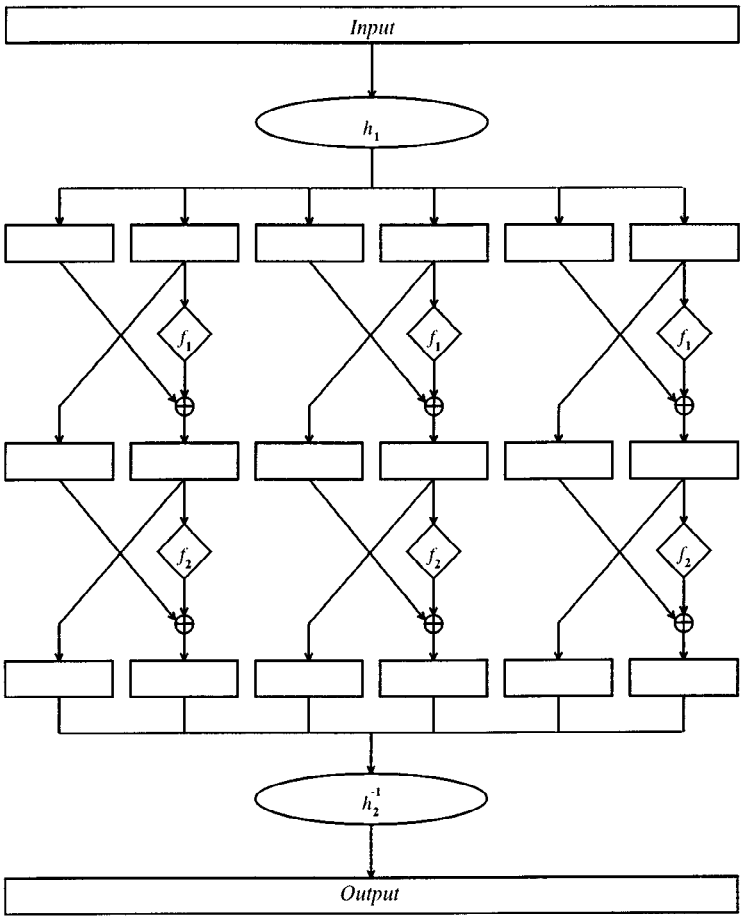


**Fig. 4.** Construction of a strong pseudorandom permutation on many (six in this case) blocks from a pseudorandom function on a single block.

*and*

$$\left|\Pr[M^{\hat{S},\hat{S}^{-1}}(1^{2nb}) = 1] - \Pr[M^{R,R^{-1}}(1^{2nb}) = 1]\right| \leq \frac{m^2 \cdot b^2}{2^{2n-1}}.$$

The proof of Theorem 7.1 for $S$ follows the framework described in Section 4. The set $\mathrm{BAD}(h_1, h_2)$ (Definition 3.5) is replaced with the set $\mathrm{BAD}_3(h_1, h_2)$ defined to be:

The set of all possible and consistent $M$-transcripts,

$$\sigma = \{\langle x_1, y_1\rangle, \ldots, \langle x_m, y_m\rangle\},$$

such that either there are two equal values in $\{F_i^{2j}\}_{1\leq i\leq m,\ 1\leq j\leq b}$ or there are two equal values in $\{L_i^{2j-1}\}_{1\leq i\leq m,\ 1\leq j\leq b}$, where $(F_i^1, F_i^2, \ldots, F_i^{2b}) = h_1(x_i)$

and $(L_i^1, L_i^2, \ldots, L_i^{2b}) = h_2(y_i)$ ($\left|F_i^1\right| = \left|F_i^2\right| = \cdots = \left|F_i^{2b}\right| = \left|L_i^1\right| = \left|L_i^2\right| = \cdots = \left|L_i^{2b}\right| = n$).

This guarantees that for any possible and consistent $M$-transcript $\sigma$ we have that

$$\Pr_S[T_S = \sigma \mid \sigma \notin \mathrm{BAD}_3(h_1, h_2)] = 2^{-2n \cdot b \cdot m}$$

(and, hence, it is a proper definition according to the framework). The reason is that, under the notation above,

$$\forall i, y_i = S(x_i) \iff \forall 1 \le i \le m,\ \forall 1 \le j \le b,\ f_1(F_i^{2j}) = F_i^{2j-1} \oplus L_i^{2j-1} \text{ and } f_2(L_i^{2j-1}) = F_i^{2j} \oplus L_i^{2j}.$$

Therefore, given any specific choice of $h_1$ and $h_2$ (in the definition of $S$) such that $\sigma \notin \mathrm{BAD}_3(h_1, h_2)$ the event $T_S = \sigma$ is composed of $2m \cdot b$ independent events, each of which has probability $2^{-n}$ of happening. In order to apply Theorem 4.2, it remains to note that for any such $\sigma$ we have that

$$\Pr_{h_1,h_2}[\sigma \in \mathrm{BAD}_3(h_1, h_2)] \le 2 \cdot \binom{m \cdot b}{2} \cdot 2^{-n} < \frac{m^2 \cdot b^2}{2^n}.$$

The proof of Theorem 7.1 for $\hat{S}$ slightly deviates from the framework described in Section 4 (providing yet further evidence to the claim that "nobody is perfect"). The set $\mathrm{BAD}(h_1, h_2)$ (Definition 3.5) is replaced with the set $\mathrm{BAD}_4(h_1, h_2)$ defined to be:

> The set of all possible and consistent $M$-transcripts, $\sigma = \{\langle x_1, y_1\rangle, \ldots, \langle x_m, y_m\rangle\}$, such that either there are two equal values in $\{F_i^j\}_{1 \le i \le m,\ 1 \le j \le b}$ or there are two equal values in $\{L_i^j\}_{1 \le i \le m,\ 1 \le j \le b}$, where $(F_i^1, F_i^2, \ldots, F_i^b) = h_1(x_i)$ and $(L_i^1, L_i^2, \ldots, L_i^b) = h_2(y_i)$ ($\left|F_i^1\right| = \left|F_i^2\right| = \cdots = \left|F_i^b\right| = \left|L_i^1\right| = \left|L_i^2\right| = \cdots = \left|L_i^b\right| = 2n$).

Now we have that, for any possible and consistent $M$-transcript $\sigma$,

$$\Pr_{h_1,h_2}[\sigma \in \mathrm{BAD}_4(h_1, h_2)] \le 2 \cdot \binom{m \cdot b}{2} \cdot 2^{-2n} < \frac{m^2 \cdot b^2}{2^{2n}}$$

but now, for any such $\sigma$,

$$\Pr_{\hat{S}}[T_{\hat{S}} = \sigma \mid \sigma \notin \mathrm{BAD}_4(h_1, h_2)] = \frac{2^{2n}!}{(2^{2n} - m \cdot b)!}$$

instead of $2^{-2n \cdot b \cdot m}$ as "required" by the framework. However, the difference in probabilities is rather small which results in only a minor deviation from the proof of Theorem 3.2.

### 7.1. *Relaxing the Construction*

As in Section 5.2 we would like to reduce the requirements from $h_1$ and $h_2$ in Theorem 7.1. Our main motivation in doing so is to *decrease the key-length* of the pseudorandom permutations. We would like the key-length to be of order $n$—the length of the small

subblocks—and *not* of order $2nb$—the length of the complete input (in some cases we may allow a small dependence on $b$).

We sketch a way to redefine the distributions on $h_1$ and $h_2$ in the definition of $\hat{S}$ (almost the same ideas apply to the definition of $S$). The requirement these distributions have to obey is that for any possible and consistent $M$-transcript $\sigma$ we have that $\Pr_{h_1,h_2}[\sigma \in \mathrm{BAD}_4(h_1, h_2)]$ is "small." We use the following notation: For any $(2n \cdot b)$-bit string $z = (z_1, z_2, \ldots, z_b)$ (such that $\forall j,\ |z_j| = 2n$) and for all $1 \le i \le b$, denote by $z_{|_i}$ the substring $z_i$ (the $i$th substring of $z$). The requirement above can be achieved by sampling $h_1$ and $h_2$ according to a permutation distribution $H$ such that for some small $\varepsilon \ge 2^{-2n}$ we have that:

1. For any $(2n \cdot b)$-bit string $x$, $\forall 1 \le i < j \le b$, $\Pr_{h\in H}[h(x)_{|_i} = h(x)_{|_j}] \le \varepsilon$.
2. For any $(2n \cdot b)$-bit strings $x \ne x'$, $\forall 1 \le i, j \le b$, $\Pr_{h\in H}[h(x)_{|_i} = h(x')_{|_j}] \le \varepsilon$.

We start by defining a permutation distribution $H'$ that almost achieves this: A permutation $h' = h'_{u_1,u_2}$ sampled from $H'$ is defined by two $\varepsilon'$-$\mathrm{AXU}_2$ functions, $u_1\colon I_{2n} \mapsto I_{2n}$ and $u_2\colon I_{\lceil \log b \rceil} \mapsto I_{2n}$ (see the definition of $\varepsilon$-$\mathrm{AXU}_2$ functions in Section 5.2). For any $z = (z_1, z_2, \ldots, z_b)$ (such that $\forall j,\ |z_j| = 2n$),

$$h'_{u_1,u_2}(z) \stackrel{\text{def}}{=} (z_1 \oplus u_1(z_b) \oplus u_2(1), z_2 \oplus u_1(z_b) \oplus u_2(2), \ldots, z_{b-1} \oplus u_1(z_b) \oplus u_2(b-1), z_b \oplus u_2(b)).$$

It is not hard to verify that:

1′. For any $(2n \cdot b)$-bit string $x$, $\forall 1 \le i < j \le b$, $\Pr_{h'\in H'}[h'(x)_{|_i} = h'(x)_{|_j}] \le \varepsilon'$.
2′. For any $(2n \cdot b)$-bit strings $x \ne x'$ such that $x_{|_b} \ne x'_{|_b}$, $\forall 1 \le i, j \le b$, $\Pr_{h'\in H'}[h'(x)_{|_i} = h'(x')_{|_j}] \le \varepsilon'$.

In order to eliminate the additional requirement in 2′ that $x_{|_b} \ne x'_{|_b}$, we define the permutation distribution $H$ such that a permutation $h$ sampled from $H$ is defined to be $h' \circ \mathbf{D}_g$ (see Definition 6.1), where $h'$ is sampled according to $H'$ and $g\colon I_{2n\cdot(b-1)} \mapsto I_{2n}$ is a $\varepsilon'$-$\mathrm{AXU}_2$ function (see Fig. 5 for an illustration). Using 1′ and 2′ and the fact that, for any $(2n \cdot b)$-bit strings $x \ne x'$,

$$\Pr_g[\mathbf{D}_g(x)_{|_b} = \mathbf{D}_g(x')_{|_b}] \le \varepsilon',$$

we get that $H$ satisfies 1 and 2 for $\varepsilon = 2\varepsilon'$.

Notice that the computation of a function $h \in H$ is essentially equivalent to one computation of an $\varepsilon$-$\mathrm{AXU}_2$ function, $g\colon I_{2n\cdot(b-1)} \mapsto I_{2n}$, and a few additional XOR operations per block. Using efficient constructions of $\varepsilon$-$\mathrm{AXU}_2$ functions [12], [19], [23], [40], [47], [49] we get an efficient function $h$. Krawczyk [23] shows a construction of $((m + \ell)/2^{\ell-1})$-$\mathrm{AXU}_2$ functions from $m$ bits to $\ell$ bits with $\ell$ key-bits. Using these functions we can achieve the desired goal of reducing the key-length of $h$ to $O(n)$ bits.

### 7.2. *Related Work*

The construction presented in this section is certainly not the only solution to the problem at hand. We refer in brief to some additional solutions:

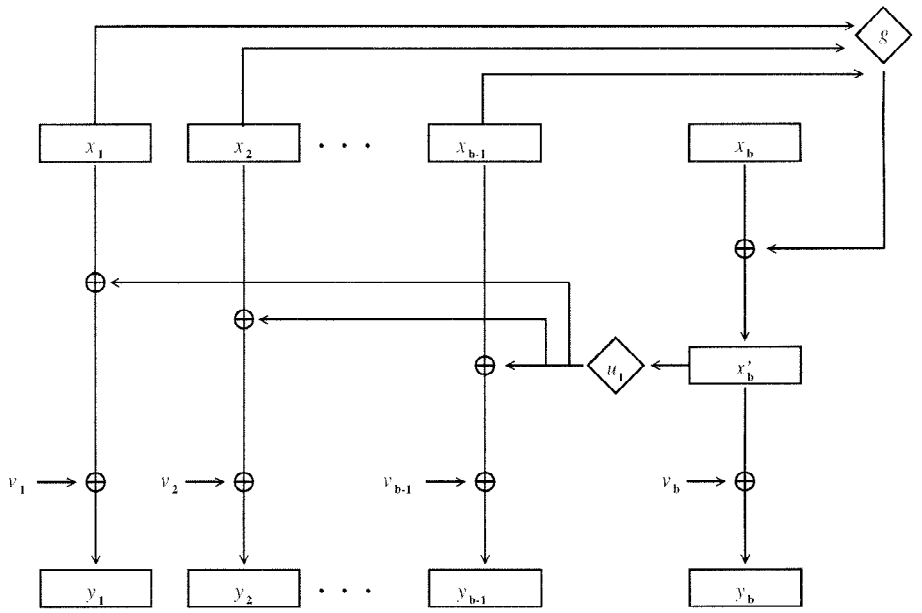


**Fig. 5.** The construction of $h \in H$. Each $v_i$ denotes the string $u_2(i)$.

As mentioned above, DES modes of operation were suggested as a way of encrypting long messages. However, none of these modes constitutes a construction of a pseudorandom permutation.[4] Note that when the encryption of a message $M$ is $f(M)$, for a pseudorandom permutation $f$, then the only information that is leaked on $M$ is whether or not $M$ is equal to a previously encrypted message. This is not true for DES modes of operation. For instance, when using the cipher block chaining mode (CBC mode), the encryptions of two messages with identical prefix will also have an identical prefix. The ECB mode leaks even more information—the existence of two identical subblocks (in two different encrypted messages or in a single message). The reason that the ECB mode leaks so much information is that every ciphertext-block solely depends on a single plaintext-block. Our construction implies that only very little and "noncryptographic" diffusion (the permutations $h_1$ and $h_2$) is required in order to overcome this flaw of the ECB mode.

Bellare and Rogaway [8] show how to convert the CBC mode in order to construct a pseudorandom permutation with large input-length (this is the only place we are aware of that explicitly refers to the problem). The amount of work in their construction is comparable with two applications of the original CBC mode (approximately twice the work of our construction, assuming that $h_1$ and $h_2$ are relatively efficient). The security of this construction is of similar order to the security of our construction. In contrast to our construction, [8] (as well as [6] and [7]) is sequential in nature.

[4] However, as shown by Bellare et al. [7], the CBC mode does define a construction of a pseudorandom *function* with small output-length. A somewhat related solution to this problem is the so-called *cascade* construction that is considered by Bellare et al. [6].

A different approach is to define a length-preserving pseudorandom function $\tilde{F}$ on $\tilde{\ell}$ bits using a length-preserving pseudorandom function $F$ on $\ell$ bits (where $\ell < \tilde{\ell}$) and then to apply our version of the LR-Construction using $\tilde{F}$ in order to get a pseudorandom *permutation* on $2 \cdot \tilde{\ell}$ bits. The function $\tilde{F}$ can be defined to be $G \circ F \circ h$, where $h$ is a pairwise independent hash function from $\tilde{\ell}$ bits to $\ell$ bits and $G$ is a pseudorandom (bit) generator from $\ell$ bits to $\tilde{\ell}$ bits. This idea may be attributed in part to Wegman and Carter [49]. Anderson and Biham [5] and Lucks [27] show how to apply similar ideas directly into the LR-Construction. A comparison between this approach and our construction relies on the specific parameters of the different primitives that are used. In particular, the parameters of the pseudorandom function $F$ versus the pseudorandom generator $G$. For instance, for this approach to be more efficient than our construction we need that one application of $G$ would be more efficient than $\lceil \tilde{\ell}/\ell \rceil$ applications of $F$.

### 7.2.1. *Reducing the Distinguishing Probability*

All the constructions of a pseudorandom permutation on many blocks from a pseudorandom function (or permutation) on a single block that are described in this subsection (including ours) have the following weakness: if the length of a single block is too small (e.g., 64-bits), then the pseudorandom permutation on many blocks is very weak even when the original pseudorandom function (or permutation) is very secure (e.g., completely random). In the following few paragraphs we discuss this problem and a way to overcome it.

Consider the permutation $S = S(h_1, f_1, f_2, h_2)$ (as in Definition 7.1), where $h_1, h_2 \in P_{2nb}$ are pairwise independent permutations and $f_1, f_2 \in F_n$ are random functions. Our analysis of the security of $S$ (Theorem 7.1) fails when the number of queries that the adversary makes is $\Omega(2^{n/2}/b)$ (in fact this analysis is tight). Having $2^{n/2}/b$ large enough forces a significant restriction on $n$. Therefore, a natural question is whether we can improve the security of the construction. A simple information-theoretic argument implies that all such constructions can be distinguished from random using $O(2^n/b)$ queries. This follows from the fact that with $O(2^n/b)$ queries the adversary gets many more bits than the length of the permutation's secret key. Hence, the distribution of the answers to these queries is statistically very different from uniform (which allows an all-powerful adversary to distinguish the permutation from random).

In order to match this bound we first note that the somewhat high distinguishing probability of $S$ is due to its vulnerability to a birthday-attack on the length of a single block. An adversary that makes $\Omega(2^{n/2}/b)$ uniformly chosen queries to $S$ will force a collision in the inputs to $f_1$ (or $f_2$) with a constant probability. Such a collision fails our analysis (and can indeed be used to distinguish $S$ from uniform). The solution lies in the following observation: the problem of foiling birthday-attacks when constructing a pseudorandom permutation on *many* blocks can be reduced to the problem of foiling birthday-attacks when constructing a pseudorandom function (or permutation) on *two* blocks. We demonstrate this using the Aiello and Venkatesan [1] construction of pseudorandom *functions*.

Let $\tilde{f}_1$ and $\tilde{f}_2$ be two independent copies of the pseudorandom *functions* on $2n$ bits we get when using truly random functions on $n$ bits in the construction of Aiello and Venkatesan. By [1] distinguishing each $\tilde{f}_i$ from a truly random function (with constant

probability) requires $\Omega(2^n)$ queries. Let $h_1$ and $h_2 \in P_{2nb}$ be pairwise independent permutations and let the permutation $\tilde{S} = S(h_1, \tilde{f}_1, \tilde{f}_2, h_2)$ be as in Theorem 7.1 (for the parameters $n' = 2n$ and $b' = b/2$). We now get that distinguishing $\tilde{S}$ from random (with constant probability) requires $O(2^n/b)$ queries which is optimal.

## 8. Constructions of $k$-Wise $\delta$-Dependent Permutations

In this section we summarize the connection between the various constructions of this paper and the task of obtaining $k$-wise $\delta$-dependent permutations. As mentioned in Section 5.1, Schnorr [46] suggested using the LR-Construction with truly random functions in order to get a pseudorandom generator that is secure as long as not too many bits of its output are accessible to the adversary. This idea is further treated by Maurer and Massey [29]. Maurer [28] suggested replacing the truly random functions with what he calls locally random (or almost random) functions. In the terminology of $k$-wise independence these ideas can be interpreted as a way of using the LR-Construction in order to obtain $k$-wise $\delta$-dependent permutations from $k$-wise $\delta'$-dependent functions (as long as $k$ is not too large). Theorem 1 in [28] implies that

> when $k$-wise $\delta'$-dependent functions are used instead of pseudorandom functions in the LR-Construction the result is a $k$-wise $\delta$-dependent permutations for $\delta = O(k^2/2^n + \delta')$.

Similar observations apply to the different constructions of our paper as discussed in this section.

**Corollary 8.1** (to Theorem 3.2). *Let $h_1, h_2 \in P_{2n}$ be pairwise independent permutations and let $f_1, f_2 \in F_n$ be $k$-wise $\delta'$-dependent functions. Then $S = S(h_1, f_1, f_2, h_2)$ (as in Definition* 3.1*) is a $k$-wise $\delta$-dependent permutation for*

$$\delta \stackrel{\text{def}}{=} \frac{k^2}{2^n} + \frac{k^2}{2^{2n}} + 2\delta'.$$

**Proof.** Let $S_1, S_2 \in P_{2n}$ have the following distributions:

- $S_1 = S(h_1, g_1, f_2, h_2)$, where $h_1, h_2 \in P_{2n}$ are pairwise independent, $f_2 \in F_n$ is a $k$-wise $\delta'$-dependent function, and $g_1 \in F_n$ is a truly random function.
- $S_2 = S(h_1, g_1, g_2, h_2)$, where $h_1, h_2 \in P_{2n}$ are pairwise independent and $g_1, g_2 \in F_n$ are truly random functions.

Let $R \in P_{2n}$ be a truly random permutation. It is enough to show that for every $k$ strings of $2n$-bits, $x_1, x_2, \ldots, x_k$, we have

1. $\|\langle S(x_1), S(x_2), \ldots, S(x_k)\rangle - \langle S_1(x_1), S_1(x_2), \ldots, S_1(x_k)\rangle\| \le \delta'$.
2. $\|\langle S_1(x_1), S_1(x_2), \ldots, S_1(x_k)\rangle - \langle S_2(x_1), S_2(x_2), \ldots, S_2(x_k)\rangle\| \le \delta'$.
3. $\|\langle S_2(x_1), S_2(x_2), \ldots, S_2(x_k)\rangle - \langle R(x_1), R(x_2), \ldots, R(x_k)\rangle\| \le k^2/2^n + k^2/2^{2n}$.

The reason 3 holds is that if we define an oracle machine $M$ such that its $i$th query is

always $(+, x_i)$ and such that

$$\begin{aligned} C_M(\{\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \ldots, \langle x_k, y_k\rangle\}) &= 1 \\ \iff \Pr[\langle S_2(x_1), \ldots, S_2(x_k)\rangle &= \langle y_1, \ldots, y_k\rangle] \\ < \Pr[\langle R(x_1), \ldots, R(x_k)\rangle &= \langle y_1, \ldots, y_k\rangle] \end{aligned}$$

we get by the definition of variation distance and from Theorem 3.2 that

$$\begin{aligned} &\|\langle S_2(x_1), S_2(x_2), \ldots, S_2(x_k)\rangle - \langle R(x_1), R(x_2), \ldots, R(x_k)\rangle\| \\ &\quad = \left|\Pr[M^{S_2, S_2^{-1}}(1^{2n}) = 1] - \Pr[M^{R, R^{-1}}(1^{2n}) = 1]\right| \\ &\quad \le \frac{k^2}{2^n} + \frac{k^2}{2^{2n}}. \end{aligned}$$

1 and 2 hold by the definition of $k$-wise $\delta'$-dependent functions. For example, if

$$\|\langle S(x_1), S(x_2), \ldots, S(x_k)\rangle - \langle S_1(x_1), S_1(x_2), \ldots, S_1(x_k)\rangle\| > \delta',$$

then we can fix $h_1, h_2 \in P_{2n}$ and $f_2 \in F_n$ in the definition of both $S$ and $S_1$ such that the inequality still holds. This defines $k$ strings of $n$-bits, $z_1, z_2, \ldots, z_k$ (not necessarily all different), and a function $V$ for which

$$\langle S(x_1), S(x_2), \ldots, S(x_k)\rangle = V(\langle f_1(z_1),\ f_1(z_2), \ldots,\ f_1(z_k)\rangle)$$

and

$$\langle S_1(x_1), S_1(x_2), \ldots, S_1(x_k)\rangle = V(\langle g_1(z_1), g_1(z_2), \ldots, g_1(z_k)\rangle).$$

We get a contradiction since, for any function $V$,

$$\begin{aligned} &\|V(\langle f_1(z_1),\ f_1(z_2), \ldots,\ f_1(z_k)\rangle) - V(\langle g_1(z_1), g_1(z_2), \ldots, g_1(z_k)\rangle)\| \\ &\quad \le \|\langle f_1(z_1),\ f_1(z_2), \ldots,\ f_1(z_k)\rangle - \langle g_1(z_1), g_1(z_2), \ldots, g_1(z_k)\rangle\| \\ &\quad \le \delta'. \end{aligned}$$

□

In a similar way we get the following two corollaries from the constructions of Sections 6 and 7:

**Corollary 8.2** (to Theorem 6.2). *Let $S$ be as in Definition* 6.2, *where $h_1$ and $h_2$ are pairwise independent permutations and $f_1,\ f_2, \ldots,\ f_t$ are $k$-wise $\delta'$-dependent functions. Then $S$ is a $k$-wise $\delta$-dependent permutation for*

$$\delta \stackrel{\text{def}}{=} \frac{t}{2} \cdot \frac{k^2}{2^{\ell - \lceil \ell/t \rceil}} + \frac{k^2}{2^\ell} + t \cdot \delta'.$$

**Corollary 8.3** (to Theorem 7.1). *Let $h_1, h_2 \in P_{2nb}$ be pairwise independent permutations, let $f_1,\ f_2 \in F_n$ be $(b \cdot k)$-wise $\delta'$-dependent functions, and let $p \in P_{2n}$ be a $(b \cdot k)$-wise $\delta'$-dependent permutation. Define $S = S(h_1,\ f_1,\ f_2, h_2)$ and $\hat{S} = \hat{S}(h_1, p, h_2)$ (as in Definition* 7.1). *Then $S$ is a $k$-wise $\delta$-dependent permutation for*

$$\delta \stackrel{\text{def}}{=} \frac{k^2 \cdot b^2}{2^n} + \frac{k^2}{2^{2nb}} + 2\delta'$$

*and $\hat{S}$ is a $k$-wise $\hat{\delta}$-dependent permutation for*

$$\hat{\delta} \stackrel{\text{def}}{=} \frac{k^2 \cdot b^2}{2^{2n-1}} + \delta'.$$

By taking $t = \ell$ in Corollary 8.2 we get a simple construction of a $k$-wise $\delta$-dependent permutation on $\ell$ bits for $\delta$ as close to $((\ell+1) \cdot k^2)/2^\ell$ as we wish. This construction requires $\ell$ applications of $k$-wise $\delta'$-dependent functions from $\ell - 1$ bits to a single bit. An interesting question is to find a simple construction of $k$-wise $\delta$-dependent permutations for an *arbitrarily small* $\delta$ and an arbitrary $k$.

An "old" proposal by the first author (see p. 17 of [41]) is to apply a card shuffling procedure that requires only few rounds and is oblivious in the sense that the location of a card after each round depends on a few random decisions. The specific card shuffling for which this idea is described in [41] was suggested by Aldous and Diaconis [2]. Unfortunately, to the best of our knowledge, this procedure was never proven to give (with few rounds) an almost uniform ordering of the cards. Nevertheless, we briefly describe it in order to demonstrate the concept of oblivious card shuffling and the way that such a procedure can be used to construct a $k$-wise $\delta$-dependent permutation. Finally we describe the main idea in the definition of another oblivious card shuffling for which we can prove that only few rounds are needed.

Each round (shuffle) in a card shuffling procedure is a permutation on the locations of the $N$ cards of a deck (i.e., a permutation on the set $[N] \stackrel{\text{def}}{=} \{1, 2, \ldots, N\}$). In the case of the Aldous and Diaconis [2] card shuffling, each such permutation is defined by a uniformly chosen $(N/2)$-bit string, $r = r_1 r_2 \cdots r_{N/2}$. Denote this permutation by $\Pi_r$ then

$$\forall 1 \le i \le N/2, \qquad \begin{cases} \Pi_r(i) = 2i - 1 \quad \text{and} \quad \Pi_r(i + N/2) = 2i & \text{if} \quad r_i = 1, \\ \Pi_r(i) = 2i \quad \text{and} \quad \Pi_r(i + N/2) = 2i - 1 & \text{otherwise.} \end{cases}$$

That is, the cards at locations $i$ and $i + N/2$ move to locations $2i - 1$ and $2i$ and their internal order is uniformly chosen independently of all other choices. Note that, $\forall x$, evaluating $\Pi_r(x)$ or $\Pi_r^{-1}(x)$ requires the knowledge of a *single* bit of $r$ and therefore this card shuffling is indeed oblivious.

Consider $s$ rounds of the card shuffling described above, $\Pi^s = \Pi_{r^1,\ldots,r^s} \stackrel{\text{def}}{=} \Pi_{r^s} \circ \Pi_{r^{s-1}} \circ \cdots \circ \Pi_{r^1}$, where $\{r^1, \ldots, r^s\}$ are uniformly distributed and independent of each other. If $\Pi^s$ is of statistical distance at most $\delta'$ from a uniform permutation, then we can construct a $k$-wise $\delta$-dependent permutation, $\tilde{\Pi}^s$, for $\delta = \delta' + \delta''$ as follows: simply take the permutation $\tilde{\Pi}^s$ to be $s$ rounds $\Pi_{r^s} \circ \Pi_{r^{s-1}} \circ \cdots \circ \Pi_{r^1}$ where the $s \cdot N/2$ bits of $\{r^1, \ldots, r^s\}$ are *the outputs of a $(k \cdot s)$-wise $\delta''$-dependent binary function, $f$*. Evaluating $\tilde{\Pi}^s$ (or its inverse) at a given point consists of $s$ invocations of $f$. Therefore, an interesting problem is to show that $\Pi^s$ is of exponentially small statistical distance from a uniform permutation for a small value of $s$. In [2] it is conjectured that this can be shown for $s = O(\log^2 N)$. While this conjecture is, to the best of our knowledge, still open we can show a different card shuffling procedure for which it can be proven that $O(\log^2 N)$ rounds are sufficient. This card shuffling is defined in a recursive manner: Split the deck into halves (locations $\{1, \ldots, N/2\}$ and locations $\{N/2, \ldots, N\}$), apply the card shuffling (recursively) on each half of the deck, and merge the (now shuffled) halves in an

almost uniform way. A permutation, $M$, on $[N]$ is a merge if for every $i$ and $j$ such that $1 \leq i < j \leq N/2$ or $N/2+1 \leq i < j \leq N$ we have that $M(i) < M(j)$. An oblivious (in the same meaning as above) merging procedure can also be defined recursively but since the construction is rather cumbersome we omit its description. This direction may become attractive given an efficient and simple merging procedure.

A different direction to solving the problem of constructing $k$-wise $\delta$-dependent permutations is to try and generalize the algebraic construction of pairwise independent permutations. Leonard Schulman (private communication) suggested such a generalization that yields 3-wise independent permutations. His suggestion is to use sharply 3-transitive permutation groups. A permutation group over the set $[n] = \{1, 2, \ldots, n\}$ is a subgroup of the symmetric group $S_n$. A permutation group $G$ over $[n]$ is $k$-transitive if for every two $k$-tuples $\{a_1, \ldots, a_k\}$ and $\{b_1, \ldots, b_k\}$ of distinct elements of $[n]$ there exists a permutation $\pi \in G$ such that, $\forall 1 \leq i \leq k,\ \pi(a_i) = b_i$. A permutation group $G$ over $[n]$ is sharply $k$-transitive if for every two such tuples there exists exactly one permutation $\pi \in G$ such that, $\forall 1 \leq i \leq k,\ \pi(a_i) = b_i$. A sharply $k$-transitive permutation group is in particular $k$-wise independent and indeed the algebraic construction of pairwise independent permutations use a sharply 2-transitive permutation group (containing all the linear permutations). Schulman suggested using the fact that there are known constructions of sharply 3-transitive permutation groups. However, this approach cannot be generalized to larger values of $k$: from the classification of finite simple groups it follows that for $k \geq 6$ there are no $k$-transitive groups over $[n]$ other than the symmetric group $S_n$ and the alternating group $A_n$ and there are only few such groups for $k = 4$ and $k = 5$ (see [11] and [39]). One should be careful not to interpret this as implying that for $k \geq 4$ there are no efficient algebraic constructions of $k$-wise independent permutations. It is however justified to deduce that for $k \geq 4$ any small family of $k$-wise independent permutations is not a permutation group (i.e., is not closed under composition and inverse).

## 9. Conclusion and Further Work

The constructions described in Sections 3 and 7 are optimal in their cryptographic work in the sense that the total number of bits on which the cryptographic functions are applied is exactly the number of bits in the input. Therefore, it seems that in order to achieve the goal of constructing efficient block-ciphers it is sufficient to concentrate on the construction of efficient pseudorandom functions. The depth of the constructions, on the other hand, is twice the depth of the cryptographic functions. It is an interesting question whether there can be a construction of similar depth. The goal of reducing the depth is even more significant in the case of the $(t+2)$-round construction in Section 6. A different question is finding a simple construction of $k$-wise $\delta$-dependent permutations for an *arbitrarily small* $\delta$ and an arbitrary $k$. This question is discussed in Section 8.

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