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STRAIGHTENING AND BOUNDED COHOMOLOGY OF HYPERBOLIC GROUPS

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Abstract

It was stated by M. Gromov [Gr2] that, for any hyperbolic group G, the map from bounded cohomology $H_b^n(G,\mathbb{R})$ to $H^n(G,\mathbb{R})$ induced by inclusion is surjective for $n \geq 2$. We introduce a homological analogue of straightening simplices, which works for any hyperbolic group. This implies that the map $H_b^n(G,V) \to H^n(G,V)$ is surjective for $n \geq 2$ when V is any bounded $\mathbb{Q}G$-module and when V is any finitely generated abelian group.

1 Introduction

The bounded cohomology of a group G, $H_b^*(G,V)$, is defined with the bar-construction the same way as the usual cohomology except that only bounded cochains are considered (see the precise definition in section 2). The bounded cohomology of a topological space is defined using bounded singular cochains.

M. Gromov showed in [Gr1] that, for a closed manifold M of negative curvature and $n \geq 2$, the map $H_b^n(M,\mathbb{R}) \to H^n(M,\mathbb{R})$ is surjective. The proof used the fact that each singular n-simplex in a simply connected manifold of negative curvature can be deformed to a "straight n-simplex", and that the straight n-simplices have uniformly bounded volumes.

In [Gr2, 8.3.T] Gromov also claimed the same surjectivity result for hyperbolic groups and gave a sketch of proof involving quasi-geodesic flows. This surjectivity result was later used by A. Connes and H. Moscovici [CM] for a proof of the Novikov conjecture for hyperbolic groups.

We use a different approach to prove the surjectivity result for *any* coefficients, namely we show

Theorem 11. *Let G be a hyperbolic group and V be a bounded $\mathbb{Q}G$-module. Then the map $H_b^n(G,V) \to H^n(G,V)$ induced by inclusion is surjective for each $n \geq 2$.*

W.D. Neumann and L. Reeves [NR] showed that, when G is hyperbolic, the map $H_b^2(G,A) \to H^2(G,A)$ is surjective for any finitely generated abelian group A. Our Theorem 11 implies the same result for higher dimensions:

Theorem 15. *Let G be a hyperbolic group and A be a finitely generated abelian group. Then the map $H_b^n(G,A) \to H^n(G,A)$ induced by inclusion is surjective for each $n \geq 2$.*

The idea of the proof of Theorem 11 is the following. For the hyperbolic group G we take X to be the universal covering of a $K(G,1)$-complex with finitely many cells in each dimension, and let Γ be the 1-skeleton of X. A $\mathbb{Q}$-*bicombing* q in Γ is a choice of a rational 1-chain $q[a,b]$ for each pair of vertices a,b such that $\partial q[a,b] = b - a$. The main step in the proof is

Theorem 10. *Let G be a hyperbolic group and Γ be a connected graph with a free cocompact G-action. Then there exists a $\mathbb{Q}$-bicombing q in Γ with the following properties:*

(1) q is quasigeodesic;
(2) q is G-equivariant;
(3) q is anti-symmetric, i.e. $q[a,b] = -q[b,a]$ for any $a,b \in \Gamma^{(0)}$;
(4) there exists a constant T such that, for any $a,b,c \in \Gamma^{(0)}$,

$$\big|q[a,b] + q[b,c] + q[c,a]\big|_1 \leq T\,.$$

Informally, one should think of $q[a,b]$ as a singular 1-simplex spanning a and b.

Hyperbolic groups satisfy linear isoperimetric inequalities for filling rational n-cycles (see [G] for $n = 1$ and [M3] for $n > 1$), hence, by Theorem 10(4), each 1-cycle of form $q[a,b] + q[b,c] + q[c,a]$ bounds a 2-chain of bounded norm. We extend this inductively to higher dimensions, i.e. we “span” each $(n+1)$-tuple of vertices in X by a cellular n-chain of bounded ℓ_1-norm. One may view this construction as a homological analogue of straightening. Then a formal homological argument is used to finish the proof of Theorem 11.

The bicombing $q[a,b]$ above may also be viewed as a generalization of global canonical representatives considered by E. Rips and Z. Sela [RS]. The question of existence of such canonical representatives for an arbitrary hyperbolic groups remains open.

Everywhere in the paper $\mathbb{Q}$ can be replaced by $\mathbb{R}$ with no change in the proofs. In [M1] a converse of Theorem 11 is shown, giving a characterization of hyperbolic groups by bounded cohomology.

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2 Definitions

2.1 Hyperbolic groups. If Γ is a graph, we always view it as a metric space with the path metric d induced by assigning length 1 to each edge. A *geodesic path* $[a,b]$ in Γ is a shortest edge path connecting the two vertices a and b. A finitely generated group G is called *hyperbolic* if, for any graph Γ with a free cocompact G-action there exists a constant $\delta \geq 0$ such that all the geodesic triangles in Γ are δ*-fine* in the following sense: if a, b, and c are vertices in $\Gamma^{(0)}$, $[a,b]$, $[b,c]$, and $[c,a]$ are geodesics from a to b, from b to c, and from c to a, respectively, and points $\bar{a} \in [b,c]$, $v, \bar{c} \in [a,b]$, $w, \bar{b} \in [a,c]$ satisfy

$$d(b,\bar{c}) = d(b,\bar{a})\,, \quad d(c,\bar{a}) = d(c,\bar{b})\,, \quad d(a,v) = d(a,w) \leq d(a,\bar{c}) = d(a,\bar{b})\,,$$

then $d(v,w) \leq \delta$. See [A&] for other equivalent definitions.

For the rest of the paper, let G be a hyperbolic group and X be a contractible cellular complex equipped with a free cellular G-action which is cocompact on the n-skeleton $X^{(n)}$ for each n. This means that the quotient of X by the G-action is a $K(G,1)$-complex with finitely many cells in each dimension. Such a complex X exists for each hyperbolic (or, more generally, combable) group (see [E&, Theorem 10.2.6] and [A]). When G is torsion free, X is the familiar Rips complex and it is finite dimensional.

In what follows, Γ will always stand for the 1-skeleton of X. Choose δ so that all geodesic triangles in Γ are δ-fine. Increase δ as needed to make it a positive integer.

For vertices a, b, and c in $\Gamma^{(0)}$, the *Gromov product* is defined by

$$(b|c)_a := \tfrac{1}{2}\big[d(a,b) + d(a,c) - d(b,c)\big]\,.$$

Note that, by the triangle inequality, this product always satisfies

$$(b|c)_a \leq d(a,b), \quad (b|c)_a \leq d(a,c), \quad (b|c)_a \geq 0, \quad d(a,b) = (b|c)_a + (a|c)_b,$$

and analogously for any permutation of letters a, b, and c.

The next lemma immediately follows from the definition of hyperbolic groups.

Lemma 1 (Fine-triangles property). *Let G and Γ be as above, and z, x, y, x', y' be vertices in $\Gamma^{(0)}$ such that x' (respectively, y') lies on a geodesic* *connecting z to x (respectively, z to y). Suppose also that*

$$d(z,x') = d(z,y') \leq (x|y)_z \,.$$

Then $d(x',y') \leq \delta$.

Given a vertex v in $\Gamma^{(0)}$ and a number r, a *sphere* $S(v,r)$ in Γ is the set of all vertices w in $\Gamma^{(0)}$ satisfying $d(v,w) = r$. A *ball* $B(v,r)$ in Γ is the set of all vertices w in $\Gamma^{(0)}$ satisfying $d(v,w) \leq r$. If S is a subset of Γ, then the *r-neighborhood* of S, $N(S,r)$, is the set of all *points* $x \in \Gamma$ such that $d(x,s) \leq r$ for some $s \in S$.

The following obvious corollary of Lemma 1 will be used several times throughout the paper.

Lemma 2. *Let G and Γ be as above, and z, x, y, x',y' be vertices in $\Gamma^{(0)}$ such that x' (respectively, y') lies on a geodesic connecting z to x (respectively, z to y). Suppose also that m is an integer, $x', y' \in S(z, 10\delta(m-1))$, $x, y \in S(z, 10\delta m)$, and $d(x,y) \leq 5\delta$. Then $d(x',y') \leq \delta$.*

2.2 Normed vector spaces. Let W be a vector space over $\mathbb{Q}$. A *norm* on W is a function $|\cdot| : W \to \mathbb{R}_+$ satisfying (1) $w = 0$ iff $|w| = 0$; (2) $|w + w'| \leq |w| + |w'|$; (3) $|\alpha w| = |\alpha| \cdot |w|$ for $w, w' \in W$, $\alpha \in \mathbb{Q}$.

Suppose now that a vector space W over $\mathbb{Q}$ has a preferred basis $\{w_i,\, i \in I\}$. The *ℓ_1-norm* on W with respect to this basis is given by

$$\Big|\sum_{i\in I} \alpha_i w_i\Big|_1 := \sum_{i\in I} |\alpha_i| \,.$$

Let $(W, |\cdot|_1)$ and $(W', |\cdot|)$ be normed vector spaces, where W is equipped with the ℓ_1-norm $|\cdot|_1$. For a linear map $\varphi : W \to W'$, the *ℓ_∞-norm* of φ, $|\varphi|_\infty$, is the operator norm of φ, i.e. $|\varphi|_\infty$ is the smallest number K (possibly infinity) such that $|\varphi(w)| \leq K|w|_1$ for each $w \in W$. One checks that

$$|\varphi|_\infty = \sup_{i\in I} |\varphi(w_i)| \,,$$

where $\{w_i,\ i \in I\}$ is the preferred basis of W.

The preferred basis on the space of cellular n-chains, $C_n(X,\mathbb{Q})$, will always be the set of n-cells in X and we always equip $C_n(X,\mathbb{Q})$ with the ℓ_1-norm.

2.3 Bounded modules. A *bounded $\mathbb{Q}G$-module* is a $\mathbb{Q}G$-module which is normed, as a vector space over $\mathbb{Q}$, and such that G acts on the module by linear maps of uniformly bounded norms. Note that this definition is more general than one may find in literature in that we do not require the module to be a Banach space, and also it is a vector space over $\mathbb{Q}$ rather than over $\mathbb{R}$. Of course, any normed vector space over $\mathbb{Q}$ may be viewed as a bounded $\mathbb{Q}G$-module with the trivial G-action.

2.4 Bounded cohomology. There are various equivalent definitions for the bounded cohomology of a group (see [I] and [N]). In the paper we will use the homogeneous bar-construction definition. Namely, for any bounded $\mathbb{Q}G$-module V, the *bounded cohomology* of G with coefficients in V, $H_b^*(G,V)$, is the homology of the cochain complex

$$0 \longrightarrow C_b^0(G,V) \xrightarrow{\delta_0} C_b^1(G,V) \xrightarrow{\delta_1} C_b^2(G,V) \xrightarrow{\delta_2} \cdots ,$$

where

$$C_b^n(G,V) := \{\alpha : G^{n+1} \to V \mid \alpha \text{ is a bounded } G\text{-map}\}$$

and the coboundary map is defined by

$$\delta_n\alpha\big(\langle x_0,\dots,x_{n+1}\rangle\big) := \sum_{i=0}^{n+1}(-1)^i\alpha\big(\langle x_0,\dots,\widehat{x_i},\dots,x_{n+1}\rangle\big) .$$

Here G^{n+1} is considered with the diagonal G-action, and "bounded" means "has bounded image with respect to the norm on V".

Equivalently, $C_b^n(G,V) \subseteq Hom_{\mathbb{Q}G}(C_n(G,\mathbb{Q}),V)$ is the subspace of all $\mathbb{Q}G$-morphisms $C_n(G,\mathbb{Q}) \to V$, which are bounded as linear maps, where $C_n(G,\mathbb{Q})$ is the space of all chains (= finite support functions) $G^{n+1} \to \mathbb{Q}$ given the ℓ_1-norm with respect to the standard basis G^{n+1}. One should think of an element of G^{n+1} as an n-simplex. Such simplices form a simplicial complex Y representing the bar-construction. For convenience, in the text we will use the notation $\mathcal{C}_n^Y = C_n(Y,\mathbb{Q})$ instead of $C_n(G,\mathbb{Q})$ and $\mathcal{C}_Y^n = C^n(Y,V)$ instead of $C^n(G,V)$.

3 Auxiliary Statements

In this section we first prove some auxiliary statements, and use them to prove Theorem 10 mentioned in the introduction. The reader may want to skip the technicalities and first go to section 4 where this result is used for bounded cohomology.

All cellular chains in X will be assumed with $\mathbb{Q}$-coefficients unless stated otherwise.

A *bicombing* p in Γ is a function assigning to each ordered pair (a,b) of vertices in $\Gamma^{(0)}$ an oriented edge-path $p[a,b]$ from a to b. A bicombing p is called *geodesic* if each path $p[a,b]$ is geodesic, i.e. a shortest edge path. A bicombing p is *G-equivariant* if $p[g\cdot a, g\cdot b] = p\cdot q[a,b]$ for each $a,b\in\Gamma$ and each $g\in G$.

For the rest of the paper, fix some G-equivariant geodesic bicombing p in Γ. So each $p[a,b]$ is an isometric embedding $p[a,b]:[0,d(a,b)]\to\Gamma$ with $p[a,b](0)=a$ and $p[a,b](d(a,b))=b$, and $p[a,b](r)$ stands for the image of $r\in[0,d(a,b)]$ via the map $p[a,b]$. Abusing the notation we will also view the path $p[a,b]$, as a 1-chain, with $\partial p[a,b]=b-a$.

A *homological bicombing* q in Γ is a function which assigns a 1-chain $q[a,b]$ to each ordered pair (a,b) of vertices in $\Gamma^{(0)}$, so that $\partial q[a,b]=b-a$. A *$\mathbb{Q}$-bicombing* is a homological bicombing with coefficients in $\mathbb{Q}$. A homological bicombing is called *quasigeodesic* if

- there exists a constant $r\geq 0$ and a geodesic bicombing p such that $supp\, q[a,b]\subseteq N(p[a,b],r)$ for each $a,b\in\Gamma^{(0)}$, and
- there exists a constant $C\geq 0$ such that $\big|q[a,b]\big|_1\leq C\,d(a,b)$ for each $a,b\in\Gamma^{(0)}$.

A homological bicombing q is *G-equivariant* if $q[g\cdot a, g\cdot b]=g\cdot q[a,b]$ for each $a,b\in\Gamma^{(0)}$ and each $g\in G$.

A *convex combination* is a (cellular) 0-chain with non-negative coefficients which sum up to 1. For $v,w\in\Gamma^{(0)}$, the *flower* at w with respect to v is the set

$$Fl(v,w):=S\big(v,d(v,w)\big)\cap B(w,\delta)\subseteq\Gamma^{(0)}.$$

First we recall a version of the dandelion construction from [M3]. For each vertex a in $\Gamma^{(0)}$, define the "one-level-lower projection toward a" $pr_a:\Gamma^{(0)}\to\Gamma^{(0)}$ as follows.

- $pr_a(a):=a$, and
- if $b\neq a$, $pr_a(b):=p[a,b](r)$, where r is the largest (integral) multiple of 10δ which is strictly less than $d(a,b)$.

Now for each pair $a,b\in\Gamma^{(0)}$ we define a (cellular) 0-chain $f(a,b)$ in Γ. The definition is inductive on the distance $d(a,b)$. For vertices a and b with $d(a,b)\leq 10\delta$, put $f(a,b):=b$. If $d(a,b)>10\delta$ and $d(a,b)$ is not a multiple of 10δ, let $f(a,b):=f(a,pr_a(b))$. If $d(a,b)>10\delta$ and $d(a,b)$ is a multiple of 10δ, let

$$f(a,b):=\frac{1}{\#Fl(a,b)}\sum_{x\in Fl(a,b)} f\big(a,pr_a(x)\big)\,.$$

Proposition 3 (Canceling convex combinations). *The function f defined above satisfies the following properties:*

(1) $f(a,b)$ is a convex combination.

(2) If $d(a,b) \geq 10\delta$, then $\operatorname{supp} f(a,b) \subseteq Fl(a, p[a,b](10\delta))$.
(3) If $d(a,b) \leq 10\delta$, then $f(a,b) = b$.
(4) f is G-equivariant, i.e. $f(g \cdot a, g \cdot b) = g \cdot f(a,b)$ for any $a, b \in \Gamma^{(0)}$ and $g \in G$.
(5) There exist constants $L \geq 0$ and $0 \leq \lambda < 1$ such that, for any $a, b, c \in \Gamma^{(0)}$,

$$\left|f(a,b) - f(a,c)\right|_1 \leq L\lambda^{(b|c)_a} .$$

The proof of this and later statements may look a bit cumbersome, but the main point should be clear: use the fine-triangles property whenever possible. It is probably also worth mentioning that the number 10δ in the statement can be replaced by any "sufficiently large" integer.

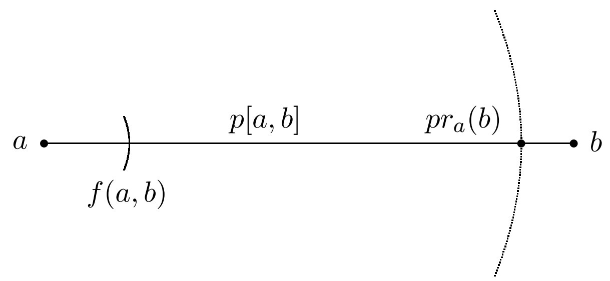

Figure 1: Convex combination $f(a,b)$.

Proof of Proposition 3. It is clear from the definition that $f(a,b)$ is a convex combination and it is G-equivariant because the definition uses only metric properties of $\Gamma^{(0)}$, which are preserved under the G-action. So properties (1) and (4) are satisfied. Property (3) follows directly from the definition. To prove (2) we need the following lemma.

LEMMA 4. *Let $a, b \in \Gamma^{(0)}$ and let m be an integer satisfying $10\delta \leq 10\delta m \leq d(a,b)$. Put $v := p[a,b](10\delta m)$. Then*

$$f(a,b) = \sum_{x \in Fl(a,v)} \alpha_x f(a,x) ,$$

for some non-negative coefficients α_x with $\sum_{x\in Fl(a,v)} \alpha_x = 1$.

Proof. This follows almost immediately from the definition of f. The main tools here are the fine-triangles property and the fact that "a convex combination of convex combinations is again a convex combination". Fix an arbitrary pair of vertices a and b in $\Gamma^{(0)}$. We prove the assertion by the *inverse* induction on m. Let $m_{\max}$ be the maximal integer among all m satisfying $10\delta m \le d(a,b)$. Since $10\delta \le d(a,b)$ by the hypotheses of the lemma, then $m_{\max} \ge 1$.

$\boldsymbol{m = m_{\max}}$**.** If $10\delta m_{\max} = d(a,b)$, then $b = v$ and the 0-chain $f(a,b) = f(a,v)$ can be represented as the *trivial* linear combination $\sum_{x\in Fl(a,v)} \alpha_x f(a,x)$, where $\alpha_v = 1$ and $\alpha_x = 0$ for all $x \ne v$.

If $10\delta m_{\max} < d(a,b)$, then, by the definition of f,

$$f(a,b) = f\big(a, pr_a(b)\big) = f(a,v)\,,$$

which is again the trivial linear combination.

$\boldsymbol{m+1 \mapsto m}$**.** If an integer m satisfies $1 \le m < m_{max}$, then $10\delta \le 10\delta(m+1) \le d(a,b)$, so, by induction hypotheses,

$$f(a,b) = \sum_{x\in Fl(a,v')} \alpha_x f(a,x)\,,$$

where $v' := p[a,b](10\delta(m+1))$ and α_x are some non-negative coefficients satisfying $\sum_{x\in Fl(a,v')} \alpha_x = 1$. By definition, each $f(a,x)$ in the last sum has the form

$$f(a,x) = \frac{1}{\#Fl(a,x)} \sum_{y\in Fl(a,x)} f\big(a, pr_a(y)\big)\,,$$

therefore

$$\begin{aligned} f(a,b) &= \sum_{x\in Fl(a,v')} \alpha_x \left[\frac{1}{\#Fl(a,x)} \sum_{y\in Fl(a,x)} f\big(a, pr_a(y)\big)\right] \\ &= \sum_{x\in Fl(a,v')} \sum_{y\in Fl(a,x)} \frac{\alpha_x}{\#Fl(a,x)} f\big(a, pr_a(y)\big) \qquad (1) \end{aligned}$$

Now collect like terms in the last double sum. It amounts to grouping the coefficients $\alpha_x/\#Fl(a,x)$. Their sum is 1:

$$\begin{aligned} \sum_{x\in Fl(a,v')} \sum_{y\in Fl(a,x)} \frac{\alpha_x}{\#Fl(a,x)} &= \sum_{x\in Fl(a,v')} \left[\frac{\alpha_x}{\#Fl(a,x)} \sum_{y\in Fl(a,x)} 1\right] \\ &= \sum_{x\in Fl(a,v')} \alpha_x = 1\,, \end{aligned}$$

and after the grouping the coefficients will still sum up to 1. We have

$$d(v',y) \le d(v',x) + d(x,y) \le 2\delta\,,$$

hence, by Lemma 2, $d(pr_a(y), p[a,b](10\delta m)) \le \delta$. This implies that all the points $pr_a(y)$ mentioned in formula (1) belong to $Fl(a, p[a,b](10\delta m))$. Lemma 4 is proved. □

Now part (2) in Proposition 3 is immediate: if $d(a,b) \geq 10\delta$, then, by taking $m := 1$ in Lemma 4, we obtain $v = p[a,b](10\delta)$ and

$$f(a,b) = \sum_{x \in Fl(a,v)} \alpha_x f(a,x) = \sum_{x \in Fl(a,v)} \alpha_x x\,.$$

To finish the proof of Proposition 3 it only remains to show part (5). Let

$$\omega := \max\left\{\#B(v,\delta) \;\middle|\; v \in \Gamma^{(0)}\right\}.$$

Obviously, $\omega \geq 1$, and also $\omega < \infty$ because, up to the G-action, there are only finitely many balls of radius δ in $\Gamma^{(0)}$. Note that the cardinality of each flower $Fl(a,b)$ does not exceed ω.

LEMMA 5. *Suppose that vertices a, b, c in $\Gamma^{(0)}$ and an integer n satisfy $d(b,c) \leq \delta$ and $d(a,b) = d(a,c) = 10\delta n$. Then*

$$\left|f(a,b) - f(a,c)\right|_1 \leq 2\left(1 - \frac{1}{\omega^2}\right)^{n-1}.$$

Proof. Induction on n.

$n \leq 1$. In this case

$$\left|f(a,b) - f(a,c)\right|_1 \leq \left|f(a,b)\right|_1 + \left|f(a,c)\right|_1 = 2 \leq 2\left(1 - \frac{1}{\omega^2}\right)^{n-1}.$$

$n - 1 \mapsto n$. Suppose $d(b,c) \leq \delta$ and $d(a,b) = d(a,c) = 10\delta n$, where $n \geq 2$. Then

$$\begin{aligned}
&\left|f(a,b) - f(a,c)\right|_1 \\
&= \left|\frac{1}{\#Fl(a,b)} \sum_{x \in Fl(a,b)} f\big(a, pr_a(x)\big) - \frac{1}{\#Fl(a,c)} \sum_{y \in Fl(a,c)} f\big(a, pr_a(y)\big)\right|_1 \\
&= \left|\frac{1}{\#Fl(a,b)\cdot\#Fl(a,c)} \sum_{x \in Fl(a,b)} \sum_{y \in Fl(a,c)} \big[f(a, pr_a(x)) - f(a, pr_a(y))\big]\right|_1 \\
&\leq \frac{1}{\#Fl(a,b)\cdot\#Fl(a,c)} \sum_{x \in Fl(a,b)} \sum_{y \in Fl(a,c)} \left|f(a, pr_a(x)) - f(a, pr_a(y))\right|_1 .
\end{aligned} \tag{2}$$

By the hypotheses, $d(b,c) \leq \delta$, so $b \in Fl(a,b) \cap Fl(a,c)$, and therefore there is a term in the last double sum corresponding to $x := y := b$. This term is obviously zero. The remaining $\#Fl(a,b)\cdot\#Fl(a,c) - 1$ terms in this double sum can be bounded as follows. Since

$$d(x,y) \leq d(x,b) + d(b,c) + d(c,y) \leq \delta + \delta + \delta = 3\delta\,,$$

then, by Lemma 2, $d(pr_a(x), pr_a(y)) \le \delta$. The induction hypotheses now apply to the vertices a, $pr_a(x)$, and $pr_a(y)$ giving the bound

$$\left|f(a, pr_a(x)) - f(a, pr_a(y))\right|_1 \le 2\left(1 - \frac{1}{\omega^2}\right)^{(n-1)-1} = 2\left(1 - \frac{1}{\omega^2}\right)^{n-2}$$

for each $x \in Fl(a,b)$ and $y \in Fl(a,c)$. So continuing inequality (2) we have

$$\begin{aligned}
&\left|f(a,b) - f(a,c)\right|_1 \\
&\quad \le \frac{1}{\#Fl(a,b)\cdot\#Fl(a,c)}\big(\#Fl(a,b)\cdot\#Fl(a,c) - 1\big)\cdot 2\left(1 - \frac{1}{\omega^2}\right)^{n-2} \\
&\quad = \left(1 - \frac{1}{\#Fl(a,b)\cdot\#Fl(a,c)}\right)\cdot 2\left(1 - \frac{1}{\omega^2}\right)^{n-2} \le 2\left(1 - \frac{1}{\omega^2}\right)^{n-1}.
\end{aligned}$$

Lemma 5 is proved. □

Now we finish the proof of Proposition 3(5). Pick any triple of vertices a, b, c in $\Gamma^{(0)}$. Let

$$\lambda := \left(1 - \frac{1}{\omega^2}\right)^{1/10\delta} \quad \text{and} \quad L := 2\left(1 - \frac{1}{\omega^2}\right)^{-3}.$$

Recall that $1 \le \omega < \infty$, hence $L \ge 0$ and $0 \le \lambda < 1$, as needed.

If $(b|c)_a \le 20\delta$, then

$$\begin{aligned}
\left|f(a,b) - f(a,c)\right|_1 &\le \left|f(a,b)\right|_1 + \left|f(a,c)\right|_1 = 2 \\
&= 2\left(1 - \frac{1}{\omega^2}\right)^{-3}\cdot\left(1 - \frac{1}{\omega^2}\right)^{3} = L\lambda^{30\delta} \le L\lambda^{20\delta} \le L\lambda^{(b|c)_a}.
\end{aligned}$$

We can now assume $(b|c)_a > 20\delta$. Let m be the maximal integer among those satisfying $10\delta m \le (b|c)_a$. It easily follows that

$$\tfrac{(b|c)_a}{10\delta} - 1 \le m \tag{3}$$

and $20\delta \le 10\delta m \le (b|c)_a \le d(a,b)$, hence, by Lemma 4,

$$f(a,b) = \sum_{x \in Fl(a,v)} \alpha_x f(a,x)\,,$$

where $v := p[a,b](10\delta m)$ and α_x are some non-negative coefficients summing up to 1. In the same way,

$$f(a,c) = \sum_{y \in Fl(a,w)} \beta_y f(a,y)\,,$$

where $w := p[a,c](10\delta m)$ and β_y are some non-negative coefficients summing up to 1 (see Fig. 2).

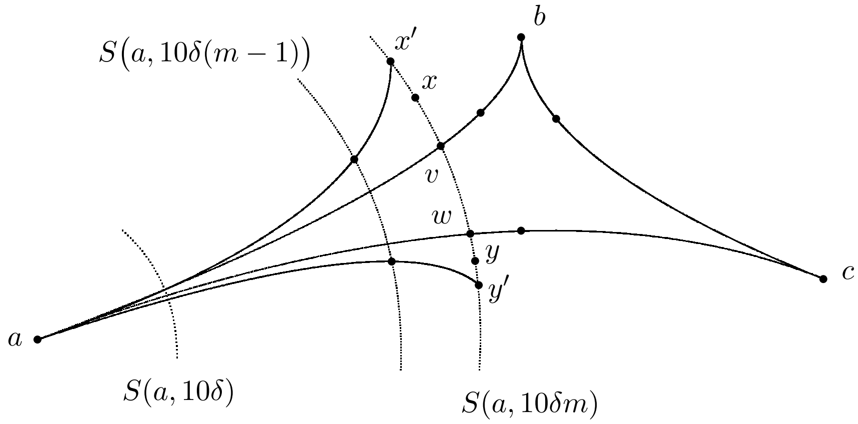

Figure 2: Proof of Proposition 3(5).

Then

$$\begin{aligned}
\left|f(a,b) - f(a,c)\right|_1 &= \Big|\sum_{x\in Fl(a,v)} \alpha_x f(a,x) - \sum_{y \in Fl(a,w)} \beta_y f(a,y)\Big|_1 \\
&= \Big|\sum_{x\in Fl(a,v)} \alpha_x f(a,x) \cdot \sum_{y\in Fl(a,w)} \beta_y - \sum_{x\in Fl(a,v)} \alpha_x \cdot \sum_{y\in Fl(a,w)} \beta_y f(a,y)\Big|_1 \\
&= \Big|\sum_{x\in Fl(a,v)} \sum_{y\in Fl(a,w)} \alpha_x\beta_y\big[f(a,x) - f(a,y)\big]\Big|_1 \\
&\le \sum_{x\in Fl(a,v)} \sum_{y\in Fl(a,w)} \alpha_x\beta_y\big|f(a,x) - f(a,y)\big|_1 ,
\end{aligned} \tag{4}$$

whereas, by the definition of f,

$$\begin{aligned}
&\big|f(a,x) - f(a,y)\big|_1 \\
&= \left|\frac{1}{\#Fl(a,x)} \sum_{x'\in Fl(a,x)} f\big(a, pr_a(x')\big) - \frac{1}{\#Fl(a,y)} \sum_{y'\in Fl(a,y)} f\big(a, pr_a(y')\big)\right|_1 \\
&\le \frac{1}{\#Fl(a,x)\cdot \#Fl(a,y)} \sum_{x'\in Fl(a,x)} \sum_{y'\in Fl(a,y)} \big|f(a, pr_a(x')) - f(a, pr_a(y'))\big|_1 .
\end{aligned} \tag{5}$$

By the choice of v, w, and m, $d(a,v) = d(a,w) = 10\delta m \le (b|c)_a$, then the fine-triangles property yields $d(v,w) \le \delta$. Since $x \in Fl(a,v)$, $y \in Fl(a,w)$,

$x' \in Fl(a,x)$, $y' \in Fl(a,y)$,

$d(x',y') \leq d(x',x)+d(x,v)+d(v,w)+d(w,y)+d(y,y') \leq \delta+\delta+\delta+\delta+\delta = 5\delta$,

then, by Lemma 2, $d(pr_a(x'),pr_a(y')) \leq \delta$, and by Lemma 5,

$$\big|f(a,pr_a(x')) - f(a,pr_a(y'))\big|_1 \leq 2\left(1-\frac{1}{\omega^2}\right)^{(m-1)-1} = 2\left(1-\frac{1}{\omega^2}\right)^{m-2}. \tag{6}$$

Combining inequalities (4), (5), (6) and (3),

$$\begin{aligned}
&\big|f(a,b) - f(a,c)\big|_1 \\
&\quad\leq \sum_{x\in Fl(a,v)} \sum_{y\in Fl(a,w)} \frac{\alpha_x\beta_y}{\#Fl(a,x)\cdot\#Fl(a,y)} \sum_{x'\in Fl(a,x)} \sum_{y'\in Fl(a,y)} 2\left(1-\frac{1}{\omega^2}\right)^{m-2} \\
&\quad= 2\left(1-\frac{1}{\omega^2}\right)^{m-2} \leq 2\left(1-\frac{1}{\omega^2}\right)^{\left[\frac{(b|c)_a}{10\delta}-1\right]-2} \\
&\quad= 2\left(1-\frac{1}{\omega^2}\right)^{-3}\cdot\left(1-\frac{1}{\omega^2}\right)^{\frac{(b|c)_a}{10\delta}} = L\lambda^{(b|c)_a}.
\end{aligned}$$

Proposition 3 is proved. □

Now we use the function f to construct another function $\bar{f}$ with additional properties.

For each $a \in \Gamma^{(0)}$ we define a 0-chain $star(a)$ by

$$star(a) := \frac{1}{\#B(a,7\delta)} \sum_{x\in B(a,7\delta)} x\,.$$

In other words, $star(a)$ is "the uniform spread" of a to all the vertices that are 7δ-close to a. Also, $star(a)$ can be defined for any 0-chain a, by linearity:

$$star\Big(\sum_{x\in\Gamma^{(0)}} \alpha_x x\Big) := \sum_{x\in\Gamma^{(0)}} \alpha_x star(x)\,.$$

Lemma 6. *star satisfies the following properties:*

- *If a is a convex combination, then $star(a)$ is a convex combination.*
- *$supp\, star(a)$ lies in the 7δ-neighborhood of $supp\, a$, for any 0-chain a.*
- *star is a linear operator $C_0(\Gamma,\mathbb{Q}) \to C_0(\Gamma,\mathbb{Q})$.*
- *This operator is of norm 1, i.e. $|star(a)|_1 \leq |a|_1$ for any 0-chain a.*
- *star is G-equivariant, i.e. $star(g\cdot a) = g\cdot star(a)$ for any 0-chain a and any $g \in G$.*

The proof is immediate.

Now if f is the function from Proposition 3 and $a, b \in \Gamma^{(0)}$, define

$$\bar{f}(a,b) := star\big(f(a,b)\big)\,.$$

PROPOSITION 7. *The function $\bar{f}$ defined above satisfies the following properties:*

(1) $\bar{f}(a,b)$ is a convex combination.
(2) If $d(a,b) \geq 10\delta$, then $supp\,\bar{f}(a,b) \subseteq B(p[a,b](10\delta), 8\delta)$.
(3) If $d(a,b) \leq 10\delta$, then $supp\,\bar{f}(a,b) \subseteq B(b, 7\delta)$.
(4) $\bar{f}$ is G-equivariant, i.e. $\bar{f}(g\cdot a, g\cdot b) = g\cdot \bar{f}(a,b)$ for any $a, b \in \Gamma^{(0)}$ and $g \in G$.
(5) There exist constants $L \geq 0$ and $0 \leq \lambda < 1$ such that, for any $a, b, c \in \Gamma^{(0)}$,

$$\big|\bar{f}(a,b) - \bar{f}(a,c)\big|_1 \leq L\lambda^{(b|c)_a}.$$

(6) There exists a constant $0 \leq \lambda' < 1$ such that if $a, b, c \in \Gamma^{(0)}$ satisfy $(a|b)_c \leq 10\delta$ and $(a|c)_b \leq 10\delta$, then

$$\big|\bar{f}(b,a) - \bar{f}(c,a)\big|_1 \leq 2\lambda'\,.$$

(7) Let $a, b, c \in \Gamma^{(0)}$, γ be a geodesic path from a to b, and let $c \in N(\gamma, 9\delta)$. Then $supp\,\bar{f}(c,a) \subseteq N(\gamma, 9\delta)$.

Proof. Lemma 6 and Proposition 3(1)–(5) imply Proposition 7(1)–(5). We show Proposition 7(6) now.

Let $\omega_7 := \max\{\#B(v, 7\delta) \mid v \in \Gamma^{(0)}\}$, and $\lambda' := 1 - \frac{1}{\omega_7^2}$. We have $1 \leq \omega_7 < \infty$, and hence $0 \leq \lambda' < 1$. Let us assume that $a, b, c \in \Gamma^{(0)}$ satisfy the hypotheses

$$(a|b)_c \leq 10\delta \quad \text{and} \quad (a|c)_b \leq 10\delta\,.$$

This implies that

$$d(b,c) = (a|b)_c + (a|c)_b \leq 20\delta\,. \tag{7}$$

Without loss of generality, $d(a,b) \leq d(a,c)$ (interchange b and c otherwise). Additionally we assume for the moment that

$$d(a,b) \geq 10\delta\,.$$

Let $v := p[b,a](10\delta)$ and $w := p[c,a](10\delta)$. By Proposition 3(1,2),

$$f(b,a) = \sum_{x\in Fl(b,v)} \alpha_x x \quad \text{and} \quad f(c,a) = \sum_{y \in Fl(b,w)} \beta_y y\,,$$

where $\alpha_x \geq 0$, $\sum_{x\in Fl(b,v)} \alpha_x = 1$, $\beta_y \geq 0$, $\sum_{y\in Fl(b,w)} \beta_y = 1$ (see Fig. 3). Then

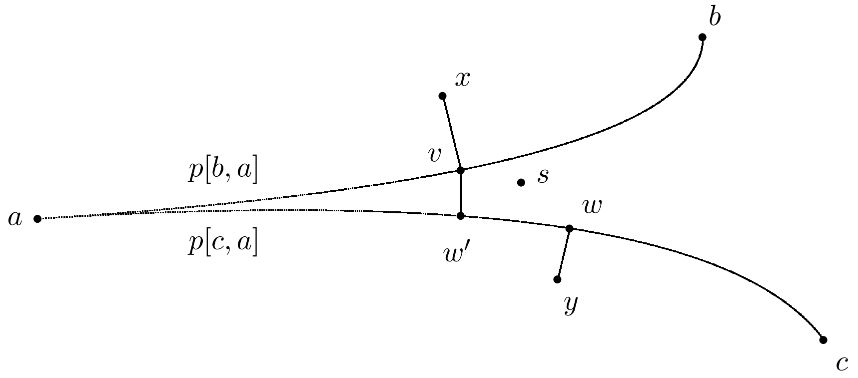

Figure 3: Proof of Proposition 7(6).

$$\begin{aligned}\left|\bar f(b,a)-\bar f(c,a)\right|_1 &= \left|star(f(b,a))-star(f(c,a))\right|_1\\ &= \Big|star\Big(\sum_{x\in Fl(b,v)}\alpha_x x\Big)-star\Big(\sum_{y\in Fl(c,w)}\beta_y y\Big)\Big|_1\\ &= \Big|\sum_{x\in Fl(b,v)}\alpha_x star(x)-\sum_{y\in Fl(c,w)}\beta_y star(y)\Big|_1\\ &\le \sum_{x\in Fl(b,v)}\sum_{y\in Fl(c,w)}\alpha_x\beta_y\left|star(x)-star(y)\right|_1 .\end{aligned} \tag{8}$$

Let w' be the vertex on the geodesic $p[c,a]$ satisfying $d(a,w')=d(a,v)$. We have

$$d(a,w')=d(a,v)=d(a,b)-d(b,v)=d(a,b)-10\delta\le d(a,c)-10\delta=d(a,w)\,,$$

hence, using inequality (7),

$$\begin{aligned}d(a,w')=d(a,v) &\le \tfrac12\big[(d(a,b)-10\delta)+\big(d(a,c)-10\delta\big)\big]\\ &= \tfrac12\big[d(a,b)+d(a,c)-20\delta\big]\le\tfrac12\big[d(a,b)+d(a,c)-d(b,c)\big]=(b|c)_a\,,\end{aligned}$$

therefore, by the fine-triangles property, $d(v,w')\le\delta$. Also

$$\begin{aligned}d(w',w) &= d(a,c)-d(c,w)-d(a,w')=d(a,c)-10\delta-d(a,v)\\ &= d(a,c)-d(a,b)=\big[(a|b)_c+(b|c)_a\big]-\big[(a|c)_b+(b|c)_a\big]\\ &= (a|b)_c-(a|c)_b\le(a|b)_c\le 10\delta\,.\end{aligned}$$

So we have

$$d(v,w)\le d(v,w')+d(w',w)\le\delta+10\delta=11\delta\,.$$

If $x \in Fl(b,v)$ and $y \in Fl(c,w)$, then using the last formula we get

$$d(x,y) \le d(x,v) + d(v,w) + d(w,y) \le \delta + 11\delta + \delta = 13\delta\,.$$

This implies that, for each such a pair of vertices x and y, there is a vertex $s \in B(x,7\delta) \cap B(y,7\delta)$. (Take s to be a vertex on a geodesic edge path between x and y nearest to the midpoint.) Then

$$\begin{aligned}
&\bigl|star(x) - star(y)\bigr|_1 \\
&= \Bigl|\frac{1}{\#B(x,7\delta)} \sum_{x' \in B(x,7\delta)} x' - \frac{1}{\#B(y,7\delta)} \sum_{y' \in B(y,7\delta)} y'\Bigr|_1 \\
&\le \frac{1}{\#B(x,7\delta)\cdot \#B(y,7\delta)} \sum_{x' \in B(x,7\delta)} \sum_{y' \in B(y,7\delta)} |x'-y'|_1 \qquad (9)\\
&\le \frac{1}{\#B(x,7\delta)\cdot \#B(y,7\delta)} \cdot 2\bigl(\#B(x,7\delta)\cdot \#B(y,7\delta) - 1\bigr) \\
&= 2\left(1 - \frac{1}{\#B(x,7\delta)\cdot \#B(y,7\delta)}\right) \le 2\left(1 - \frac{1}{\omega_7^2}\right) = 2\lambda'\,.
\end{aligned}$$

Combining inequalities (8) and (9),

$$\bigl|\bar f(b,a) - \bar f(c,a)\bigr|_1 \le \sum_{x \in Fl(b,v)} \sum_{y \in Fl(c,w)} \alpha_x \beta_y \cdot 2\lambda' = 2\lambda'\,.$$

This was proved assuming that $d(a,b) \ge 10\delta$. Also recall that $d(a,b) \le d(a,c)$ holds.

If $d(a,b) \le d(a,c) \le 10\delta$, then take $v := w := a$. If $d(a,b) \le 10\delta \le d(a,c)$, then take $v := a$ and $w := p[c,a](10\delta)$. In the latter case we have

$$\begin{aligned}
d(v,w) = d(a,w) &= d(a,c) - d(c,w) = d(a,c) - 10\delta \\
&\le d(a,c) - (a|b)_c = (b|c)_a \le 10\delta\,.
\end{aligned}$$

Therefore $d(v,w) \le 10\delta$ in either case, hence, for any $x \in Fl(b,v)$ and any $y \in Fl(c,w)$,

$$d(y,x) \le 10\delta + 2\delta = 12\delta\,,$$

so the same argument using formulas (8) and (9) works. Part (6) is proved.

Part (7) of Proposition 7 is almost immediate. If $d(a,c) \le 10\delta$, then $supp\,\bar f(c,a) \subseteq B(a,7\delta) \subseteq N(\gamma,9\delta)$ by Proposition 7(3). Suppose now $d(a,c) > 10\delta$. Let b' be a vertex on γ with $d(b',c) \le 9\delta$ (see Fig. 4). Let also $v := p[c,a](10\delta)$ and w be the vertex on γ with $d(a,w) = d(a,v)$. Such a vertex w always exists because

$$d(a,b') \ge d(a,c) - d(c,b') \ge d(a,c) - 9\delta \ge d(a,c) - 10\delta = d(a,v)\,.$$

Then

$$d(a,w) = d(a,v) = d(a,c) - 10\delta = \tfrac{1}{2}\bigl[d(a,c) + d(a,c) - 20\delta\bigr]$$

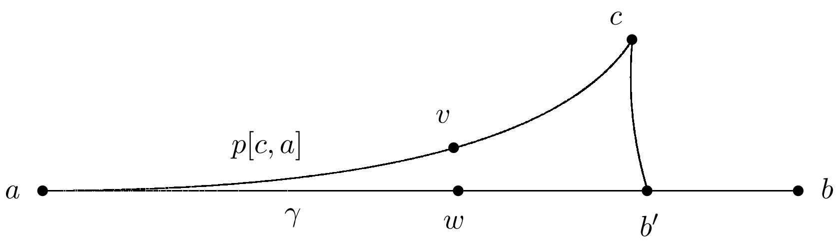

Figure 4: Proof of Proposition 7(7).

$$\leq \tfrac{1}{2}\big[d(a,c)+d(a,b')+d(b',c)-20\delta\big] \leq \tfrac{1}{2}\big[d(a,c)+d(a,b')-d(b',c)\big] = (b|c)_a\,,$$

and, by the fine-triangles property, $d(v,w) \leq \delta$. Since $supp\,\bar{f}(c,a) \subseteq B(v,8\delta)$, then

$$supp\,\bar{f}(c,a) \subseteq B(w,9\delta) \subseteq B(\gamma,9\delta)\,.$$

Proposition 7 is proved. □

First we will construct a homological bicombing q' in Γ with certain properties. Recall that p was a choice of a geodesic bicombing in Γ. The notation $p[a,b]$ makes sense not only when a and b are vertices in $\Gamma^{(0)}$, but it also can be defined when a is any 0-chain, by linearity:

$$p\Big[\sum_{x\in\Gamma^{(0)}} \alpha_x x, b\Big] := \sum_{x\in\Gamma^{(0)}} \alpha_x p[x,b]\,.$$

One easily checks that $\partial p[a,b] = b-a$ if a is a convex combination.

The 1-chain $q'[a,b]$ is defined inductively on $d(a,b)$. If $d(a,b) \leq 10\delta$, put $q'[a,b] := p[a,b]$. Assume now that $d(a,b) > 10\delta$. By Proposition 7(2),

$$supp\,\bar{f}(b,a) \subseteq B\big(p[b,a](10\delta), 8\delta\big)\,,$$

hence, for each vertex $x \in supp\,\bar{f}(b,a)$,

$$d(a,x) \leq d\big(a,p[b,a](10\delta)\big)+d\big(p[b,a](10\delta),x\big) \leq \big[d(a,b)-10\delta\big]+8\delta < d(a,b)\,,$$

so $q'[a,x]$ is defined by the induction hypotheses. Now we define $q'[a,\bar{f}(b,a)]$ by linearity over the second variable, and put

$$q'[a,b] := q'\big[a,\bar{f}(b,a)\big] + p\big[\bar{f}(b,a),b\big]\,.$$

One easily checks that $\partial q'[a,b] := b-a$, so q' is a homological bicombing in Γ.

PROPOSITION 8. *The $\mathbb{Q}$-bicombing q' constructed above satisfies the following conditions.*

(1) *q' is G-equivariant.*
(2) *q' is quasigeodesic.*
(3) *There exist constants $M \geq 0$ and $N \geq 0$ such that, for all $a, b, c \in \Gamma^{(0)}$,*

$$\big|q'[a,b] - q'[a,c]\big|_1 \leq M\, d(b,c) + N\,.$$

Proof. (1) is obvious because the definition of q' used p and $\bar f$, and they are G-equivariant.

(2) First we define a sequence of sets of vertices $V_i(a,b)$ for each pair $a, b \in \Gamma^{(0)}$. Put $V_0(a,b) := \{b\}$ and

$$V_{i+1}(a,b) := V_i(a,b) \cup \bigcup_{c \in V_i(a,b)} \mathit{supp}\, \bar f(c,a)\,.$$

This sequence is increasing and stabilizes at a certain vertex set which we denote by $V(a,b)$. Tracing the definitions of $q'[a,b]$ and $V(a,b)$ we see that $q'[a,b]$ is a linear combination of geodesic paths of length at most 18δ whose endpoints lie in $V(a,b)$. Hence, to show that q' is quasigeodesic, it is enough to show that $V(a,b)$ lies uniformly close to $p[a,b]$.

We prove that $V_i(a,b) \subseteq N(p[a,b], 9\delta)$ inductively on i. Firstly, $V_0(a,b) = \{b\} \subseteq N(p[a,b], 9\delta)$. Secondly, if $V_i(a,b) \subseteq N(p[a,b], 9\delta)$, then, by Proposition 7(7),

$$V_{i+1}(a,b) = V_i(a,b) \cup \bigcup_{c \in V_i(a,b)} \mathit{supp}\, \bar f(c,a) \subseteq N\big(p[a,b], 9\delta\big)\,.$$

This implies $V(a,b) \subseteq N(p[a,b], 9\delta)$.

The inequality $|q'[a,b]|_1 \leq 18\delta\, d(a,b)$ follows by induction on $d(a,b)$, using the definition of q', so part (2) is proved.

(3) Up to the G-action, there are only finitely many triples of vertices a, b, c, satisfying $d(a,b) + d(a,c) \leq 60\delta$, hence there exists a uniform bound N' for the norms

$$\big|q'[a,b] - q'[a,c]\big|_1$$

for such vertices a, b, and c. Let λ' be the constant from Proposition 7(6),

$$M := 18\delta \quad \text{and} \quad N := \max\left\{N',\ \frac{\lambda' \cdot 56\delta M + 36\delta}{1 - \lambda'}\right\}.$$

We prove the statement by induction on $d(a,b) + d(a,c)$.

If $d(a,b) + d(a,c) \leq 60\delta$, then

$$\big|q'[a,b] - q'[a,c]\big|_1 \leq N' \leq N \leq M\, d(b,c) + N$$

just by the choice of N' and N. We assume now that $d(a,b) + d(a,c) > 60\delta$. Consider the following two cases.

Case 1. $(a|c)_b > 10\delta$ *or* $(a|b)_c > 10\delta$.
Assume, for example, that $(a|c)_b > 10\delta$ (see Fig. 5). Then, in particular,

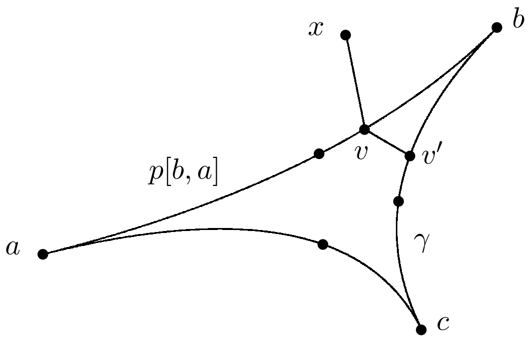

Figure 5: Case 1.

$d(a,b) > 10\delta$, hence, by definition,

$$q'[a,b] = q'i\big[a, \bar f(b,a)\big] + p\big[\bar f(b,a), b\big]$$

and $supp\, \bar f(b,a) \subseteq B(v, 8\delta)$, where $v := p[b,a](10\delta)$. Also, $d(b,c) \geq (a|c)_b > 10\delta$, so there exists a geodesic γ between b and c, and a vertex v' on γ with $d(b,v') = d(b,v) = 10\delta$. By the fine-triangles property, $d(v,v') \leq \delta$. If $x \in supp\, \bar f(b,a)$, then

$$d(x,b) \leq d(x,v) + d(v,b) \leq 8\delta + 10\delta = 18\delta\,, \tag{10}$$

$$d(x,c) \leq d(x,v) + d(v,v') + d(v',c) \leq 8\delta + \delta + \big[d(b,c) - 10\delta\big] \leq d(b,c) - 1\,,$$

and

$$d(a,x) \leq d(a,v) + d(v,x) \leq \big[d(a,b) - 10\delta\big] + 8\delta < d(a,b)\,,$$

therefore $d(a,x) + d(a,c) < d(a,b) + d(a,c)$, so the induction hypotheses apply to the vertices a, x, and c, giving

$$\begin{aligned} \big|q'[a,x] - q'[a,c]\big|_1 &\leq M\, d(x,c) + N \leq M\big(d(b,c) - 1\big) + N \\ &= M\, d(b,c) - M + N\,. \end{aligned} \tag{11}$$

For some non-negative coefficients α_x summing up to 1,

$$\bar f(b,a) = \sum_{x \in B(v,8\delta)} \alpha_x x\,.$$

Then, by the definition of $q'[a,b]$ and M and inequalities (10) and (11),

$$\big|q'[a,b] - q'[a,c]\big|_1 = \big|q'[a, \bar f(b,a)] + p[\bar f(b,a), b] - q'[a,c]\big|_1$$

$$
\begin{aligned}
&= \Big| \sum_{x\in B(v,8\delta)} \alpha_x q'[a,x] + \sum_{x\in B(v,8\delta)} \alpha_x p[x,b] - q'[a,c]\Big|_1 \\
&\le \Big| \sum_{x\in B(v,8\delta)} \alpha_x \big(q'[a,x] - q'[a,c]\big)\Big|_1 + \Big| \sum_{x\in B(v,8\delta)} \alpha_x p[x,b]\Big|_1 \\
&\le \sum_{x\in B(v,8\delta)} \alpha_x \big|q'[a,x] - q'[a,c]\big|_1 + \sum_{x\in B(v,8\delta)} \alpha_x \big|p[x,b]\big|_1 \\
&\le \sum_{x\in B(v,8\delta)} \alpha_x \cdot \big(M\,d(b,c) - M + N\big) + \sum_{x\in B(v,8\delta)} \alpha_x\, d(x,b) \\
&\le M\,d(b,c) - M + N + 18\delta = M\,d(b,c) + N\,.
\end{aligned}
$$

Case 2. $(a|c)_b \le 10\delta$ *and* $(a|b)_c \le 10\delta$.
In this case Proposition 7(6) applies. Since $d(a,b) + d(a,c) > 60\delta$ and $d(b,c) = (a|c)_b + (a|b)_c \le 20\delta$, then $d(a,b) > 10\delta$ and $d(a,c) > 10\delta$. Then, by the definition of $q'[a,b]$ and $q'[a,c]$,

$$
\begin{aligned}
\big|q'[a,b] - q'[a,c]\big|_1 &= \big|q'[a,\bar f(b,a)] + p[\bar f(b,a),b] - q'[a,\bar f(c,a)] - p[\bar f(c,a),c]\big|_1 \\
&\le \big|q'[a,\bar f(b,a)] - q'[a,\bar f(c,a)]\big|_1 + \big|p[\bar f(b,a),b]\big|_1 + \big|p[\bar f(c,a),c]\big|_1 \\
&= \big|q'[a,\bar f(b,a) - \bar f(c,a)]\big|_1 + \big|p[\bar f(b,a),b]\big|_1 + \big|p[\bar f(c,a),c]\big|_1\,. \qquad (12)
\end{aligned}
$$

The 0-chain $\bar f(b,a) - \bar f(c,a)$ (as any other) can be represented in the form $f_+ - f_-$, where f_+ and f_- are 0-chains with non-negative coefficients and disjoint supports. By Proposition 7(6),

$$|f_+|_1 + |f_-|_1 = |f_+ - f_-|_1 = \big|\bar f(b,a) - \bar f(c,a)\big|_1 \le 2\lambda'\,.$$

The coefficients of the 0-chain $f_+ - f_- = \bar f(b,a) - \bar f(c,a)$ sum up to 0, because $\bar f(b,a)$ and $\bar f(c,a)$ are convex combinations. It follows that

$$|f_+|_1 = |f_-|_1 \le \lambda'\,. \qquad (13)$$

Also,

$$\mathit{supp}\, f_+ \subseteq \mathit{supp}\, \bar f(b,a) \subseteq B\big(p[b,a](10\delta), 8\delta\big)$$

and

$$\mathit{supp}\, f_- \subseteq \mathit{supp}\, \bar f(c,a) \subseteq B\big(p[c,a](10\delta), 8\delta\big)\,,$$

hence, for each $x \in \mathit{supp}\, f_+$ and $y \in \mathit{supp}\, f_-$, we have (by the hypotheses of Case 2)

$$d(x,y) \le d(x,b) + d(b,c) + d(c,y) \le 18\delta + 20\delta + 18\delta = 56\delta\,.$$

Also $d(a,x) + d(a,y) < d(a,b) + d(a,c)$, so, by the induction hypotheses for the vertices a, x, and y,

$$\big|q'[a,x] - q'[a,y]\big|_1 \le M\,d(x,y) + N \le 56\delta M + N \qquad (14)$$

for each $x \in supp\, f_+$ and $y \in supp\, f_-$. Then we continue inequality (12) using inequalities (13) and (14) and the definition of N:

$$\begin{aligned}\big|q'[a,b]-q'[a,c]\big|_1 &\le \big|q'[a,\bar f(b,a)-\bar f(c,a)]\big|_1+\big|p[\bar f(b,a),b]\big|_1+\big|p[\bar f(c,a),c]\big|_1\\ &= \big|q'[a,f_+]-q'[a,f_-]\big|_1+\big|p[\bar f(b,a),b]\big|_1+\big|p[\bar f(c,a),c]\big|_1\\ &\le \lambda'\cdot[56\delta M+N]+18\delta+18\delta \le N \le M\,d(b,c)+N\,.\end{aligned}$$

Proposition 8 is proved. □

Given a triple of vertices a, b, c in $\Gamma^{(0)}$, a vertex z is called *a center* of the triple $\{a,b,c\}$ if there exist geodesics $[a,b]$, $[b,c]$, $[c,a]$, points $\bar a \in [b,c]$, $\bar b \in [c,a]$, and $\bar c \in [a,b]$ satisfying

$$d(b,\bar c)=d(b,\bar a)\,,\quad d(c,\bar a)=d(c,\bar b)\,,\quad d(a,\bar c)=d(a,\bar b)\,,$$

and such that $d(z,\bar a)\le\delta$, $d(z,\bar b)\le\delta$, and $d(z,\bar c)\le\delta$. Since geodesic triangles in Γ are δ-fine, such a center always exists.

LEMMA 9. *For the bicombing q' constructed above, there exist constants $K\ge 0$ and $0\le\lambda<1$ with the following property. If $a',a,b,c\in\Gamma^{(0)}$, $z\in\Gamma$ is a center of the triple $\{a,b,c\}$, and $a'\in N(p[z,a],10\delta)$, then*

$$\big|q'[b,a']-q'[c,a']-q'[b,z]+q'[c,z]\big|_1 \le K(1+\lambda+\lambda^2+\dots+\lambda^{d(z,a')})$$

(*see Fig. 6*).

The expression on the right can be bounded by a universal constant, and this will be used later in Theorem 10. We present the lemma in this form to enable the inductive proof.

Proof of Lemma 9. Let λ and L be the constants from Proposition 7(5) and M and N be the constants from Proposition 8(3). We take K to be sufficiently large, namely

$$K:=\max\big\{44\delta M+2N,\ L\lambda^{-4\delta}(26\delta M+N+18\delta)\big\}\,.$$

Note that all these constants are universal, i.e. they depend only on the choice of Γ.

We prove the lemma by induction on $d(z,a')$. If $d(z,a')\le 22\delta$, then, by Proposition 8(3),

$$\begin{aligned}&\big|q'[b,a']-q'[c,a']-q'[b,z]+q'[c,z]\big|_1\\ &\quad\le \big|q'[b,a']-q'[b,z]\big|_1+\big|q'[c,a']-q'[c,z]\big|_1\\ &\quad\le \big[M\,d(z,a')+N\big]+\big[M\,d(z,a')+N\big]\\ &\quad\le 44\delta M+2N\le K\le K(1+\dots+\lambda^{d(z,a')})\,.\end{aligned}$$

We now assume that $d(z,a')\ge 22\delta$. Since z is a center of $\{a,b,c\}$, there exist a geodesic γ from b to a and a point $\bar c\in\gamma$ with $d(a,\bar c)=(b|c)_a$ and

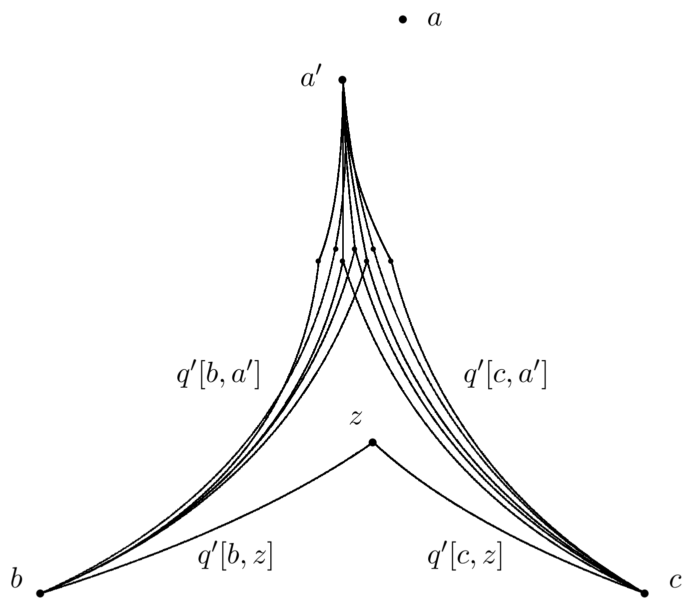

Figure 6: Lemma 9.

$d(z,\bar c) \le \delta$ (see Fig. 7). Denote $v := p[a',b](10\delta)$ and pick an arbitrary $x \in B(v, 8\delta)$. We want to use the induction hypotheses for the vertex x, so our first goal is to show that $d(z,x) < d(z,a')$ and $x \in N(p[z,a], 10\delta)$. This will be possible to do because $d(z,a')$ is large enough. By the hypotheses of the lemma there is a vertex u on $p[z,a]$ with $d(a',u) \le 10\delta$. Then

$$d(z,u) \ge d(z,a') - d(a',u) \ge 22\delta - 10\delta \ge \delta \ge d(z,\bar c) \ge (a|\bar c)_z\,,$$

hence

$$d(a,u) = d(a,z) - d(z,u) = \big[(z|\bar c)_a + (a|\bar c)_z\big] - d(z,u) \le (z|\bar c)_a\,.$$

The last inequality implies that there is a vertex u' on γ with $d(a,u') = d(a,u)$ and, by the fine-triangles property, $d(u,u') \le \delta$. This implies that

$$\begin{aligned} d(a',u') &\le d(a',u) + d(u,u') \le 10\delta + \delta = 11\delta\,, \\ d(a,\bar c) &\ge d(a,z) - d(z,\bar c) \ge \big[d(a,u) + d(u,z)\big] - \delta \\ &\ge d(a,u) + \big[d(z,a') - d(a',u)\big] - \delta \\ &\ge d(a,u) + 22\delta - 10\delta - \delta = d(a,u) + 11\delta = d(a,u') + 11\delta\,. \end{aligned}$$

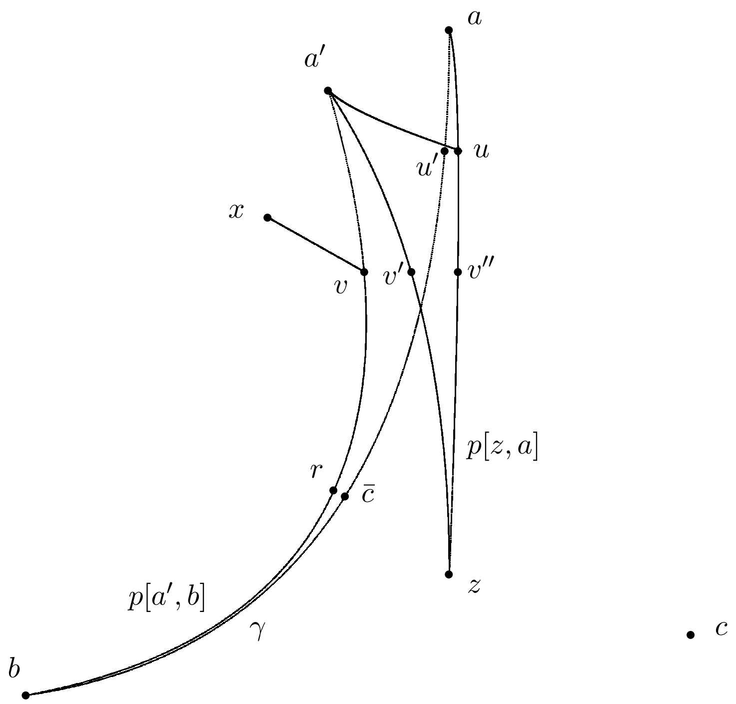

Figure 7: Proof of Lemma 9.

This means that $\bar{c}$ lies between b and u' on the geodesic γ and

$$d(u', \bar{c}) = d(a, \bar{c}) - d(a, u') \geq 11\delta\,.$$

Further,

$$\begin{aligned} d(b, \bar{c}) &= d(b, u') - d(u', \bar{c}) \leq d(b, u') - 11\delta \\ &\leq d(b, u') - d(a', u') \leq d(b, u') - (a'|b)_{u'} = (a'|u')_b\,. \end{aligned}$$

Therefore there exists a point r on $p[a', b]$ with $d(b, r) = d(b, \bar{c})$ and, by the fine-triangles property, $d(r, \bar{c}) \leq \delta$, so

$$d(z, r) \leq d(z, \bar{c}) + d(\bar{c}, r) \leq \delta + \delta = 2\delta\,. \tag{15}$$

Recall that v was defined as $p[a', b](10\delta)$, then we have

$$d(a', v) = 10\delta \leq 22\delta - 2\delta \leq d(a', z) - d(z, r) \leq d(a', z) - (a'|r)_z = (z|r)_{a'}\,,$$

so there exists a vertex v' on $p[z, a']$ with $d(a', v') = d(a', v) = 10\delta$ and, by the fine-triangles property, $d(v, v') \leq \delta$.

$$(z|u)_{a'} = \tfrac{1}{2}\big[d(z, a') + d(a', u) - d(z, u)\big]$$

$$\begin{aligned}&\le \tfrac{1}{2}\big[d(z,a')+d(a',u)-d(z,a')+d(a',u)\big]\\&= d(a',u)\le 10\delta = d(a',v')\,,\end{aligned}$$

then

$$d(z,v') = d(z,a') - d(a',v') \le d(z,a') - (z|u)_{a'} = (a'|u)_z\,,$$

hence there exists a vertex $v''\in p[z,a]$ with $d(z,v'')=d(z,v')$, and, by the fine-triangles property, $d(v',v'')\le\delta$.

For any vertex $x\in B(v,8\delta)$,

$$d(v'',x)\le d(v'',v')+d(v',v)+d(v,x)\le \delta+\delta+8\delta=10\delta\,,$$

so

$$x\in N\big(p[z,a],10\delta\big)$$

and

$$\begin{aligned}d(z,x)&\le d(z,v')+d(v',x)=\big[d(z,a')-10\delta\big]+d(v',x)\\&\le \big[d(z,a')-10\delta\big]+9\delta\le d(z,a')-1\,.\end{aligned}$$

The last two formulas say that each vertex $x\in B(v,8\delta)$ satisfies the induction hypotheses, hence

$$\begin{aligned}\big|q'[b,x]-q'[c,x]-q'[b,z]+q'[c,z]\big|_1 &\le K(1+\lambda+\lambda^2+\cdots+\lambda^{d(z,x)})\\&\le K(1+\lambda+\lambda^2+\cdots+\lambda^{d(z,a')-1})\,. \quad (16)\end{aligned}$$

The convex combinations $\bar f(a',b)$ and $\bar f(a',c)$ have the form

$$\bar f(a',b)=\sum_{x\in\Gamma^{(0)}}\alpha'_x x \quad\text{and}\quad \bar f(a',c)=\sum_{x\in\Gamma^{(0)}}\alpha''_x x$$

for some coefficients α'_x and α''_x. Define a 0-chain f_0 by

$$f_0:=\sum_{x\in\Gamma^{(0)}}\alpha_x x\,,$$

where $\alpha_x:=\min\{\alpha'_x,\alpha''_x\}$. Put

$$f_+:=\bar f(a',b)-f_0 \quad\text{and}\quad f_-:=\bar f(a',c)-f_0\,.$$

Then we have

$$\mathit{supp}\, f_0=\mathit{supp}\,\bar f(a',b)\cap \mathit{supp}\,\bar f(a',c)\,,$$

and f_+ and f_- are with non-negative coefficients and disjoint supports. Also

$$\bar f(a',b)-\bar f(a',c)=f_+-f_-\,,$$

hence the coefficients of f_+-f_- sum up to 0, so $|f_+|_1=|f_-|_1$. By Proposition 7(5),

$$|f_+|_1+|f_-|_1=|f_+-f_-|_1=\big|\bar f(a',b)-\bar f(a',c)\big|_1\le L\lambda^{(b|c)_{a'}}\,.$$

We recently proved the existence of a vertex $r \in p[a', b]$ which is 2δ-close to z (see inequality (15)). The same argument with c in place of b shows that there exists a vertex $s \in p[a', c]$ which is 2δ-close to z. It follows that

$$\begin{aligned}
(b|c)_{a'} &= \tfrac{1}{2}\big[d(a',b)+d(a',c)-d(b,c)\big]\\
&\geq \tfrac{1}{2}\big[d(a',b)+d(a',c)-(d(b,r)+d(r,z)+d(z,s)+d(s,c))\big]\\
&= \tfrac{1}{2}\big[(d(a',b)-d(b,r))+(d(a',c)-d(c,s))-d(r,z)-d(z,s)\big] \qquad (17)\\
&= \tfrac{1}{2}\big[d(a',r)+d(a',s)-d(r,z)-d(z,s)\big]\\
&\geq \tfrac{1}{2}\big[(d(a',z)-d(z,r))+(d(a',z)-d(z,s))-d(r,z)-d(z,s)\big]\\
&\geq \tfrac{1}{2}\big[d(a',z)-2\delta+d(a',z)-2\delta-2\delta-2\delta\big]=d(a',z)-4\delta\,.
\end{aligned}$$

Thus,

$$|f_+|_1=|f_-|_1\leq \tfrac{1}{2}L\lambda^{(b|c)_{a'}}\leq \tfrac{1}{2}L\lambda^{d(z,a')-4\delta}\,. \qquad (18)$$

Since $d(z,a')$ is large enough, then $d(a',b)>10\delta$ and $d(a',c)>10\delta$, so, by the definition of q', we have

$$\begin{aligned}
&\big|q'[b,a']-q'[c,a']-q'[b,z]+q'[c,z]\big|_1\\
&\quad=\big|(q'[b,\bar f(a',b)]+p[\bar f(a',b),a'])-(q'[c,\bar f(a',c)]\\
&\qquad\quad+p[\bar f(a',c),a'])-q'[b,z]+q'[c,z]\big|_1\\
&\quad\leq\big|q'[b,\bar f(a',b)]-q'[c,\bar f(a',c)]-q'[b,z]+q'[c,z]\big|_1\\
&\qquad\quad+\big|p[\bar f(a',b),a']-p[\bar f(a',c),a']\big|_1\\
&\quad=\big|q'[b,f_0+f_+]-q'[c,f_0+f_-]-q'[b,z]+q'[c,z]\big|_1\\
&\qquad\quad+\big|p[f_+-f_-,a']\big|_1\\
&\quad\leq\big|q'[b,f_0]-q'[c,f_0]-|f_0|_1\cdot q'[b,z]+|f_0|_1\cdot q'[c,z]\big|_1\\
&\qquad\quad+\big|q'[b,f_+]-q'[c,f_-]-(1-|f_0|_1)\cdot q'[b,z]+(1-|f_0|_1)\cdot q'[c,z]\big|_1\\
&\qquad\qquad+\big|p[f_+-f_-,a']\big|_1\,.
\end{aligned}$$

We are going to bound each of the three terms in the last sum, let us call them A_1, A_2, and A_3, respectively.

Bound for A_1. The 0-chain f_0 is supported in the ball $B(v,8\delta)$, so, for some α_x,

$$f_0=\sum_{x\in B(v,8\delta)}\alpha_x x\,,$$

and for each $x\in B(v,8\delta)$, inequality (16) holds, hence

$$A_1=\bigg|q'\Big[b,\sum_{x\in B(v,8\delta)}\alpha_x x\Big]-q'\Big[c,\sum_{x\in B(v,8\delta)}\alpha_x x\Big]-|f_0|_1\cdot q'[b,z]+|f_0|_1\cdot q'[c,z]\bigg|_1$$

$$= \Big| \sum_{x\in B(v,8\delta)} \alpha_x \big(q'[b,x] - q'[c,x] - q'[b,z] + q'[c,z]\big)\Big|_1$$
$$\le \sum_{x\in B(v,8\delta)} \alpha_x \cdot K(1+\lambda+\lambda^2+\dots+\lambda^{d(z,a')-1})$$
$$= |f_0|_1 \cdot K(1+\lambda+\lambda^2+\dots+\lambda^{d(z,a')-1}) .$$

Bound for A_2. Pick any $x_0 \in B(v,8\delta)$, so inequality (16) holds for x_0 as well:

$$\big|q'[b,x_0] - q'[c,x_0] - q'[b,z] + q'[c,z]\big|_1 \le K(1+\lambda+\lambda^2+...+\lambda^{d(z,a')-1}) . \tag{19}$$

Informally speaking, we are going to move both f_+ and f_- to x_0, and this move will not change the picture much. As before, $v = p[a',b](10\delta)$, and we denote $w := p[a',c](10\delta)$. The 0-chains f_+ and f_- have forms

$$f_+ = \sum_{x\in B(v,8\delta)} \beta'_x x \qquad \text{and} \qquad f_- = \sum_{y\in B(w,8\delta)} \beta''_y y .$$

Note that

$$\sum_{x\in B(v,8\delta)} \beta'_x = \sum_{y\in B(w,8\delta)} \beta''_y = |f_+|_1 = |f_-|_1 = 1 - |f_0|_1 .$$

Also, for each $x \in B(v,8\delta)$,

$$d(x,x_0) \le d(x,v) + d(v,x_0) \le 8\delta + 8\delta = 16\delta ,$$

and, for each $y \in B(w,8\delta)$,

$$d(y,x_0) \le d(y,w)+d(w,a')+d(a',v)+d(v,x_0) \le 8\delta+10\delta+10\delta+8\delta = 36\delta .$$

Using these observations, Proposition 8(3), and formulas (18) and (19), we obtain a bound for the second term:

$$A_2 = \big|q'[b,f_+] - q'[c,f_-] - |f_+|_1 \cdot q'[b,z] + |f_-|_1 \cdot q'[c,z]\big|_1$$
$$\le \big|q'[b,f_+] - q'[c,f_-] - (|f_+|_1 \cdot q'[b,x_0] + |f_-|_1 \cdot q'[c,x_0])\big|_1$$
$$+ \big|(|f_+|_1 \cdot q'[b,x_0] - |f_-|_1 \cdot q'[c,x_0]) - |f_+|_1 \cdot q'[b,z] + |f_-|_1 \cdot q'[c,z]\big|_1$$
$$= \Big|\sum_{x\in B(v,8\delta)} \beta'_x\big(q'[b,x] - q'[b,x_0]\big) - \sum_{y\in B(w,8\delta)} \beta''_y\big(q'[c,y] - q'[c,x_0]\big)\Big|_1$$
$$+ |f_+|_1 \cdot \big|q'[b,x_0] - q'[c,x_0] - q'[b,z] - q'[c,z]\big|_1$$
$$\le \sum_{x\in B(v,8\delta)} \beta'_x \big|q'[b,x] - q'[b,x_0]\big|_1 + \sum_{y\in B(w,8\delta)} \beta''_y \big|q'[c,y] - q'[c,x_0]\big|_1$$
$$+ |f_+|_1 \cdot \big|q'[b,x_0] - q'[c,x_0] - q'[b,z] + q'[c,z]\big|_1$$

$$\leq \sum_{x\in B(v,8\delta)} \beta'_x \cdot \big(M\cdot 16\delta + N\big) + \sum_{y\in B(w,8\delta)} \beta''_y \cdot \big(M\cdot 36\delta + N\big)$$
$$+ |f_+|_1 \cdot K(1+\lambda+\lambda^2+\dots+\lambda^{d(z,a')-1})$$
$$= |f_+|_1(M\cdot 52\delta + 2N) + |f_+|_1 \cdot K(1+\lambda+\lambda^2+\dots+\lambda^{d(z,a')-1})$$
$$\leq \tfrac{1}{2} L\lambda^{d(z,a')-4\delta}\cdot (M\cdot 52\delta+2N)$$
$$+ |f_+|_1 \cdot K(1+\lambda+\lambda^2+\dots+\lambda^{d(z,a')-1})$$
$$= L\lambda^{d(z,a')-4\delta}\cdot(26\delta M+N) + |f_+|_1\cdot K(1+\lambda+\lambda^2+\ldots+\lambda^{d(z,a')-1}).$$

Bound for A_3. For each vertex $x \in supp\, f_+ \cup supp\, f_-$,

$$\big|p[a',x]\big|_1 = d(a',x) \leq 18\delta\,,$$

then, using formula (18),

$$A_3 = \big|p[f_+ - f_-, a']\big|_1 \leq \big|f_+ - f_-\big|_1 \cdot 18\delta$$
$$= \big(|f_+|_1 + |f_-|_1\big)\cdot 18\delta \leq L\lambda^{d(z,a')-4\delta}\cdot 18\delta\,.$$

Combining the three bounds above and the definition of K, we obtain

$$\big|q'[b,a'] - q'[c,a'] - q'[b,z] + q'[c,z]\big|_1 \leq A_1 + A_2 + A_3$$
$$\leq |f_0|_1\cdot K(1+\lambda+\lambda^2+\dots+\lambda^{d(z,a')-1}) + L\lambda^{d(z,a')-4\delta}\cdot(26\delta M+N)$$
$$+ |f_+|_1\cdot K(1+\lambda+\lambda^2+\dots+\lambda^{d(z,a')-1}) + L\lambda^{d(z,a')-4\delta}\cdot 18\delta$$
$$= K(1+\lambda+\lambda^2+\dots+la^{d(z,a')-1}) + L\lambda^{-4\delta}(26\delta M+N+18\delta)\cdot\lambda^{d(z,a')}$$
$$\leq K(1+\lambda+\lambda^2+\dots+\lambda^{d(z,a')-1}) + K\lambda^{d(z,a')}$$
$$= K(1+\lambda+\lambda^2+\dots+\lambda^{d(z,a')})\,.$$

Lemma 9 is proved. □

Theorem 10. *Let G be a hyperbolic group and Γ be a connected graph with a free cocompact G-action. Then there exists a $\mathbb{Q}$-bicombing q in Γ with the following properties:*

(1) q is quasigeodesic.
(2) q is G-equivariant.
(3) q is anti-symmetric, i.e. $q[a,b] = -q[b,a]$ for any $a,b\in\Gamma$.
(4) There exists a constant T such that, for any $a,b,c\in\Gamma$,

$$\big|q[a,b]+q[b,c]+q[c,a]\big|_1 \leq T\,.$$

Proof. Define q by "anti-symmetrizing" q', namely,

$$q[a,b] := \tfrac{1}{2}\big(q'[a,b] - q'[b,a]\big)\,.$$

The first three properties follow directly from the definition of q and the fact that q' is quasigeodesic and G-equivariant. Now we prove property (4).

Let a, b, and c be arbitrary vertices in $\Gamma^{(0)}$, and z be a center of the triple $\{a,b,c\}$. Then, by Lemma 9, taking $a' := a$,

$$\begin{aligned}\big|q'[b,a]-q'[c,a]-q'[b,z]+q'[c,z]\big|_1 &\le K(1+\lambda+\lambda^2+\cdots+\lambda^{d(z,a)})\\ &\le K\sum_{i=0}^{\infty}\lambda^i=\frac{K}{1-\lambda}\,.\end{aligned}$$

The same argument for cyclic permutations of the vertices a, b, and c yields

$$\big|q'[c,b]-q'[a,b]-q'[c,z]+q'[a,z]\big|_1\le\frac{K}{1-\lambda}$$

and

$$\big|q'[a,c]-q'[b,c]-q'[a,z]+q'[b,z]\big|_1\le\frac{K}{1-\lambda}\,.$$

The three inequalities above provide just what is needed:

$$\begin{aligned}&\big|q[a,b]+q[b,c]+q[c,a]\big|_1\\ &\quad=\tfrac{1}{2}\big|(q'[a,b]-q'[b,a])+(q'[b,c]-q'[c,b])+(q'[c,a]-q'[a,c])\big|_1\\ &\quad=\tfrac{1}{2}\big|-(q'[b,a]-q'[c,a]-q'[b,z]+q'[c,z])\\ &\qquad-(q'[c,b]-q'[a,b]-q'[c,z]+q'[a,z])\\ &\qquad\quad-(q'[a,c]-q'[b,c]-q'[a,z]+q'[b,z])\big|_1\\ &\quad\le\tfrac{1}{2}\big(|q'[b,a]-q'[c,a]-q'[b,z]+q'[c,z]|_1\\ &\qquad+|q'[c,b]-q'[a,b]-q'[c,z]+q'[a,z]|_1\\ &\qquad\quad+|q'[a,c]-q'[b,c]-q'[a,z]+q'[b,z]|_1\big)\le\frac{1}{2}\cdot 3\frac{K}{1-\lambda}\,,\end{aligned}$$

so we put $T := \frac{3K}{2(1-\lambda)}$. Theorem 10 is proved. □

4 Bounded Cohomology

In this section we prove

Theorem 11. *Let G be a hyperbolic group and V be a bounded $\mathbb{Q}G$-module. Then the map $H_b^n(G,V)\to H^n(G,V)$ induced by inclusion is surjective for each $n\ge 2$.*

Let X be as in section 2 and Y be the geometric realization of the homogeneous bar-construction for G. This means that Y is the simplicial complex whose n-simplices are labeled by ordered $(n+1)$-tuples $(x_0,\dots,x_n)$ of elements of the group G, and the i-th face of $(x_0,\dots,x_n)$ is identified with the simplex labeled $(x_0,\dots,\widehat{x_i},\dots,x_n)$ in the obvious way. The action of G on Y is diagonal:

$$g \cdot (x_0, \dots, x_n) := (g \cdot x_0, \dots, g \cdot x_n).$$

Let $\mathcal{C}^X$ and $\mathcal{C}^Y$ be the augmented chain complexes of cellular $\mathbb{Q}$-chains in X and Y, respectively, i.e. $\mathcal{C}^X$ is

$$\dots \xrightarrow{\partial_3} C_2(X,\mathbb{Q}) \xrightarrow{\partial_2} C_1(X,\mathbb{Q}) \xrightarrow{\partial_1} C_0(X,\mathbb{Q}) \xrightarrow{\epsilon} \mathbb{Q} \longrightarrow 0$$

and $\mathcal{C}^Y$ is

$$\dots \xrightarrow{\partial_3} C_2(Y,\mathbb{Q}) \xrightarrow{\partial_2} C_1(Y,\mathbb{Q}) \xrightarrow{\partial_1} C_0(Y,\mathbb{Q}) \xrightarrow{\epsilon} \mathbb{Q} \longrightarrow 0,$$

where ϵ is the augmentation map taking each 0-chain to the sum of its coefficients. Both X and Y are contractible, hence $\mathcal{C}^X$ and $\mathcal{C}^Y$ are acyclic. Both $\mathcal{C}^X$ and $\mathcal{C}^Y$ have free $\mathbb{Q}G$-modules in each non-negative dimension (and those of $\mathcal{C}^X$ are finitely generated). Once again, $\mathcal{C}_n^X$ and $\mathcal{C}_n^Y$ are normed vector spaces with the ℓ_1-norm.

Proposition 12. *Given a hyperbolic group G and chain complexes $\mathcal{C}^X$ and $\mathcal{C}^Y$ as above, there exist G-equivariant chain maps $\varphi_* : \mathcal{C}^Y \to \mathcal{C}^X$ and $\psi_* : \mathcal{C}^X \to \mathcal{C}^Y$ such that*

(1) φ_* and ψ_* are identities in each negative dimension, and
(2) φ_* is bounded in each dimension at least 2.

Remark. The existence of φ_* and ψ_* satisfying condition (1) follows from standard arguments for *any* group G (see below). Since each $\mathcal{C}_n^X$ is finitely generated, it follows automatically that ψ_* is bounded in each dimension. Property (2) is what requires a new argument, and hyperbolicity of G is essential here. We give a formal homological proof, but the main idea is that, when G is hyperbolic, it is possible to represent n-simplices of the bar-construction, $n \geq 2$, as n-chains in X of bounded ℓ_1-norm.

Recall the following standard fact from homological algebra (see [B, Lemma I.7.4] for the proof).

Lemma 13. *Suppose that $(\mathcal{C}, \partial)$ is a chain complex having free modules in dimensions $n \geq 0$, $(\mathcal{C}', \partial')$ is an acyclic chain complex, and homomorphisms $\psi_n : \mathcal{C}_n \to \mathcal{C}'_n$ are defined for $n \leq -1$ such that $\partial'_n \circ \psi_n = \psi_{n-1} \circ \partial_n$ for each $n \leq -1$. Then the maps ψ_n extend to a chain map $\psi_* : \mathcal{C} \to \mathcal{C}'$. This extension is unique up to a chain homotopy.*

Of course, by a "module" here we mean a $\mathbb{Q}G$-module. The following theorem was proved in [M3] using [M2, Theorem 5.4].

Theorem 14. *Hyperbolic groups satisfy linear isoperimetric inequalities in each positive dimension (over $\mathbb{Q}$ and over $\mathbb{R}$). More precisely, if G is* *a hyperbolic group, X is the universal cover of a $K(G,1)$ complex with finitely many cells in each dimension, and i is a positive integer, then there exists a constant S_i such that, for any cellular i-cycle b in X, there exists an $(i+1)$-chain a with $\partial a = b$ and $|a|_1 \leq S_i |b|_1$.*

It was shown by S. Gersten [G] that, for finitely presented groups, linearity of the isoperimetric inequalities for 1-cycles is equivalent to hyperbolicity.

Proof of Proposition 12. Define φ_n and ψ_n to be the identity maps in all dimensions $n \leq -1$. Let ψ_* be an arbitrary extension of the maps ψ_n guaranteed by Lemma 13.

The chain map φ_* is constructed inductively on dimension as follows. $C_0(Y,\mathbb{Q})$ is isomorphic to $\mathbb{Q}G$, so we can define $\varphi_0 : C_0(Y,\mathbb{Q}) \to C_0(X,\mathbb{Q})$ by mapping the unit element of G to some vertex in X and extending by G-equivariance and by linearity over $\mathbb{Q}$. Define $\varphi_1 : C_1(Y,\mathbb{Q}) \to C_1(X,\mathbb{Q})$ on the 1-simplices (x_0, x_1) by

$$\varphi_1(x_0,x_1) := q\big[\varphi_0(x_0), \varphi_0(x_1)\big]\,,$$

and extending to $C_1(Y,\mathbb{Q})$ by linearity over $\mathbb{Q}$. In other words, each 1-simplex in Y maps to an element of the homological bicombing q from Theorem 10. Since q and φ_0 are G-equivariant, then φ_1 is a $\mathbb{Q}G$-module morphism. For each 2-simplex (x_0,x_1,x_2) of Y,

$$\begin{aligned}\varphi_1\big(\partial(x_0,x_1,x_2)\big) &= \varphi_1\big((x_1,x_2) - (x_0,x_2) + (x_0,x_1)\big)\\ &= q\big[\varphi_0(x_1),\varphi_0(x_2)\big] - q\big[\varphi_0(x_0),\varphi_0(x_2)\big] + q\big[\varphi_0(x_0),\varphi_0(x_1)\big]\\ &= q\big[\varphi_0(x_1),\varphi_0(x_2)\big] + q\big[\varphi_0(x_2),\varphi_0(x_0)\big] + q\big[\varphi_0(x_0),\varphi_0(x_1)\big]\,,\end{aligned}$$

because q is antisymmetric, hence, by Theorem 10(4),

$$\begin{aligned}&\big|\varphi_1(\partial(x_0,x_1,x_2))\big|_1\\ &\qquad \leq \big|q[\varphi_0(x_1),\varphi_0(x_2)] + q[\varphi_0(x_2),\varphi_0(x_0)] + q[\varphi_0(x_0),\varphi_0(x_1)]\big|_1 \leq T\,,\end{aligned}$$

where the constant T is independent of the choice of the triple (x_0,x_1,x_2). Since $\varphi_1(\partial(x_0,x_1,x_2))$ is a 1-cycle, then, by Theorem 14, there exists a 2-chain $c = c(x_0,x_1,x_2)$ in X with $\partial c = \varphi_1(\partial(x_0,x_1,x_2))$ and

$$|c|_1 \leq S_1 \big|\varphi_1(\partial(x_0,x_1,x_2))\big|_1 \leq S_1 \cdot T\,.$$

Since G acts freely on Y, this 2-chain $c(x_0,x_1,x_2)$ can be chosen G-equivariantly so that the map $\varphi_2 : C_2(Y,\mathbb{Q}) \to C_2(X,\mathbb{Q})$ defined by

$$\varphi_2(x_0,x_1,x_2) := c(x_0,x_1,x_2)$$

is a homomorphism of $\mathbb{Q}G$-modules and it is bounded, by the inequality above.

The further inductive steps are similar. If a $\mathbb{Q}G$-module morphism

$$\varphi_n : C_n(Y,\mathbb{Q}) \to C_n(X,\mathbb{Q})$$

is constructed for some $n \geq 2$ and has norm bounded by some constant R_n, then we define $\varphi_{n+1}(x_0,\dots,x_{n+1})$ to be an equivariant choice of a filling for the n-cycle $\varphi_n(\partial(x_0,\dots,x_{n+1}))$. Since

$$\begin{aligned}\big|\varphi_n(\partial(x_0,\dots,x_{n+1}))\big|_1 &= \Big|\sum_{i=0}^{n+1}(-1)^i\varphi_n(x_0,\dots,\widehat{x_i},\dots,x_{n+1})\Big|_1 \\ &\leq \sum_{i=0}^{n+1}\big|\varphi_n(x_0,\dots,\widehat{x_i},\dots,x_{n+1})\big|_1 \leq (n+2)R_n\,,\end{aligned}$$

then, by Theorem 14, the filling can be chosen to satisfy

$$\big|\varphi_{n+1}(x_0,\dots,x_{n+1})\big|_1 \leq S_n\big|\varphi_n(\partial(x_0,\dots,x_{n+1}))\big|_1 \leq S_n(n+2)R_n\,,$$

i.e. the norm of the map $\varphi_{n+1} : C_{n+1}(Y,\mathbb{Q}) \to C_{n+1}(X,\mathbb{Q})$ is bounded by

$$R_{n+1} := S_n(n+2)R_n\,.$$

One easily checks that the maps φ_n constructed this way form a chain map $\varphi_* : \mathcal{C}^Y \to \mathcal{C}^X$. Proposition 12 is proved. □

Proof of Theorem 11. Apply functor $Hom_{\mathbb{Q}G}(\cdot,V)$ to the chain maps ψ_* and φ_* from Proposition 12. With the notation

$$\begin{aligned}\mathcal{C}_X &:= Hom_{\mathbb{Q}G}(\mathcal{C}^X,V)\,, & \mathcal{C}_Y &:= Hom_{\mathbb{Q}G}(\mathcal{C}^Y,V)\,,\\ \varphi^* &:= Hom_{\mathbb{Q}G}(\varphi_*,V)\,, & \psi^* &:= Hom_{\mathbb{Q}G}(\psi_*,V)\,,\end{aligned}$$

we have two cochain complexes, $\mathcal{C}_X$ and $\mathcal{C}_Y$, and two cochain maps, $\varphi^* : \mathcal{C}_X \to \mathcal{C}_Y$ and $\psi^* : \mathcal{C}_Y \to \mathcal{C}_X$. In the positive dimensions, the homology of the cochain complexes $\mathcal{C}_X$ and $\mathcal{C}_Y$ is equal to the cohomology of G, $H^*(G,V)$, and in these dimensions both φ^* and ψ^* induce endomorphisms of $H^*(G,V)$.

By the definition of ψ_* and φ_*, the chain map $\psi_* \circ \varphi_* : \mathcal{C}^Y \to \mathcal{C}^Y$ and the identity chain map $id_* : \mathcal{C}^Y \to \mathcal{C}^Y$ coincide in each negative dimension, hence, by Lemma 13, they are chain homotopic. Hence their duals $\varphi^* \circ \psi^* : \mathcal{C}_Y \to \mathcal{C}_Y$ and $id^* : \mathcal{C}_Y \to \mathcal{C}_Y$ are chain homotopic, so the map $\varphi^* \circ \psi^*$ induces the identity map on $H^*(G,V)$ in each positive dimension.

Let $n \geq 2$. Given any element of $H^n(G,V)$, we represent it by an n-cocycle $\alpha \in \mathcal{C}_Y^n$. Then the cocycle

$$(\varphi^n \circ \psi^n)(\alpha) = \varphi^n(\psi^n(\alpha)) \in \mathcal{C}_Y^n$$

represents the same element of $H^n(G,V)$. It remains to show that $\varphi^n(\psi^n(\alpha))$ is bounded (with respect to the ℓ_∞-norm on $\mathcal{C}_Y^n$, or, equivalently, as a linear map $\mathcal{C}_n^Y \to V$).

$$\psi^n(\alpha) \in \mathcal{C}_X^n = Hom_{\mathbb{Q}G}(\mathcal{C}_n^X, V) = Hom_{\mathbb{Q}G}\big(C_n(X,\mathbb{Q}), V\big),$$

i.e. $\psi^n(\alpha)$ is a $\mathbb{Q}G$-module morphism $C_n(X,\mathbb{Q}) \to V$. Since V is a bounded $\mathbb{Q}G$-module, each G-orbit of n-cells in X is mapped by this morphism to a bounded set in V. There are only finitely many such orbits in X, hence $|\psi^n(\alpha)|_\infty < \infty$. Also, by Lemma 12(2), $|\varphi_n|_\infty < \infty$, hence

$$\big|\varphi^n(\psi^n(\alpha))\big|_\infty = \big|\psi^n(\alpha) \circ \varphi_n\big|_\infty \le |\psi^n(\alpha)|_\infty \cdot |\varphi_n|_\infty < \infty\,.$$

This shows that each element of $H^n(G,V)$, $n \ge 2$, is represented by a bounded cocycle in the bar-construction. Theorem 11 is proved. □

It was not needed for the proof, but (using the explicit cone-off procedure from [M2]) it is possible to refine the argument above so that each k-simplex σ in the bar-construction maps to a "quasi-straight" k-chain in X, in the sense that the support of this k-chain lies uniformly close to a union of geodesics connecting the images of the vertices of σ. This is a combinatorial analogue of the fact that straight simplices in $\mathbb{H}^n$ lie close to their 1-skeleta.

5 Abelian Groups as Coefficients

In this section we deduce surjectivity for finitely generated abelian coefficients. The argument is similar in spirit to the one by W.D. Neumann and L. Reeves [NR].

For an arbitrary abelian group A, it is not clear what the definition of the bounded cohomology, $H_b^*(G,A)$, should be. One may use a "norm" $|\cdot|$ on A to make sense of boundedness for cochains (in the bar-construction). In this case either one calls a function $|\cdot| : A \to \mathbb{R} \cup \{\infty\}$ a *norm* if

- $|a| = 0$ iff $a = 0$, and
- $|a+b| \le |a| + |b|$ for all $a, b \in A$,

or one requires an additional condition that

- $|na| = |n| \cdot |a|$ for all $n \in \mathbb{Z}$ and $a \in A$.

There is a third possible definition. Given an arbitrary abelian group A, one may call an n-cochain (in the bar-construction) with coefficients in A *bounded* if it takes only finitely many values on n-simplices. Then the set of bounded cochains has a $\mathbb{Z}G$-module structure and it defines the bounded cohomology of G with coefficients in A. (Unfortunately, this makes the notation $H_b^*(G,V)$ ambiguous, because one needs to say whether V is a vector space or it is only viewed as an abelian group.)

Whatever definition we accept, the surjectivity statement always holds for *finitely generated* coefficients:

Theorem 15. *Let G be a hyperbolic group and A be a finitely generated abelian group. Then the map $H^n_b(G,A) \to H^n(G,A)$ induced by inclusion is surjective for each $n \geq 2$.*

Proof. Let $n \geq 2$. First we prove the theorem in the special case $A = \mathbb{Z}$. Pick any element of $H^n(G,\mathbb{Z})$ and represent it by a(n integer-valued G-invariant) cocycle α in the bar-construction. This cocycle can be viewed as a real-valued cocycle, so, taking $V := \mathbb{R}$ in Theorem 11,

$$\alpha = \delta\beta + \alpha',$$

where $\delta\beta$ is the coboundary of a real-valued $(n-1)$-cochain β, and α' is a bounded real-valued n-cocycle. Both β and α' are G-invariant. Let $int(\beta)$ be the $(n-1)$-cochain whose value on a simplex σ is the integer part of $\beta(\sigma)$. Then $int(\beta)$ is G-invariant. We have

$$\alpha - \delta\big(int(\beta)\big) = \delta\big(\beta - int(\beta)\big) + \alpha'.$$

The left-hand side of this equality is integer-valued and the right-hand side is bounded (with respect to the usual norm on $\mathbb{R}$), hence the cochain $\alpha - \delta\big(int(\beta)\big)$ is a bounded integer-valued cocycle (i.e. it takes only finitely many values in $\mathbb{Z}$). Obviously, it represents the same cohomology class in $H^n(G,\mathbb{Z})$ as α.

Now assume that A is any finitely generated abelian group. In this case $A = A_0 \oplus \mathbb{Z}^m$, where A_0 is a finite abelian group. According to this decomposition each cocycle $\alpha \in C^n(Y,A)$ decomposes as a sum of integer-valued *cocycles* $\alpha = \alpha_0 + \alpha_1 + ... + \alpha_m$. Obviously, α_0 takes only finitely many values. By case $A = \mathbb{Z}$, each component α_k, $k \geq 1$, can be replaced by a cohomologous cocycle α'_k which takes only finitely many values. Then the cocycle $\alpha_0 + \alpha'_1 + ... + \alpha'_m$ is cohomologous to α and it takes finitely many values, therefore it is bounded (in any sense). □

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