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DESCRIPTION OF TRANSLATION INVARIANT VALUATIONS ON CONVEX SETS WITH SOLUTION OF P. MCMULLEN'S CONJECTURE

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Abstract

The main result of this paper is the description of translation invariant continuous valuations on convex sets. In particular, it provides an affirmative solution of P. McMullen's conjecture, and in a stronger form.

1 Introduction

Let $\mathcal{K}^n$ denote the family of all convex compact subsets of $\mathbb{R}^n$. Equipped with the Hausdorff metric, $\mathcal{K}^n$ becomes a locally compact metric space.

Definition 1.1. *A scalar valued function*

$$\varphi : \mathcal{K}^n \longrightarrow \mathbb{C}$$

is called a valuation if, for all convex compact sets $K_1, K_2 \in \mathcal{K}^n$ such that their union $K_1 \cup K_2$ is also convex, one has

$$\varphi(K_1 \cup K_2) = \varphi(K_1) + \varphi(K_2) - \varphi(K_1 \cap K_2)\,.$$

A valuation φ is called continuous if it is continuous with respect to the Hausdorff metric on $\mathcal{K}^n$.

A valuation φ is called translation invariant if for every convex compact set $K \in \mathcal{K}^n$ and for every vector $x \in \mathbb{R}^n$

$$\varphi(K + x) = \varphi(K)\,.$$

Let us make a few historical remarks on valuations, referring for more details and references to the surveys [MS] and [M3]. Valuations on convex bodies (polytopes) played an important initial role in Dehn's solution in 1900 of Hilbert's third problem on non-equidecomposability of convex

Partially supported by the Minkowski Center.

polytopes of equal volume in $\mathbb{R}^3$. A first attempt at classifying (in three-dimensional space) the rigid-motion invariant valuations (satisfying a suitable additional assumption) was made by Blaschke in the 1930s (see [Bl]). However, his result was not satisfactory since in the course of the proof he had to introduce an additional invariance assumption. Probably the most famous result on valuations is Hadwiger's characterization of rigid motion invariant valuations continuous with respect to the Hausdorff metric as linear combinations of quermassintegrals (see [H1,2,3]). It has numerous applications to integral geometry (see, e.g. Ch. 9 of [KR], or [H3], or [S1]).

The basic examples of valuations we would like to mention are as follows:

1. $\varphi(K) = \mathrm{vol}(K)$ for every $K \in \mathcal{K}^n$ (here $\mathrm{vol}(K)$ denotes the usual volume of the body K).
2. $\varphi(K) = 1$ for every $K \in \mathcal{K}^n$.
3. To state this example let us recall the definition of mixed volumes (see [S1] for details). Let $K_1, \dots, K_n \in \mathcal{K}^n$. For non-negative scalars $\lambda_1, \dots, \lambda_n$ let us denote by $\lambda_1 K_1 + \dots + \lambda_n K_n$ the Minkowski linear combination of K_i's, i.e. the set $\left\{ \sum_{i=1}^n \lambda_i x_i \;\middle|\; x_i \in K_i \right\}$. This is also a convex compact set. By the Minkowski theorem the volume $\mathrm{vol}(\lambda_1 K_1 + \dots + \lambda_n K_n)$ is a polynomial in $\lambda_i \geq 0$. The coefficient of $\lambda_1 \cdot \dots \cdot \lambda_n$ divided by $n!$ is called the mixed volume of the sets $K_1, \dots, K_n$ and is denoted by $V(K_1, \dots, K_n)$.
   Let us now fix a non-negative integer number $j \leq n$. Fix convex compact sets $A_1, \dots , A_j$. Then the mixed volume
   $$\varphi(K) = V(\underbrace{K, \dots , K}_{n-j \text{ times}}, A_1, \dots , A_j)$$
   is a translation invariant continuous valuation (see [S1]).

Note that if in example 3 one takes $j = 0$ (resp. $j = n$), one gets example 1 (resp. 2). Clearly every finite linear combination of the mixed volumes is a translation invariant continuous valuation.

The linear space of all continuous translation invariant valuations has the natural topology given by a sequence of semi-norms:

$$\|\varphi\|_N = \sup_{K \subset N \cdot B} |\varphi(K)|,$$

where $N \cdot B$ denotes the Euclidean ball of radius N. Note that the supremum is finite since it is taken over a compact subset of $\mathcal{K}^n$ (by the Blaschke selection theorem).

This sequence of semi-norms defines the Frechet space structure on the space of translation invariant continuous valuations (in fact it is even a

Banach space structure).

P. McMullen [M2] has conjectured that *the mixed volumes span a dense subspace in the space of translation invariant continuous valuations*.

The goal of this paper is to prove Theorem 1.3 below, which implies this conjecture in particular. This conjecture was known to be true in $\mathbb{R}^n$ with $n \leq 3$ (for references see the comments after Theorem 1.3 in this section). Recently it was proved for *even* valuations in $\mathbb{R}^4$ ([A]). This paper further develops the method of [A]. To state our main result we will need the following theorem due to McMullen.

Theorem 1.2 ([M1]). *Every continuous translation invariant valuation φ on $\mathbb{R}^n$ can be presented uniquely as a sum*

$$\varphi = \sum_{i=0}^{n} \varphi_i \,,$$

where φ_i are homogeneous valuations of degree i, $0 \leq i \leq n$, i.e. for every $K \in \mathcal{K}^n$ and a scalar $\lambda \geq 0$

$$\varphi_i(\lambda K) = \lambda^i \varphi_i(K) \,.$$

Furthermore, every valuation φ can be decomposed uniquely into even and odd parts

$$\varphi = \varphi^{\text{even}} + \varphi^{\text{odd}} \,,$$

where $\varphi^{\text{even}}(-K) = \varphi^{\text{even}}(K)$, $\varphi^{\text{odd}}(-K) = -\varphi^{\text{odd}}(K)$, for every $K \in \mathcal{K}^n$.

On the Frechet space of translation invariant continuous valuations we have the natural continuous representation π of the linear group $GL(n, \mathbb{R})$. Namely, for every $g \in GL(n, \mathbb{R})$, $K \in \mathcal{K}^n$

$$(\pi(g)\varphi)(K) = \varphi(g^{-1}K) \,.$$

Our main result is

Theorem 1.3. *The natural representation of $GL(n, \mathbb{R})$ on the space of even (resp. odd) translation invariant continuous valuations of a given degree of homogeneity is irreducible.*

Note that this theorem immediately implies McMullen's conjecture. Indeed by Theorem 1.2 it is sufficient to prove it for valuations, of a given degree of homogeneity, which are even or odd. The linear subspace spanned by the mixed volumes is $GL(n, \mathbb{R})$-invariant, hence by Theorem 1.3 it must be either zero or dense everywhere. But obviously it is nonzero.

Moreover Theorem 1.3 implies in the same way that the linear combinations of valuations of the form $\varphi(K) = V(K, \dots, K; \mathcal{E}, \dots, \mathcal{E})$, where $\mathcal{E}$ is an ellipsoid, are dense in the space of even valuations, and valuations of the form $\varphi(K) = V(K, \dots, K; \Delta, \dots, \Delta)$, where Δ is a simplex, are dense in the space of all translation invariant continuous valuations.

McMullen's conjecture was known to be true for valuations, which are homogeneous of degree 0 (the trivial case), 1 (see [GW]), n (which follows from [H3]; see Theorem 2.1(b) of this paper), and $n-1$ (see [M2]).

REMARKS. 1. It follows from the proof of Theorem 1.3 that the space of homogeneous even (resp. odd) translation invariant valuations is an admissible $GL(n,\mathbb{R})$-module with rather small characteristic variety, which is equal to the set of complex symmetric nilpotent matrices of rank at most 1.

2. There is a series of results providing some representations of translation invariant valuations on convex sets or polytopes with different assumptions on continuity: [GW], [H4,5], [M2,4]. For more details and references, see §16 of the survey [MS].

We would also like to mention a recent nice result [LR] characterizing *semi-continuous* valuations, which are invariant with respect to all volume preserving affine transformations.

3. There is an interesting class of *monotone* valuations. Recall that valuation φ is called monotone if for all convex compact sets such that $K_1 \subset K_2$ one has an inequality $\varphi(K_1) \leq \varphi(K_2)$ (monotone translation invariant valuations must be continuous, see [M1]). It is not clear whether the methods of this paper can be applied to studying this class of valuations.

Let us add a few words about the method of the proof. In the case of even valuations, we use two different embeddings, constructed in [A], of the $GL(n,\mathbb{R})$-module of even translation invariant continuous valuations, of a given degree of homogeneity, into two $GL(n,\mathbb{R})$-modules induced from certain parabolic subgroups of $GL(n,\mathbb{R})$ (i.e. these $GL(n,\mathbb{R})$-modules can be studied from a purely representation theoretical point of view). The next step which is solved in this paper is the proof that these two $GL(n,\mathbb{R})$-modules can have only one common irreducible $GL(n,\mathbb{R})$-submodule (which implies the main Theorem 1.3 in the even case). This step is based on some computations using $\mathcal{D}$-modules (following the suggestions of J. Bernstein and A. Braverman).

The idea is as follows: from the existence of one of the embeddings mentioned it follows that our $GL(n,\mathbb{R})$-module of valuations has a very small characteristic variety – just complex symmetric nilpotent matrices of rank at most 1. Next it is shown that the second $GL(n,\mathbb{R})$-module can have at most one submodule with such a property.

The plan of the proof in the odd case is similar. Here we also construct two different embeddings of our $GL(n,\mathbb{R})$-module of odd valuations, of a given degree of homogeneity. (One of the constructions is taken from [A] and is the same as in the even case.) The injectivity of the other embedding follows from the characterization of odd simple translation invariant valuations due to R. Schneider [S2]. (Note that in the even case the injectivity of the analogous embedding followed from the corresponding characterization of even simple translation invariant valuations due to D. Klain [K1].)

Next an analogous computation with $\mathcal{D}$-modules shows that these two $GL(n,\mathbb{R})$-modules can have only one common irreducible submodule.

The paper is organized as follows. In section 2 we recall necessary facts about valuations and construct the embeddings we need. In section 3 we compute the associated variety of the $(\mathfrak{g},K)$-module of valuations. In section 4 we prove the main theorem for even valuations, and in section 5 for odd ones. Section 6 discusses applications to unitarily invariant translation invariant valuations. Thus we compute the dimension of the space of these valuations. Section 7 contains a few comments on further possible applications and raises some questions.

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2 Preliminaries and Construction of the Embeddings

In this section we review some basic facts about valuations and then present constructions of embeddings of $GL(n,\mathbb{R})$-modules of translation invariant continuous valuations into other $GL(n,\mathbb{R})$-modules.

Theorem 2.1. (a) *Every translation invariant valuation φ, which is homogeneous of degree 0, is constant, i.e.* $\varphi(K) = \text{const}$ *for every* $K \in \mathcal{K}^n$.

(b) [H3] *Every translation invariant valuation on $\mathbb{R}^n$, which is homogeneous of degree n, is a density (i.e. proportional to Lebesgue measure).*

Note that part (a) is trivial. Recall that a valuation φ is called *simple* if it vanishes on degenerate convex sets, i.e. if $\dim K < n$ then $\varphi(K) = 0$.

Theorem 2.2 ([K1])**.** *Every even simple translation invariant continuous* *valuation is a density.*

To state the next result let us fix some Euclidean structure on $\mathbb{R}^n$.

Theorem 2.3 ([S2]). *Every odd simple translation invariant continuous valuation φ has the form:*

$$\varphi(K) = \int_{S^{n-1}} f(\omega) dS_{n-1}(K, \omega) \, ,$$

where $dS_{n-1}(K;\omega)$ is the surface area measure of K on the unit sphere S^{n-1}, $f : S^{n-1} \to \mathbb{C}$ is a continuous odd function. Moreover the function f is defined uniquely by the valuation φ up to a linear functional on S^{n-1}.

From now on, we will denote by $\mathrm{Val}_{n,k}^{\mathrm{ev}}$ (resp. $\mathrm{Val}_{n,k}^{\mathrm{odd}}$) the Frechet space of even (resp. odd) translation invariant continuous valuations on $\mathbb{R}^n$, which are homogeneous of degree k. Because of Theorem 2.1 we will consider only the case $1 \leq k \leq n-1$.

Let us denote by $Gr_{n,k}$ the Grassmannian of linear k-subspaces in $\mathbb{R}^n$. Let $L_{n,k}$ denote the line bundle $L_{n,k} \to Gr_{n,k}$ of densities over $Gr_{n,k}$, i.e. the fiber of $L_{n,k}$ over each $E \in Gr_{n,k}$ is the (one-dimensional) space of complex densities (= complex Lebesgue measures) on E.

Let us recall the construction of an embedding of $\mathrm{Val}_{n,k}^{\mathrm{ev}}$ into the space $\Gamma(L_{n,k})$ of global continuous sections of $L_{n,k}$.

Fix any valuation $\varphi \in \mathrm{Val}_{n,k}^{\mathrm{ev}}$. For every subspace $E \in Gr_{n,k}$ consider a restriction of φ on the class of convex compact subsets of E. Clearly it is a translation invariant continuous valuation, homogeneous of degree k. By Theorem 2.1(b) it is a density on E. Hence we get a linear map commuting with the natural action of $GL(n,\mathbb{R})$

$$\mathrm{Val}_{n,k}^{\mathrm{ev}} \longrightarrow \Gamma(L_{n,k}) \, .$$

The following result was used in [K2] and independently in [A] (see Proposition 3.1 of [A]). It is an easy consequence of the nontrivial Theorem 2.2 due to D. Klain.

Theorem 2.4. *The constructed map*

$$\mathrm{Val}_{n,k}^{\mathrm{ev}} \longrightarrow \Gamma(L_{n,k})$$

is injective.

Now we are going to describe an analogous embedding of odd homogeneous valuations. This space will be realized as a subquotient of the space of sections of a certain line bundle over the partial flag manifold (of type $(k, k+1)$, where k is the degree of the homogeneity of valuations). First let us do it under additional assumptions that $n = k+1$. In this case we actually just give a reformulation of Theorem 2.3 in invariant terms without fixing any Euclidean structure.

Consider for $\mathbb{R}^{k+1}$ the manifold of *co-oriented* hyperplanes. We have the line bundle of densities over this manifold. By Theorem 2.3 every valuation from $\mathrm{Val}^{\mathrm{odd}}_{k+1,k}$ defines an *odd* continuous section of this bundle (i.e. section which changes its sign if one changes the co-orientation of the hyperplane) modulo some $(k+1)$-dimensional subspace.

Thus odd sections of the bundle of densities over the manifold of co-oriented hyperplanes can be identified with sections of a certain line bundle $L'_{k+1,k}$ over the manifold of hyperplanes $Gr_{k+1,k}$ (without any (co-)orientation) modulo a $(k+1)$-dimensional subspace. Clearly the $L'_{k+1,k}$ is just a twist of the bundle of densities $L_{k+1,k}$ by some flat line bundle. It is easy to see that the group $GL(k+1,\mathbb{R})$ acts naturally on $L'_{k+1,k}$, and the corresponding representation in the space of continuous sections of $L'_{k+1,k}$ is (nonunitarily) induced from the character χ of the parabolic subgroup P, where P consists of the matrices of the form

$$\left[\begin{array}{c|c} A & * \\ \hline 0 & b \end{array}\right] \quad \text{with} \quad A \in GL(k,\mathbb{R}), \quad b \in \mathbb{R}^\times$$

and

$$\chi\left(\left[\begin{array}{c|c} A & * \\ \hline 0 & b \end{array}\right]\right) = |\det A|^{-1} \cdot \operatorname{sgn} b\,. \tag{2.1}$$

LEMMA 2.5. *The constructed map from* $\mathrm{Val}^{\mathrm{odd}}_{k+1,k}$ *to the quotient space of continuous sections of* $L'_{k+1,k}$ *by a* $(k+1)$*-dimensional subspace is a continuous map of linear topological spaces.*

Proof. Now it is convenient to fix a Euclidean structure on $\mathbb{R}^{k+1}$. Let $\varphi \in \mathrm{Val}^{\mathrm{odd}}_{k+1,k}$. In the statement of Theorem 2.3 one can assume the function f to be orthogonal to any linear functional (with respect to the standard Lebesgue measure on the sphere S^k). Then f is defined uniquely. Under this normalization we have the map

$$\Psi : \mathrm{Val}^{\mathrm{odd}}_{k+1,k} \longrightarrow \widetilde{C}(S^k)\,,$$

where $\widetilde{C}(S^k)$ denotes the Banach space of continuous odd functions on S^k orthogonal to every linear functional and equipped with the usual topology of uniform convergence on S^k. We have to prove the continuity of this map Ψ. Note that the inverse map $\Psi^{-1} : \widetilde{C}(S^k) \to \mathrm{Val}^{\mathrm{odd}}_{k+1,k}$ is defined

$$(\Psi^{-1} f)(K) = \int_{S^k} f(\omega)\, dS_k(K,\omega)\,.$$

The map Ψ^{-1} is continuous:

$$\sup_{K\subset N\cdot B}|\Psi^{-1}(K)|\leq\|f\|\cdot\sup_{K\subset N\cdot B}|\partial K|_k\leq\|f\|\cdot N^k\cdot|\partial B|_k\,,$$

where $|\partial K|_k$ is the surface area of K. Since $\mathrm{Val}^{\mathrm{odd}}_{k+1,k}$ is a Frechet space, by Banach's inverse operator theorem Ψ is continuous as well. □

Let us now consider the general case of $\mathrm{Val}^{\mathrm{odd}}_{n,k}$, $1\leq k\leq n-1$. For every $\varphi\in\mathrm{Val}^{\mathrm{odd}}_{n,k}$ we consider the restriction of φ on all $(k+1)$-dimensional subspaces of $\mathbb{R}^n$. Thus we get a linear map

$$\Theta:\mathrm{Val}^{\mathrm{odd}}_{n,k}\longrightarrow X_{n,k}\,,$$

where $X_{n,k}$ is the linear space of continuous sections of the (infinite dimensional) vector bundle over the Grassmannian $Gr_{n,k+1}$, whose fiber over $E\in Gr_{n,k+1}$ is the space of odd valuations on E homogeneous of degree k (as usual all valuations are assumed to be continuous and translation invariant). The space $X_{n,k}$ has the obvious structure of Frechet space. It easily follows from Lemma 2.5 that the map Θ is a continuous operator.

Proposition 2.6. *The map Θ defines an injection of $\mathrm{Val}^{\mathrm{odd}}_{n,k}$ into $X_{n,k}$.*

Proof. Assume that $\varphi\in\operatorname{Ker}\Theta$. For every $(k+2)$-dimensional linear subspace $F\subset\mathbb{R}^n$ the restriction of φ on F defines a *simple odd* translation invariant valuation. Hence by Theorem 2.3 it must be homogeneous of degree $k+1$. Hence it vanishes on all $(k+2)$-dimensional sets. By an induction argument φ vanishes identically. □

Let us fix a $(k+1)$-dimensional linear subspace $E_0\subset\mathbb{R}^n$. Let us denote by Ω the representation of $GL(E_0)$ in the space of odd valuations on E_0 homogeneous of degree k. Let P_1 denote the (parabolic) subgroup of $GL(n,\mathbb{R})$ fixing E_0. Then Proposition 2.6 can be reformulated as

Theorem 2.7. *The space $\mathrm{Val}^{\mathrm{odd}}_{n,k}$ embeds injectively as $GL(n,\mathbb{R})$-module into the induced representation $\mathrm{Ind}_{P_1}^{GL(n,\mathbb{R})}\Omega$.*

Now let us fix a pair of linear subspaces $E_1\subset E_0$, where $\dim E_1=k$, $\dim E_0=k+1$. Let P_2 denote the (parabolic) subgroup which fixes this flag. Clearly P_2 consists of matrices of the form:

$$X=\left[\begin{array}{c|c|c} A & \multicolumn{2}{c}{*} \\ \hline O & b & \\ \cline{2-3} & & C \end{array}\right]\ \text{with } A\in GL(k,\mathbb{R}),\ b\in\mathbb{R}^\times,\ C\in GL(n-k-1,\mathbb{R})\,.$$

Consider the character ξ of P_2:

$$\xi(X)=|\det A|^{-1}\cdot\operatorname{sgn}b\,.$$

From Theorem 2.7 and a previous description of the embedding of $\mathrm{Val}^{\mathrm{odd}}_{k+1,k}$ (see (2.1)) we obtain

Corollary 2.8. *The space* $\mathrm{Val}^{\mathrm{odd}}_{n,k}$ *can be realized as a subquotient of* $\mathrm{Ind}^{GL(n,\mathbb{R})}_{P_2}\,\xi$.

Now we will state a result proved in [A] (see Section 3 of [A]). Let $\mathbb{P}^{n-1}_{+}$ denote the manifold of oriented lines in the dual space $\mathbb{R}^{n*}$ through the origin (clearly $\mathbb{P}^{n-1}_{+}$ is diffeomorphic to the sphere).

Theorem 2.9 ([A])**.** *Let $1 \le k \le n-1$. The space of translation invariant continuous even (resp. odd) valuations homogeneous of degree k can be embedded into the space of even (resp. odd) distributions on $\mathbb{P}^{n-1}_{+} \times \cdots \times \mathbb{P}^{n-1}_{+}$ (k times) with support on the diagonal and with values in a certain line bundle (equipped with the natural action of $GL(n,\mathbb{R})$). The order of these distributions is uniformly bounded by a constant depending on n and k only. The embedding commutes with the natural action of $GL(n,\mathbb{R})$ on each space.*

We will not describe here the construction of this embedding. We need only to know the existence of this embedding and the following property of distributions mentioned in Theorem 2.9 (see [A] for details).

Proposition 2.10. *The space of even (odd) distributions on $\mathbb{P}^{n-1*}_{+} \times \cdots \times \mathbb{P}^{n-1*}_{+}$ (k times) of given order, which have support on the diagonal, is infinitesimally equivalent as $(\mathfrak{g}, K)$-module to a representation of $GL(n,\mathbb{R})$, which is induced from a finite dimensional representation ρ of the parabolic subgroup $P \subset GL(n,\mathbb{R})$. The subgroup P consists of matrices of the form*

$$T = \left[\begin{array}{c|c} A & * \\ \hline 0 & b \end{array}\right], \quad \text{where} \quad A \in GL(n-1,\mathbb{R}), \quad b \in \mathbb{R}^{\times}. \tag{2.2}$$

Furthermore the representation ρ has the form

$$\rho(T) = \rho_0(T) \cdot \big(\mathrm{sgn}(\det A)\big)^{\varepsilon_1} \cdot (\mathrm{sgn}\, b)^{\varepsilon_2}, \tag{2.3}$$

where $\varepsilon_1, \varepsilon_2$ are equal to 0 or 1 (the precise values are not important, they depend on the parity of distributions), and ρ_0 is a finite dimensional algebraic representation of the Levi factor of P.

3 Computation of the Associated Variety of $\mathrm{Val}^{\mathrm{ev}}_{n,k}$ and $\mathrm{Val}^{\mathrm{odd}}_{n,k}$

For the general theory of algebraic $\mathcal{D}$-modules we refer to [BoGKHME]. First recall a few definitions (for details see [BorB1]). Let $\mathfrak{g}$ be a complex Lie algebra. Let M be a finitely generated module over the universal enveloping algebra $U(\mathfrak{g})$. Then one can choose a filtration of M by finite dimensional subspaces $M_0 \subset M_1 \subset M_2 \subset \cdots \subset M$ such that $\mathfrak{g}M_i \subset M_{i+1}$ with the equality $\mathfrak{g}M_i = M_{i+1}$ for large i. Such a filtration is called *good*. The *associated variety* (*Bernstein variety*) of M is the support in $\mathfrak{g}^*$ of the associated graded module grM over the symmetric algebra $S(\mathfrak{g}) \simeq grU(\mathfrak{g})$. The associated variety does not depend on a choice of a good filtration; it will be denoted by $V_{\mathfrak{g}}M$. Let us fix once and forever a positive definite quadratic form on $\mathbb{R}^n$. We will denote for brevity $\mathfrak{g} = g\ell(n,\mathbb{C})$ and $K = O(n,\mathbb{R})$ the real orthogonal group.

Let $P \subset GL(n,\mathbb{R})$ be the parabolic subgroup consisting of matrices of the form (2.2). We will study the representation $V = \mathrm{Ind}_P^{GL(n,\mathbb{R})}\,\rho$, where ρ has the form (2.3). We will write $\rho = \rho_0 \otimes \nu$, where ν is the product of the last two terms on the right hand side of (2.3).

We are going to prove:

Theorem 3.1. *The associated variety of V is equal to the variety of complex symmetric nilpotent matrices of rank at most one.*

Note that $GL(n,\mathbb{C})/{}^{\mathbb{C}}P = \mathbb{C}P^{n-1*} =: X$. Let ${}^{\mathbb{C}}K = O(n,\mathbb{C})$. Let us denote by $U \subset X$ the open subset consisting of those hyperplanes in $\mathbb{C}^n$, on which the restriction of our quadratic form is non-degenerate (this quadratic form on $\mathbb{C}^n = \mathbb{R}^n \otimes \mathbb{C}$ is the complexification of our fixed positive definite quadratic form on $\mathbb{R}^n$). The set U is an *affine* variety since $U = O(n,\mathbb{C})/(O(n-1,\mathbb{C}) \times O(1,\mathbb{C}))$, i.e. it is the quotient of an affine variety $O(n,\mathbb{C})$ by the reductive group $O(n-1,\mathbb{C}) \times O(1,\mathbb{C})$.

Let L denote the algebraic bundle over X corresponding to ρ_0 (see (2.3)). Let $\mathcal{D}$ denote the (quasicoherent) sheaf of differential operators on X acting on sections of L.

Observe that the K-finite vectors of V correspond to regular (algebraic) sections of a certain algebraic bundle L_1 over U. Moreover, $L_1 = N \otimes_{\mathcal{O}_U} L|_U$, where $N \to U$ is an algebraic *line* bundle whose sections over U correspond to K-finite vectors of the representation $\mathrm{Ind}_P^{GL(n,\mathbb{R})}\,\nu$, where, by the definition of ν,

$$\nu(T) = \big(\mathrm{sgn}(\det A)\big)^{\varepsilon_1}(\mathrm{sgn}\, b)^{\varepsilon_2},$$

for $T = \left[\begin{array}{c|c} A & * \\ \hline 0 & b \end{array}\right]$ with $A \in GL(n-1,\mathbb{R})$ and $b \in \mathbb{R}^*$. Hence we deduce

Claim 3.2. The ring $\mathcal{D}_0$ of (untwisted) differential operators on U acts on $\Gamma(U,N)$.

Hence $V \otimes_{\mathbb{C}} \mathcal{O}_U = L_1$ becomes a $\mathcal{D}_U$-module on U (here $\mathcal{D}_U$ is the restriction of the sheaf $\mathcal{D}$ on U). We have the natural morphism of rings $U(\mathfrak{g}) \to \Gamma(X, \mathcal{D})$. The composition of it with the action of $\Gamma(U, \mathcal{D})$ on $\Gamma(U, L_1) = V$ gives the original action of $U(\mathfrak{g})$ on V.

Since V is finitely generated over $U(\mathfrak{g})$, L_1 is a coherent $\mathcal{D}_U$-module. Also L_1 is a holonomic $\mathcal{D}_U$-module (since it is K-equivariant). Let $i : U \hookrightarrow X$ be the identical embedding.

Then it is well known that $\mathcal{F} := i_* L_1$ is a coherent $\mathcal{D}$-module. Since U is an affine subvariety, the map i is an affine embedding. Hence $\Gamma(X, \mathcal{F}) = \Gamma(U, L_1) = V$ (which is a finitely generated $U(\mathfrak{g})$-module). The following proposition is essentially taken from [BorB2].

Proposition 3.3. *Let $\mathcal{M}$ be a coherent $\mathcal{D}$-module on X. Consider $M = \Gamma(X, \mathcal{M})$ as $U(\mathfrak{g})$-module. Let $\pi : T^*X \to \mathfrak{g}^*$ be the moment map. Then $V_{\mathfrak{g}} M \subset \pi(Ch\mathcal{M})$, where $V_{\mathfrak{g}} M$ denotes the associated variety of M, and $Ch\mathcal{M} \subset T^*X$ denotes the singular support of $\mathcal{M}$.*

Proof. Let $\mathcal{D}_j$ denote the coherent $\mathcal{O}_X$-module of differential operators on X of order at most j. We can choose a coherent $\mathcal{O}_X$-submodule $\mathcal{M}_0 \subset \mathcal{M}$ which generates $\mathcal{M}$ as a $\mathcal{D}$-module and which is $(\operatorname{End} L)$-stable. Set $\mathcal{M}_j = \mathcal{D}_j \mathcal{M}_0$. We obtain a filtration of $\mathcal{M}$ by coherent $\mathcal{O}_X$-submodules $\mathcal{M}_0 \subset \mathcal{M}_1 \subset \cdots$. This is a good filtration: $\mathcal{D}_j \mathcal{M}_i = \mathcal{M}_{i+j}$. Consider $gr\mathcal{M} = \bigoplus_{j=0}^{\infty} \mathcal{M}_j / \mathcal{M}_{j-1}$. This is a coherent sheaf on T^*X, whose support coincides with $Ch\mathcal{M}$ by definition. Let us denote $M_j = \Gamma(X, \mathcal{M}_j)$. This is a filtration of M: $M_0 \subset M_1 \subset \cdots$, $M = \bigcup_{j=0}^{\infty} M_j$. Clearly $U^i(\mathfrak{g}) M_j \subset M_{i+j}$. Since we have an exact sequence $0 \to \mathcal{M}_{j-1} \to \mathcal{M}_j \to \mathcal{M}_j/\mathcal{M}_{j-1} \to 0$, then the sequence $0 \to M_{j-1} \to M_j \to \Gamma(\mathcal{M}_j/\mathcal{M}_{j-1})$ is exact too. Hence M_j/M_{j-1} embeds into $\Gamma(\mathcal{M}_j/\mathcal{M}_{j-1})$. Note that $\Gamma(\mathfrak{g}^*, \pi_* gr\mathcal{M}) = \bigoplus_{j=0}^{\infty} \Gamma(X, \mathcal{M}_j/\mathcal{M}_{j-1})$. Since π is proper this is a coherent $grU(\mathfrak{g}) = S(\mathfrak{g})$-module (see [DG, III Théorème 3.2.1] or [Ha, III, Theorem 8.8(b)]). Hence grM is an $S(\mathfrak{g})$-submodule of the coherent $S(\mathfrak{g})$-module $\Gamma(\mathfrak{g}^*, \pi_* gr\mathcal{M})$; thus grM is coherent too. Hence $\{M_j\}$ is a good filtration of M and $V_{\mathfrak{g}} M = \operatorname{supp}(grM) \subset \operatorname{supp}(\pi_* gr\mathcal{M}) \subset \overline{\pi(\operatorname{supp}(gr\mathcal{M}))} = \pi(ch\mathcal{M})$ since π is proper. □

Let us finish the proof of Theorem 3.1. In our situation we have a ${}^{\mathbb{C}}K$-equivariant $\mathcal{D}$-module on $\mathbb{C}\mathbb{P}^{n-1}$. Its singular support is contained in $\bigcup_Y \overline{T^*_Y \mathbb{C}\mathbb{P}^{n-1}}$, where the union is taken over all ${}^{\mathbb{C}}K$-orbits Y in $\mathbb{C}\mathbb{P}^{n-1}$, and $T^*_Y \mathbb{C}\mathbb{P}^{n-1}$ denotes the co-normal bundle to Y (see [BorB2]). But on $\mathbb{C}\mathbb{P}^{n-1}$ there are just two ${}^{\mathbb{C}}K$-orbits: the open orbit consisting of those lines, on which the restriction of our quadratic form is non-zero, and the closed orbit consisting of lines, the restriction on which of our quadratic form vanishes. The co-normal bundle to the open orbit is zero. The image under the moment map of the co-normal bundle to the closed orbit consists of symmetric nilpotent matrices of rank at most 1. This and Proposition 3.3 imply Theorem 3.1. □

4 Proof of the Main Result in the Even Case

Let us consider the case of even valuations. It was shown (Theorem 2.4) that for $1 \le k \le n-1$ the space $\mathrm{Val}^{\mathrm{ev}}_{n,k}$ can be realized as the $GL(n,\mathbb{R})$-submodule of $\mathrm{Ind}_P^{GL(n,\mathbb{R})}\chi$, where P is the parabolic subgroup consisting of matrices of the form

$$T = \left[\begin{array}{c|c} A & * \\ \hline 0 & B \end{array}\right], \quad A \in GL(k,\mathbb{R})\,, \quad B \in GL(n-k,\mathbb{R})\,,$$

and $\chi(T) = |\det A|^{-1}$.

Let us denote by F the space of K-finite vectors of $\mathrm{Ind}_P^{GL(n,\mathbb{R})}\chi$. Let X denote the complex Grassmannian ${}^{\mathbb{C}}Gr_{n,k} = GL(n,\mathbb{C})/{}^{\mathbb{C}}P$. Let $U \subset X$ be the open subvariety consisting of those k-dimensional subspaces, on which the restriction of the given quadratic form is non-degenerate. Clearly $U = {}^{\mathbb{C}}K \cdot {}^{\mathbb{R}}Gr_{n,k}$. Moreover U is an affine variety since

$$U = O(n,\mathbb{C})/\big(O(k,\mathbb{C}) \times O(n-k,\mathbb{C})\big)\,.$$

Let $\mathcal{D}_\chi$ denote the sheaf of twisted differential operators on X corresponding to χ. The character χ satisfies the conditions of the Beilinson–Bernstein theorem (see [Bi, Th. I.6.3]). We will not list here these conditions, just explain why they are satisfied in our situation. Consider a different $GL(n,\mathbb{R})$-module $N := \mathrm{Ind}_P^{GL(n,\mathbb{R})}\chi_1$, where $\chi_1(T) = (\det(A))^{-1}$ (here T denotes the same matrix as at the beginning of the section). Clearly the sheaf $\mathcal{D}_{\chi_1}$ of twisted differential operators on X corresponding to χ_1 is equal to $\mathcal{D}_\chi$. So it is sufficient to check the conditions of the Beilinson–Bernstein theorem for $\mathcal{D}_{\chi_1}$. But the module N coincides with the space of sections of the line bundle over the real Grassmannian ${}^{\mathbb{R}}Gr_{n,k}$, whose fiber over subspace $E \in {}^{\mathbb{R}}Gr_{n,k}$ is equal to translation invariant k-forms on E (i.e. to $\Lambda^k E^* \otimes \mathbb{C}$). But this module has a finite dimensional submodule which is just the image of $\Lambda^k\mathbb{R}^n \otimes \mathbb{C}$ under the natural restriction map $\Lambda^k\mathbb{R}^n \otimes \mathbb{C} \to \Lambda^k E^* \otimes \mathbb{C}$ for each E. But it is well known that the existence of such finite dimensional submodule implies that the conditions of the Beilinson–Bernstein theorem are satisfied.

Recall that the Beilinson–Bernstein theorem states that the functor of global sections is exact and faithful on the category of $\mathcal{D}_\chi$-modules.

Note that the space F can be considered as the space of global sections of a certain algebraic line bundle $\mathcal{F}$ over U. This $\mathcal{F}$ is a $\mathcal{D}_\chi|_U$-module. Let $i : U \to X$ be the identical embedding. Consider $\mathcal{F}_1 := i_*\mathcal{F}$.

Lemma 4.1. $\Gamma(X, \mathcal{F}_1) = F$, $H^i(X, \mathcal{F}_1) = 0$ *for* $i > 0$.

Proof. This immediately follows from the fact that $i : U \to X$ is an affine embedding. □

Thus by the Beilinson–Bernstein theorem the ${}^{\mathbb{C}}K$-equivariant $\mathcal{D}_\chi$-submodules of $\mathcal{F}_1$ are in bijective correspondence with the $(\mathfrak{g}, K)$-submodules of F.

Using the translation functor we will replace the study of the $\mathcal{D}_\chi$-module $\mathcal{F}_1$ by the study of another $\mathcal{D}$-module on X, where $\mathcal{D}$ is the sheaf of untwisted differential operators. Consider the space H of K-finite vectors of the representation $\operatorname{Ind}_P^{GL(n,\mathbb{R})} \chi_1$, where $\chi_1(T) = \operatorname{sgn}(\det A)$, and

$$T = \left[\begin{array}{c|c} A & * \\ \hline 0 & B \end{array}\right].$$

Then H is the space of global sections of a certain algebraic line bundle $\mathcal{H}$ over U. Set $\mathcal{H}_1 = i_*\mathcal{H}$. Then $\mathcal{H}_1$ is a coherent holonomic $\mathcal{D}$-module, and for the category of $\mathcal{D}$-modules on X the Beilinson–Bernstein theorem is satisfied as well. Next ${}^{\mathbb{C}}K$-equivariant $\mathcal{D}$-submodules of $\mathcal{H}_1$ are in bijective correspondence with the ${}^{\mathbb{C}}K$-equivariant $\mathcal{D}_\chi$-submodules of $\mathcal{F}_1$, and the corresponding subquotients have the same singular supports on T^*X.

Let $\pi : T^*X \to \mathfrak{g}^*$ be the moment map. Recall that we denote by $V_\mathfrak{g}M$ the associated variety of a $\mathfrak{g}$-module M, and by $Ch(\mathcal{M})$ the singular support of a $\mathcal{D}$-module $\mathcal{M}$.

Proposition 4.2. *Let* $\mathcal{M}$ *be a coherent* $\mathcal{D}$*-module* (*or* $\mathcal{D}_\chi$*-module*) *on* X. *Let* $M = \Gamma(X, \mathcal{M})$. *Then* $V_\mathfrak{g}(M) = \pi(Ch\mathcal{M})$.

Proof. Let us prove it for $\mathcal{D}$-modules; for $\mathcal{D}_\chi$-modules the proof is exactly the same. In Proposition 3.3 it was shown that $V_\mathfrak{g}(M) \subset \pi(Ch\mathcal{M})$. Let us prove the converse inclusion. The argument is taken from [BorB2]. The sheaf of differential operators on X of order $\leq j$ is generated by its global sections as $\mathcal{O}_X$-module (see [BorB2, Section 1.7, Lemma 2]); actually it is generated by differential operators from $U^j(\mathfrak{g})$, where $U^j(\mathfrak{g})$ denotes the subspace of $U(\mathfrak{g})$ spanned by the expression of the form $Y_1 \dots Y_l$ with $Y_i \in \mathfrak{g}$ and $l \leq j$. In other words, if $\mathcal{D}^j$ denotes the sheaf of differential operators of order at most j, then $\mathcal{D}^j = \mathcal{O}_{\mathcal{X}} \cdot U^j(\mathfrak{g})$. Since M is a finitely generated $U(\mathfrak{g})$-module we can choose a good filtration of M by $M_0 \subset M_1 \subset \cdots$, i.e. $U^i(\mathfrak{g})M_j = M_{i+j}$. Set $\mathcal{M}_j = \mathcal{O}_X M_j$. Let us show that $\{\mathcal{M}_j\}$ is a good filtration of $\mathcal{M}$. We have

$$\mathcal{D}^1\mathcal{M}_j = \mathcal{O}_X U^1(\mathfrak{g})\mathcal{O}_X M_j \supset \mathcal{O}_X U^1(\mathfrak{g}) M_j = \mathcal{O}_X M_{j+1} = \mathcal{M}_{j+1}\,.$$

The inclusion $\mathcal{D}^1\mathcal{M}_j \subset \mathcal{M}_{j+1}$ is obvious, hence $\mathcal{D}^1\mathcal{M}_j = \mathcal{M}_{j+1}$. It is sufficient to show that if the principal symbol of the operator $P \in U(\mathfrak{g})$ annihilates grM, then it annihilates $gr\mathcal{M}$. Let us assume that P has degree m, and $PM_j \subset M_{j+m-1}$ for all j. Clearly $[P, \mathcal{O}_X] \subset \mathcal{D}^{m-1}$. Thus $P\mathcal{M}_j = P\mathcal{O}_X M_j \subset \mathcal{D}^{m-1}M_j + \mathcal{O}_X P M_j \subset \mathcal{O}_X U^{m-1}(\mathfrak{g})M_j + \mathcal{O}_X M_{j+m-1} \subset \mathcal{M}_{j+m-1}$, namely, P annihilates $gr\mathcal{M}$. □

Now let us return to the study of $\mathcal{D}$-module $\mathcal{H}_1$.

Theorem 4.3. *The $\mathcal{D}$-module $\mathcal{H}_1$ has at most one ${}^{\mathbb{C}}K$-equivariant subquotient whose associated variety is equal to the variety of symmetric nilpotent matrices of rank at most 1.*

So let $X = {}^{\mathbb{C}}Gr_{n,k}$. Let Z denote the (locally closed) subset of X consisting of the subspaces, on which the restriction of our quadratic form has rank $k-1$. Z is an $O(n,\mathbb{C})$-orbit. Fix a certain subspace $\mathbb{C}^{k+1} \subset \mathbb{C}^n$ such that the restriction of the quadratic form on $\mathbb{C}^{k+1}$ is non-degenerate. Set $X_1 = {}^{\mathbb{C}}Gr_{k+1,k}$, i.e. the Grassmannian of k-subspaces in this fixed copy of $\mathbb{C}^{k+1}$. Easily one has the following

Claim 4.4. Z intersects X_1 transversally.

Let $f : X_1 \hookrightarrow X$ be the identical closed embedding. Let $U_1 = X_1 \cap U$, where $U \subset X$ is, as previously, the open affine subset consisting of the k-subspaces of $\mathbb{C}^n$, on which the restriction of our quadratic form is non-degenerate. Let

$$\begin{aligned} Z_1 &:= X_1 \cap Z\,, \\ W &:= U \cup Z\,, \\ W_1 &:= U_1 \cup Z_1 = W \cap X_1\,. \end{aligned}$$

Clearly W and W_1 are open subsets of X and X_1, respectively. Let us denote by

$$\begin{aligned} g &: U \hookrightarrow W\,, \\ g_1 &: U_1 \hookrightarrow W_1\,, \\ h &: U_1 \hookrightarrow U\,, \end{aligned}$$

the identical embeddings. First we will prove

PROPOSITION 4.5. *The $\mathcal{D}_W$-module $g_*\mathcal{H}$ is simple.*

LEMMA 4.6. *If the $\mathcal{D}_{W_1}$-module $g_{1*}h^!\mathcal{H}$ is simple then the $\mathcal{D}_W$-module $g_*\mathcal{H}$ is simple.*

This lemma follows from the previous claim and the $O(n,\mathbb{C})$-equivariance of $\mathcal{H}$.

Thus it suffices to show that $g_{1*}h^!\mathcal{H}$ is a simple $\mathcal{D}_{W_1}$-module. First observe that $h^!\mathcal{H}$ is a $\mathcal{D}_{U_1}$-module on U_1 which is equal to the space of global sections of an algebraic line bundle over U_1 coinciding with $O(k+1,\mathbb{R})$-finite vectors of the representation of $GL(k+1,\mathbb{R})$ induced from the character ξ of the parabolic subgroup with the Levi factor $GL(k,\mathbb{R})\times GL(1,\mathbb{R})$ and

$$\xi\left(\left[\begin{array}{c|c} A & * \\ \hline 0 & b\end{array}\right]\right) = \operatorname{sgn}(\det A)\,, \quad A\in GL(k,\mathbb{R})\,, \quad b\in\mathbb{R}^\times\,.$$

Let us fix a $(k-1)$-dimensional subspace $M\subset\mathbb{C}^{k+1}$ such that the restriction of our quadratic form on M is non-degenerate. Consider a closed embedding

$$s:\mathbb{C}\,\mathbb{P}^1=\mathbb{P}(M^\perp)\to {}^{\mathbb{C}}Gr_{k+1,k}$$

given by $\ell\mapsto\ell\oplus M$. The intersection $Z_1\cap\mathbb{C}\,\mathbb{P}^1$ consists of two points $\{0,\infty\}$, and Z_1 intersects $\mathbb{C}\,\mathbb{P}^1$ transversally. Next $U_1\cap\mathbb{C}\,\mathbb{P}^1=\mathbb{C}^\times$. Let $t:\mathbb{C}^\times\hookrightarrow\mathbb{C}P^1$. As previously we can conclude that if $t_*(s^!(h^!\mathcal{H}))$ is a simple $\mathcal{D}_{\mathbb{C}\,\mathbb{P}^1}$-module then $g_{1*}h^!\mathcal{H}$ is a simple $\mathcal{D}_{W_1}$-module. Hence let us check that the former module is simple. It is easy to see that the $\mathcal{D}_{\mathbb{C}^\times}$-module $s^!(h^!\mathcal{H})$ is equal to $\{z^{\frac{1}{2}+\mathbb{Z}}\}$. By direct (and well known) computation one shows that $t_*(\{z^{\frac{1}{2}+\mathbb{Z}}\})$ is a simple $\mathcal{D}_{\mathbb{C}\,\mathbb{P}^1}$-module. Combining this with Lemma 4.6 we conclude that $g_*\mathcal{H}$ is a simple $\mathcal{D}_W$-module, i.e. Proposition 4.5 is proved.

Let us recall that we are studying $\mathcal{H}_1=i_*\mathcal{H}$, where $i:U\hookrightarrow Y$ is the open affine embedding. Let us denote $\mathcal{H}_2:=i_{*!}\mathcal{H}$ the simple submodule of $\mathcal{H}_1$, which is also obviously ${}^{\mathbb{C}}K$-equivariant. By Proposition 4.5, $\mathcal{H}_2|_W=\mathcal{H}_1|_W$. Hence $\mathcal{H}_1/\mathcal{H}_2$ is concentrated on k-subspaces of $\mathbb{C}^n$, on which the restriction of our quadratic form has rank at most $k-2$. Let us denote by S_j ($\subset X$) the (locally closed) subvariety consisting of k-subspaces of $\mathbb{C}^n$ such that the restriction of our quadratic form has rank j. Then the singular support of any subquotient of $\mathcal{H}_1/\mathcal{H}_2$ is a union of some of the closures of the conormal bundles $T^*_{S_j}X$ with $0\le j\le k-2$.

In order to prove Theorem 4.3 it remains to prove

LEMMA 4.7. *Under the moment map π the sets $T^*_{S_j}X$, $0\le j\le k-2$, are mapped onto symmetric nilpotent matrices of rank at least two.*

Proof. Fix j, $0 \le j \le k-2$, and any $E \in S_j$. Then the conormal vectors to S_j at E are identified (using the Killing form on $\mathfrak{g}\ell_n(\mathbb{C})$) with the set of linear operators $A : \mathbb{C}^n \to \mathbb{C}^n$ such that

(i) A is symmetric;
(ii) $A(\mathbb{C}^n) \subset E$;
(iii) $A(E) = \{0\}$.

Let us show that if $0 \le j \le k-2$ then there exists A with properties (i)–(iii) and $\operatorname{rk} A \ge 2$. By assumption we can choose $\ell = k - j \ge 2$ linearly independent vectors $e_1, \dots, e_l \in E$ such that $\langle e_i, x\rangle = 0$ for any $x \in E$ and any $i = 1, \dots, l$.

Consider $Ax = \sum_{i=1}^{\ell} \langle x, e_i\rangle e_i$.

Clearly A satisfies (i)–(iii) and $\operatorname{rk} A = \ell \ge 2$. ◻

Thus Theorem 4.3 is proved. It and Theorem 3.1 imply our main result in the case of even valuations.

5 Proof of the Main Theorem in the Odd Case

Recall some notation from section 2. We fix a $(k+1)$-dimensional linear subspace $E_0 \subset \mathbb{R}^n$. We denote by Ω the representation of $GL(E_0) = GL(k+1,\mathbb{R})$ in the space of odd valuations on E_0 homogeneous of degree k. Let P_1 denote the (parabolic) subgroup of $GL(n,\mathbb{R})$ fixing E_0. By Theorem 2.7 the space $\mathrm{Val}_{n,k}^{\mathrm{odd}}$ embeds as a $GL(n,\mathbb{R})$-module into the representation $\mathrm{Ind}_{P_1}^{GL(n,\mathbb{R})}\,\Omega$. Let P_2 be the parabolic subgroup consisting of matrices of the form

$$T = \begin{bmatrix} A & \multicolumn{2}{c}{*} \\ O & b & \\ & & C \end{bmatrix},$$

where $A \in GL(k,\mathbb{R})$, $b \in \mathbb{R}^\times$, $C \in GL(n-k-1,\mathbb{R})$. Consider the character ξ of P_2

$$\xi(T) = |\det A|^{-1} \cdot \operatorname{sgn} b\,.$$

Corollary 2.8 states that the $GL(n,\mathbb{R})$-module $\mathrm{Val}_{n,k}^{\mathrm{odd}}$ can be realized as a subquotient of $\mathrm{Ind}_{P_2}^{GL(n,\mathbb{R})}\,\xi$.

First let us construct an $O(k+1,\mathbb{C})$-equivariant $\mathcal{D}$-module on ${}^{\mathbb{C}}Gr_{k+1,k}$ satisfying the assumptions of the Beilinson–Bernstein theorem such that the $(g\ell(k+1,\mathbb{C}), O(k+1,\mathbb{C}))$-module of its global sections coincides with the Harish-Chandra module of Ω.

For a real subspace $E_1 \subset E_0$, $\dim E_1 = k$, let us denote by $P \subset GL(E_0,\mathbb{R})$ the parabolic subgroup of transformations preserving E_1. It was shown in section 2 that Ω can be realized as a subquotient of the representation of $GL(E_0,\mathbb{R})$ induced from the character χ of P, where

$$\chi\left(\left[\begin{array}{c|c} A & * \\ \hline 0 & b \end{array}\right]\right) = |\det A|^{-1}\cdot \operatorname{sgn} b\,,$$

and $A \in GL(k,\mathbb{R})$, $b \in \mathbb{R}^\times$. More precisely Ω is isomorphic to a quotient module M_1/M_2, where M_1 is the Harish-Chandra module of $\operatorname{Ind}_P^{GL(E_0,\mathbb{R})}\chi$, and M_2 is a finite dimensional submodule of it.

Let Y denote the complex Grassmannian ${}^{\mathbb{C}}Gr_{k+1,k}$ of k-dimensional complex subspaces of $E_0 \otimes \mathbb{C}$. Let $V \subset Y$ denote the open affine subvariety of Y consisting of those k-dimensional subspaces, on which the restriction of the given quadratic form is non-degenerate (again we have fixed some positive definite quadratic form on $\mathbb{R}^n$ and consider its complexification on $\mathbb{C}^n = \mathbb{R}^n \otimes \mathbb{C}$). Clearly $V = O(k+1,\mathbb{C}) \cdot {}^{\mathbb{R}}Gr_{k,k+1}$ (under the natural embedding ${}^{\mathbb{R}}Gr_{k+1,k} \subset {}^{\mathbb{C}}Gr_{k+1,k}$). Similarly to section 4 we observe that the space of $O(k+1,\mathbb{C})$-finite vectors of Ω can be realized as a space of regular sections of a certain algebraic line bundle T over V, and the sheaf of regular sections of T is a $\mathcal{D}_\chi|_V$-module, where $\mathcal{D}_\chi$ is the sheaf of twisted differential operators corresponding to χ. Again χ satisfies the assumptions of the Beilinson–Bernstein theorem ([Bi, Th. I.6.3]). The reasoning is similar to that for the even case. Namely consider a new $GL(k+1,\mathbb{R})$-module $\operatorname{Ind}_P^{GL(E_0,\mathbb{R})}\chi_1$, where

$$\chi_1\left(\left[\begin{array}{c|c} A & * \\ \hline 0 & b \end{array}\right]\right) = (\det A)^{-1}.$$

Clearly the sheaf of twisted differential operators $\mathcal{D}_{\chi_1}$ is equal to $\mathcal{D}_\chi$. So it is sufficient to check the conditions of the Beilinson–Bernstein theorem for $\mathcal{D}_{\chi_1}$. But this module has a finite dimensional submodule (for the reason exactly as in the even case, see the beginning of section 4). Hence $\mathcal{D}_\chi$ satisfies the conditions of the Beilinson–Bernstein theorem.

Let $\mathcal{M} := i_*T$ be the $\mathcal{D}_\chi$-module on Y. As in section 4 we have

Claim 5.1. The embedding $i : V \hookrightarrow Y$ is affine, $H^i(Y,\mathcal{M}) = 0$ if $i > 0$, and $\Gamma(Y,\mathcal{M})$, considered as a $(U(g\ell_{k+1}(\mathbb{C})), O(k+1,\mathbb{C}))$-module, is the Harish-Chandra module of Ω.

Let us denote by $\mathcal{M}_1$ and $\mathcal{M}_2$ the $\mathcal{D}_\chi$-submodules of $\mathcal{M}$ corresponding to M_1 and M_2, respectively (by the Beilinson–Bernstein theorem). Since M_1 is finite dimensional, $\mathcal{M}_1$ is a coherent $\mathcal{O}_Y$-module and its singular support $Ch(\mathcal{M}_1)$ coincides with the zero section of T^*Y. Hence the restriction $\mathcal{M}_1|_V$ is not zero. But $\mathcal{M}|_V = T$ is an irreducible $\mathcal{D}_\chi|_V$-module. Hence $\mathcal{M}_1|_V = \mathcal{M}|_V$. But $\mathcal{M}_1$ is a submodule of $\mathcal{M}_2$. Thus we obtain

Claim 5.2. $(\mathcal{M}_2/\mathcal{M}_1)|_V = 0$.

On the complex Grassmannian $Y = {}^{\mathbb{C}}Gr_{k+1,k}$ there are just two $O(k+1,\mathbb{C})$-orbits: V and the subvariety of subspaces of $E_0 \otimes \mathbb{C}$, on which the restriction of our quadratic form has a one-dimensional kernel. In what follows we will denote by $\mathcal{N}$ the $\mathcal{D}_\chi$-module $\mathcal{M}_2/\mathcal{M}_1$ on ${}^{\mathbb{C}}Gr_{k+1,k}$.

Lemma 5.3. *$\mathcal{N}$ is a simple $O(k+1,\mathbb{C})$-equivariant $\mathcal{D}_\chi$-module.*

Proof. We will reduce the statement to the case $k = 1$. Assume first that $k > 1$. Fix a $(k-1)$-dimensional subspace L such that the restriction of the quadratic form on L is non-degenerate. Consider the subvariety $\{E \in {}^{\mathbb{C}}Gr_{k+1,k} | E \supset L\}$. It is isomorphic to $\mathbb{C}P^1$. Denote by j the embedding

$$j : \mathbb{C}P^1 \hookrightarrow {}^{\mathbb{C}}Gr_{k+1,k}\,.$$

This $\mathbb{C}P^1$ is transversal to the orbit ${}^{\mathbb{C}}Gr_{k+1,k}\backslash V$. Hence $\mathcal{N}$ is a simple $O(k+1,\mathbb{C})$-equivariant $\mathcal{D}_\chi$-module iff $j^!\mathcal{N}$ is a simple $O(2,\mathbb{C})$-equivariant $j^{\cdot}\mathcal{D}_\chi$-module. Hence let us assume that $k = 1$. In this case, the $(g\ell(2,\mathbb{C}), O(2,\mathbb{C}))$-module M_1 is $\mathrm{Ind}_Q^{GL(2,\mathbb{R})}\chi$, where Q consists of 2×2 matrices of the form

$$X = \begin{bmatrix} a & * \\ 0 & b \end{bmatrix}, \quad a,b \in \mathbb{R}^\times, \quad \chi(X) = |a|^{-1}\cdot \mathrm{sgn}\, b\,.$$

Using the translation functor we can change χ to χ_1 defined as

$$\chi_1(X) = |a|^{-1}|b|^{-1} = |\det X|^{-1}\,.$$

Moreover we can replace $\mathrm{Ind}_Q^{GL(2,\mathbb{R})}\chi_1$ by $|\det\cdot|\otimes \mathrm{Ind}_Q^{GL(2,\mathbb{R})}\chi_1$, which is isomorphic to $\mathrm{Ind}_Q^{GL(2,\mathbb{R})}\mathbb{1}$. It is well known that the last module has just one proper irreducible submodule (which is finite dimensional) and the quotient module is irreducible. In fact the last statement can be seen immediately using $\mathcal{D}$-modules. Indeed if for the moment $\mathcal{D}$ denotes the ring of (untwisted) differential operators on $\mathbb{C}P^1$ then $\mathrm{Ind}_Q^{GL(2,\mathbb{R})}\mathbb{1}$ corresponds to the following $\mathcal{D}$-module on $\mathbb{C}P^1$. Let $f : \mathbb{C}^\times \hookrightarrow \mathbb{C}P^1$ be the identical embedding (i.e. $\mathbb{C}P^1\backslash\mathbb{C}^\times = \{0,\infty\}$). Then the $\mathcal{D}$-module we are looking for is $f_*(\mathcal{O}_{\mathbb{C}^\times})$. It has a simple submodule $\mathcal{O}_{\mathbb{C}P^1}$, and the quotient module $f_*(\mathcal{O}_{\mathbb{C}^\times})/\mathcal{O}_{\mathbb{C}P^1}$ is a simple $O(2,\mathbb{C})$-equivariant $\mathcal{D}$-module. □

By Theorem 2.7 the space $\mathrm{Val}_{n,k}^{\mathrm{odd}}$ embeds into $\mathrm{Ind}_{P_1}^{GL(n,\mathbb{R})}\Omega$, where P_1 is the parabolic subgroup of $GL(n,\mathbb{R})$ fixing the given $(k+1)$-dimensional real subspace $E_0 \subset \mathbb{R}^n$. Next we are going to construct a $\mathcal{D}$-module on the complex variety of partial flags on $\mathbb{C}^n$ of type $(k, k+1)$ satisfying the assumptions of the Beilinson–Bernstein theorem and whose space of global sections, considered as a $(U(g\ell(n,\mathbb{C}), O(n,\mathbb{C}))$-module, is isomorphic to the Harish-Chandra module of $\mathrm{Ind}_{P_1}^{GL(n,\mathbb{R})} \Omega$.

Note that the Levi factor of P_1 is $GL(k+1,\mathbb{R}) \times GL(n-k-1,\mathbb{R})$. Let us for brevity denote by H the subgroup $O(k+1,\mathbb{C}) \times O(n-k-1,\mathbb{C})$ of $O(n,\mathbb{C})$.

Let us consider the varieties

$$Z = O(n,\mathbb{C}) \times {}^{\mathbb{C}}Gr_{k+1,k},$$
$$W = O(n,\mathbb{C}) \times_H {}^{\mathbb{C}}Gr_{k+1,k} = Z/H ,$$

where H acts on ${}^{\mathbb{C}}Gr_{k+1,k}$ through the first factor $O(k+1,\mathbb{C})$ while the second factor $O(n-k-1,\mathbb{C})$ acts trivially on ${}^{\mathbb{C}}Gr_{k+1,k}$; and H acts on $O(n,\mathbb{C})$ by right translations. We have the projections

$$p : Z \to {}^{\mathbb{C}}Gr_{k+1,k} ,$$
$$q : Z \to W .$$

W is the geometric factor of Z by the free action of H. We have constructed an $O(k+1,\mathbb{C})$-equivariant $\mathcal{D}_\chi$-module $\mathcal{N}$ on ${}^{\mathbb{C}}Gr_{k+1,k}$.

On Z we have naturally the ring of twisted differential operators $\mathcal{D} := p^{\cdot}\mathcal{D}_\chi$ (see [BB2, Section 1]). Consider $\mathcal{N}_1 := p^!\mathcal{N}$. $\mathcal{N}_1$ is a $\mathcal{D}$-module, which is naturally $(O(n,\mathbb{C}) \times H)$-equivariant (the H-equivariance comes from the $O(k+1,\mathbb{C})$-equivariance of $\mathcal{N}$).

By [BB2] the category of H-equivariant $\mathcal{D}_\chi$-modules on ${}^{\mathbb{C}}Gr_{k+1,k}$ is equivalent to the category of $(O(n,\mathbb{C}) \times H)$-equivariant $p^{\cdot}\mathcal{D}_\chi$-modules on Z. Hence since $\mathcal{N}$ was simple, $\mathcal{N}_1$ is simple too as $O(n,\mathbb{C}) \times H$-equivariant $\mathcal{D}$-module.

Since the action of H is free on Z, again by [BB2], the category of $(O(n,\mathbb{C}) \times H)$-equivariant rings of twisted differential operators on Z is equivalent to the category of $O(n,\mathbb{C})$-equivariant rings of twisted differential operators on $Z/H = W$. Thus we have a ring $\mathcal{D}'$ of $O(n,\mathbb{C})$-equivariant twisted differential operators on W corresponding to $\mathcal{D}$. Moreover the category of $(O(n,\mathbb{C}) \times H)$-equivariant $\mathcal{D}$-modules on Z is equivalent to the category of $O(n,\mathbb{C})$-equivariant $\mathcal{D}'$-modules on W. Thus we obtain an $O(n,\mathbb{C})$-equivariant $\mathcal{D}'$-module $\mathcal{N}_2$ on W corresponding to $\mathcal{N}_1$. Since $\mathcal{N}_1$ is simple, $\mathcal{N}_2$ is also simple.

It is easy to see that the representation of $O(n,\mathbb{C})$ in the space $\Gamma(W,\mathcal{N}_2)$ is isomorphic to the representation of it in the Harish-Chandra module of

$\operatorname{Ind}_{P_1}^{GL(n,\mathbb{R})}\Omega$.

Next W can be identified with an open subvariety of the variety ${}^{\mathbb{C}}F_{n,k+1,k}$ of complex partial flags of type $(k, k+1)$ in $\mathbb{C}^n$. Namely under this identification W consists of those pairs of subspaces $A \subset B$ ($\dim A = k$, $\dim B = k+1$) such that the restriction of our quadratic form on B is non-degenerate.

On ${}^{\mathbb{C}}F_{n,k+1,k}$ there exists a ring of twisted differential operators $\widetilde{\mathcal{D}}$ such that $\widetilde{\mathcal{D}}|_W = \mathcal{D}'$. Indeed such a $\widetilde{\mathcal{D}}$ corresponds to the character ξ of the parabolic subgroup $P_2 \subset GL(n,\mathbb{R})$, where P_2 consists of matrices of the form

$$X = \begin{bmatrix} A & * & * \\ O & b & * \\ O & O & C \end{bmatrix},$$

where $A \in GL(k,\mathbb{R})$, $b \in \mathbb{R}^\times$, $C \in GL(n-k-1,\mathbb{R})$, and $\xi(X) = |\det A|^{-1} \cdot \operatorname{sgn} b$ (see section 2). Also $\widetilde{\mathcal{D}}$ satisfies the assumptions of the Beilinson–Bernstein theorem.

It follows from the construction that the support of $\mathcal{N}_2$ is the subvariety of W consisting of partial flags $A \subset B$ such that the restriction of our quadratic form on B is non-degenerate and the restriction of it on A has a one-dimensional kernel.

Let $i : W \hookrightarrow {}^{\mathbb{C}}F_{n,k+1,k}$ denote the identical embedding. One can easily see that i is an *affine* embedding. The sheaf $\mathcal{N}_3 := i_*\mathcal{N}_2$ is clearly an $O(n,\mathbb{C})$-equivariant $\widetilde{\mathcal{D}}$-module, and $\Gamma({}^{\mathbb{C}}F_{n,k+1,k}, \mathcal{N}_3) = \Gamma(W, \mathcal{N}_2)$. It can be readily seen that this space $\Gamma({}^{\mathbb{C}}F_{n,k+1,k}, \mathcal{N}_3)$ considered as $(U(g\ell(n,\mathbb{C}))$, $O(n,\mathbb{C}))$-module is isomorphic to the Harish-Chandra module of $\operatorname{Ind}_{P_1}^{GL(n,\mathbb{R})}\Omega$.

Next $\operatorname{supp}\mathcal{N}_3$ is the closure of $\operatorname{supp}\mathcal{N}_2$ in ${}^{\mathbb{C}}F_{n,k+1,k}$. By construction, $\operatorname{supp}\mathcal{N}_2$ consists of flags $A \subset B$ such that the restriction of the quadratic form on B is non-degenerate, and whose restriction to A has a one-dimensional kernel. Hence we have

CLAIM 5.4. $\operatorname{supp}\mathcal{N}_3 \subset {}^{\mathbb{C}}F_{n,k+1,k}$ consists of the flags $A \subset B$ such that the restriction of the quadratic form on A has a kernel of dimension at least one.

For any subquotient $\mathcal{R}$ of $\mathcal{N}_3$ we have (similarly to Proposition 4.2)

$$V_{\mathfrak{g}}(\Gamma(\mathcal{R})) = \pi(Ch\mathcal{R}),$$

where $V_{\mathfrak{g}}(\Gamma(\mathcal{R}))$ is the associated variety of $\Gamma(\mathcal{R})$, $Ch\mathcal{R}$ is the singular support of $\mathcal{R}$, and

$$\pi : T^*({}^{\mathbb{C}}F_{n,k+1,k}) \to \mathfrak{g}\ell^*(n,\mathbb{C})$$

is the moment map.

Let $\mathcal{M} := i_{!*}\mathcal{N}_2$ be the minimal extension of $\mathcal{N}_2$. $\mathcal{M}$ is a simple $\widetilde{\mathcal{D}}$-submodule of $\mathcal{N}_3$ (since $\mathcal{N}_2$ is simple). Clearly $\mathrm{supp}(\mathcal{N}_3/\mathcal{M}) \subset {}^{\mathbb{C}}F_{n,k+1,k}\backslash W$.

Now the proof of our main result in the odd case will follow from the next lemma.

Lemma 5.5. *Let $\mathcal{R}$ be a nonzero $O(n,\mathbb{C})$-equivariant $\widetilde{\mathcal{D}}$-module on ${}^{\mathbb{C}}F_{n,k+1,k}$ such that* $\mathrm{supp}\,\mathcal{R}$ *is contained in the subvariety of ${}^{\mathbb{C}}F_{n,k+1,k}$ consisting of pairs $A \subset B$ such that the restriction of our quadratic form on A and on B is degenerate. Then the image of the singular support of $\mathcal{R}$ under the moment map $\pi(Ch\mathcal{R})$ cannot be contained in the variety of nilpotent matrices of rank at most one.*

(This lemma indeed implies the main result by the computation in section 3 of the associated variety of the space of valuations and its equality to $\pi(Ch\mathcal{R})$.)

Proof of Lemma 5.5. Choose an orbit S of maximal dimension which is contained in $\mathrm{supp}\,\mathcal{R}$. Then in particular $S \subset {}^{\mathbb{C}}F_{n,k+1,k}\backslash W$, and the conormal bundle $T^*_S({}^{\mathbb{C}}F_{n,k+1,k})$ is contained in $Ch\mathcal{R}$. Fix an arbitrary point $y \in S$. Then y is a pair $A \subset B$ of subspaces with $\dim A = k$, $\dim B = k+1$. The fiber of $T^*_S({}^{\mathbb{C}}F_{n,k+1,k})$ over y consists of symmetric nilpotent matrices Λ such that

$$\Lambda(\mathbb{C}^n) \subset B\,, \quad \Lambda(B) \subset A\,, \quad \Lambda(A) = 0\,.$$

It is sufficient to show that under the assumptions of the lemma there exists Λ with these properties and $\mathrm{rk}\,\Lambda \geq 2$.

Since $\mathrm{supp}\,\mathcal{R} \subset {}^{\mathbb{C}}F_{n,k+1,k}\backslash W$ and taking into account Claim 5.4 we can consider the following cases (where we denote by Q our quadratic form on $\mathbb{C}^n$):

1. $\mathrm{rk}\,Q|_A \leq k-2$;
2. $\mathrm{rk}\,Q|_A = k-1$, $\mathrm{rk}\,Q|_B \leq k$.

First consider case 1. Choose $v_1, v_2 \in A$, two linearly independent vectors belonging to the kernel of $Q|_A$. Consider the operator $\varphi : \mathbb{C}^n \to \mathbb{C}^n$ defined as

$$\varphi(x) = Q(x,v_1)v_1 + Q(x,v_2)v_2\,.$$

Since the form Q is non-degenerate then $\mathrm{rk}\,\varphi = 2$. Also φ is symmetric and $\mathrm{Im}\,\varphi \subset A$, $\varphi(A) = 0$.

Now consider case 2. By assumption the kernel of $Q|_B$ is nontrivial. Assume first that $\operatorname{Ker} Q|_B \not\subset A$. Then we can choose $v_1 \in \operatorname{Ker} Q|_B$ such that $v_1 \notin A$. Then choose $v_2 \in \operatorname{Ker} Q|_A \backslash \{0\}$. Consider the operator $\varphi(x) = Q(v_1, x)v_1 + Q(v_2, x)v_2$. Clearly φ is symmetric of rank two and $\operatorname{Im} \varphi \subset B$. For any $b \in B$, $\varphi(b) = Q(v_2, b)v_2 \in A$, namely $\varphi(B) \subset A$. For any $a \in A$, $\varphi(a) = 0$. Hence in this situation Lemma 5.5 is proved.

Assume now that $\operatorname{Ker} Q|_B \subset A$. Since by the assumption of case 2, $\operatorname{rk} Q|_A = k - 1$, then $\operatorname{Ker} Q|_B$ is one-dimensional and assume that it is spanned by $v_1 \in A$. It is easy to see that the orthogonal complement L of A in B is two-dimensional (i.e. $L := \{x \in B | Q(x, a) = 0\,, \forall a \in A\}$). Indeed the form Q is defined on the quotient space $B/(\operatorname{Ker} Q|_B)$ and is nondegenerate on it; the orthogonal complement of $A/(KerQ|_B) = A/L$ is one-dimensional. So let us choose $v_2 \in L$ to be linearly independent of v_1. Consider the operator $\varphi(x) = Q(v_1, x)v_2 + Q(v_2, x)v_1$. Then φ is symmetric of rank two and $\operatorname{Im} \varphi \subset B$. For any $b \in B$, $\varphi(b) = Q(v_2, b)v_1 \in A$, namely $\varphi(B) \subset A$. For any $a \in A$, $\varphi(a) = 0$. Hence Lemma 5.5 is proved. This finishes the proof of Theorem 1.3. □

6 Unitarily Invariant Valuations

In this section we deal with translation invariant continuous valuations on convex compact subsets of $\mathbb{C}^n$, which are invariant with respect to the unitary group $U(n)$. This space of valuations turns out to be finite dimensional, and we compute the dimension of this space. Namely, we have

Theorem 6.1. *For every k, $0 \leq k \leq 2n$, the dimension of the space of translation invariant $U(n)$-invariant continuous valuations on $\mathbb{C}^n$, homogeneous of degree k, is equal to*

$$1 + \left[\frac{\min(k, 2n - k)}{2} \right].$$

REMARK. Recall that by McMullen's Theorem 1.2 every translation invariant continuous valuation can be uniquely presented as a sum of homogeneous valuations. The uniqueness implies that if the valuation was in addition unitarily invariant then all homogeneous summands must also be unitarily invariant.

Note that the cases $k = 0$ and $k = 2n$ of the above theorem follow immediately from Theorem 2.1. The cases $k = 1$ and $k = 2n - 1$ were proved in [A] (see Lemmas 7.4 and 7.1 respectively) using results of [GW] and [M2]. So from now on we will assume that $2 \leq k \leq 2n - 2$.

The proof of this result will use representation theoretical computations of Howe and Lee [HoL]. Now we are going to formulate the particular case of their result we need. First let us recall that irreducible representations of the group $SO(2n)$ are paramatrized by n-tuples of integer numbers $(\mu_1,\dots,\mu_n)$ (called highest weights) such that

$$\mu_1 \geq \dots \geq \mu_{n-1} \geq |\mu_n|.$$

Let $1 \leq m \leq 2n-1$. Let P_m be the parabolic subgroup of $GL(2n,\mathbb{R})$ consisting of matrices of the form

$$\left[\begin{array}{c|c} A & * \\ \hline 0 & B \end{array}\right], \text{ where} A \in GL(2n-m,\mathbb{R}),\ B \in GL(m,\mathbb{R}).$$

Let $\delta : P_m \to \mathbb{C}$ be the character

$$\delta\left(\left[\begin{array}{c|c} A & * \\ \hline 0 & B \end{array}\right]\right) = |\det B|.$$

For $1 \leq l \leq n$ let us denote by $\Lambda(l)$ the subset of highest weights of $SO(2n)$ of the form

(a) if $1 \leq l < n$ set

$$\Lambda(l) := \big\{(\mu_1,\dots,\mu_l,0,\dots,0) \ \big| \ \mu_1 \geq \dots \geq \mu_l \geq 0\big\};$$

(b) if $l = n$ let $\Lambda(l)$ be the set of all heighest weights of $SO(2n)$.

The following result was proved in [HoL] (see Theorem 3.4.4 (a)(ii) in [HoL] and the discussion on p. 288 of that paper).

Theorem 6.2. (i) *Let $2 \leq m \leq n$. Then the length of the $GL(2n,\mathbb{R})$-module $\mathrm{Ind}_{P_m}^{GL(2,\mathbb{R})}\delta$ is equal to $1+[m/2]$.*

(ii) *The representation of $SO(2n)$ in $\mathrm{Ind}_{P_m}^{GL(2,\mathbb{R})}\delta$ is multiplicity free. Moreover if we denote (as in [HoL]) by $R^+(1,j)$, $j = 1,3,5,\dots,1+2[m/2]$ all the subquotients of $\mathrm{Ind}_{P_m}^{GL(2,\mathbb{R})}\delta$ then the representation of $SO(2n)$ in $R^+(1,j)$ decomposes into irreducible components which have highest weights $(\mu_1,\mu_2,\dots,\mu_n) \in \Lambda(m)$ precisely of the following form:*

(a) *if $j = 1$ then μ_i is even $\forall i$, and $|\mu_2| \leq 2$;*
(b) *if $2 \leq j \leq k-1$ then μ_i is even $\forall i$, and $\mu_{j-1} \geq 2 \geq \mu_{j+1}$;*
(c) *if $j = k, k+1$ then μ_i is even $\forall i$, and $\mu_{j-1} \geq 2$.*

(iii) *In the $GL(2n,\mathbb{R})$-module $\mathrm{Ind}_{P_m}^{GL(2n,\mathbb{R})}\delta$ there is only one irreducible submodule. This submodule is isomorphic to $R^+(1,1)$.*

We will need the following

Proposition 6.3. *Under the action of $SO(2n)$ the space $\mathrm{Val}_{2n,k}^{ev}$ is multiplicity free and is a direct sum of irreducible components with highest* *weights* $\mu = (\mu_1, \dots, \mu_n)$ *precisely of the form* $\mu \in \Lambda(\min(k, 2n-k))$ *and* $|\mu_2| \le 2$.

Proof. First recall that the space $\mathrm{Val}^{ev}_{2n,k}$ naturally embeds into the space $I := \mathrm{Ind}^{GL(2n,\mathbb{R})}_{P_{2n-k}} \chi$, where as above P_{2n-k} consists of matrices of the form

$$\left[\begin{array}{c|c} A & * \\ \hline 0 & B \end{array}\right], \text{ where } A \in GL(k,\mathbb{R}),\ B \in GL(2n-k,\mathbb{R}),$$

and χ is the character $\chi : P_{2n-k} \to \mathbb{C}$

$$\chi\left(\left[\begin{array}{c|c} A & * \\ \hline 0 & B \end{array}\right]\right) = |\det A|^{-1}.$$

We will denote by $|\det|$ the character of $GL(2n,\mathbb{R})$ equal to the absolute value of the determinant.

LEMMA 6.4. *Assume* $n \le k \le 2n-2$. *Then* $\mathrm{Val}^{ev}_{2n,k} \otimes |\det|$ *is infinitesimally isomorphic to* $R^+(1,1)$

Clearly this lemma implies Proposition 6.3 in the case $n \le k \le 2n-2$. Let us prove it.

Clearly $\mathrm{Val}^{ev}_{2n,k} \otimes |\det|$ is also an irreducible $GL(2n,\mathbb{R})$-module with the same K-type structure as $\mathrm{Val}^{ev}_{2n,k}$. It can be embedded into $\mathrm{Ind}^{GL(2n,k)}_{P} \chi_1$, where

$$\chi_1\left(\left[\begin{array}{c|c} A & * \\ \hline 0 & B \end{array}\right]\right) = |\det B|.$$

Hence by Theorem 6.2 (iii) the space $\mathrm{Val}^{ev}_{2n,k} \otimes |\det|$ is infinitesimally isomorphic to $R^+(1,1)$. So Lemma 6.4 is proved.

Now we are going to prove Proposition 6.3 in the case of $\mathrm{Val}^{ev}_{2n,k}$, $2 \le k < n$. It is easy to see that the representation $I = \mathrm{Ind}^{GL(2n,\mathbb{R})}_{P_{2n-k}} \chi$ is *unitarily* induced from the character $\theta : P_{2n-k} \to \mathbb{C}$, where

$$\theta\left(\left[\begin{array}{c|c} A & * \\ \hline 0 & B \end{array}\right]\right) = \chi\left(\left[\begin{array}{c|c} A & * \\ \hline 0 & B \end{array}\right]\right) \cdot \left(|\det A|^{2n-k} \cdot |\det B|^{-k}\right)^{-\frac{1}{2}}$$
$$= |\det A|^{-\frac{2n-k}{2}-1} \cdot |\det B|^{\frac{k}{2}}.$$

Now consider the parabolic subgroup P_k consisting of matrices of the form

$$\left[\begin{array}{c|c} B & * \\ \hline 0 & A \end{array}\right], \text{ where } B \in GL(2n-k,\mathbb{R}),\ A \in GL(k,\mathbb{R}).$$

Consider another representation I' of $GL(2n,\mathbb{R})$ which is *unitarily* induced from the character $\theta' : P_k \to \mathbb{C}$ defined by

$$\theta\left(\left[\begin{array}{c|c} B & * \\ \hline 0 & A \end{array}\right]\right) = |\det A|^{-\frac{2n-k}{2}-1} \cdot |\det B|^{\frac{k}{2}}.$$

It is well known that the Jordan–Holder series of I' coincides with the Jordan–Holder series of I up to a permutation of elements. It is easy to see that the dual representation $(I')^*$ is infinitesimally equivalent to the (non-unitarily) induced representation $\mathrm{Ind}_{P_k}^{GL(2n,\mathbb{R})}\lambda$, where

$$\lambda\left(\left[\begin{array}{c|c} B & * \\ \hline 0 & A \end{array}\right]\right) = |\det A|\,.$$

Since by assumption $2 \le k < n$ the last representation satisfies conditions of Theorem 6.2. Thus we conclude that, for $2 \le k < n$, $\mathrm{Val}_{2n,k}^{ev}$ is infinitesimally equivalent to the dual representations of one of the subquotients of $\mathrm{Ind}_{P_k}^{GL(2n,\mathbb{R})}\lambda$, which are $R^+(1,j)$, $j = 1,3,5,\dots,2[k/2]+1$. Since any irreducible representation of the special orthogonal group is isomorphic to its dual then the K- type structure of $\mathrm{Val}_{2n,k}^{ev}$ coincides with that of one of $R^+(1,j)$.

Recall (Proposition 2.10) that $\mathrm{Val}_{2n,k}^{ev}$ can be embedded into the $GL(2n,\mathbb{R})$-module $\mathrm{Ind}_{P_1}^{GL(2n,\mathbb{R})}\rho$, where P_1 is the parabolic subgroup consisting of matrices of the form $\left[\begin{array}{c|c} A & * \\ \hline 0 & b \end{array}\right]$, where $A \in GL(2n-1,\mathbb{R})$, $b \in \mathbb{R}^\times$, and ρ is a finite dimensional representation of the Levi subgroup of P_1.

Let us restrict this representation to $SO(2n)$. We will obtain a representation Φ of $SO(2n)$ in sections of some finite dimensional vector bundle over the unit sphere S^{2n-1}.

Claim 6.5. There is a constant C (depending on n only) such that the highest weight $(\mu_1,\dots,\mu_n)$ of any irreducible component of Φ satisfies $|\mu_2| \le C$.

Obviously Claim 6.5 and Theorem 6.2 imply Proposition 6.3. Let us prove Claim 6.5. Let us fix a vector $x_0 \in S^{2n-1}$. Let $SO(2n-1)$ denote the stabilizer of x_0 in $SO(2n)$. Let F denote the fiber at x_0 of our vector bundle over S^{2n-1}. Then F is $SO(2n-1)$-module. For any irreducible component ϕ of Φ we have the $SO(2n-1)$-equivariant map

$$f : \phi \to F$$

defined by $f(s) = s(x_0)$ for any $s \in \phi$. This is non-zero map. Indeed any non-trivial $SO(2n)$-invariant subspace of Φ has a section which does not vanish at x_0 (since $SO(2n)$ acts transitively on the unit sphere).

Hence if we consider ϕ as $SO(2n-1)$-module then it contains at least one of the irreducible components of F. Let $(\nu_1,\dots,\nu_{n-1})$ be the highest weight of such a component. If we denote by $(\mu_1,\dots,\mu_n)$ the highest weight of ϕ then it is well known (see [Z]) that there are inequalities

$$\mu_1 \ge \nu_1 \ge \mu_2 \ge \nu_2 \ge \dots \ge \nu_{n-1} \ge |\mu_n|\,.$$

Since F is finite dimensional and fixed we see that $|\mu_2|$ is bounded by some constant. So Claim 6.5 is proved. This also finishes the proof of Proposition 6.3. $\square$

Let us prove Theorem 6.1. Since $(SO(2n), U(n))$ is a symmetric pair then for every irreducible representation of $SO(2n)$ the subspace of $U(n)$-invariant vectors has dimension at most one (see [T] or [He]).

Next it can be shown (e.g. see [T, §8]) that irreducible representations of $SO(2n)$ which have $U(n)$-fixed non-zero vectors have highest weights precisely of the form $(\mu_1, \dots, \mu_n)$, where

(i) if n is even, then

$$\mu_1 = \mu_2 \geq \mu_3 = \mu_4 \geq \dots \geq \mu_{n-1} = \mu_n \geq 0\,;$$

(ii) if n is odd, then

$$\mu_1 = \mu_2 \geq \mu_3 = \mu_4 \geq \dots \geq \mu_{n-2} = \mu_{n-1} \geq \mu_n = 0\,.$$

Obviously this fact and Proposition 6.3 imply Theorem 6.1. $\square$

REMARK. In [A] we obtained an *explicit* description of unitarily invariant translation invariant continuous valuations on $\mathbb{C}^2$. It would be interesting to obtain it in general on $\mathbb{C}^n$.

7 Questions and Comments

It was shown in [A, Theorem 8.1] that if G is a compact subgroup of the orthogonal group $O(n)$ which acts transitively on the unit sphere then the space of G-invariant translation invariant continuous valuations is finite dimensional. However there is a classification of compact connected groups acting transitively and effectively on the sphere due to Montgomery and Samelson [MoS] and Borel [Bo1,2]. They have obtained the following list:

6 infinite series: $SO(n)$, $U(n)$, $SU(n)$, $Sp(n)$, $Sp(n)\cdot U(1)$, $Sp(n)\cdot Sp(1)$;

3 exceptions: G_2, $Spin(7)$, $Spin(9)$.

The action of the group G on the corresponding sphere is defined by some linear representation of G in a real vector space (cf. also the discussion in [Be, Ch. 7 B]).

In each of these cases one has a finite dimensional space of G-invariant translation invariant continuous valuations. The case of the $SO(n)$ is completely covered by the Hadwiger theorem: every $SO(n)$-invariant translation invariant continuous valuation is a linear combination of the intrinsic volumes (in particular the dimension of this space is equal to $n+1$). The case of the unitary group $U(n)$ was discussed in section 6 of this paper, where we have computed the dimension of this space. It would be interesting to obtain the *explicit* description of these valuations (the case of $\mathbb{C}^1$ coincides with Hadwiger's characterization on the plane, and the case of $\mathbb{C}^2$ was completely described in [A]). We believe that such a description would have applications to integral geometry in the complex affine space $\mathbb{C}^n$ and the complex projective space $\mathbb{C}P^n$.

We think that the next interesting case to describe is the case of the group $Sp(n)\cdot Sp(1)$. Let us recall the definition of this group. Let $\mathbb{H}^n$ be the space of n-tuples of quaternions considered as a right quaternionic space. Then the group $Sp(n)$ acts on this space from the left. The group $Sp(1)$ of quaternions of unit length acts on this space (considered as a real vector space) by multiplication from the right. These actions of $Sp(n)$ and $Sp(1)$ commute and generate the group denoted by $Sp(n)\cdot Sp(1)$. The description of valuations invariant with respect to this group would have applications to integral geometry of the quaternionic affine space $\mathbb{H}^n$ and the quaternionic projective space $\mathbb{H}P^n$. (Note that for $n = 1$ one has $Sp(1)\cdot Sp(1) = SO(4)$, and we are again reduced to the Hadwiger theorem.)

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