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HYPERBOLIC KÄHLER MANIFOLDS AND PROPER HOLOMORPHIC MAPPINGS TO RIEMANN SURFACES

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0 Introduction

In [Gro2], Gromov proved that a connected noncompact complex manifold which is a Galois covering of a compact Kähler manifold and which has infinitely many ends admits a proper holomorphic mapping to a Riemann surface. In fact, by [NaR1], any complete Kähler manifold which has at least three ends and which has bounded geometry or is weakly 1-complete admits such a mapping to a Riemann surface. This is also the case for a bounded geometry or weakly 1-complete manifold which has two ends and admits a nonconstant holomorphic function.

The main goal of this paper is the following:

Theorem 0.1. *Let M be a connected noncompact complete Kähler manifold which does not have exactly two ends and which admits a positive Green's function that vanishes at infinity. Then M admits a proper holomorphic mapping to a Riemann surface if and only if $H^1_c(M, \mathcal{O}) \neq 0$.*

Depending on the context, by an *end* of a connected manifold X we will mean either a connected component of $X \setminus K$ with noncompact closure, where K is some compact subset of X, or an element of

$$\lim_{\leftarrow} \pi_0(X \setminus K)\,,$$

where the limit is taken as K ranges over the compact subsets of X. In the latter context, we will denote the number of ends by $e(X)$.

A noncompact complex manifold M for which $H^1_c(M, \mathcal{O}) = 0$ is said to have the *Bochner–Hartogs property* (see Hartogs [H], Bochner [Bo], and Harvey and Lawson [HL]). Equivalently, for every C^∞ compactly supported

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form α of type $(0,1)$ with $\bar{\partial}\alpha = 0$ on M, there is a C^∞ compactly supported function β on M such that $\bar{\partial}\beta = \alpha$. If M has the Bochner-Hartogs property, then every holomorphic function on a neighborhood of infinity with no relatively compact connected components extends to a holomorphic function on M. For, cutting off away from infinity, one gets a C^∞ function λ on M. Taking $\alpha = \bar{\partial}\lambda$ and forming β as above, the function $\lambda - \beta$ will be the desired extension. In particular, $e(M) = 1$. Thus, in a sense, the space $H^1_c(M, \mathcal{O})$ is a function theoretic approximation of the set of (topological) ends of M. An open Riemann surface X (as well as any complex manifold which admits a proper holomorphic mapping onto X) cannot have the Bochner-Hartogs property, because X admits nonconstant meromorphic functions with finitely many poles. Examples of manifolds of dimension n which have the Bochner-Hartogs property include strongly $(n-1)$-complete complex manifolds (Andreotti and Vesentini [AV]) and strongly hyper-$(n-1)$-convex Kähler manifolds (Grauert and Riemenschneider [GrR]).

We will also obtain the following as a consequence of Theorem 0.1:

Theorem 0.2. *Let M be a compact Kähler manifold and let $\pi : \widetilde{M} \to M$ be a connected infinite Galois covering manifold which does not have exactly two ends. Then $\widetilde{M}$ admits a proper holomorphic mapping to a Riemann surface if and only if $H^1_c(\widetilde{M}, \mathcal{O}) \neq 0$.*

For a connected infinite Galois covering $\widetilde{M}$ of a compact Kähler manifold M, $e(\widetilde{M}) = 1$, 2, or ∞ and $e(\widetilde{M}) = 2$ if and only if the covering group $\Gamma = \pi_1(M)/\pi_1(\widetilde{M})$ contains an infinite cyclic subgroup of finite index (see Cohen [Co] or Scott and Wall [SW]). Cousin's [Cou] example of a $\mathbb{Z}$-covering of an Abelian variety which has no nonconstant holomorphic functions shows that Theorem 0.2 fails when $e(\widetilde{M}) = 2$. By a result of Arapura, Bressler, and the second author [ArBR], the universal covering of a compact Kähler manifold has at most one end. In fact, as shown in [NaR1], any complete noncompact connected Kähler manifold M which satisfies $H^1(M, \mathbb{R}) = 0$ and which has bounded geometry or is weakly 1-complete has exactly one end.

In [R], the second author proved Theorem 0.2 for $\widetilde{M}$ the universal covering, or a Galois covering with infinite covering group of more than quadratic growth, assuming that $\widetilde{M}$ admits a nonconstant holomorphic function (one goal of this paper is to remove the rather strong hypothesis that $\mathcal{O}(\widetilde{M}) \neq \mathbb{C}$). In [NaR2], the conclusion of Theorem 0.1 was shown to hold for M a connected noncompact weakly 1-complete Kähler manifold with exactly one end. This too may be considered to be a consequence of

Theorem 0.1, because one may apply (the proof of) Theorem 0.1 to the sublevels of the exhaustion function. Finally, it should be remarked that Theorem 0.2 may be relevant to the conjecture of Shafarevich that the universal covering of a smooth projective variety is holomorphically convex. For any holomorphically convex complex manifold either has the Bochner-Hartogs property or admits a proper holomorphic mapping to a Riemann surface.

The paper is organized as follows. Some known facts required for the proof of Theorem 0.1 are collected in sect. 1. With the exception of a required technical lemma which is proved in sect. 3, the proof of Theorem 0.1 is contained in sect. 2. Theorem 0.2 is considered in sect. 4. Moreover, by general principles, the mapping on the covering space $\widetilde{M}$ in Theorem 0.2 determines a surjective holomorphic mapping of some finite covering of the base manifold onto a compact Riemann surface. This observation and an observation for the case in which $e(\widetilde{M}) = 2$ are also considered in sect. 4. In particular, we get the following:

COROLLARY 0.3. *Let M be a connected compact Kähler manifold. Assume that there exists a connected infinite Galois covering manifold $\pi : \widetilde{M} \to M$ which satisfies one of the following:*

(i) $e(\widetilde{M}) = 1$ *and* $H^1_c(M, \mathcal{O}) \neq 0$,

(ii) $e(\widetilde{M}) = 2$ *and* $\pi_1(\widetilde{M})$ *is not finitely generated, or*

(iii) $e(\widetilde{M}) = \infty$.

Then some finite covering of M admits a surjective holomorphic mapping onto a Riemann surface.

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1 Preliminary Facts

The main known facts required for the proof of Theorem 0.1 are collected in this section. We begin with some terminology and facts from potential theory and pluripotential theory. The *Levi form* of a real-valued function φ of class C^2 on a complex manifold M is the Hermitian form

$$\mathcal{L}(\varphi) = \sum_{i,j} \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} dz_i \, d\bar{z}_j \, .$$

The function φ is called *plurisubharmonic* (*strictly plurisubharmonic*, *pluriharmonic*) if $\mathcal{L}(\varphi) \geq 0$ (respectively, $\mathcal{L}(\varphi) > 0$, $\mathcal{L}(\varphi) = 0$). A complex-valued function is called *pluriharmonic* if its real and imaginary parts are pluriharmonic. The complex manifold M is called *weakly* 1-*complete* if there exists a continuous plurisubharmonic exhaustion function $\varphi : M \to \mathbb{R}$ (i.e. φ is locally the limit of a nonincreasing sequence of C^∞ plurisubharmonic functions and the sublevel $\{ x \in M \mid \varphi(x) < a \}$ is relatively compact in M for each $a \in \mathbb{R}$).

A connected noncompact Riemannian manifold N which admits a positive symmetric Green's function $G(x,y)$ is said to be *hyperbolic*; otherwise, N is called *parabolic*. Equivalently, N is hyperbolic if, given a relatively compact C^∞ domain Ω for which no connected component of $N \setminus \Omega$ is compact, there is a connected component E of $N \setminus \overline{\Omega}$ and a (unique) C^∞ function $u_E : \overline{E} \to [0,1)$ such that u_E is harmonic on E, $u_E = 0$ on ∂E, and $\limsup_{x\to\infty} u_E(x) = 1$. We will also say that the end E is *hyperbolic*. An end E which is not hyperbolic is called *parabolic* and we set $u_E = 0$. We call the function $u : N \setminus \Omega \to [0,1)$ defined by $u|_{\overline{E}} = u_E$, for each connected component E of $N \setminus \overline{\Omega}$, the *harmonic measure of the ideal boundary of* N *with respect to* $N \setminus \overline{\Omega}$. We normalize the Green's function G so that, for each point $x_0 \in N$,

$$\Delta_{\text{distr.}} G(\cdot, x_0) = -\delta_{x_0} \, ,$$

where δ_{x_0} is the Dirac function at x_0 and $\Delta = -(d^*d + dd^*)$ is the Laplacian. We will use the same notation for the corresponding integral operator G given by

$$(G\alpha)(x) = \int_N G(x,y)\alpha(y)\, dV(y) \quad \forall\, x \in N$$

for each suitable function α on N.

If α is a C^∞ compactly supported function, then $\beta \equiv -G\alpha$ is a C^∞ bounded function with finite energy (i.e. $\int_N |\nabla\beta|^2 \, dV < \infty$) and $\Delta\beta = \alpha$. Moreover, $\beta(x_\nu) \to 0$ if $\{x_\nu\}$ is a sequence in N with $x_\nu \to \infty$ and $G(\cdot, x_\nu) \to 0$ (equivalently, $u(x_\nu) \to 1$). Such a sequence $\{x_\nu\}$ always exists (for N hyperbolic) and will be called a *regular sequence*.

As in [Gro3], [ArBR], [R], [NaR1], and [NaR2], a holomorphic mapping to $\mathbb{P}^1$ will be obtained as a quotient of holomorphic 1-forms (see Lemma 1.4). The main tool for producing such holomorphic 1-forms from a $\bar\partial$-closed compactly supported form will be the following (as in [R] and [NaR2]):

Lemma 1.1. *Let (M,g) be a connected noncompact complete hyperbolic Kähler manifold and let α be a C^∞ compactly supported form of type $(0,1)$ on M such that $\bar\partial\alpha = 0$. Then there exist an L^2 harmonic form γ of type $(0,1)$ which is closed and coclosed and a C^∞ bounded function $\beta : M \to \mathbb{C}$ with finite energy such that $\gamma = \alpha - \bar\partial\beta$ and $\beta(x_\nu) \to 0$ for every regular* *sequence $\{x_\nu\}$ in M.*

Remark. In particular, $\bar{\gamma}$ is a holomorphic 1-form on M and β is pluriharmonic on the complement of the support of α.

Proof of Lemma 1.1. Let G denote the Green's function on M as well as the associated integral operator. Since $\bar{\partial}^*\alpha$ is a C^∞ function with compact support on M, the function

$$\beta \equiv 2G(\bar{\partial}^*\alpha)$$

is a C^∞ bounded function with finite energy which vanishes at infinity along any regular sequence and which satisfies $\bar{\partial}^*\bar{\partial}\beta = -\frac{1}{2}\Delta\beta = \bar{\partial}^*\alpha$. The form

$$\gamma \equiv \alpha - \bar{\partial}\beta$$

is therefore L^2 and harmonic, and the Gaffney theorem [G] implies that γ is closed and coclosed. □

The next lemma is a special case of [NaR1, Theorem 2.6] and is contained implicitly in the work of Sario, Nakai, and their collaborators (see also Li [L] and Li and Tam [LT]). It will be applied in the proof of Theorem 0.1 in the same way it was applied in [NaR2] (i.e. to the case in which the function β in Lemma 1.1 is real-valued).

Lemma 1.2. *Let (M, g) be a connected noncompact complete hyperbolic Kähler manifold with Green's function G, let K be a compact subset of M, and let E be a connected component of $M \setminus K$ with noncompact closure. Assume that G vanishes at infinity along the end E and that $M \setminus E$ contains a hyperbolic end (in particular, $e(M) > 1$). Then there exists a real-valued pluriharmonic function ρ on M such that ρ has finite energy, $0 < \rho < 1$ on M, $\lim_{x\to\infty} \rho|_{\overline{E}}(x) = 1$, and $\lim_{\nu\to\infty} \rho(x_\nu) = 0$ for any regular sequence $\{x_\nu\}$ in $M \setminus E$.*

Sketch of the proof. Clearly, we may assume that the end E is smooth at its compact boundary ∂E. By choosing a C^∞ nondecreasing convex function $\chi : (0,1) \to (0,1)$ which vanishes near 0, which is linear near 1, and which approaches 1 at 1, and by forming the composition with the harmonic measure of the ideal boundary of M with respect to E and extending by 0, one gets a C^∞ subharmonic function $\eta : M \to [0,1)$ which vanishes on $M \setminus E$, which is harmonic on the end E outside a relatively compact neighborhood of ∂E, and which approaches 1 at infinity along E. The function $\rho = \eta + G\Delta\eta$ is harmonic with finite energy, and is therefore pluriharmonic by the Gaffney theorem [G]. Moreover, ρ has the required limits at infinity because $G\Delta\eta$ approaches zero along any regular sequence.

We also have $\rho \geq \eta \geq 0$ and, by forming an exhaustion of M by C^∞ relatively compact domains and writing G as the limit of the corresponding sequence of Green's function, one sees that $\rho \leq 1$. Therefore, by the maximum principle, we have $0 < \rho < 1$. □

The following useful observation is due to J. Wang (it will rule out the case $\gamma = 0$ when Lemma 1.1 is applied in the proof of Theorem 0.1):

LEMMA 1.3 (Wang). *Let M be a connected noncompact Kähler manifold. Suppose that, for some compact subset K of M, there exists a holomorphic function f on the complement $M \setminus K$ which vanishes at infinity but is not identically zero on any open subset. Then M is parabolic.*

Proof. The function $\varphi = -\log|f|^2$ is superharmonic on $M \setminus K$ and $\varphi(x) \to \infty$ as $x \to \infty$ in M. In particular, we may assume that φ is positive. Fix a C^∞ relatively compact domain Ω in M such that $K \subset \Omega$ and such that each connected component of $M \setminus \overline{\Omega}$ has noncompact closure. Let u be the harmonic measure of the ideal boundary of M with respect to $M \setminus \overline{\Omega}$. Given $\epsilon > 0$, we have $\epsilon\varphi > 0 = u$ on $\partial\Omega$ and $\epsilon\varphi > 1 > u$ on the complement of a sufficiently large compact subset of M. Therefore $0 \leq u < \epsilon\varphi$ on $M \setminus \Omega$ for every $\epsilon > 0$ and hence $u \equiv 0$. □

The next lemma will be used for the production of a holomorphic mapping to $\mathbb{P}^1$ (as in [Gro3], [ArBR], [NaR1], and [NaR2]).

LEMMA 1.4. *If θ_1 and θ_2 are two linearly independent closed holomorphic 1-forms satisfying $\theta_1 \wedge \theta_2 \equiv 0$ on a connected complex manifold M, then the meromorphic function $f \equiv \theta_1/\theta_2$ has no points of indeterminacy in M. In other words, f is a holomorphic mapping of M into $\mathbb{P}^1$.*

REMARKS. 1. One can prove this by considering holomorphic equivalence relations (for an elementary proof, see [NaR2, Lemma 2.2]).

2. Let $S_i \equiv \{ x \in M \mid (\theta_i)_x = 0 \}$ for $i = 1, 2$. Then f is locally constant on the analytic set $S_1 \cup S_2$ (see [NaR2]).

3. If f is nonconstant and $A = S_1 \cap S_2$, then the levels of $f|_{M\setminus A}$ are precisely the (smooth) leaves of the holomorphic foliation determined by θ_1 and θ_2 on $M \setminus A$. For, on a neighborhood U of a point at which $\theta_2 \neq 0$, we have local coordinates $z = (z_1, \dots, z_n)$ in which $\theta_2 = dz_1$ and $f = f(z_1)$. Hence the levels of $f|_U$ are smooth coordinate slices which are leaves of the foliation in U. A similar argument applies when $\theta_1 \neq 0$ and the claim follows. In particular, if L is a level of f which does not meet A (for example, if L lies over a point not in the countable set $f(A)$), then L is precisely a leaf of the foliation.

4. If f is nowhere locally constant and, for $i = 1, 2$, $\theta_i = \partial\rho_i$, where ρ_i is a *real-valued* pluriharmonic function on M, then ρ_i is constant on each level L of f. For ρ_i is clearly locally constant on the zero set of θ_i (since $d\rho_i = 0$ at each point in this set), and, outside this set, ρ_i is constant on the leaves of the foliation described above and these leaves locally agree with levels of f.

After Lemma 1.4 gives a mapping of an open set to $\mathbb{P}^1$, the following lemma, which is essentially contained in [NaR1], will yield a proper mapping to a Riemann surface:

LEMMA 1.5. *Let (M, g) be a connected noncompact complete hyperbolic Kähler manifold. Assume that the Green's function vanishes at infinity and that there exists an open subset $U \subset M$ and a holomorphic mapping $f : U \to \mathbb{P}^1$ with at least one (nonempty) compact level. Then there exists a proper holomorphic mapping of M onto a Riemann surface.*

Sketch of the proof. By hypothesis, the set V of points which lie in a compact level of f is nonempty. This set is open for point-set topological reasons. Applying Stein factorization [St], one gets a Riemann surface X and a proper surjective holomorphic mapping $\Phi : V \to X$ whose fibers are precisely the compact levels of f.

We now apply arguments from the proof of [NaR1, Theorem 4.6]. By replacing X by a small coordinate neighborhood of some regular value of Φ, we may assume that X is a region in the plane and Φ is a proper holomorphic submersion with connected fibers. Let D_1 and D_2 be two disks which are relatively compact in X and which have disjoint closures. Then the domain $N \equiv M \setminus \Phi^{-1}(\overline{D}_1 \cup \overline{D}_2)$ has at least three ends. Moreover, N admits a complete Kähler metric for which there is a positive Green's function which vanishes at infinity in M and at $\partial N = \Phi^{-1}(\partial D_1 \cup \partial D_2)$. Therefore, by Lemma 1.2, there exists a pluriharmonic function $\rho : N \to (0, 1)$ which approaches 1 at infinity in M and 0 at ∂N. Thus $-\log\rho - \log(1 - \rho)$ is a C^∞ plurisubharmonic exhaustion function on N. The main result of [NaR1] now implies that there exists a proper holomorphic mapping Ψ of N onto a Riemann surface Y; again, with connected fibers. The maps Φ and Ψ determine a proper holomorphic mapping of M onto the Riemann surface Z obtained by identifying each point x in $X \setminus (\overline{D}_1 \cup \overline{D}_2)$ with the point $\Psi(\Phi^{-1}(x))$ in Y. □

REMARKS. 1. The results of [NaR1] together with the above arguments imply that the lemma also holds if, in place of the existence of a positive Green's function which vanishes at infinity, one assumes that M has bounded geometry or M is weakly 1-complete.

2. The weakly 1-complete case is a special case of a theorem of Nishino [Ni] who proved it without the assumption that M is Kähler.

3. For the proof of the above, one may instead apply the theory of Barlet spaces [B], because the graph over some 1-dimensional irreducible component of the Barlet space of compact analytic $(\dim M - 1)$-cycles will map properly and surjectively onto M.

2 Proof of the Main Result

The proof of Theorem 0.1 will require the following two lemmas which may be thought of as versions of Gromov's cup product lemma [Gro3]:

LEMMA 2.1. *Suppose M is a connected noncompact complex manifold of dimension n, ρ and τ are two real-valued pluriharmonic functions on M, and ρ has a compact fiber F. Then $\partial\rho \wedge \partial\tau \equiv 0$. In fact, if the differentials $d\rho$ and $d\tau$ are linearly independent for some such pair of functions, then there exists a surjective proper holomorphic mapping of some open subset of M onto a Riemann surface.*

REMARK. Two real-valued pluriharmonic functions ρ and τ have linearly dependent differentials on a connected complex manifold M (i.e. the functions 1, ρ, and τ are linearly dependent) if and only if $d\rho \wedge d\tau \equiv 0$.

LEMMA 2.2. *Let (M, g) be a connected noncompact complete hyperbolic Kähler manifold of dimension n with exactly one end, let $K \subset M$ be a compact subset with connected complement $E = M \setminus K$, and let ρ_1 and ρ_2 be real-valued pluriharmonic functions on E which vanish at infinity in M. Then $\partial\rho_1 \wedge \partial\rho_2 \equiv 0$.*

REMARK. Lemma 2.2 also holds if M is weakly 1-complete or M has bounded geometry.

The proof of Lemma 2.1 is given below. The proof of Lemma 2.2 is slightly technical and will be postponed until the next section.

Proof of Lemma 2.1. Since the image of the nowhere dense complex analytic set

$$\{\, x \in M \mid (d\rho)_x = 0 \,\} = \{\, x \in M \mid (\partial\rho)_x = 0 \,\}$$

under ρ is countable and since the restriction of ρ to the complement is an open mapping, there exist points arbitrarily close to the compact fiber

F which lie over regular values of ρ. Moreover, levels of ρ through such points near F are compact. Thus ρ has a compact level N which lies over a regular value. The restriction of τ to N assumes its maximum at some point $x_0 \in N$ and, since ρ is real-valued, the leaf L of the foliation determined by $\partial\rho$ through x_0 is contained in N. Therefore $\tau|_L$ attains its maximum at x_0 and hence $\tau \equiv \tau(x_0)$ on L. Thus $\partial\rho = \partial\tau = 0$ on $T^{1,0}L$ and hence, since $\dim L = n-1$, we have $\partial\rho \wedge \partial\tau = 0$ at each point in L.

Now if the analytic set $A = \{\, x \in M \mid (\partial\rho \wedge \partial\tau)_x = 0 \,\}$ is nowhere dense in M, then L contains some connected open subset V of A_{reg} (again, since $\dim L = n-1$) and, therefore, the pluriharmonic function ρ is constant on the irreducible component A_1 containing V. Thus $A_1 \subset N$ and hence A_1 is compact. On the other hand, ρ is constant on every compact irreducible component of A, so we may choose the level N so as to avoid the compact irreducible components of A. Thus we have arrived at a contradiction and we may conclude that $\partial\rho \wedge \partial\tau \equiv 0$ on M as claimed.

Suppose now that $d\rho$ and $d\tau$ are linearly independent. Then, by Lemma 1.4, the quotient $\partial\rho/\partial\tau$ determines a holomorphic mapping $f : M \to \mathbb{P}^1$. If f is nonconstant, then ρ is constant on the level of f through x_0. It follows that this level is compact and hence that the set of points which lie in a compact level of f is a nonempty open subset of M which admits a proper holomorphic mapping onto a Riemann surface. If f is constant, then $\partial\rho$ and $\partial\tau$ are linearly dependent and therefore, for some nonzero constant $a \in \mathbb{C}$, the function $h \equiv \rho + a\tau$ is holomorphic and nonconstant on M. Since $\tau = (\operatorname{Im} a)^{-1}(\operatorname{Im} h)$ and $\rho = h - a\tau$ are constant on the level of h through x_0, we again get the required proper holomorphic mapping to a Riemann surface on some open set. □

For the proof of Theorem 0.1, we will consider the set up $\gamma = \alpha - \bar\partial\beta$ as in Lemma 1.1. The case in which β is real-valued is contained in the next lemma. This lemma was essentially proved in [NaR2, pp. 1361–1363], but, for the convenience of the reader, the proof is reproduced here.

LEMMA 2.3. *Let (M, g) be a connected noncompact hyperbolic complete Kähler manifold. Assume that the Green's function G vanishes at infinity and that there exist a compact subset K with connected complement $E = M \setminus K$, a nonzero closed harmonic form γ of type $(0,1)$ on M, and a real-valued pluriharmonic function τ on E such that*

$$\gamma|_E = -\bar\partial\tau \quad \text{and} \quad \tau \to 0 \text{ at infinity in } M\,.$$

Then M admits a proper holomorphic mapping to a Riemann surface.

Proof. Fix a point $x_0 \in E$ and let Γ be the image of $\pi_1(E, x_0)$ in $\pi_1(M, x_0)$. Then Γ is a proper subgroup. For if $\Gamma = \pi_1(M, x_0)$, then, since the C^∞ closed real 1-form $\theta = -\gamma - \bar{\gamma}$ on M is equal to $d\tau$ on E, we get a well-defined extension of τ to a C^∞ function τ_0 on M by setting

$$\tau_0(x) = \tau(x_0) + \int_{x_0}^{x} \theta \quad \forall\, x \in M\,.$$

Moreover, τ_0 is pluriharmonic because $d\tau_0 = \theta$ and hence $\partial\bar{\partial}\tau_0 = -\partial\gamma = 0$. On the other hand, $\tau = \tau_0|_E$ vanishes at infinity in M, so we must have $\tau_0 \equiv 0$. Since γ is not everywhere 0, this is impossible. Thus $\Gamma \neq \pi_1(M, x_0)$.

Now let $\pi : \widetilde{M} \to M$ be a connected covering space with $\pi_*(\pi_1(\widetilde{M}, x_1)) = \Gamma$ for some point $x_1 \in \pi^{-1}(x_0)$. Clearly, we may assume that E is smooth at its boundary and hence that π maps a neighborhood of the closure $\overline{E}_1$ of the connected component E_1 of $\widetilde{E} = \pi^{-1}(E)$ containing x_1 isomorphically onto a neighborhood of $\overline{E}$. In particular, E_1 is a hyperbolic end of $\widetilde{M}$ with respect to the complete Kähler metric $\tilde{g} = \pi^* g$ and the Green's function $\widetilde{G}$ on $\widetilde{M}$ vanishes at infinity along E_1. Since Γ is a proper subgroup, $\widetilde{M} \setminus E_1$ is noncompact (i.e. $e(\widetilde{M}) > 1$). Moreover, the lifting to $\widetilde{M}$ of a negative subharmonic function on M which vanishes at infinity must vanish at infinity along some sequence in $\widetilde{M} \setminus E_1$. So $\widetilde{M} \setminus E_1$ must contain a hyperbolic end. Therefore, by Lemma 1.2, there exists a pluriharmonic function $\tilde{\rho}$ on $\widetilde{M}$ such that $0 < \tilde{\rho} < 1$, $\tilde{\rho}$ has finite energy, $\lim_{x\to\infty} \tilde{\rho}|_{\overline{E}_1}(x) = 1$, and $\lim_{\nu\to\infty} \tilde{\rho}(x_\nu) = 0$ for any regular sequence $\{x_\nu\}$ in $\widetilde{M} \setminus E_1$. Since π maps E_1 isomorphically onto E, the restriction $\tilde{\rho}|_{E_1}$ determines a pluriharmonic function ρ on E such that $0 < \rho < 1$ and $\rho \to 1$ at infinity.

Because ρ has a compact fiber, Lemma 2.1 and Lemma 1.5 imply that it suffices to show that the differentials $d\rho$ and $d\tau$ are linearly independent. If this is not the case, then there exist real constants r and s such that $\tau = r\rho + s$ on E. The lifting $\tilde{\gamma} = \pi^*\gamma$ of γ is then a closed form of type $(0,1)$ on $\widetilde{M}$ which is equal to the form $-\bar{\partial}(r\tilde{\rho} + s)$ on the nonempty open set E_1 and hence on the entire manifold $\widetilde{M}$. Therefore, on the nonempty open set $\widetilde{E} \setminus \overline{E}_1$, we have

$$-\bar{\partial}(\tau \circ \pi) = \tilde{\gamma} = -\bar{\partial}(r\tilde{\rho} + s)\,.$$

Hence the restriction of the function $(\tau \circ \pi) - (r\tilde{\rho} + s)$ to $\widetilde{E} \setminus \overline{E}_1$ is real-valued and holomorphic and is therefore locally constant. Thus if E_2 is a connected component of $\widetilde{E}$ which is not equal to E_1, then, for some real constant s', we have $\tau \circ \pi = r\tilde{\rho} + s'$ on E_2. Now since $\pi(E_1) = \pi(E_2) = E$, we may choose a sequence $\{x_\nu\}$ in E with $x_\nu \to \infty$ in M and sequences $\{y_\nu\}$ and $\{z_\nu\}$ in E_1 and E_2, respectively, such that $\pi(y_\nu) = \pi(z_\nu) = x_\nu$ for each ν. The sequences $\{y_\nu\}$ and $\{z_\nu\}$ are then regular sequences in $\widetilde{M}$, because the lifting v of the function $-G(x_0,\cdot)$ to $\widetilde{M}$ is a negative subharmonic function and $v(y_\nu), v(z_\nu) \to 0$. Therefore $\tilde{\rho}(y_\nu) \to 1$ and $\tilde{\rho}(z_\nu) \to 0$. Since τ vanishes at infinity, we get

$$\begin{aligned} 0 &= \lim \tau(x_\nu) = \lim \big(r\tilde{\rho}(y_\nu) + s\big) = r + s \\ \text{and}\quad 0 &= \lim \tau(x_\nu) = \lim \big(r\tilde{\rho}(z_\nu) + s'\big) = s'\,. \end{aligned}$$

Therefore, for each point $x \in E$ and each pair of points $y \in E_1 \cap \pi^{-1}(x)$ and $z \in E_2 \cap \pi^{-1}(x)$, we have $\tilde{\rho}(y) - 1 = r^{-1}\tau(x) = \tilde{\rho}(z)$. But this is impossible because $0 < \tilde{\rho} < 1$ on $\widetilde{M}$. Thus $d\rho$ and $d\tau$ are linearly independent and the lemma follows. □

Proof of Theorem 0.1. If $e(M) \geq 3$, then we may proceed as in the proof of Lemma 1.5. More precisely, there exists a smooth relatively compact domain Ω in M such that each of the (finitely many) connected components $E_1, \dots, E_m$ of $M \setminus \overline{\Omega}$ has noncompact closure and such that $m \geq 3$. By Lemma 1.2, there exists a pluriharmonic function $\xi : M \to (0,1)$ which approaches 1 at infinity along E_1 and 0 at infinity along $E_2, \dots, E_m$. Therefore, the function $-\log(1-\xi) - \log\xi$ is a C^∞ plurisubharmonic exhaustion function on M and hence, by the main result of [NaR1], M admits a proper holomorphic mapping onto a Riemann surface.

Assume now that $e(M) = 1$ and $H^1_c(M, \mathcal{O}) \neq 0$. Then there is a C^∞ compactly supported form α of type $(0,1)$ on M such that $\bar{\partial}\alpha = 0$ and such that $\alpha \notin \bar{\partial}C^\infty_c(M)$. We may fix a connected compact set K such that the complement $E = M \setminus K$ is connected and such that $\alpha \equiv 0$ on E. Applying Lemma 1.1, we get a (closed and coclosed) L^2 harmonic form γ of type $(0,1)$ on M and a C^∞ bounded function $\beta : M \to \mathbb{C}$ with finite energy such that

$$\gamma = \alpha - \bar{\partial}\beta \quad \text{and} \quad \beta \to 0 \text{ at infinity}\,.$$

In particular, $\bar{\gamma}$ is a holomorphic 1-form on M and β is pluriharmonic on the end E. Observe also that γ is not identically zero on M. For if this were the case, then β would be holomorphic on E and therefore, since $e(M) = 1$ and M is hyperbolic, Lemma 1.3 would imply that β has compact support; which contradicts the choice of α.

We have $\beta|_E = \rho_1 + i\rho_2$, where ρ_1 and ρ_2 are real-valued pluriharmonic functions on the connected set E which vanish at infinity in M. We may assume that ρ_1 and ρ_2 are linearly independent (i.e. that $d\rho_1$ and $d\rho_2$ are linearly independent). For if this is not the case, then a suitable nonzero complex multiple of $\beta|_E$ is real-valued and Lemma 2.3 applies.

By Lemma 2.2, we have $\partial\rho_1 \wedge \partial\rho_2 \equiv 0$ on E. Therefore (for ρ_1 and ρ_2 linearly independent), Lemma 1.4 implies that the quotient $\partial\rho_1/\partial\rho_2$ determines a holomorphic mapping

$$f : E \to \mathbb{P}^1 .$$

This mapping is nonconstant. For if f were equal to a constant $a \in \mathbb{C}\setminus\{0\}$, then the function $\rho_1 - \bar{a}\rho_2$ would be a nonconstant holomorphic function on E which vanishes at infinity (in M). Since M is hyperbolic, this would contradict Lemma 1.3.

According to Lemma 1.5, it suffices to show that f has at least one compact level. As a first step, we will show that f has a compact level or f has a *fiber* which is *relatively* compact in M. This will be obtained as a consequence of the following observation: the image under f of the set

$$Z = \{\, x \in E \mid \rho_1(x) = \rho_2(x) = 0 \,\} = \beta^{-1}(0) \setminus K$$

is a set of measure 0 in $\mathbb{P}^1$. For the proof of this observation, note that, by the remarks following Lemma 1.4, ρ_1 and ρ_2 (and hence β) are constant on the levels of f. If x is a point in Z at which $d\rho_1 \neq 0$, then there is a neighborhood U of x in which the set $N \equiv \rho_1^{-1}(0) \cap U$ is a real-analytic submanifold of real dimension $2n-1$ and the levels of $f|_N$ (which are open sets in levels of f) are complex analytic sets of real dimension $2n-2$. Hence the set $f(N)$, which contains $f(Z \cap U)$, is a set of measure 0 in the real 2-dimensional manifold $\mathbb{P}^1$ (by Sard's theorem). The same argument applies near any point at which $d\rho_2 \neq 0$. On the other hand, f is locally constant on the (complex) analytic set

$$A \equiv \{\, x \in E \mid (d\rho_1)_x = (d\rho_2)_x = 0 \,\} = \{\, x \in E \mid (\partial\rho_1)_x = (\partial\rho_2)_x = 0 \,\},$$

so $f(A)$ is countable. Thus $f(Z)$ is a set of measure 0 in $\mathbb{P}^1$. In particular, since f is an open mapping, the set $E \setminus f^{-1}(f(Z))$ is dense in E. If $\zeta \in f(E) \setminus (f(Z) \cup \{\infty\})$ and L is a level of f over ζ, then L lies in some fiber of the mapping (ρ_1, ρ_2) over a point in $\mathbb{R}^2 \setminus \{(0,0)\}$ and therefore, since $\rho_1, \rho_2 \to 0$ at infinity, we get $L \Subset M$. Hence, if L is noncompact, then $\overline{L} \cap K \neq \emptyset$. Fixing an open set Ω with $K \subset \Omega \Subset M$, we see that only finitely many noncompact connected components $L_1, \dots, L_m$ of $f^{-1}(\zeta)$ can meet $M \setminus \Omega$, because all such connected components must meet $\partial\Omega$ and the collection of all connected components is locally finite in E. Therefore, if f has no compact level over ζ, then

$$f^{-1}(\zeta) \subset \overline{\Omega} \cup L_1 \cup \dots \cup L_m \Subset M .$$

In particular, by replacing K by a larger compact set if necessary (containing $\overline{\Omega} \cup L_1 \cup \dots \cup L_m$ for some ζ as above), we may assume that there exists a point

$$\zeta_0 \in \mathbb{P}^1 \setminus \big(f(E) \cup \{\infty\}\big) .$$

Hence the function

$$h \equiv (f - \zeta_0)^{-1} : E \to \mathbb{C}$$

is a nonconstant holomorphic function on E.

Fix a C^∞ function $\lambda : M \to \mathbb{C}$ and a connected compact subset K' such that $K \subset \overset{\circ}{K'}$, the set $E' \equiv M \setminus K'$ is connected, and $\lambda|_{E'} = h|_{E'}$. Then the form

$$\alpha' \equiv \bar{\partial}\lambda$$

is a C^∞ compactly supported form of type $(0,1)$ on M satisfying $\bar{\partial}\alpha' = 0$. Thus we may apply Lemma 1.1 to get an L^2 harmonic form γ' and a C^∞ function β' (with finite energy) such that

$$\gamma' = \alpha' - \bar{\partial}\beta' = \bar{\partial}(\lambda - \beta') \quad \text{and} \quad \beta' \to 0 \text{ at infinity} .$$

The function $\tau \equiv \lambda - \beta' : M \to \mathbb{C}$ is pluriharmonic. Moreover, τ is nonconstant because β' vanishes at infinity and, by Lemma 1.3, the nonconstant holomorphic function $h|_{E'} = \lambda|_{E'}$ cannot approach a constant at infinity.

If $\tau_1 \equiv \operatorname{Re}\tau$ is nonconstant, then the maximum principle implies that we may choose a point $x_0 \in E' \subset E$ at which

$$\tau_1(x_0) > \max_{K'} \tau_1 .$$

Furthermore, we may choose x_0 to lie in the dense subset $E' \setminus f^{-1}(f(Z))$. The level L of f through x_0 is then relatively compact in M. Hence $K' \cup L$ is a compact set and the restriction of τ_1 to this set attains its maximum at some point $x_1 \in K' \cup L$. But then

$$\tau_1(x_1) = \max_{K' \cup L} \tau_1 \geq \tau_1(x_0) > \max_{K'} \tau_1 .$$

Thus $\tau_1(x_1) = \max_L \tau_1$ and hence τ_1 is constant on L. Therefore

$$L \subset \{\, x \in M \mid \tau_1(x) \geq \tau_1(x_0) \,\} .$$

But the set on the right-hand side is a (closed) subset of E', the closed subset L of E is relatively compact in M, and $\overline{E'} \subset E$, so L must be a compact level of f. If τ_1 is constant, then the imaginary part $\tau_2 \equiv \operatorname{Im}\tau$ is nonconstant and the same argument applied to τ_2 again yields a compact level of f. Therefore, by Lemma 1.5, M admits a proper holomorphic mapping onto a Riemann surface. $\square$

3 Hyper q-convexity and the Proof of the Technical Lemma

As in [NaR2], we will say that a real-valued C^2 function φ on a Kähler manifold (M, g) is *q-plurisubharmonic* (*strictly q-plurisubharmonic*) if, for each point $x_0 \in M$, the trace of the restriction of the Levi form $\mathcal{L}(\varphi)$ to any complex subspace of $T^{1,0}_{x_0}M$ of dimension q is nonnegative (respectively, positive). The Kähler manifold (M, g) is said to be *hyper-q-complete* if M admits a C^∞ strictly q-plurisubharmonic exhaustion function. If there exists a C^∞ exhaustion function which is strictly q-plurisubharmonic on the complement of some compact subset of M, then (M, g) is said to be *strongly hyper-q-convex*. The following fact is contained implicitly in the work of Richberg [Ri], Greene and Wu [GreW], Ohsawa [O], Colţoiu [Col], and Demailly [D2] (see [NaR2]):

Proposition 3.1 (Richberg, Greene-Wu, Ohsawa, Colţoiu, Demailly). *Suppose M is a Kähler manifold of dimension n and X is a nowhere dense analytic subset with no compact irreducible components. Then there exists a C^∞ strictly $(n-1)$-plurisubharmonic function φ on a neighborhood V of X such that φ exhausts X.*

The following lemma is a consequence of the work of Grauert and Riemenschneider [GrR] (for a relatively compact domain E), of Gromov [Gro3] and of Li [L] (for $E = M$), and of Siu [Si] (for a harmonic mapping of a relatively compact domain into a manifold satisfying certain curvature conditions).

Lemma 3.2 (Grauert–Riemenschneider, Gromov, Li, Siu). *Let (M, g) be a connected complete Hermitian manifold of dimension n, let E be a (not necessarily relatively compact) domain with smooth compact (possibly empty) boundary in M, let φ be C^∞ real-valued function on M such that $d\varphi \neq 0$ at every point in ∂E and such that $E = \{ x \in M \mid \varphi(x) < 0 \}$, and, for each point $x \in \partial E$, let*

$$\tau(x) = \mathrm{tr}\left(\mathcal{L}(\varphi)|_{T^{1,0}_x(\partial E)}\right).$$

Assume that g is Kähler on E and that $\tau \geq 0$ on ∂E. Then we have the following:

(a) If β is a C^∞ function on $\overline{E}$ such that β is harmonic on E, β satisfies the tangential Cauchy-Riemann equation $\bar{\partial}_b \beta = 0$ on ∂E, and there is a sequence of positive real numbers $R_m \to \infty$ and a point $p \in M$

such that

$$\|\nabla\beta\|^2_{L^2(B_p(R_m)\cap E)} = \int_{B_p(R_m)\cap E} |\nabla\beta|^2\,dV = o(R_m^2) \quad \text{as} \quad m\to\infty\,,$$

then β is pluriharmonic on E.

(b) *If E is a hyperbolic end of M, then $\tau\equiv 0$ on ∂E.*

Remark. If φ' is another smooth defining function for E, then

$$\mathcal{L}(\varphi')|_{T^{1,0}(\partial E)} = (\varphi'/\varphi)|_{\partial E}\cdot\mathcal{L}(\varphi)|_{T^{1,0}(\partial E)}\,.$$

So the conditions $\tau\geq 0$ and $\tau=0$ are independent of the choice of the defining function.

Sketch of the proof. For part (a), let $\gamma=\bar\partial\beta$ and let $\eta=\overline{*\gamma}$, where $*$ denotes the Hodge star operator. Then η may be thought of as a C^∞ form of type $(n,n-1)$ or as a form of type $(0,n-1)$ with values in the canonical bundle K_M on $\overline{E}$. A computation in normal coordinates implies that it suffices to show that $\overline{\nabla}\eta=0$ (see [Si, Proof of Lemma 5.6(d)]). Applying the Gaffney construction [G], one gets a sequence of C^∞ functions $\{\lambda_m\}$ on M such that, for each m, $\lambda_m\equiv 0$ on $M\setminus B_p(R_m)$, $\lambda_m\equiv 1$ on $B_p(R_m/2)$, and $|d\lambda_m|\leq 3/R_m$ on M. Since ∂E is compact, we may also assume that $\partial E\subset B_p(R_m/2)$. A straightforward computation in normal coordinates then shows that, because $\lambda_m^2\eta=\eta=\overline{*\bar\partial\beta}$ on a neighborhood of ∂E and $\lambda_m^2\eta$ has compact support in $\overline{E}$, the form $\lambda_m^2\eta$ lies in the domain of the adjoint operator $\bar\partial^*$. The Bochner-Kodaira formula is then (see [GrR] or [Si, formula 2.1.4])

$$\begin{aligned}&\left\|\overline{\nabla}(\lambda_m^2\eta)\right\|^2_{L^2(E)}\\ &\quad= \left\|\bar\partial(\lambda_m^2\eta)\right\|^2_{L^2(E)} + \left\|\bar\partial^*(\lambda_m^2\eta)\right\|^2_{L^2(E)} - \int_{\partial E}|\gamma|^2\cdot|\nabla\varphi|^{-1}\cdot\tau\,d\sigma\,, \qquad (3.1)\end{aligned}$$

where $d\sigma$ is the volume element on ∂E.

Remark. Here, the curvature terms drop out because η is of type $(n,n-1)$ and we have used the fact that $\bar\partial\beta\wedge\bar\partial\varphi$ vanishes at each point in ∂E. The authors mistakenly left out the normalizing factor $|\nabla\varphi|^{-1}$ in the proof of the analogous version [NaR2, Theorem 1.6].

If K is a compact subset of E, then, for m sufficiently large, we have

$$\|\overline{\nabla}\eta\|^2_{L^2(K)} = \left\|\overline{\nabla}(\lambda_m^2\eta)\right\|^2_{L^2(K)} \leq \left\|\overline{\nabla}(\lambda_m^2\eta)\right\|^2_{L^2(E)}\,.$$

Therefore, since $\tau\geq 0$, it suffices to show that

$$\left\|\bar\partial(\lambda_m^2\eta)\right\|^2_{L^2(E)} + \left\|\bar\partial^*(\lambda_m^2\eta)\right\|^2_{L^2(E)} \to 0 \quad \text{as} \quad m\to\infty\,.$$

Since β is harmonic and g is Kähler on E, we have $\bar\partial\eta=0$ and $\bar\partial^*\eta=0$. We also have $|\eta|\leq|\nabla\beta|$. Therefore, for some positive constant C, we have $\left\|\bar\partial(\lambda_m^2\eta)\right\|_{L^2(E)} = \|2\lambda_m\bar\partial\lambda_m\wedge\eta\|_{L^2(E)} \leq \frac{C}{R_m}\|\nabla\beta\|_{L^2(B_p(R_m)\cap E)} = \frac{o(R_m)}{R_m}\to 0$

and

$$\begin{aligned}\big\|\bar\partial^*(\lambda_m^2\eta)\big\|_{L^2(E)} &= \big\|-2\lambda_m *(\partial\lambda_m \wedge *\eta)\big\|_{L^2(E)} \le \tfrac{C}{R_m}\|\nabla\beta\|_{L^2(B_p(R_m)\cap E)}\\ &= \tfrac{o(R_m)}{R_m} \to 0\,.\end{aligned}$$

Thus part (a) is proved.

Assume now that E is a hyperbolic end and let β be the harmonic measure of the ideal boundary of M with respect to E. Thus β is a C^∞ function on $\overline{E}$, β is harmonic on E, $\beta \equiv 0$ on ∂E, $0 < \beta < 1$ on E, and $\limsup_{x\to\infty}\beta(x) = 1$. Moreover, β has finite energy and therefore, by part (a), β is pluriharmonic on E. With the notation from the proof of part (a), we see that each of the terms in equation (3.1) approaches 0 as $m \to \infty$. In particular,

$$\int_{\partial E} |\gamma|^2 \cdot |\nabla\varphi|^{-1} \cdot \tau\, d\sigma = 0\,.$$

Hence if $\tau > 0$ at some point in ∂E, then $|\gamma| = |\bar\partial\beta| = 0$ on $V \cap \partial E$ for some nonempty connected open subset V of M which meets ∂E. If f is a C^∞ function on $V \cap \overline{E}$ which is holomorphic on $V \cap E$ and which vanishes on $V \cap \partial E$, then one may extend f to a continuous function h on V which vanishes outside E. But then $\bar\partial h = 0$ in the weak sense, so h is holomorphic. It follows that h, and therefore f, must vanish identically. Shrinking V and letting f be a coefficient of the holomorphic 1-form $\bar\gamma$ with respect to some local holomorphic frame, we see that γ vanishes on a nonempty open subset of E and hence on the entire end E. Thus β is a real-valued holomorphic function on E and must therefore be a constant function. This contradicts the choice of β, so the claim (b) follows. □

Remark. Part (b) also holds if M is weakly 1-complete or if M has bounded geometry. In the former case, one replaces M by a sublevel of the exhaustion function. In the latter case, when M is parabolic, one produces a pluriharmonic function β as in [NaR1]. Similarly, Part (b) holds if $E \Subset M$ and ∂E is not connected. For one may take β to be a harmonic function which is equal to 1 on one boundary component and 0 on the other components.

Proof of Lemma 2.2. Let $n = \dim M$. Assume that $\partial\rho_1 \wedge \partial\rho_2$ is not everywhere 0. To obtain a contradiction, we will construct an end with an $(n-1)$-plurisubharmonic defining function and we will then apply Lemma 3.2. Let $Z = \{\, x \in E \mid \rho_1(x) = \rho_2(x) = 0 \,\}$ and $A = \{\, x \in E \mid (\partial\rho_1 \wedge \partial\rho_2)_x = 0 \,\}$ (A is a nowhere dense complex analytic set), and fix open sets Ω_1, Ω_2, and Ω_3 such that

$$K \subset \Omega_1 \Subset \Omega_2 \Subset \Omega_3 \Subset M$$

and such that $A \cap \Omega_3 \setminus \overline{\Omega}_1$ is empty or has no compact irreducible components (one can form these sets by choosing open sets $\Omega_1 \Subset \Omega'_3 \Subset M$ and setting $\Omega_3 = \Omega'_3 \setminus H$, where H is a finite subset of $E \setminus \overline{\Omega}_1$ which contains a point from each irreducible component of A which meets $\Omega'_3 \setminus \overline{\Omega}_1$). Let $\Omega_0 = \Omega_3 \setminus \overline{\Omega}_1$, let $Z_0 = Z \cap \Omega_0$, and let $A_0 = A \cap \Omega_0$. By applying Proposition 3.1 (Richberg, Greene-Wu, Ohsawa, Colţoiu, Demailly) to A_0 and forming a suitable extension of the resulting function to Ω_0 (possibly after shrinking the neighborhood of A_0 on which the function is initially defined), one obtains a positive C^∞ exhaustion function α on Ω_0 which is strictly $(n-1)$-plurisubharmonic on a neighborhood of A_0. Suppose

$$\psi = \alpha + \gamma \cdot (\rho_1^2 + \rho_2^2)\,,$$

where γ is a C^∞ positive function on Ω_0. Then ψ exhausts Ω_0 because $\psi \geq \alpha$. Moreover, at each point $x_0 \in Z_0$, we have

$$\mathcal{L}(\psi)_{x_0} = \mathcal{L}(\alpha)_{x_0} + \gamma(x_0) \cdot \mathcal{L}(\rho_1^2 + \rho_2^2)_{x_0}\,.$$

Therefore, since α is strictly $(n-1)$-plurisubharmonic on a neighborhood of A_0 and since $\rho_1^2 + \rho_2^2$ is $(n-1)$-plurisubharmonic on E and strictly $(n-1)$-plurisubharmonic on $E \setminus A$, we may choose γ so that the trace of the restriction of $\mathcal{L}(\psi)$ to each $(n-1)$-dimensional complex subspace of $T^{1,0}_{x_0}M$ is positive for each point $x_0 \in Z_0$. The restriction of ψ to some neighborhood V of Z_0 is then strictly $(n-1)$-plurisubharmonic.

Fix constants a and b with $a > b > \max_{\partial\Omega_2}\psi$. For $\epsilon > 0$ sufficiently small, the set $\{\, x \in \Omega_0 \mid \rho_1^2 + \rho_2^2 \leq \epsilon \text{ and } \psi(x) \leq a \,\}$ is compact (and possibly empty) and is contained in V. Choosing a C^∞ nondecreasing convex function $\chi : \mathbb{R} \to \mathbb{R}$ which vanishes on the interval $(-\infty, -\log(a-b)]$ and which approaches $+\infty$ at $+\infty$, we obtain a C^∞ $(n-1)$-plurisubharmonic function φ on the open set

$$N = \{\, x \in M \setminus \Omega_2 \mid \rho_1^2 + \rho_2^2 < \epsilon \,\} \cup \{\, x \in V \cap \Omega_2 \mid \rho_1^2 + \rho_2^2 < \epsilon \text{ and } \psi(x) < a \,\}$$

by defining

$$\varphi(x) = \begin{cases} \rho_1^2 + \rho_2^2 - \log(\epsilon - (\rho_1^2 + \rho_2^2)) & \text{if } x \in N \setminus \Omega_2 \\ \rho_1^2 + \rho_2^2 - \log(\epsilon - (\rho_1^2 + \rho_2^2)) + \chi(-\log(a - \psi(x))) & \text{if } x \in N \cap \Omega_2\,. \end{cases}$$

Moreover, $\overline{N} \subset E$, $\varphi \to \infty$ at ∂N, $\varphi \geq -\log \epsilon$ on N, and, since $\rho_1^2 + \rho_2^2$ vanishes at infinity in M, $M \setminus N$ is compact and $\varphi \to -\log\epsilon$ at infinity in M. Therefore, if c is a regular value of φ with $c > -\log \epsilon$ and E_0 is the (unique) connected component of the set $\{\, x \in N \mid \varphi(x) < c \,\}$ with noncompact closure in M, then E_0 is a C^∞ domain in M which has nonempty compact boundary and which admits a C^∞ $(n-1)$-plurisubharmonic defining function $\varphi - c$ with nonvanishing differential at each boundary point.

Now observe that φ is strictly $(n-1)$-plurisubharmonic on $N \setminus A$ because, on this set, $\varphi - (\rho_1^2 + \rho_2^2)$ is $(n-1)$-plurisubharmonic and $\rho_1^2 + \rho_2^2$ is strictly $(n-1)$-plurisubharmonic. Moreover, $(\partial E_0) \setminus A \neq \emptyset$ because $\dim_{\mathbb{R}} A \leq 2n-2 < \dim_{\mathbb{R}} \partial E_0$. Since (M, g) is hyperbolic, this contradicts Lemma 3.2, part (b). Thus we must have $\partial\rho_1 \wedge \partial\rho_2 \equiv 0$ on M. □

REMARKS. 1. In place of Lemma 3.2, part (b), one may instead apply part (a) to the harmonic measure and then apply Lemma 2.1.

2. By the remarks following the proof of Lemma 3.2, Lemma 2.2 also holds if M is weakly 1-complete or M has bounded geometry.

4 Compact Kähler Manifolds and their Coverings

Theorem 0.2 and Corollary 0.3 are proved in this section.

Proof of Theorem 0.2. Suppose (M, g) is a connected compact Kähler manifold of dimension n and $\pi : \widetilde{M} \to M$ is a connected infinite Galois covering with covering group

$$\Gamma = \pi_1(M)/\pi_1(\widetilde{M})\,.$$

We equip $\widetilde{M}$ with the complete Kähler metric $\tilde{g} = \pi^* g$ lifted from M. We have $e(\widetilde{M}) = 1$, 2, or ∞ (see Cohen [Co] or Scott and Wall [SW]). According to a theorem of Gromov [Gro2], if $e(\widetilde{M}) = \infty$, then $\widetilde{M}$ admits a proper holomorphic mapping to a Riemann surface (see also [NaR1]). By another theorem of Gromov [Gro1], if $e(\widetilde{M}) = 1$, then either Γ has more than quadratic growth or Γ contains $\mathbb{Z}^2$ as a subgroup of finite index. In the former case, the work of Varopoulos [V1,V2] and the work of Chavel and Feldman [CF] (see [R]) imply that $(\widetilde{M}, \tilde{g})$ admits a positive Green's function which vanishes at infinity. Hence one may apply Theorem 0.1 in this case. Thus it remains to address the latter case in which Γ contains $\mathbb{Z}^2$ as a subgroup of finite index.

By replacing M by a finite covering, we may assume that $\Gamma = \mathbb{Z}^2$. Choosing loops σ_1 and σ_2 in M mapping to $(1, 0)$ and $(0, 1)$, respectively, in $\mathbb{Z}^2$, there exists, for $j = 1, 2$, a unique (real) harmonic 1-form θ_j on M which integrates to δ_{ij} on σ_i for $i = 1, 2$ and which integrates to 0 on the image of $\pi_1(\widetilde{M})$ in $\pi_1(M)$. The liftings $\pi^*\theta_1$ and $\pi^*\theta_2$ integrate to pluriharmonic functions $\tau_1 : \widetilde{M} \to \mathbb{R}$ and $\tau_2 : \widetilde{M} \to \mathbb{R}$, respectively, which satisfy

$$\tau_1((r, s) \cdot x) = \tau_1(x) + r \quad \text{and} \quad \tau_2((r, s) \cdot x) = \tau_2(x) + s$$
$$\forall\, (r, s) \in \Gamma = \mathbb{Z}^2, x \in \widetilde{M}\,.$$

In particular, $\varphi \equiv \tau_1^2 + \tau_2^2$ is a C^∞ plurisubharmonic exhaustion function on $\widetilde{M}$.

REMARK. The theorem now follows from [NaR2], but a direct proof is provided below.

If $H_c^1(\widetilde{M}, \mathcal{O}) \neq 0$, then there exists a C^∞ compactly supported form α of type $(0,1)$ on $\widetilde{M}$ such that $\bar\partial\alpha = 0$ and such that $\alpha \notin \bar\partial C_c^\infty(\widetilde{M})$. Since $e(\widetilde{M}) = 1$, we may choose a connected compact set K such that $\alpha \equiv 0$ on $\widetilde{M} \setminus K$ and such that $\widetilde{M} \setminus K$ is connected. Fix a regular value a of φ with $a > \max_K \varphi$, and let Ω be the connected component of $\{\, x \in \widetilde{M} \mid \varphi(x) < a \,\}$ containing K. We may choose K and a so that $\Omega \setminus K$ is connected. By a theorem of Nakano [N] (see also Demailly [D1]), Ω admits a complete Kähler metric g'. Furthermore, since $\varphi - a$ is a negative (pluri)subharmonic function on Ω which approaches 0 at each point in $\partial\Omega$, (Ω, g') is hyperbolic and the Green's function vanishes at $\partial\Omega$.

Observe that $\alpha \notin \bar\partial C_c^\infty(\Omega) \subset \bar\partial C_c^\infty(M)$. Therefore, if $\partial\Omega$ is connected (i.e. $e(\Omega) = 1$), then Theorem 0.1 implies that Ω admits a proper holomorphic mapping with connected fibers onto a Riemann surface. If $\partial\Omega$ is *not* connected, then, by Lemma 1.2, there exists a pluriharmonic function ρ on Ω such that $0 < \rho < 1$ on Ω, $\rho \to 1$ at some boundary component C, and $\rho \to 0$ at $(\partial\Omega) \setminus C \neq \emptyset$. In particular, ρ has compact fibers. Moreover, since the functions $d\tau_1$ and $d\tau_2$ are linearly independent, $d\rho$ and $d\tau_i$ must be linearly independent for $i = 1$ or for $i = 2$. Therefore, Lemma 2.1 implies that some open subset of Ω admits a proper holomorphic mapping onto a Riemann surface. Hence, by Lemma 1.5, Ω again admits a proper holomorphic mapping with connected fibers onto a Riemann surface. Exhausting $\widetilde{M}$ by such domains, we get a proper holomorphic mapping (with connected fibers) of $\widetilde{M}$ onto the limit Riemann surface. □

The following observation, which is well known (at least for the case of a Galois covering), was used in [NaR1] (along with the main result of [NaR1]) to show that a connected compact Kähler manifold whose fundamental group is an amalgamated free product along $\mathbb{Z}$ admits a finite covering manifold which maps holomorphically onto a Riemann surface [NaR1, Theorem 4.1]:

PROPOSITION 4.1. *Suppose (M, g) is a connected compact Kähler manifold for which some connected covering space admits a surjective proper holomorphic mapping onto a Riemann surface. Then some finite covering of M admits a surjective holomorphic mapping onto a (compact) Riemann* *surface.*

Proof. By hypothesis, there exists a connected covering space $\pi : \widetilde{M} \to M$, a Riemann surface $\widetilde{X}$, and a surjective proper holomorphic mapping $\tilde{\Phi} : \widetilde{M} \to \widetilde{X}$. Clearly, we may assume that $\widetilde{M}$ is noncompact. By Stein factorization, we may also assume that $\tilde{\Phi}$ has connected fibers. Observe that if $\rho : \widetilde{M}' \to \widetilde{M}$ is the covering space with fundamental group equal to the image of the fundamental group of some generic fiber of $\tilde{\Phi}$, then the composite mapping $\tilde{\Phi}' : \widetilde{M}' \to \widetilde{X}$ has compact levels. For this is clear for the nonsingular fibers, because $\tilde{\Phi}$ is locally C^∞ trivial over the complement of the (discrete) set C of critical values in $\widetilde{X}$. Given a level F_0' over a point $x_0 \in C$, we may form a coordinate unit disk Δ centered at $0 = x_0$ such that $\Delta \cap C = \{x_0\}$. Let $F_0 = \rho(F_0') = \tilde{\Phi}^{-1}(0)$, let $U = \tilde{\Phi}^{-1}(\Delta)$ (a connected neighborhood of F_0), and let U' be the connected component of $\rho^{-1}(U)$ containing F_0'. Then the restriction mapping $U' \setminus F_0' \to \Delta^*$ is a C^∞ fiber bundle with compact levels and hence, aftering Stein factoring, we get $U' \setminus F_0' \xrightarrow{\alpha} Y \xrightarrow{\beta} \Delta^*$; where α is a proper holomorphic submersion with connected fibers and β is a covering map. In particular, we have $Y = \Delta$ or Δ^*, and hence the bounded holomorphic function α extends to a holomorphic mapping $\gamma : U' \to \Delta$ (the image must lie in the interior Δ of $\overline{\Delta}$ by the maximum principle). Since $\tilde{\Phi}'(F_0') = \{0\}$ and the image of the covering map $\Delta \to \Delta^*$ does not contain 0, we must have $Y = \Delta^*$ (and $\gamma(F_0') = 0$). Hence the covering map $Y \to \Delta^*$ is a finite mapping and therefore the covering map $U' \to U$ must have finite fibers. Thus F_0' is compact and the claim follows. Therefore, by replacing $\widetilde{M}$ by the covering space $\widetilde{M}'$ and $\tilde{\Phi}$ by the Stein factorization of $\tilde{\Phi}'$, we may assume that, for a generic fiber F, the mapping $\pi_1(F) \to \pi_1(\widetilde{M})$ is surjective. In particular, $\widetilde{X} = \Delta$ or $\mathbb{C}$ because $\pi_1(\widetilde{M})$ maps surjectively onto $\pi_1(\widetilde{X})$ and $\pi_1(F)$ maps to 0.

Now by a result of Kollár [K, Proposition 1.2.11] (one gets the proposition for compact Kähler manifolds by using the Barlet space in place of the Douady space in order to get the required family of normal cycles in the proof), for a generic fiber F of $\tilde{\Phi}$, the normalizer of the group $\Lambda = \mathrm{im}\,(\pi_1(\widetilde{M}) \to \pi_1(M)) = \mathrm{im}\,(\pi_1(F) \to \pi_1(M))$ is of finite index in $\pi_1(M)$. Therefore, by passing to the associated finite covering of M, we may assume that $\pi : \widetilde{M} \to M$ is a Galois covering and hence that $M = \widetilde{M}/\Gamma$, where $\Gamma = \pi_1(M)/\Lambda$.

Each automorphism $\sigma \in \Gamma$ maps fibers (of $\tilde{\Phi}$) to fibers, because $\tilde{\Phi}$ is constant on every connected compact analytic set. Thus σ descends to an automorphism of $\widetilde{X}$. The image Θ of the homomorphism $\Gamma \to \mathrm{Aut}\,(\widetilde{X})$ acts properly discontinuously on $\widetilde{X}$ and the kernel is a finite normal subgroup of Γ. If $\widetilde{X} = \mathbb{C}$, then $\Theta' = \Theta$ is torsion free and, if $\widetilde{X} = \Delta$, then, by a well known theorem of Selberg, there exists a torsion-free normal subgroup Θ' of finite index in Θ. Thus we get a commutative diagram of holomorphic mappings

$$\begin{tikzcd}
 & \widetilde{M} \arrow[r, "\tilde{\Phi}"] \arrow[dl, "\pi"'] \arrow[d] & \widetilde{X} \arrow[d] \\
M & \widetilde{M}/\Gamma' = M' \arrow[l, "\pi'"] \arrow[r, "\Phi'"'] & X' = \widetilde{X}/\Theta'
\end{tikzcd}$$

where Γ' is the inverse image of Θ', $\pi' : M' \to M$ is a connected finite covering space, X' is a Riemann surface, and Φ' is a surjective holomorphic map. $\square$

For example, the main result of [NaR1] and Proposition 4.1 together give the following:

Corollary 4.2. *Suppose (M, g) is a connected compact Kähler manifold for which there is a connected covering space with at least three ends. Then some finite covering of M admits a surjective holomorphic mapping onto a Riemann surface.*

Theorem 0.2 and the above together give Corollary 0.3 for the cases (i) and (iii). Finally, we address the case of a covering with two ends (in particular, case (ii)). By Lemma 1.2, Lemma 1.5, and Lemma 2.1, the conclusion of Theorem 0.1 also holds if $e(M) = 2$ and there exists a nowhere locally constant holomorphic function (or two pluriharmonic functions with locally linearly independent differentials) on the complement of a compact set. Similarly, one gets the corresponding versions of Theorem 0.2, Corollary 0.3, and Corollary 4.2. We also have the following fact which gives Corollary 0.3 for the case (ii):

Theorem 4.3. *Suppose (M, g) is a connected compact Kähler manifold for which there exists a surjective homomorphism $\pi_1(M) \to \mathbb{Z}$ whose kernel is not finitely generated. Then M admits a surjective holomorphic mapping onto a Riemann surface X and the homomorphism factors through $\pi_1(X)$.*

Proof. Let $\pi : \widetilde{M} \to M$ be a connected Galois covering space with

$$\pi_*(\pi_1(\widetilde{M})) = \Lambda = \ker\left(\pi_1(M) \to \mathbb{Z}\right).$$

Then $\mathbb{Z} = \Gamma = \pi_1(M)/\Lambda$ acts (fixed-point freely and properly discontinuously) by deck transformations on $\widetilde{M}$. The main step is, of course, to show that $\widetilde{M}$ admits a proper holomorphic mapping onto a Riemann surface $\widetilde{X}$. One can then pass to the quotient $\widetilde{X}/\mathbb{Z}$.

We first observe that standard arguments imply that $\widetilde{M}$ admits a proper pluriharmonic function into $\mathbb{R}$. For example, given a loop σ in M which maps to $1 \in \mathbb{Z}$, there is a unique real harmonic 1-form θ on M which integrates to 1 on σ and to 0 on elements of Λ. The lifting $\pi^*\theta$ integrates to a pluriharmonic function $\rho : \widetilde{M} \to \mathbb{R}$ which is proper because $\rho(m \cdot x) = \rho(x) + m$ for all $x \in \widetilde{M}$ and $m \in \mathbb{Z} = \Gamma$.

Fix a regular value c of ρ, a connected component N of $\rho^{-1}(c)$, and a connected covering space $\hat{\pi} : \widehat{M} \to \widetilde{M}$ with

$$\hat{\pi}_*(\pi_1(\widehat{M})) = \mathrm{im}\big(\pi_1(N) \to \pi_1(\widetilde{M})\big)\,.$$

Then $\hat{\pi}$ maps a relatively compact neighborhood V_1 of some connected component N_1 of $\hat{\pi}^{-1}(N)$ biholomorphically onto a neighborhood V of N in $\widetilde{M}$, and, since $\pi_1(\widetilde{M}) = \Lambda$ is not finitely generated, N_1 is a boundary component of some connected component $\widehat{P}$ of $\widehat{M} \setminus \hat{\pi}^{-1}(N)$ which has at least one other boundary component N_2. By replacing ρ by $-\rho$ and c by $-c$, if necessary, and by shrinking V_1, we may assume that, for some connected component P of $\{\, x \in \widetilde{M} \mid \rho(x) < c \,\}$, $\widehat{P}$ is a connected component of $\hat{\pi}^{-1}(P)$ and $\widehat{P} \setminus \overline{V}_1$ is connected. We may also assume that $V_1 \cap \widehat{P}$, and hence $V \cap P$, is connected. By a theorem of Nakano [N] (and Demailly [D1]), there exists a complete Kähler metric on P and, therefore, on $\widehat{P}$. Moreover, $\widehat{P}$ is hyperbolic and the Green's function vanishes at $\partial\widehat{P}$, because $\rho \circ \hat{\pi}|_{\widehat{P}} - c$ is a negative subharmonic function which vanishes at the boundary. Since the boundary component N_1 is compact, we may apply Lemma 1.2 to $\widehat{P}$ to get a nonconstant bounded real-valued pluriharmonic function $\hat{\tau}$ with finite energy on $\widehat{P}$. In particular, since $\rho(x) \to -\infty$ as $x \to \infty$ in $\overline{P}$, the functions 1, $\rho \circ \hat{\pi}|_{\widehat{P}}$, and $\hat{\tau}$ are linearly independent. Hence $\hat{\tau}$ determines a pluriharmonic function

$$\tau = \hat{\tau} \circ (\hat{\pi}|_{V_1 \cap \widehat{P}})^{-1} : V \cap P \to \mathbb{R}$$

such that 1, $\rho|_{V \cap P}$, and τ are linearly independent. Since $\rho|_{V \cap P}$ has a compact level, Lemma 2.1 implies that some open subset of $\widetilde{M}$ admits a proper holomorphic mapping onto a Riemann surface. On the other hand, each of the sublevels Ω of the plurisubharmonic exhaustion function ρ^2 on $\widetilde{M}$ admits a complete hyperbolic Kähler metric whose Green's function vanishes at $\partial\Omega$. Therefore, by Lemma 1.5, Ω admits a proper holomorphic mapping with connected fibers to a Riemann surface. Exhausting $\widetilde{M}$ by such domains, we get a proper holomorphic mapping $\tilde{\Phi} : \widetilde{M} \to \widetilde{X}$ with connected fibers of $\widetilde{M}$ onto the limit Riemann surface $\widetilde{X}$.

As in the proof of Proposition 4.1, the action of $\mathbb{Z}$ on $\widetilde{M}$ descends to a properly discontinuous action of $\mathbb{Z}$ on $\widetilde{X}$. This action is also fixed-point free because $\mathbb{Z}$ is torsion free. Thus $\tilde{\Phi}$ descends to a surjective holomorphic mapping $\Phi : M \to X$ of M onto the Riemann surface $X = \widetilde{X}/\mathbb{Z}$. $\square$

Remarks. 1. The above theorem generalizes part of a result of Arapura [Ar].

2. The theorem can also be proved using techniques of Gromov and Schoen [GroS].

Proof of Corollary 0.3. The conclusion for the cases (i) and (iii) follows from Theorem 0.2, Proposition 4.1, and Corollary 4.2. For the case (ii), suppose $e(\widetilde{M}) = 2$ and $\pi_1(\widetilde{M})$ is not finitely generated. Then the covering group $\Gamma = \pi_1(M)/\pi_1(\widetilde{M})$ contains $\mathbb{Z}$ as a subgroup of finite index. The inverse image H of $\mathbb{Z}$ in $\pi_1(M)$ is a subgroup of finite index and the image of $\pi_1(\widetilde{M})$ in $\pi_1(M)$ is a normal subgroup of H. Thus we may form the finite covering space M' with $\operatorname{im}(\pi_1(M') \to \pi_1(M)) = H$ and the Galois covering $\pi' : \widetilde{M} \to M'$. We then get a surjective homomorphism $\pi_1(M') \to \Gamma' = \pi_1(M')/\pi_1(\widetilde{M}) = \mathbb{Z}$ and the kernel $\pi_1(\widetilde{M})$ is *not* finitely generated. Theorem 4.3 now gives the claim. $\square$

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