GAFA, Geom. funct. anal.
Vol. 10 (2000) 1507 – 1526
1016-443X/00/0601507-20 $ 1.50+0.20/0

GAFA Geometric And Functional Analysis

A FUNCTIONAL ANALYTIC APPROACH TO INTERSECTION BODIES

A. Koldobsky

Abstract

We consider several generalizations of the concept of an intersection body and show their connections with the Fourier transform and embeddings in L_p-spaces. These connections lead to generalizations of the recent solution to the Busemann–Petty problem on sections of convex bodies.

1 Introduction

The concept of an intersection body was introduced by Lutwak [Lu] in 1988. Let K and L be symmetric star bodies in $\mathbb{R}^n$. We say that K is the intersection body of L if the radius of K in every direction is equal to the $(n-1)$-dimensional volume of the central hyperplane section of L perpendicular to this direction, i.e. for every vector ξ from the unit sphere S^{n-1} in $\mathbb{R}^n$,

$$\|\xi\|_K^{-1} = vol_{n-1}(L \cap \xi^\perp)\,. \tag{1}$$

Here $\|\xi\|_K$ is the Minkowski functional of K and $\xi^\perp = \{x \in \mathbb{R}^n :\ (x,\xi) = 0\}$. A star body K in $\mathbb{R}^n$ is called an *intersection body of a star body* if there exists a star body L satisfying (1).

A more general concept was introduced in [GoLW]. A star body K is called an *intersection body* if there exists a finite Borel measure μ on S^{n-1} so that, for every $\phi \in C(S^{n-1})$,

$$\int_{S^{n-1}} \|x\|_K^{-1}\phi(x)dx = \int_{S^{n-1}} d\mu(\xi) \int_{S^{n-1}\cap\xi^\perp} \phi(x)dx\,.$$

It is easily seen that intersection bodies of star bodies are those intersection bodies for which the corresponding measure μ has positive continuous density.

Research supported in part by the NSF Grants DMS-9820848 and DMS-9996431

Intersection bodies are closely related to the following Busemann–Petty problem (BP-problem): suppose that K, L are symmetric convex bodies in $\mathbb{R}^n$ so that for every $\xi \in S^{n-1}$

$$vol_{n-1}(K \cap \xi^\perp) \le vol_{n-1}(L \cap \xi^\perp)\,.$$

Does it follow that $vol_n(K) \le vol_n(L)$? The answer is negative if $n > 4$ and affirmative if $n \le 4$, and the solution appeared as the result of work of many people (see [Z2] and [GKS] for the history of the problem and solution in all dimensions). Lutwak [Lu] found a connection between intersection bodies and the BP-problem that played an important role in the solution: if K is an intersection body then the answer to the question is affirmative for every L, and if L is not an intersection body one can perturb it to construct a body K which together with L gives a counterexample. Therefore, the answer to the BP-problem in $\mathbb{R}^n$ is affirmative if and only if every symmetric convex body in $\mathbb{R}^n$ is an intersection body. The unified solution to the BP-problem in [GKS] also made use of the following Fourier transform characterization of intersection bodies found in [K2]: a symmetric star body K in $\mathbb{R}^n$ is an intersection body if and only if $\|x\|_K^{-1}$ is a positive definite distribution.

More general classes of bodies were introduced in [K5] and [Z1]. Each of these classes is related to a certain generalization of the BP-problem in the same way, as intersection bodies are related to the original BP-problem, i.e. the answer is affirmative if the body K belongs to this class, and if L does not belong to the corresponding class one can use it to construct a counterexample. Therefore, the answer is affirmative in $\mathbb{R}^n$ if and only if every symmetric convex body belongs to the corresponding class of bodies.

First, let us consider the class of bodies introduced in [K5].

Definition 1. *Let K and L be star bodies in $\mathbb{R}^n$, and $1 \le k < n$. We say that K is a k-intersection body of L if for every $H \in G(n, n-k)$*

$$vol_k(K \cap H^\perp) = vol_{n-k}(L \cap H)\,. \tag{2}$$

We say that K is a k-intersection body of a star body if there exists L satisfying (2) *for every H.*

It was shown in [K5] that an infinitely smooth symmetric star body K in $\mathbb{R}^n$ is a k-intersection body of a star body if and only if the Fourier transform of $\|x\|_K^{-k}$ is a positive continuous function outside of the origin in $\mathbb{R}^n$. The corresponding generalization of the BP-problem is as follows. Let K, L be symmetric convex bodies in $\mathbb{R}^n$ that are $(k-1)$-smooth for some integer $k,\ 1 \le k < n$ (see section 2). For each $\xi \in S^{n-1}$ define the *parallel section function* of a body K in the direction of ξ by

$$A_{K,\xi}(t) = vol_{n-1}\big(K \cap \{x \in \mathbb{R}^n :\ (x,\xi) = t\}\big)\,, \quad t \in \mathbb{R}\,.$$

Suppose that k is an odd integer and for every $\xi \in S^{n-1}$

$$(-1)^{(k-1)/2} A^{(k-1)}_{K,\xi}(0) \le (-1)^{(k-1)/2} A^{(k-1)}_{L,\xi}(0)\,,$$

where $A^{(k-1)}_{K,\xi}(0)$ stands for the derivative of the order $k-1$ at zero. Does it follow that $vol_n(K) \le vol_n(L)$? It was proved in [K5] that the answer is affirmative if $k \ge n-3$ and negative if $k < n-3$. The case $k=1$ represents the answer to the original BP-problem.

Another generalization of intersection bodies was introduced in [Z1].

Definition 2. *For $1 \le k < n$, we say that a star body K in R^n is a generalized k-intersection body if there exists a finite Borel measure μ on the Grassmanian $G(n, n-k)$ such that for every $f \in C(S^{n-1})$*

$$\int_{S^{n-1}} \|x\|_K^{-k} f(x)dx = \int_{G(n,n-k)} d\mu(H) \int_{S^{n-1}\cap H} f(x)dx\,. \tag{3}$$

This class of bodies is related to the so-called generalized Busemann–Petty problem (GBP-problem). Suppose that $1 \le k < n$ and symmetric convex bodies K, L in $\mathbb{R}^n$ satisfy

$$vol_{n-k}(K \cap H) \le vol_{n-k}(L \cap H)$$

for every $(n-k)$-dimensional subspace H of $\mathbb{R}^n$. Does it follow that $vol_n(K) \le vol_n(L)$? It was proved by Bourgain and Zhang [BZ] that if $n-k > 3$ the answer to the GBP-problem is negative. The problem is still open in the cases where $n-k = 2, 3$.

In this article we present an interpretation of these classes of bodies in the language of functional analysis. The concept of embedding of a normed space in L_p was extended in [K4] (analytically with respect to p) to negative values of p. In section 3, we modify the definition of k-intersection bodies of star bodies and introduce a more general class of k-intersection bodies. We then prove that a symmetric star body in $\mathbb{R}^n$ is a k-intersection body if and only if the space $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_{-k}. Both conditions in the latter statement are equivalent to $\|x\|_K^{-k}$ being a positive definite distribution. We also give a purely geometric criterion for the existence of a k-intersection body for a given body L. In section 4, we introduce the concept of embedding in $L_{-p}(\mathbb{R}^k)$ by extending (analytically with respect to p) the property of embedding in the spaces $L_p(\mathbb{R}^k)$ of vector valued functions. We prove that a symmetric star body K is a generalized k-intersection body if and only if the space $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in $L_{-k}(\mathbb{R}^k)$. We then show that if a space $(\mathbb{R}^n, \|\cdot\|)$ embeds in $L_{-p}(\mathbb{R}^k)$ then it also embeds in L_{-p}. This immediately implies that every generalized k-intersection body is a k-intersection body. In section 5, we use the latter fact to give a new proof of the result of Bourgain and Zhang that the answer to the GBP-problem is negative if the dimension of sections is greater than 3. Another result in section 5 generalizes the positive part of the solution to the original Busemann–Petty problem. We prove that if K, L are symmetric convex bodies in $\mathbb{R}^n$ and, for every three-dimensional subspace H of $\mathbb{R}^n$,

$$\int_{S^{n-1}\cap H} \|x\|_K^{-n+1} dx \le \int_{S^{n-1}\cap H} \|x\|_L^{-n+1} dx \,,$$

then $vol_n(K) \le vol_n(L)$. The question of whether it is possible to replace the three-dimensional subspaces in the latter result by subspaces of higher dimension (with an absolute constant in the inequality; the solution to the original BP-problem shows that this is impossible without a constant even for four dimensional integrals) is related to the famous hyperplane (or slicing) problem, see [MP]. Our main tool is the Fourier transform of distributions and its connection with volumes of lower dimensional sections, established in section 2 and generalizing the formula for hyperplane sections from [GKS, Th.2].

2 Volumes of Sections and the Fourier Transform

We recall some notations related to the Fourier transform of distributions. As usual, we denote by $\mathcal{S}(\mathbb{R}^n)$ the space of rapidly decreasing infinitely differentiable functions (test functions) in $\mathbb{R}^n$, and $\mathcal{S}'(\mathbb{R}^n)$ is the space of distributions over $\mathcal{S}(\mathbb{R}^n)$. The Fourier transform $\hat{f}$ of a distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ is defined by $\langle \hat{f}, \phi\rangle = \langle f, \hat{\phi}\rangle$ for every test function ϕ. A distribution is called even homogeneous of degree $p \in \mathbb{R}$ if $\langle f(x), \phi(x/\alpha)\rangle = |\alpha|^{n+p}\langle f, \phi\rangle$ for every test function ϕ and every $\alpha \in \mathbb{R}$, $\alpha \ne 0$. The Fourier transform of an even homogeneous distribution of degree p is an even homogeneous distribution of degree $-n-p$. If $f \in C(S^{n-1})$, we denote by $f(\theta)r^{-k}$ the extension of f to a homogeneous of degree $-k$ function on $\mathbb{R}^n \setminus \{0\}$. If $p > -1$ and p is not an even integer, then the Fourier transform of the function $h(z) = |z|^p$, $z \in \mathbb{R}$ is equal to $(|z|^p)^\wedge(t) = c_p|t|^{-1-p}$ (see [GeS, p. 173]), where $c_p = \frac{2^{p+1}\sqrt{\pi}\,\Gamma((p+1)/2)}{\Gamma(-p/2)}$. The well-known connection between the Radon transform and the Fourier transform is that, for every $\xi \in S^{n-1}$, the function $t \to \hat{\phi}(t\xi)$ is the Fourier transform of the function $z \to \int_{(x,\xi)=z} \phi(x)\, dx$. A distribution f is called positive definite if, for every test function ϕ, $\langle f, \phi * \overline{\phi(-x)}\rangle \ge 0$. According to L.Schwartz's generalization of Bochner's theorem, a distribution is positive definite if and only if it is the Fourier transform of a tempered measure in $\mathbb{R}^n$ ([GeV, p. 152]).

Let K be a body that is star-shaped with respect to the origin. The *Minkowski functional* of K is given by

$$\|x\|_K = \min\{a \geq 0 :\ x \in aK\}\,, \quad x \in \mathbb{R}^n\,.$$

We call K a *star body* if $\|\cdot\|_K$ is continuous *and positive* on S^{n-1}. We say that a star body K is $(k-1)$-smooth (infinitely smooth) if the restriction of $\|x\|_K$ to the sphere S^{n-1} belongs to $C^{k-1}(S^{n-1})$ (correspondingly, $C^\infty(S^{n-1})$). We denote by $G(n,k)$ the Grassmanian of k-dimensional subspaces of $\mathbb{R}^n$. Throughout the paper, dH denotes integration with respect to the normalized Haar measure on the Grassmanian. However, integrating over the sphere or its sections with respect to the uniform measure, we do not normalize this measure. If $H \in G(n,k)$ then

$$vol_{n-k}(K \cap H) = \frac{1}{n-k}\int_{S^{n-1}\cap H} \|x\|_K^{-n+k}dx\,. \tag{4}$$

The first result connecting the volume of sections of star bodies with the Fourier transform was established in [K1, Th. 1]. It was based on the following observation ([K1, Lemma 1]):

Proposition 1. *Let f be an even continuous homogeneous function of degree $-n+1$ on $\mathbb{R}^n \setminus \{0\}$, $n > 1$. Then, for every $\xi \in S^{n-1}$,*

$$\hat{f}(\xi) = \pi \int_{S^{n-1}\cap\{(\theta,\xi)=0\}} f(\theta)d\theta\,.$$

Putting $f(x) = \|x\|_K^{-n+1}$ in Proposition 1 and using (4), we immediately see that for every symmetric star body K in $\mathbb{R}^n$ and every $\xi \in S^{n-1}$,

$$vol_{n-1}(K \cap \xi^\perp) = \frac{1}{\pi(n-1)}\big(\|x\|_K^{-n+1}\big)^\wedge(\xi)\,.$$

This formula was generalized in [GKS, Th. 2]:

Theorem 1. *Let K be an origin-symmetric infinitely smooth star body in $\mathbb{R}^n$, and let $k \in \mathbb{N}\cup\{0\}$, $k \neq n$. Suppose that $\xi \in S^{n-1}$, and let $A_{K,\xi}$ be the corresponding parallel section function of K.*

(a) *If k is odd, then*

$$\big(\|x\|_K^{-n+k}\big)^\wedge(\xi) = (-1)^{k/2}\pi(n-k)A_{K,\xi}^{(k-1)}(0)\,;$$

(b) *if k is even, then*

$$\big(\|x\|_K^{-n+k}\big)^\wedge(\xi) = c_k\int_0^\infty \frac{A_{K,\xi}(z) - A_{K,\xi}(0) - A''_{K,\xi}(0)\frac{z^2}{2} - \cdots - A_{K,\xi}^{(k-2)}(0)\frac{z^{k-2}}{(k-2)!}}{z^k}\,dz\,,$$

where $c_k = (-1)^{k/2}\,2(n-k)(k-1)!$.

REMARK 1. The condition of Theorem 1 that K is infinitely smooth can be weakened. It is enough to assume that the body K is $(k-1)$-smooth to prove the statements (a) and (b) in exactly the same way as was done in [GKS]. Also if K is $(k-1)$-smooth the expressions in the right-hand side of (a) and (b) are continuous functions of ξ on the sphere. In particular, if K is infinitely smooth, then for every $k \in \mathbb{N} \cup \{0\}$, $(\|x\|_K^{-n+k})^\wedge$ is a continuous function on S^{n-1}. Moreover, if K and $K_m,\ m \in \mathbb{N}$ are $(k-1)$-smooth star bodies such that the distance between the functions $\|\cdot\|_K$ and $\|\cdot\|_{K_m}$ in the space $C^{(k-1)}(S^{n-1})$ approaches zero as $m \to \infty$, then the distance between the functions $(\|x\|_K^{-n+k})^\wedge$ and $(\|x\|_{K_m}^{-n+k})^\wedge$ in the space $C(S^{n-1})$ also has limit zero. We can choose the bodies K_m to be infinitely smooth.

We now generalize the result of Theorem 1 to the case of lower dimensional sections. Let $H \in G(n, n-k)$, $1 \le k < n$ and let $\xi_1, \dots, \xi_k$ be an orthonormal basis in $H^\perp$. For a star body K in $\mathbb{R}^n$, the $(n-k)$-dimensional parallel section function $A_{K,H}$ is a function on $\mathbb{R}^k$ defined by

$$\begin{aligned} A_{K,H}(u) &= vol_{n-k}\big(K \cap \{H + u_1\xi_1 + \dots + u_k\xi_k\}\big) \\ &= \int_{\{x \in \mathbb{R}^n : (x,\xi_1) = u_1, \dots, (x,\xi_k) = u_k\}} \chi(\|x\|_K)\,dx\,, \quad u \in \mathbb{R}^k\,, \end{aligned} \tag{5}$$

where χ is the indicator function of the interval $[0,1]$.

Let $\|u\|_2$ be the Euclidean norm of $u \in \mathbb{R}^k$. For every $q \in \mathbb{C}$, the value of the distribution $\|u\|_2^{-q-k}/\Gamma(-q/2)$ on a test function $\phi \in \mathbb{R}^k$ can be defined in the usual way (see [GeS, p. 71]) and represents an entire function of $q \in \mathbb{C}$. If K is infinitely smooth the function $A_{K,H}$ is infinitely differentiable at the origin, and the same regularization procedure can be applied to define the action of these distributions on the function $A_{K,H}$. The function

$$q \mapsto \left\langle \frac{\|u\|_2^{-q-k}}{\Gamma(-q/2)}, A_{K,H}(u) \right\rangle \tag{6}$$

is an entire function of $q \in \mathbb{C}$. In particular, if $q = 2m$, $m \in \mathbb{N} \cup \{0\}$, then

$$\left\langle \frac{\|u\|_2^{-q-k}}{\Gamma(-q/2)}\bigg|_{q=2m}, A_{K,H}(u) \right\rangle = \frac{(-1)^m \Omega_k}{2^{m+1}k(k+2)\dots(k+2m-2)} \Delta^m A_{K,H}(0)\,, \tag{7}$$

where $\Omega_k = 2\pi^{k/2}/\Gamma(k/2)$ is the surface area of the unit sphere S^{k-1} in $\mathbb{R}^k$, and $\Delta = \sum_{i=1}^k \partial^2/\partial u_i^2$ is the k-dimensional Laplace operator (for details, see [GeS, p. 71-74] and [GKS, p. 698-700]). According to the theory of fractional derivatives (see for example [SKM]), the value of the function (6) at q is equal (up to a constant) to $\Delta^{q/2} A_{K,H}(0)$. Note that these quantities do not depend on the choice of an orthonormal basis in $H^\perp$.

We also use the following well-known fact (see for example [GeS, p. 76]): for any $v \in \mathbb{R}^k$ and $q < -k+1$,

$$(v_1^2 + \cdots + v_k^2)^{(-q-k)/2} = \frac{\Gamma(-q/2)}{2\Gamma((-q-k+1)/2)\pi^{(k-1)/2}} \int_{S^{k-1}} \left|(v,u)\right|^{-q-k} du\,. \tag{8}$$

Theorem 2. *Let K be an infinitely smooth symmetric star body in $\mathbb{R}^n$, $1 \le k < n$. Then for every $H \in G(n, n-k)$ and every $m \in \mathbb{N} \cup \{0\}$, $m \neq (n-k)/2$,*

$$\Delta^m A_{K,H}(0) = \frac{(-1)^m}{2^k \pi^k (n-2m-k)} \int_{S^{n-1}\cap H^\perp} (\|x\|_K^{-n+2m+k})^\wedge(\theta) d\theta\,.$$

Proof. Let $q \in (-k, -k+1)$. Then,

$$\left\langle \frac{\|u\|_2^{-q-k}}{\Gamma(-q/2)}, A_{K,H}(u) \right\rangle = \frac{1}{\Gamma(-q/2)} \int_{\mathbb{R}^k} \|u\|_2^{-q-k} A_{K,H}(u) du\,. \tag{9}$$

Using the expression (5) for the function $A_{K,H}$, writing the integral in polar coordinates and then using (8), we see that the right-hand side of (9) is equal to

$$\frac{1}{\Gamma(-q/2)} \int_{\mathbb{R}^n} \big((x,\xi_1)^2 + \cdots + (x,\xi_k)^2\big)^{(-q-k)/2} \chi(\|x\|_K) dx$$
$$= \frac{1}{\Gamma(-q/2)(n-q-k)} \int_{S^{n-1}} \big((\theta,\xi_1)^2 + \cdots + (\theta,\xi_k)^2\big)^{(-q-k)/2} \|\theta\|_K^{-n+q+k} d\theta$$
$$= \frac{1}{2\Gamma(\frac{-q-k+1}{2})\pi^{(k-1)/2}} \int_{S^{n-1}} \|\theta\|_K^{-n+q+k} d\theta \int_{S^{k-1}} \left| \left(\sum_{i=1}^k u_i \xi_i, \theta \right) \right|^{-q-k} du$$
$$= \frac{1}{2\Gamma(\frac{-q-k+1}{2})\pi^{(k-1)/2}} \int_{S^{k-1}} du \int_{S^{n-1}} \|\theta\|_K^{-n+q+k} \left| \left(\sum_{i=1}^k u_i \xi_i, \theta \right) \right|^{-q-k} d\theta\,. \tag{10}$$

Let us show that the function under the integral over S^{k-1} is the Fourier transform of $\|x\|_K^{-n+q+k}$ at the point $\sum u_i \xi_i$. For any even test function $\phi \in \mathcal{S}(\mathbb{R}^n)$, using the connection between the Fourier and Radon transforms and the expression for the Fourier transform of the distribution $|z|^{q+k-1}$ (see the beginning of section 2), we get

$$\langle (\|x\|_K^{-n+q+k})^\wedge, \phi \rangle = \langle \|x\|_K^{-n+q+k}, \hat\phi \rangle = \int_{\mathbb{R}^n} \|x\|_K^{-n+q+k} \hat\phi(x) dx$$
$$= \int_{S^{n-1}} \|\theta\|_K^{-n+q+k} d\theta \int_0^\infty z^{q+k-1} \hat\phi(z\theta) dz$$

$$= \frac{1}{2}\int_{S^{n-1}} \|\theta\|_K^{-n+q+k}\langle |z|^{q+k-1}, \hat{\phi}(z\theta)\rangle d\theta$$
$$= \frac{2^{q+k}\sqrt{\pi}\ \Gamma((q+k)/2)}{2\Gamma((-q-k+1)/2)}\int_{S^{n-1}} \|\theta\|_K^{-n+q+k}\Big\langle |t|^{-q-k}, \int_{(x,\theta)=t}\phi(x)dx\Big\rangle d\theta$$
$$= \frac{2^{q+k}\sqrt{\pi}\Gamma((q+k)/2)}{2\Gamma((-q-k+1)/2)}\int_{\mathbb{R}^n}\bigg(\int_{S^{n-1}} \big|(\theta,\xi)\big|^{-q-k}\|\theta\|_K^{-n+q+k}d\theta\bigg)\phi(\xi)d\xi\,.$$

Since ϕ is an arbitrary test function, this proves that, for every $\xi\in\mathbb{R}^n$,

$$\big(\|x\|_K^{-n+q+k}\big)^\wedge(\xi) = \frac{2^{q+k}\sqrt{\pi}\Gamma((q+k)/2)}{2\Gamma((-q-k+1)/2)}\int_{S^{n-1}}\big|(\theta,\xi)\big|^{-q-k}\|\theta\|_K^{-n+q+k}d\theta\,.$$

Together with (10), the latter equality shows that

$$\Big\langle \frac{\|u\|_2^{-q-k}}{\Gamma(-q/2)}, A_{K,H}(u)\Big\rangle = \frac{2^{-q-k}\pi^{-k/2}}{\Gamma(\frac{q+k}{2})(n-q-k)}\int_{S^{n-1}\cap H^\perp}\big(\|x\|_K^{-n+q+k}\big)^\wedge(\theta)d\theta\,. \tag{11}$$

We have proved (11) under the assumption that $q\in(-k,-k+1)$. However, both sides of (11) are analytic functions of $q\in\mathbb{C}$ in the domain where $Re(q)>-k$, $q\neq n-k$. (For every $q\in\mathbb{C}$, $Re(q)>-k$, one can see from the proof of Theorem 1 in [GKS] that the Fourier transform of $\|x\|_K^{-n+q+k}$ is a continuous function on the sphere S^{n-1}.) This implies that the equality (11) holds for every q from this domain.

Putting $q=2m$, $m\in\mathbb{N}\cup\{0\}$, $m\neq(n-k)/2$ in (11) and applying (7) and the fact that $\Gamma(x+1)=x\Gamma(x)$, we get the desired formula. □

Theorem 2 (with $m=0$) and the expression (4) for the volume of central sections imply the following fact that was proved in [K5, Lemma 7] using other methods.

Corollary 1. *Let K be an infinitely smooth symmetric star body in $\mathbb{R}^n$, $1\le k<n$. Then for every $(n-k)$-dimensional subspace H of $\mathbb{R}^n$,*

$$\int_{S^{n-1}\cap H}\|x\|_K^{-n+k}dx = \frac{1}{(2\pi)^k}\int_{S^{n-1}\cap H^\perp}(\|x\|_K^{-n+k})^\wedge(\theta)d\theta\,.$$

We need one more application of Theorem 2.

Corollary 2. *Let K be an infinitely smooth symmetric* convex *body in $\mathbb{R}^n$, $1\le k\le n-3$. Then for every k-dimensional subspace H of $\mathbb{R}^n$,*

$$\int_{S^{n-1}\cap H}\big(\|x\|_K^{-n+k+2}\big)^\wedge(\theta)d\theta\ge 0\,.$$

Proof. Since K is symmetric and convex, so is $K_i = K\cap span(H,\xi_i)$ for every $i=1,\dots,n$, where ξ_i, $i=1,\dots,k$ is an orthonormal basis in $H^\perp$.

By the Brunn-Minkowski theorem (see for example [Sc, Th. 6.1.1]), the central hyperplane section has maximal volume among all hyperplane sections perpendicular to a given direction. Therefore, the parallel section function $u_i \mapsto A_{K\cap H,\xi_i}(u_i)$ of the body $K \cap H$ in the direction of ξ_i has maximum at zero, and $A''_{K\cap H,\xi_i}(0) \le 0$ for every i. This implies that $\Delta A_{K,H}(0) \le 0$, and the result follows from Theorem 2 with $m = 1$. ◻

Finally, we prove a formula that will allow us to work with the integrals appearing in Theorem 2.

LEMMA 1. *Let $1 \le k < n$, and $f, g \in C(S^{n-1})$. Then*

$$\int_{S^{n-1}} g(x)dx \int_{S^{n-1}\cap x^\perp} f(\xi)d\xi$$
$$= c(n,k) \int_{G(n,k)} \left(\int_{S^{n-1}\cap H^\perp} f(\xi)d\xi \right) \left(\int_{S^{n-1}\cap H} g(x)dx \right) dH\,, \quad (12)$$

where $c(n,k) = (\Omega_n \Omega_{n-1})/(\Omega_k \Omega_{n-k})$, and $\Omega_k = 2\pi^{k/2}/\Gamma(k/2)$ is the surface area of the unit sphere S^{k-1} in $\mathbb{R}^k$.

Proof. Let $O(n)$ be the group of linear isometries of the n-dimensional Euclidean space. We identify every element U of $O(n)$ with the n-tuple $(u_1, \dots, u_n)$ of orthonormal vectors $u_i = Ue_i$, where e_i, $i = 1, \dots, n$, is the standard basis in $\mathbb{R}^n$. By the conditional expectation theorem (see for example [F, Ch. 5]), there exists a probability measure ν on $G(n,k)$ so that, for every continuous function F on $O(n)$, the expectation of F with respect to the normalized Haar measure dU on $O(n)$ is equal to the expectation with respect to $d\nu(H)$ of the conditional expectations $\mathbb{E}(F|u_1, \dots, u_k \in H)$. In other words,

$$\int_{O(n)} F(U)dU = \int_{G(n,k)} \int_{u_{m+1},\dots u_n \in H^\perp} dU_{n-k} \int_{u_1,\dots,u_k \in H} F(U)dU_k\, d\nu(H)\,,$$

where dU_k, dU_{n-k} are the Haar measures on $O(k)$ and $O(n-k)$, respectively. Since the measure ν is invariant with respect to isometries, it is the Haar measure on $G(n,k)$. Putting $F(U) = f(u_1)g(u_n)$ we get the desired formula. ◻

3 k-intersection Bodies

In view of (2) and (4), a star body K is the k-intersection body of a star body L if and only if for every $H \in G(n, n-k)$

$$\frac{1}{k} \int_{S^{n-1}\cap H^\perp} \|x\|_K^{-k} dx = \frac{1}{n-k} \int_{S^{n-1}\cap H} \|x\|_L^{-n+k} dx\,. \quad (13)$$

The uniqueness theorem for the Radon transforms (see, for example, [G, Th. 7.2.3]), implies that, for a given L, the k-intersection body is unique, if it exists. The existence of the k-intersection body (with $k > 1$) is a more complicated matter. It was proved in [K5,Th. 4] that an infinitely smooth symmetric star body K is an intersection body of a star body if and only if the Fourier transform of $\|x\|_K^{-k}$ is a positive continuous function on $\mathbb{R}^n \setminus \{0\}$. The proof of this fact is a combination of Corollary 1 and the uniqueness theorem for the Radon transforms. We now present an approximation argument that allows to avoid the condition that K is infinitely smooth.

PROPOSITION 2. *Let K be a symmetric star body in $\mathbb{R}^n$ that is a k-intersection body of a star body. Then $(\|x\|_K^{-k})^\wedge$ is a positive continuous function on $\mathbb{R}^n \setminus \{0\}$. The corresponding body L is then given by*

$$\|\theta\|_L^{-n+k} = \frac{n-k}{(2\pi)^{n-k}k}\big(\|x\|_K^{-k}\big)^\wedge(\theta)\,, \quad \theta \in S^{n-1}\,. \tag{14}$$

Proof. Suppose that K, L are symmetric star bodies in $\mathbb{R}^n$ so that K is the k-intersection body of L. Let $K_m \subset K$, $m \in \mathbb{N}$ be infinitely smooth symmetric star bodies in $\mathbb{R}^n$ approximating K in the uniform metric on the sphere. Then, as $m \to \infty$,

$$\int_{S^{n-1}\cap H^\perp} \|x\|_{K_m}^{-k}dx \to \int_{S^{n-1}\cap H^\perp} \|x\|_K^{-k}dx$$

uniformly with respect to $H \in G(n, n-k)$. By Corollary 1 and the equality (13),

$$\int_{S^{n-1}\cap H} \big(\|x\|_{K_m}^{-k}\big)^\wedge(\theta)d\theta \to \frac{(2\pi)^{n-k}k}{n-k}\int_{S^{n-1}\cap H} \|x\|_L^{-n+k}dx \tag{15}$$

uniformly with respect to H. Denote the constant in the right-hand side by c. Let $\psi \in C^\infty(S^{n-1})$ be an even infinitely differentiable function on the sphere. Then, by Remark 1 or [K5, Lemma 5], $(\psi(\theta)r^{-1})^\wedge$ is an infinitely differentiable function on $\mathbb{R}^n \setminus \{0\}$. The uniform convergence in (15) implies that

$$\int_{G(n,n-k)} \bigg(\int_{S^{n-1}\cap H} \big(\|x\|_{K_m}^{-k}\big)^\wedge\bigg)\bigg(\int_{S^{n-1}\cap H^\perp} \big(\psi(\theta)r^{-1}\big)^\wedge\bigg)dH$$
$$= c\int_{G(n,n-k)} \bigg(\int_{S^{n-1}\cap H} \|x\|_L^{-n+k}\bigg)\bigg(\int_{S^{n-1}\cap H^\perp} \big(\psi(\theta)r^{-1}\big)^\wedge\bigg)dH\,.$$

Applying Lemma 1 to both sides of the latter equality, we get

$$\int_{S^{n-1}} \big(\|x\|_{K_m}^{-k}\big)^\wedge(\xi)\bigg(\int_{S^{n-1}\cap\xi^\perp} \big(\psi(\theta)r^{-1}\big)^\wedge\bigg)d\xi$$
$$= c\int_{S^{n-1}} \|\xi\|_L^{-n+k}\bigg(\int_{S^{n-1}\cap\xi^\perp} \big(\psi(\theta)r^{-1}\big)^\wedge\bigg)d\xi\,.$$

Since the function $(\psi(\theta)r^{-1})^\wedge$ is homogeneous of degree $-n+1$, by Proposition 1,

$$\int_{S^{n-1}\cap\xi^\perp}(\psi(\theta)r^{-1})^\wedge = \frac{1}{\pi}\big((\psi(\theta)r^{-1})^\wedge\big)^\wedge(\xi) = \frac{(2\pi)^n}{\pi}\psi(\xi)\,.$$

Therefore,

$$\int_{S^{n-1}}\big(\|x\|_{K_m}^{-k}\big)^\wedge(\xi)\psi(\xi)\ d\xi \to c\int_{S^{n-1}}\|\xi\|_L^{-n+k}\psi(\xi)d\xi\,, \tag{16}$$

as $m\to\infty$.

Let $\phi\in\mathcal{S}(\mathbb{R}^n)$ be any even test function. For every m, $\|x\|_{K_m}^{-k}|\hat\phi(x)|\le\|x\|_K^{-k}|\hat\phi(x)|$, and the latter function belongs to $L_1(\mathbb{R}^n)$ as the product of a locally integrable, bounded outside of the origin function $\|x\|_K^{-k}$ and an L_1-function $|\hat\phi|$. By the dominated convergence theorem and formula (16) applied to the function $\psi(\xi)=\int_0^\infty t^{k-1}\phi(t\xi)dt$,

$$\begin{aligned}\big\langle(\|x\|_K^{-k})^\wedge,\phi\big\rangle &= \int_{\mathbb{R}^n}\|x\|_K^{-k}\hat\phi(x)dx\\ &= \lim_{m\to\infty}\int_{\mathbb{R}^n}\|x\|_{K_m}^{-k}\hat\phi(x)dx = \lim_{m\to\infty}\int_{\mathbb{R}^n}\big(\|x\|_{K_m}^{-k}\big)^\wedge(y)\phi(y)dy\\ &= \lim_{m\to\infty}\int_{S^{n-1}}\big(\|x\|_{K_m}^{-k}\big)^\wedge(\xi)d\xi\int_0^\infty t^{k-1}\phi(t\xi)dt\\ &= c\int_{S^{n-1}}\|\xi\|_L^{-n+k}d\xi\int_0^\infty t^{k-1}\phi(t\xi)dt = c\int_{\mathbb{R}^n}\|y\|_L^{-n+k}\phi(y)dy\,.\end{aligned}$$

Since ϕ is an arbitrary even test function, we get $(\|x\|_K^{-k})^\wedge(y)=c\|y\|_L^{-n+k}$. □

Suppose that K is a k-intersection body of a star body. By Proposition 2, $(\|x\|_K^{-k})^\wedge$ is a positive continuous, homogeneous of degree $-n+k$ function on $\mathbb{R}^n\setminus\{0\}$. For every even test function $\phi\in\mathcal{S}(\mathbb{R}^n)$, we have

$$\begin{aligned}\int_{\mathbb{R}^n}\|x\|_K^{-k}\phi(x)dx &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\big(\|x\|_K^{-k}\big)^\wedge(y)\hat\phi(y)dy\\ &= \frac{1}{(2\pi)^n}\int_{S^{n-1}}\big(\|x\|_K^{-k}\big)^\wedge(\xi)\int_0^\infty t^{k-1}\hat\phi(t\xi)dt\,.\end{aligned}$$

Similar to how it was done in the case $k=1$ in [GoLW], we define a more general class of bodies by replacing the positive continuous function $(\|x\|_K^{-k})^\wedge$ in the latter equality by any measure on the sphere S^{n-1}.

Definition 3. *Let $1\le k<n$. We say that a symmetric star body K in $\mathbb{R}^n$ is a k-intersection body if there exists a measure μ on S^{n-1} such that, for every even test function ϕ in $\mathbb{R}^n$,*

$$\int_{\mathbb{R}^n}\|x\|_K^{-k}\phi(x)dx = \int_{S^{n-1}}d\mu(\xi)\int_0^\infty t^{k-1}\hat\phi(t\xi)dt\,. \tag{17}$$

The calculation before Definition 3 immediately shows that every symmetric k-intersection body of a star body is also a k-intersection body.

Let us look at k-intersection bodies from a different point of view. Assume that $p > 0$ and $E = (\mathbb{R}^n, \|\cdot\|)$ is an n-dimensional subspace of $L_p(\Omega, \sigma)$, where (Ω, σ) is a space with finite measure. Let $f_1, \dots, f_n$ be a basis in E and, for every $\omega \in \Omega$, denote by $f(\omega) = (f_1(\omega), \dots, f_n(\omega)) \in \mathbb{R}^n$. Let μ be the image of the measure $\|f(\omega)\|_2^p d\sigma(\omega)$ under the mapping $\omega \mapsto f(\omega)/\|f(\omega)\|_2$. This means that μ is a finite Borel measure on S^{n-1} such that, for every Borel set A in $\mathbb{R}^n$, $\mu(A)$ is equal to the measure $\|f(\omega)\|_2^p d\sigma(\omega)$ of the set $\{\omega \in \Omega : f(\omega)/\|f(\omega)\|_2 \in A\}$.

For every $x \in \mathbb{R}^n$, we have

$$\|x\|^p = \|x_1 f_1 + \dots x_n f_n\|_{L_p}^p = \int_\Omega \big|x_1 f_1(\omega) + \dots + x_1 f_1(\omega)\big|^p d\sigma(\omega)$$

$$= \int_{S^{n-1}} \big|(x,\xi)\big|^p d\mu(\xi)\,. \tag{18}$$

The fact that embedding in L_p is equivalent to the representation (18) was known to P. Lèvy [L]. Now we write (18) in a different form. Suppose that p is not an even integer. Let us apply both sides of (18) to a test function $\phi \in \mathcal{S}(\mathbb{R}^n)$ and use the connection between the Fourier and Radon transforms and the formula for the Fourier transform of the function $|t|^p$ (see the beginning of section 2). We get

$$\int_{\mathbb{R}^n} \|x\|^p \phi(x)dx = \int_{S^{n-1}} d\mu(\xi) \int_{\mathbb{R}^n} \big|(x,\xi)\big|^p \phi(x)dx$$

$$= \int_{S^{n-1}} d\mu(\xi) \int_{\mathbb{R}} |t|^p \bigg(\int_{(x,\xi)=t} \phi(x)dx \bigg) dt$$

$$= \int_{S^{n-1}} \bigg\langle |t|^p, \int_{(x,\xi)=t} \phi(x)dx \bigg\rangle d\mu(\xi) = c_p \int_{S^{n-1}} \big\langle |z|^{-1-p}, \hat{\phi}(z\xi) \big\rangle d\mu(\xi)\,.$$

This calculation gives another condition for embedding of a space in L_p that looks more complicated than (18). However, if $p < 0$ the action of the distribution $|z|^{-1-p}$ can be written as an integral, and the latter calculation allows us to extend the concept of embedding of a finite dimensional space in L_p to negative values of p. The following definition was first given in [K4].

Definition 4. *Suppose that $x \to \|x\|$ is a homogeneous of degree 1, even continuous, positive outside of the origin function on $\mathbb{R}^n$. We say that $(\mathbb{R}^n, \|\cdot\|)$ embeds in L_{-p}, $0 < p < n$, if there exist a finite symmetric measure μ on the sphere S^{n-1} so that, for every even test function ϕ,*

$$\int_{\mathbb{R}^n} \|x\|^{-p} \phi(x)dx = \int_{S^{n-1}} d\mu(\xi) \int_0^\infty t^{p-1} \hat{\phi}(t\xi)dt\,.$$

The condition that $0 < p < n$ guarantees absolute convergence of the integral in the left-hand side. It was proved in [K4, Th. 1] that $(\mathbb{R}^n, \|\cdot\|)$ embeds in L_{-p}, $0 < p < n$ if and only if $\|x\|^{-p}$ is a positive definite distribution. Using this result and comparing Definitions 3 and 4 we get

Theorem 3. *Let K be a symmetric star body in $\mathbb{R}^n$, $1 \le k < n$. The following are equivalent:*

(i) *K is a k-intersection body;*
(ii) *The function $\|x\|_K^{-k}$ is a positive definite distribution on $\mathbb{R}^n$;*
(iii) *The space $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_{-k}.*

In particular, the fact that K is a k-intersection body if and only if the Fourier transform of $\|x\|_K^{-k}$ is a tempered measure on $\mathbb{R}^n$ extends the result of Proposition 2 to the class of k-intersection bodies.

Let us translate some known results on embedding in L_{-p} into the language of k-intersection bodies. First, the unit ball of any n-dimensional subspace of L_q with $0 < q \le 2$ is a k-intersection body for every $1 \le k < n-1$ (see [K4, Th .2]). Every symmetric convex body in $\mathbb{R}^n$ is a k-intersection body for $k = n-3, n-2, n-1$ (see [K5, Corollary 3]). The unit balls of the spaces ℓ_q^n, $2 < q \le \infty$ are k-intersection bodies if and only if $k = n-3, n-2, n-1$ (see [K3, Theorems 1,2]).

We conclude this section with a translation of the Fourier transform characterization of k-intersection bodies into the language of geometry. This purely geometric formulation suggests that there may be a proof not using the Fourier transform.

Theorem 4. *Let L be a $(k-1)$-smooth symmetric star body in $\mathbb{R}^n$, where k is an odd integer, $1 \le k < n$. If $(-1)^{(k-1)/2} A_{L,\xi}^{(k-1)}(0) > 0$ for every $\xi \in S^{n-1}$, then the body K defined by*

$$\|\xi\|_K = \tfrac{2\pi}{(\pi k)^{1/k}} \big((-1)^{(k-1)/2} A_{L,\xi}^{(k-1)}(0)\big)^{-1/k}$$

has the property that the k-dimensional volume of the section of K by any k-dimensional subspace is equal to the $(n-k)$-dimensional volume of the section of L by the orthogonal subspace. In other words, K is the k-intersection body of L.

Proof. By Theorem 1 (one needs only $(k-1)$-smoothness in the proof), the Fourier transform of $\|x\|_L^{-n+k}$ is a positive continuous function on S^{n-1}, and for every $\xi \in S^{n-1}$,

$$\|\xi\|_K^{-k} = \frac{(2\pi)^{n-k} k}{n-k} \big(\|x\|_L^{-n+k}\big)^\wedge(\xi) \, .$$

Approximate the body L in the norm of the space $C^{k-1}(S^{n-1})$ by a sequence of infinitely smooth bodies L_m. By Remark 1, the functions $(\|x\|_{L_m}^{-n+k})^\wedge$ converge to $(\|x\|_L^{-n+k})^\wedge$ in the uniform metric on S^{n-1}. Therefore, by Corollary 1 and the equality (4), for every $H \in G(n, n-k)$,

$$\begin{aligned} vol_{n-k}(L \cap H) &= \frac{1}{n-k} \int_{S^{n-1}\cap H} \|x\|_L^{-n+k} dx \\ &= \frac{1}{n-k} \lim_{m\to\infty} \int_{S^{n-1}\cap H} \|x\|_{L_m}^{-n+k} dx \\ &= \frac{(2\pi)^{n-k}}{n-k} \lim_{m\to\infty} \int_{S^{n-1}\cap H^\perp} \big(\|x\|_{L_m}^{-n+k}\big)^\wedge(\xi) d\xi \\ &= \frac{(2\pi)^{n-k}}{n-k} \int_{S^{n-1}\cap H^\perp} \big(\|x\|_L^{-n+k}\big)^\wedge(\xi) d\xi \\ &= \frac{1}{k} \int_{S^{n-1}\cap H^\perp} \|\xi\|_K^{-k} d\xi = vol_k(K \cap H^\perp)\,. \end{aligned}$$

□

It follows from Proposition 2 and Theorem 1(a) that if $(-1)^{(k-1)/2} A_{L,\xi}^{(k-1)}(0) \le 0$ at any point ξ then the k-intersection body of L does not exist. Also if k is even the statement of Theorem 4 holds if one replaces $(-1)^{(k-1)/2} A_{L,\xi}^{(k-1)}(0)$ by the expression appearing in part (b) of Theorem 1.

4 Generalized *k*-intersection Bodies

The class of generalized k-intersection bodies (see Definition 2) also admits a functional analytic interpretation. Suppose that $p > 0$ and k is an integer, $1 \le k < n$. If (Ω, σ) is a space with finite measure then $L_p(\mathbb{R}^k)$ is the space of (classes of equivalence of) measurable functions $f : \Omega \mapsto \mathbb{R}^k$ such that

$$\|f\|^p = \int_\Omega \big\|f(\omega)\big\|_2^p d\sigma(\omega) < \infty\,.$$

Consider an n-dimensional subspace $(\mathbb{R}^n, \|\cdot\|)$ of $L_p(\mathbb{R}^k)$. An argument, similar to that in the scalar case, shows that there exists a finite Borel measure ν on $\mathbb{R}^{nk}$ such that, for every $x \in \mathbb{R}^n$,

$$\|x\|^p = \int_{\mathbb{R}^{nk}} \big((x,\xi_1)^2 + \cdots + (x,\xi_k)^2\big)^{p/2} d\nu(\xi)\,, \tag{19}$$

where we arrange vectors $\xi_1, \dots, \xi_k \in \mathbb{R}^n$ consequently to form a vector $\xi \in \mathbb{R}^{nk}$.

Let us assume that p is not an even integer and apply both sides of (19) to an even test function $\phi \in \mathcal{S}(\mathbb{R}^n)$. For every $u \in \mathbb{R}^k$, denote by

$H_\xi(u) = \{x \in \mathbb{R}^n : (x,\xi_i) = u_i,\ i = 1,\dots,k\}$. Note that the Fourier transform of the function $u \mapsto \int_{H_\xi(u)} \phi(x)\ dx$ at a point $v \in \mathbb{R}^k$ is equal to $\hat\phi(v_1\xi_1 + \cdots + v_k\xi_k)$ (see for example the proof of Lemma 6 in [K5]). Also, since p is not an even integer, $(\|u\|_2^p)^\wedge(v) = C_p\|v\|_2^{-p-k}$, where $C_p = 2^{p+k}\pi^{k/2}\Gamma((p+k)/2)/\Gamma(-p/2)$ (see [GeS, p. 192]). We have

$$\begin{aligned}\int_{\mathbb{R}^n} \|x\|^p\phi(x)dx &= \int_{\mathbb{R}^{nk}} d\nu(\xi)\int_{\mathbb{R}^n}\big((x,\xi_1)^2+\cdots+(x,\xi_k)^2\big)^{p/2}\phi(x)dx\\ &= \int_{\mathbb{R}^{nk}} d\nu(\xi)\int_{\mathbb{R}^k}\|u\|_2^p\bigg(\int_{H_\xi(u)}\phi(x)\ dx\bigg)du\\ &= \frac{C_p}{(2\pi)^k}\int_{\mathbb{R}^{nk}}\big\langle\|v\|_2^{-p-k},\hat\phi(v_1\xi_1+\cdots+v_k\xi_k)\big\rangle d\nu(\xi)\,.\end{aligned}$$

If $p < 0$ the function $\|v\|_2^{-p-k}$ is locally integrable on $\mathbb{R}^k$, so we can extend the concept of embedding in $L_p(\mathbb{R}^k)$ to negative values of p.

Definition 5. *Let $x \to \|x\|$ be a homogeneous of degree 1, even continuous, positive outside of the origin function on $\mathbb{R}^n$. For any $0 < p < n$ and an integer k, $1 \le k < n$, we say that the space $(\mathbb{R}^n, \|\cdot\|)$ embeds in $L_{-p}(\mathbb{R}^k)$ if there exists a finite Borel measure ν on $\mathbb{R}^{nk}$ such that for every test function $\phi \in \mathcal{S}(\mathbb{R}^n)$*

$$\int_{\mathbb{R}^n}\|x\|^{-p}\phi(x)\ dx = \int_{\mathbb{R}^{nk}} d\nu(\xi)\int_{\mathbb{R}^k}\|v\|_2^{p-k}\hat\phi(v_1\xi_1+\cdots+v_k\xi_k)dv\,. \quad (20)$$

Note that the integral in the left-hand side converges absolutely, since $\|x\|^{-p}$ is a locally integrable $(0 < p < n)$ bounded outside of the origin function and $|\phi|$ is an L_1-function on $\mathbb{R}^n$.

It is well known that, for $p > 0$, the spaces L_p and $L_p(\mathbb{R}^k)$ can be embedded isometrically in each other. One of these embeddings extends to the negative case.

Theorem 5. *Suppose that $0 < p < n$, k is an integer, $1 \le k < n$. If a space $(\mathbb{R}^n, \|\cdot\|)$ embeds in $L_{-p}(\mathbb{R}^k)$ then it also embeds in L_{-p}.*

Proof. Let us write the inner integral in the right-hand side of (20) in polar coordinates $v = t\theta,\ t \in [0,\infty),\ \theta \in S^{k-1}$:

$$\int_{\mathbb{R}^n}\|x\|^{-p}\phi(x)\ dx = \int_{\mathbb{R}^{nk}} d\nu(\xi)\int_{S^{k-1}} d\theta\int_0^\infty t^{p-1}\hat\phi\big(t(\theta_1\xi_1+\cdots+\theta_k\xi_k)\big)dt\,. \quad (21)$$

Denote by μ the measure on S^{n-1} that is the image of the measure

$$\|\theta_1\xi_1+\cdots+\theta_k\xi_k\|_2^{-p}d\nu(\xi)\times d\theta$$

under the mapping $\tau : \mathbb{R}^{nk} \times S^{k-1} \mapsto S^{n-1}$

$$\tau(\xi,\theta) = \frac{\theta_1\xi_1 + \cdots + \theta_k\xi_k}{\|\theta_1\xi_1 + \cdots + \theta_k\xi_k\|_2}\,.$$

Making a substitution $r = t\|\theta_1\xi_1 + \cdots + \theta_k\xi_k\|_2$ in the inner integral in the right-hand side of (21) and then the substitution corresponding to the mapping τ, we get the equality (17). The fact that μ is a finite measure on S^{n-1} immediately follows from (21) with $\phi(x) = \exp(-\|x\|_2^2)$. □

The author does not know whether every space that embeds in L_{-p} also embeds in $L_{-p}(\mathbb{R}^k)$.

Our next result shows the connection between generalized k-intersection bodies and embedding in $L_{-k}(\mathbb{R}^k)$.

Theorem 6. *Let $1 \le k < n$. A symmetric star body K is a generalized k-intersection body if and only if $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in $L_{-k}(\mathbb{R}^k)$.*

Proof. It is easily seen that K is a generalized k-intersection body if and only if there exists a finite Borel measure μ on $G(n, n-k)$ so that, for every test function $\phi \in \mathcal{S}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \|x\|_K^{-k}\phi(x)dx = \int_{G(n,n-k)} d\mu(H) \int_H \phi(x)dx\,. \tag{22}$$

In fact, writing the integrals over $\mathbb{R}^n$ and H in (22) in polar coordinates one gets condition (3). Note that every infinitely differentiable function g on S^{n-1} can be represented in the form $g(\theta) = \int_0^\infty t^{n-k-1}\phi(t\theta)$, where $\phi \in \mathcal{S}(\mathbb{R}^n)$. Take for example the function $\phi(x) = u(t)g(\theta)$, $x = t\theta$, $t \in \mathbb{R}$, $\theta \in S^{n-1}$, where $u \in \mathcal{S}(\mathbb{R})$, $\int_0^\infty t^{n-k-1}u(t)dt = 1$.

Suppose that $(\mathbb{R}^n, \|\cdot\|)$ embeds in $L_{-k}(\mathbb{R}^k)$. Let us now modify condition (20) of Definition 5 (we put $p = k$). Putting $\phi(x) = \exp(-\|x\|_2^2)$ in (20), we see that the measure ν of the set of those ξ, for which $\xi_1, \dots, \xi_k$ are linearly dependent, is equal to zero. If $\xi_1, \dots, \xi_k$ are linearly independent then

$$\int_{\mathbb{R}^k} \hat{\phi}(v_1\xi_1 + \cdots + v_k\xi_k)\ dv = c(\xi) \int_{H^\perp} \hat{\phi}(x)dx\,,$$

where $H^\perp = span(\xi_1, \dots, \xi_k)$, integration over $H^\perp$ is with respect to an orthonormal basis in $H^\perp$, and $c(\xi)$ are positive constants appearing as the result of changing the coordinates of integration. On the other hand, by [K5, Lemma 6],

$$\int_{H^\perp} \hat{\phi}(x)\ dx = (2\pi)^k \int_H \phi(x)dx\,. \tag{23}$$

Therefore, condition (20) with $p = k$ is equivalent to (22) with the measure μ on $G(n, n-k)$ defined by

$$\mu(A) = \int_{\{\xi : (span(\xi_1,\dots,\xi_k))^\perp \in A\}} c(\xi) d\nu(\xi)$$

for every Borel set $A \subset G(n, n-k)$. Thus, K is a generalized k-intersection body.

Conversely, suppose that K is a generalized k-intersection body, and let μ be the corresponding measure on $G(n, n-k)$. Let $\tau : G(n, n-k) \to \mathbb{R}^{nk}$ be a continuous mapping so that for every $H \in G(n, n-k)$, $\tau(H) = \xi = (\xi_1, \dots, \xi_k)$, where $\xi_1, \dots, \xi_k$ form an orthonormal basis in $H^\perp$. Define a measure ν on $\mathbb{R}^{nk}$ as the image of μ under the mapping τ, i.e. for every Borel set $A \subset \mathbb{R}^{nk}$

$$\nu(A) = \mu\big\{H \in G(n, n-k) : \tau(H) \in A\big\}\,.$$

Then, by (22) and (23),

$$\int_{\mathbb{R}^n} \|x\|_K^{-k} \phi(x)dx = \int_{G(n,n-k)} d\mu(H) \int_H \phi(x)dx$$

$$= \int_{\mathbb{R}^{nk}} d\nu(\xi) \int_{(span(\xi_1,\dots,\xi_k))^\perp} \phi(x)dx = \frac{1}{(2\pi)^k} \int_{\mathbb{R}^{nk}} d\nu(\xi) \int_{span(\xi_1,\dots,\xi_k)} \hat{\phi}(x)dx$$

$$= \int_{\mathbb{R}^{nk}} d\nu(\xi) \int_{\mathbb{R}^k} \hat{\phi}(v_1\xi_1 + \cdots + v_k\xi_k)dv\,. \quad \square$$

Corollary 3. *Let $1 \le k < n$, and let K be a symmetric star body in $\mathbb{R}^n$. If K is a generalized k-intersection body then it is also a k-intersection body.*

Proof. By Theorem 6, if K is a generalized k-intersection body then $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in $L_{-k}(\mathbb{R}^k)$. Then, by Theorem 5, $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_{-k}, and the result follows from Theorem 3. $\square$

Our next result gives a sufficient condition for generalized k-intersection bodies in terms of the Fourier transform.

Proposition 3. *Let K be an infinitely smooth star body in $\mathbb{R}^n$, and $1 \le k < n$. For $H \in G(n, n-k)$ put*

$$\mu(H) = \frac{\pi c(n,k)}{(2\pi)^n} \int_{S^{n-1}\cap H^\perp} \big(\|x\|_K^{-k}\|x\|_2^{k-1}\big)^\wedge(\theta)d\theta\,,$$

where $c(n,k)$ is the constant from Lemma 1. Then the signed measure $\mu(H)dH$ satisfies the equation (3). Therefore, if $\mu(H) \ge 0$ for every $H \in G(n, n-k)$ then K is a generalized k-intersection body.

Proof. By a version of Parseval's formula from [K5, Lemma 3], for every $f \in C(S^{n-1})$

$$(2\pi)^n \int_{S^{n-1}} \|x\|_K f(x)dx = \int_{S^{n-1}} \big(\|x\|_K^{-k}\|x\|_2^{k-1}\big)^\wedge(\xi)\big(f(\theta)r^{-n+1}\big)^\wedge(\xi)d\xi\,.$$

On the other hand, by Proposition 1 and Lemma 1, the right-hand side of the latter formula is equal to

$$\pi \int_{S^{n-1}} \left(\|x\|_K^{-k}\|x\|_2^{k-1}\right)^\wedge(\xi)\left(\int_{S^{n-1}\cap\xi^\perp} f(\theta)\ d\theta\right)d\xi$$
$$= \pi c(n,k)\int_{G(n,n-k)}\left(\int_{S^{n-1}\cap H^\perp}\left(\|x\|_K^{-k}\|x\|_2^{k-1}\right)^\wedge(\xi)d\xi\right)$$
$$\cdot\left(\int_{S^{n-1}\cap H} f(\theta)d\theta\right)dH\,. \quad \square$$

5 Generalizations of the Busemann–Petty Problem

First, we give a new proof of the result of Bourgain and Zhang [BZ] that the solution to the generalized Busemann–Petty problem is negative if the dimension of sections is greater than 3.

Theorem 7 ([BZ]). *If $n-k>3$ then the answer to the GBP-problem is negative.*

Proof. A result of Zhang [Z1, Th. 7] is that the answer to GBP is affirmative if and only if every symmetric convex body in $\mathbb{R}^n$ is a generalized k-intersection body. In view of Theorem 5, this means that every symmetric convex body in $\mathbb{R}^n$ must also be a k-intersection body. However, if $k<n-3$ then for every $2<q\le\infty$, the unit ball of the space ℓ_q^n is not a k-intersection body, as mentioned in section 3 after Theorem 3. $\square$

Our techniques allow us to generalize the positive solution to the original BP-problem in dimension 4 from [Z2] in the following way:

Theorem 8. *Let K, L be symmetric star bodies in $\mathbb{R}^n$ so that K is convex and, for every $H\in G(n,3)$,*

$$\int_{S^{n-1}\cap H}\|x\|_K^{-n+1}dx \le \int_{S^{n-1}\cap H}\|x\|_L^{-n+1}dx\,.$$

Then $vol_n(K)\le vol_n(L)$.

Proof. We can assume without loss of generality that K, L are infinitely smooth. In fact, one can approximate K in the uniform metric on the sphere by a sequence of infinitely smooth convex bodies contained in K, and approximate L by a sequence of infinitely smooth star bodies containing L, then the result for approximating bodies easily implies the general result. By Corollary 2 with $k=n-3$, for every $H\in G(n,3)$

$$\int_{S^{n-1}\cap H^\perp}(\|x\|_K^{-1})^\wedge(\theta)d\theta\ge 0\,.$$

Therefore,

$$\int_{G(n,3)} \left(\int_{S^{n-1}\cap H^\perp} \left(\|x\|_K^{-1}\right)^\wedge(\theta)\ d\theta \right) \left(\int_{S^{n-1}\cap H} \|x\|_K^{-n+1} dx \right) dH$$
$$\leq \int_{G(n,3)} \left(\int_{S^{n-1}\cap H^\perp} \left(\|x\|_K^{-1}\right)^\wedge(\theta) d\theta \right) \left(\int_{S^{n-1}\cap H} \|x\|_L^{-n+1} dx \right) dH \,.$$

Applying Lemma 1 to both sides of this inequality, we get

$$\int_{S^{n-1}} \left(\|x\|_K^{-1}\right)^\wedge(\theta) d\theta \int_{S^{n-1}\cap\theta^\perp} \|x\|_K^{-n+1} dx$$
$$\leq \int_{S^{n-1}} \left(\|x\|_K^{-1}\right)^\wedge(\theta) d\theta \int_{S^{n-1}\cap\theta^\perp} \|x\|_L^{-n+1} dx \,.$$

By Proposition 1,

$$\int_{S^{n-1}} \left(\|x\|_K^{-1}\right)^\wedge(\theta) \left(\|x\|_K^{-n+1}\right)^\wedge(\theta) d\theta \leq \int_{S^{n-1}} \left(\|x\|_K^{-1}\right)^\wedge(\theta) \left(\|x\|_L^{-n+1}\right)^\wedge(\theta) d\theta \,.$$

Now we can apply a version of Parseval's formula from [K5, Lemma 3] to remove the Fourier transforms and then use Hölder's inequality:

$$\begin{aligned} n\, vol_n(K) &= \int_{S^{n-1}} \|x\|_K^{-n} dx \leq \int_{S^{n-1}} \|x\|_K^{-1} \|x\|_L^{-n+1} dx \\ &\leq \left(\int_{S^{n-1}} \|x\|_K^{-n} dx \right)^{1/n} \left(\int_{S^{n-1}} \|x\|_L^{-n} dx \right)^{(n-1)/n} \\ &= n\big(vol_n(K)\big)^{1/n} \big(vol_n(L)\big)^{(n-1)/n} \,. \end{aligned}$$

The result follows. □

Acknowledgements. I wish to thank S. Alesker, P. Goodey and A. Vershik for explaining to me some facts about integration on the Grassmanians. The proof of Lemma 1 was suggested by M. Rudelson and replaces my more complicated geometric argument.

References

[BZ] J. Bourgain, Gaoyong Zhang, On a generalization of the Busemann–Petty problem, Convex Geometric Analysis (Berkeley, CA, 1996), Cambridge Univ. Press, Cambridge, Math. Sci. Res. Inst. Publ. 34 (1999), 65–76.

[F] W. Feller, An Introduction to Probability Theory and its Applications, Wiley & Sons, New York, 1971.

[G] R.J. Gardner, Geometric Tomography, Cambridge University Press, New York, 1995.

[GKS] R.J. Gardner, A. Koldobsky, T. Schlumprecht, An analytic solution to the Busemann–Petty problem on sections of convex bodies, Annals of Math. 149 (1999), 691–703.

[GeS] I.M. Gelfand, G.E. Shilov, Generalized Functions, vol. 1. Properties and Operations, Academic Press, New York, 1964.

[GeV] I.M. Gelfand, N.Ya. Vilenkin, Generalized Functions, vol. 4. Applications of Harmonic Analysis, Academic Press, New York, 1964.

[GoLW] P.R. Goodey, E. Lutwak, W. Weil, Functional analytic characterization of classes of convex bodies, Math. Z. 222 (1996), 363–381.

[K1] A. Koldobsky, An application of the Fourier transform to sections of star bodies, Israel J. Math. 106 (1998), 157–164.

[K2] A. Koldobsky, Intersection bodies, positive definite distributions and the Busemann–Petty problem, Amer. J. Math. 120 (1998), 827–840.

[K3] A. Koldobsky, Intersection bodies in $\mathbb{R}^4$, Advances in Math. 136 (1998), 1–14.

[K4] A. Koldobsky, Positive definite distributions and subspaces of L_{-p} with applications to stable processes, Canad. Math. Bull. 42 (1999), 344–353.

[K5] A. Koldobsky, A generalization of the Busemann–Petty problem on sections of convex bodies, Israel J. Math. 110 (1999), 75–91.

[L] P. Lèvy, Théorie de l'addition de variable aléatoires, Gauthier-Villars, Paris, 1937.

[Lu] E. Lutwak, Intersection bodies and dual mixed volumes, Advances in Math. 71 (1988), 232–261.

[MP] V.D. Milman, A. Pajor, Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space, in "Geometric Aspects of Functional Analysis" (J. Lindenstrauss, V.D. Milman, eds.), Springer Lecture Notes in Mathematics 1376 (1989), 64–104.

[SKM] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives, Gordon and Breach Sci. Publ., Amsterdam, 1993.

[Sc] R. Schneider, Convex Bodies: the Brunn-Minkowski Theory, Cambridge University Press, Cambridge, 1993.

[Z1] Gaoyong Zhang, Sections of convex bodies, Amer. J. Math. 118 (1996), 319–340.

[Z2] Gaoyong Zhang, A positive answer to the Busemann–Petty problem in four dimensions, Annals of Math. 149 (1999), 535–543.

Alexander Koldobsky, Department of Mathematics, University of Missouri-Columbia, Columbia MO 65211, USA koldobsk@math.missouri.edu

Submitted: October 1999
Revision: December 1999