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SALEM SETS AND RESTRICTION PROPERTIES OF FOURIER TRANSFORMS

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1 Introduction

The aim of this paper is to give on the real line an analogue of restriction phenomena of Fourier transforms first discovered in the late sixties by E.M. Stein for higher dimensions. In fact, we will establish a result on the real line which is almost as sharp as the Stein–Tomas theorem. This says, for a function from $L^p(\mathbf{R}^n)$, $1 \leq p \leq 2(n+1)/(n+3)$, $n > 1$, its Fourier transform restricted to the unit sphere S^{n-1} in $\mathbf{R}^n$ is well defined and square integrable against the uniform measure on S^{n-1}. Since the seventies there were many generalizations of this result, mainly for situations where the unit sphere is replaced by some smooth submanifold of $\mathbf{R}^n$ satisfying suitable curvature conditions, but similar questions have also been discussed in the case where the Fourier transform is replaced by a more general oscillatory integral operator (see e.g. [St], [Ch], [D], [Mü], [SeS]). More recently, J. Bourgain [B1,2] developed a method improving the Stein–Tomas result for $n \geq 3$ – the case $n = 2$ was settled by Stein in [F1]. The literature on the subject might suggest that these restriction phenomena are genuinely n-dimensional aspects of Fourier Analysis (see, e.g. [F3]). We will see however, that a proper analogue of restriction phenomena for Fourier transforms can be developed on the real line. In part this point of view is motivated by a recent result of Bourgain [B3], [W], showing that the bounds conjectured by H. Montgomery [Mo, p. 142] on finite Dirichlet sums imply that Kakeya sets, i.e. sets containing a line segment in every direction in $\mathbf{R}^n$, have Hausdorff dimension n. This conclusion is known to follow in a more natural way from the conjectured optimal restriction estimates for spheres, which in dual form state that

$$\int_{\mathbf{R}^n} |\widehat{fd\sigma}|^p dx \leq C\|f\|^p_{L^q(S^{n-1},d\sigma)} \tag{1}$$

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for $p > 2n/(n-1)$, $q = \infty$ and $d\sigma$ being the uniform measure on S^{n-1}.

The question we ask here is whether we can replace the measure $d\sigma$ in (1) by a probability measure $d\mu$ supported on a fractional dimensional compact set E on $\mathbf{R}$ and still obtain the estimate (1) with appropriate nontrivial exponents p, q. This, of course, requires that the Fourier transform of a measure $d\mu$ supported on a set E of Hausdorff dimension $\alpha \in (0,1)$ lies in $L^p(\mathbf{R})$ for some $p < \infty$. It is known that this implies strong conditions on the structure of the set E and $p > 2/\alpha$. Now, by answering a question of A. Buerling, R. Salem in [S] constructed for a fixed $\alpha \in (0,1)$ and each $\epsilon > 0$ measures $d\mu$ supported on a set of Hausdorff dimension α whose Fourier transform lies in $L^p(\mathbf{R})$ for $p > 2/\alpha + \epsilon$ (see [KL] for a nice historical background).

It seems natural to conjecture for the measures constructed by Salem that the following (L^q, L^p)-estimate holds:

$$\int_{\mathbf{R}} |\widehat{f d\mu}|^p dx \leq C \|f\|^p_{L^q(d\mu)} \,, \tag{2}$$

with $p > 2/\alpha + \epsilon$ and $q = \infty$. By multiplying the measure μ with a nonnegative weight function a factorization argument would allow us to lower the q-exponent. An interpolation argument with the trivial (L^1, L^∞)-estimate would then give an (L^2, L^p)-estimate which is essentially what we are going to proof here for the original measure $d\mu$.

We would like to point out that we cannot exclude the possibility that the inequality in (2) does hold for all $p > 2/\alpha + \epsilon$ and $q = 2$. Note that in case of the sphere the sharpness of the Stein–Tomas result follows from the curvature of the sphere, or, by considering the equivalent problem for the n-torus, from the fact that $\mathbf{Z}^n \cap \{R < |x| \leq R+1\}$ contains an $n-1$-dimensional arithmetic progression of size $R^{(n-1)/2}$. The analogue of this *curvature condition* for a $1/R$ neighborhood of the set $E = supp(\mu)$ would be a suitable estimate of the size of an arithmetic progression contained in it.

Restriction theorems for the Fourier transform are known to be an important tool to exploit cancellation properties of convolution operators (see [F1], [St], [B1]). The restriction theorems we show here will allow us to construct new $L^p(\mathbf{R})$-multipliers which in a sense play the same role that the Bochner–Riesz multipliers do in $\mathbf{R}^n$ for $n > 1$.

2 Hausdorff and Fourier dimension

We will begin with a short review of two notions of dimension. For details we refer to Kahane's book [K]. By a theorem of Frostman it is known that if $E \subset \mathbf{R}^n$ is a compact set of Hausdorff dimension α, then there is a probability measure μ supported on E satisfying $\mu(B_r(x)) \le C\, r^\alpha$, where $B_r(x)$ denotes a ball of radius r centered at x. Therefore, the β-potential $(0 < \beta < n)$ of μ at a point x defined as

$$I_\beta(\mu)(x) = \int \frac{d\mu(y)}{|x-y|^\beta}$$

is a bounded function as long as $\beta < \alpha$ and this implies the finiteness of the β-energy of μ defined as

$$I_\beta(\mu) = \iint \frac{d\mu(y)d\mu(x)}{|x-y|^\beta}$$

for $\beta < \alpha$. On the other hand the theorem of Frostman shows that if $I_\alpha(\nu) < \infty$, for some probability measure ν supported on a compact set E, then its Hausdorff dimension $\dim_H E \ge \alpha$. Since the Fourier transform of $|x|^{-\alpha}$, $0 < \alpha < n$, is $C|x|^{\alpha-n}$, one can show (see [C]):

$$I_\alpha(\mu) = c \int_{\mathbf{R}^n} \frac{|\widehat{d\mu}(y)|^2}{|y|^{n-\alpha}}\, dy\,.$$

Thus $I_\alpha(\mu) < \infty$ provides some information on the size of $\widehat{d\mu}$, although it does not even imply that $\widehat{d\mu}(x) \to 0$ as $x \to \infty$ (consider, e.g. $E = [0,1] \subset \mathbf{R}^2$).

We define the Fourier dimension of a compact set $E \subset \mathbf{R}^n$, denoted by $\dim_F E$, as the supremum of $\beta \ge 0$ such that for some probability measure $d\mu$ supported on E

$$\big|\widehat{d\mu}(x)\big| \le C|x|^{-\beta/2}.$$

Since the last inequality – or even the weaker assumption that $\widehat{d\mu} \in L^p(\mathbf{R}^n)$ for $p > 2n/\beta$ – implies $I_\alpha(\mu) < \infty$ for $\alpha < \beta$, we always have $\dim_F E \le \dim_H E$. In case E is a compact smooth α-dimensional submanifold of $\mathbf{R}^n$ and $d\mu$ is the measure induced by the Lebesgue measure on $\mathbf{R}^n$, we may expect an isotropic decay of the form $|x|^{-\alpha/2}$ only under some conditions on the curvature and on the dimension of E. Here are some examples:

(1) The unit sphere in $\mathbf{R}^n$ has Fourier dimension $n-1$.
(2) The boundary of the unit cube in $\mathbf{R}^n$ has Fourier dimension 0.
(3) The Cantor middle third set has Fourier dimension 0 and Hausdorff dimension $\log 2/\log 3$.

(4) R. Kaufman [Ka] has shown that for $t > 0$ the set E_t of those real numbers $x \in [0,1]$ such that

$$||qx|| \leq q^{-1-t}$$

has solutions for arbitrarily large integers q, $||x||$ denotes the distance to the nearest integer, has Fourier dimension and Hausdorff dimension equal to $\frac{2}{2+t}$.

These examples show that Hausdorff dimension and Fourier dimension do not agree in general. This is not surprising since Hausdorff dimension measures a metric property of a set, whereas the Fourier dimension measures an arithmetic property of a set. However, the sets in examples (1) and (4) do have the property that their Fourier dimension and Hausdorff dimension agree. Prior to the examples of Kaufman above the existence of subsets on the real line having this property was first shown by Salem [S] and they are named after him. Later J.-P. Kahane [K] provided a rich class of Salem sets by showing that images of compact sets of a given Hausdorff dimension under Brownian motion are almost surely Salem sets.

3 Salem's Construction

We begin with a generalization of the classical Cantor type construction (see [Z, p. 194]). Let $M > 2$ be an even integer and put $N = M^M$. Choose $\xi \in (0, 1/N)$ and let points $0 < a_1 < a_2 < \cdots < a_N < 1$ be given such that they are linearly independent over the rational numbers and such that

$$0 < a_1 < 1/N - \xi \quad \text{and} \quad \xi < a_k - a_{k-1} < 1/N\,, \quad k = 2, \ldots, N\,. \tag{3}$$

On an interval $[A, B]$ of length L, a dissection of type $(N, a_1, \ldots, a_N, \xi)$ is performed by calling the closed intervals

$$\big[A + La_k, A + L(a_k + \xi)\big]\,, \quad k = 1, \ldots, N\,,$$

white and the complementary intervals *black*. Let $\Xi = (\xi_k)_{k\geq 1}$ be a sequence such that

$$\big(1 - \tfrac{1}{2k^2}\big)\,\xi \leq \xi_k \leq \xi\,, \quad k \geq 1\,.$$

Starting with $E_0 = [0,1]$, we perform a dissection of type $(N, a_1, \ldots, a_N, \xi_1)$ and remove the black intervals, thereby obtaining a set E_1 which is a union of N intervals each of length ξ_1. On each remaining interval we perform a dissection of type $(N, a_1, \ldots, a_N, \xi_2)$, remove the black intervals and so obtain a set E_2 of N^2 intervals each of length $\xi_1\xi_2$. After n steps we obtain a set E_n of N^n intervals each of length $\xi_1\xi_2 \ldots \xi_n$. Put $E = \bigcap_{n\geq 0} E_n$. Then

E is a perfect set having measure 0 if $N^n\xi^n \to 0$ and Hausdorff dimension α if we choose $\xi = N^{-1/\alpha}$.

For each $n \in \mathbf{N}$, let F_n be a continuous nondecreasing function satisfying:

- $F_n(0) = 0$ for $x \le 0$ and $F_n(1) = 1$ for $x \ge 1$.
- F_n increases linearly by $1/N^n$ on each white interval in E_n.
- F_n is constant on each black interval in E_n.

Let $F = \lim_{n\to\infty} F_n$. Then F is a nondecreasing continuous function and the Fourier transform of the corresponding measure is given by

$$\widehat{dF}(x) = P(x) \prod_{n\ge 1} P(x\xi_1 \dots \xi_n)\,.$$

where $P(x) = \frac{1}{N}\sum_{1\le k\le N}\ e^{ia_k x}$ (see [Z]). Note that E and F do depend on Ξ. The mentioned result of Salem [S] is the following

Proposition 3.1. *Given $\epsilon > 0$, there is $M > 2$ and a sequence Ξ as above such that*

$$\left|\widehat{dF}(x)\right| \le C_\epsilon\ |x|^{-\frac{\alpha}{2}+\epsilon}\,.$$

In fact, this estimate holds for a.e. Ξ w.r.t. a suitable measure. Moreover, one can take ϵ a fixed multiple of $1/\sqrt{M}$. Besides Salem's result we will need the following regularity property of the function $F = F_\Xi$ which holds for all $\Xi \in [0,1]$.

Proposition 3.2. *There is $C > 0$, depending only on M such that for $x, y \in \mathbf{R}$,*

$$\left|F(x) - F(y)\right| \le C\ |x-y|^\alpha\,.$$

To prove this let $x, y \in [0,1]$ and suppose that $y > x$. Since F is constant on black intervals complementary to E, we may assume that x, y lie in E. Let k be the smallest integer such that after k dissections at least two black intervals lie in $[x,y]$. Then $[x,y]$ contains a white interval and

$$y - x \ge \xi_1 \dots \xi_k \ge \xi^k \prod_{1\le m\le k} \left(1 - \tfrac{1}{2m^2}\right) \ge C\xi^k\,.$$

Now, after $k-1$ dissections there is at most one black interval (a,b) contained in $[x,y]$. Hence using $N = 1/\xi^\alpha$ we find

$$F(y) - F(x) = F(y) - F(b) + F(a) - F(x) \le \tfrac{2}{N^{k-1}} \le C\xi^{k\alpha} \le C(y-x)^\alpha.$$

4 A Restriction Theorem on the Real Line

As an application of Young's inequality and the fact that the Fourier transform of the measures constructed by Salem lie in a nontrivial $L^p(\mathbf{R})$, one can get a restriction result for Fourier transforms of functions in $L^p(\mathbf{R})$ for p close to 1 (by following E.M. Stein's original argument [F2]). To improve on this range we will follow the Stein–Tomas argument (see [St]).

Theorem 4.1. *Let μ be a compactly supported positive measure on $\mathbf{R}^n$ which satisfies the following properties.*

(i) *There is $\beta > 0$ such that $|\widehat{d\mu}(x)| \leq C\,|x|^{-\beta/2}$.*

(ii) *There is $\alpha > 0$ such that $\mu(B_r(x)) \leq C\,r^\alpha$ for every ball $B_r(x)$ of radius r centered at x.*

Then, for $1 \leq p < \frac{2(2n-2\alpha+\beta)}{4(n-\alpha)+\beta}$, we have

$$\left(\int |\widehat{f}|^2 d\mu\right)^{1/2} \leq C\|f\|_{L^p(\mathbf{R}^n)}\,. \tag{4}$$

We have $\|\widehat{f}\|^2_{L^2(d\mu)} \leq \|f*\widehat{d\mu}\|_{p'}\|f\|_p$ with p' the dual exponent of p. The theorem follows if we can show that the convolution operator $Tf = \widehat{d\mu}*f$ is bounded from $L^p \to L^{p'}$, for $p' > 2(2n-2\alpha+\beta)/\beta$. Let $\varphi_k = \varphi(\,\cdot\,/2^k) \in C_0^\infty$, $0 \leq \varphi_k \leq 1$, have support in the spherical annulus $\{2^{k-1} \leq |x| \leq 2^{k+1}\}$ and define $\varphi_0 \in C_0^\infty(|x| \leq 2)$ such that $\sum_{k\geq 0}\ \varphi_k = 1$. We decompose T according to this partition: $Tf = \sum_{k\geq 0} T_k f$, where $T_k f = (\varphi_k\widehat{d\mu}) * f$. Then by (i) we have

$$\|T_k\|_{L^1\to L^\infty} \leq \|\varphi_k\,\widehat{d\mu}\|_\infty \leq C2^{-k\frac{\beta}{2}}\,. \tag{5}$$

By Plancherel's Theorem we get for the norm of the operators T_k on L^2

$$\begin{aligned}\|T_k\|_{L^2\to L^2} &\leq C\sup_{x\in\mathbf{R}^n}\left|\widehat{\varphi_k} * d\mu(x)\right| \\ &\leq C2^{kn}\sup_{x\in\mathbf{R}^n}\int \frac{1}{(1+2^k|x-y|)^N}\,d\mu(y) \\ &= C'2^{kn}\sup_{x\in\mathbf{R}^n}\int_0^\infty \mu\big(B_{r/2^k}(x)\big)(1+r)^{-N-1}dr\,.\end{aligned}$$

Using the regularity of the measure μ, i.e. $\mu(B_{r/2^k}(x)) \leq Cr^\alpha 2^{-k\alpha}$, we get

$$||T_k||_{L^2\to L^2} \leq C2^{k(n-\alpha)}\,. \tag{6}$$

Interpolating the bounds (5) and (6) gives $||T_k||_{L^p\to L^{p'}} \leq C2^{k(\frac{2n-2\alpha+\beta}{p'}-\frac{\beta}{2})}$. Hence, $T = \sum T_k$ is bounded from L^p to $L^{p'}$ for $p' > 2\frac{2n-2\alpha+\beta}{\beta}$.

Now, let $E \subset [0,1]$ und dF be as in the previous paragraph and $\alpha = \dim_H E$. We get the following

COROLLARY 4.2. *Let $p_0 < \frac{2(2-\alpha)}{4-3\alpha}$ and choose M in Proposition 3.1 sufficiently large. Then there is a constant C depending only on M such that*

$$\int_E |\widehat{f}|^2 dF \leq C\|f\|^2_{L^{p_0}(\mathbf{R})} .$$

5 Application to Multiplier Theory

We proceed to construct L^p-multipliers on $\mathbf{R}$ which may serve as analogues of the Bochner–Riesz multipliers in $\mathbf{R}^n, n > 1$. Suppose that $E \subset [0,1]$ is a compact set of Hausdorff dimension α supporting a measure μ satisfying both $|\widehat{d\mu}(x)| \leq C_\beta |x|^{-\beta/2}$ and the regularity estimates $\mu(B_r(x)) \leq Cr^\alpha$. Let $\psi \in C_0^\infty([-1,1])$, $\psi(0) \neq 0$, be an even function and define $k_z(x) = \psi(x)/|x|^{\alpha-z}$, $z > 0$. For $z > 0$, we will consider the multipliers given by $m_z = k_z * d\mu$. Let T_z be the convolution operator corresponding to the multiplier m_z. Obviously, since k_z and μ have compact support, the same holds for m_z. Furthermore, the regularity assumption on μ makes m_z a bounded measurable function provided $z > 0$. Hence, T_z is bounded on $L^2(\mathbf{R})$. Since m_z has compact support a necessary condition to be a p-multiplier is $\widehat{m_z} \in L^p(\mathbf{R})$. Assuming $\widehat{d\mu}$, $\widehat{k}_z \in L^p(\mathbf{R})$ and using $\widehat{k}_z(x) \approx C\psi(0)|x|^{-(1+z-\alpha)}$ and $|\widehat{d\mu}(x)| \leq C|x|^{-\beta/2}$ we find that

$$\int |\widehat{d\mu}|^{p(1+\frac{2}{\beta}(1+z-\alpha))} dx \leq C \int |\widehat{d\mu}|^p |\hat{k}_z|^p dx < \infty .$$

Hence, Hölder's inequality gives $I_\sigma(\mu) < \infty$, for all $\sigma < \frac{2}{p(1+\frac{2}{\beta}(1+z-\alpha))}$. Since $\alpha = \dim_H E \geq \sigma$ we get the necessary condition $p \geq \frac{2\beta}{\alpha(2+2z-2\alpha+\beta)}$ ($\to \frac{2}{2z+2-\alpha}$ as $\beta \to \alpha$).

Theorem 5.1. *Let $1 \leq p < \frac{2(2-2\alpha+\beta)}{4-4\alpha+\beta}$. Then T_z is a bounded operator on $L^p(\mathbf{R})$ for $p > \frac{2}{2z+2-\alpha}$.*

We sketch the proof here (see [F2]). First we decompose T_z in dyadic pieces as in the proof of Theorem 4.1: $T_z f = \sum_{k\geq 0} T_k f$. It is enough to get bounds for T_k on an interval I_k of length 2^{k+1}. Applying Hölders inequality we bound $\|T_k f\|_{L^p(I_k)}$ against the L^2-norm times $|I_k|^{1/p-1/2}$. Using Plancherel's theorem we bound the remaining L^2-norm by

$$\big\|(\varphi_k \widehat{k_z})^\vee * d\mu \widehat{f}\big\|_2^2 \leq \big\||(\varphi_k \widehat{k_z})^\vee| * d\mu\big\|_\infty \||(\varphi_k \widehat{k_z})^\vee| * d\mu |\widehat{f}|^2\|_1 .$$

We bound the above L^1-norm by $\|(\varphi_k\widehat{k_z})^\vee\|_1 \sup_x \int |\widehat{f}(x-y)|^2 d\mu(y)$. Using translation invariance Theorem 4.1 implies

$$\|T_k f\|_2^2 \le C\big\||(\varphi_k\widehat{k_z})^\vee| * d\mu\big\|_\infty \big\|(\varphi_k\widehat{k_z})^\vee\big\|_1 \|f\|_p^2 .$$

We deal with the L^∞-term as in the proof of Theorem 4.1 resulting in a bound of order 2^{-zk}. The L^1-term is easily seen to be of order $2^{k(1+z-\alpha)}$. Collecting terms gives $\|T_k f\|_{L^p} \le 2^{k(-1-z+\alpha/2+\frac{1}{p})}\|f\|_p$. Hence, T_z is bounded for $p > \frac{2}{2z+2-\alpha}$.

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