GAFA, Geom. funct. anal.
Vol. 10 (2000) 679 – 701
1016-443X/00/040679-23 $ 1.50+0.20/0

GAFA Geometric And Functional Analysis

MINIMAL SURFACES IN FLAT TORI

C. Arezzo and M.J. Micallef

1 Introduction

Calibrated geometries were initially introduced by Harvey and Lawson to construct many examples of volume minimizing submanifolds. The simplest of these is Kähler geometry and it has been known for a long time from the work of Wirtinger and Federer that complex submanifolds of Kähler manifolds minimize volume in their homology class. The converse of this fact can be loosely stated as: *is an even-dimensional volume minimizing submanifold of a Kähler manifold necessarily holomorphic?*

Of course, this cannot be true in this generality. To start with, if the dimension of the submanifold is $2p$, then it had better be of type (p,p) (that is, its homology class should be Poincaré dual to a cohomology class of type $(n-p,n-p)$) in order for it to stand a chance of being holomorphic. For instance, let g_1 and g_2 be two different hyperbolic metrics on a surface Σ of genus $\geqslant 2$. Let $M=(\Sigma,g_1)\times(\Sigma,g_2)$ with the Kähler form dA_1-dA_2 where dA_j, $j\in\{1,2\}$, is the oriented element of area on Σ with the metric g_j. M is then a Kähler Einstein manifold of negative scalar curvature and, by a theorem of Schoen, the diagonal class Δ in M is represented by a unique embedded Lagrangian area-minimizing surface S. (This result is contained in the proof of Proposition 2.12 in [S].) Being Lagrangian, S is as far as possible from being holomorphic, but S does not provide a suitable counterexample to the converse of the Wirtinger–Federer theorem because it is not of type $(1,1)$. Now a Kähler Einstein manifold of nonzero scalar curvature is algebraic and therefore, by the Lefschetz theorem, every integral cohomology class of type $(1,1)$ is Poincaré dual to a divisor. However, the Wirtinger–Federer theorem applies only to *effective* algebraic cycles and therefore, it may still be possible to find an example of a stable minimal surface of type $(1,1)$ in a Kähler Einstein 4-manifold of

The first author was supported by European Community Grant TRM N.ERBFMBICT 950079.

negative scalar curvature which is not plus or minus holomorphic. Indeed, the present authors expect plenty of such examples (including Lagrangian ones) to exist, though they are not aware of any.

Another possible obstruction to the converse of the Wirtinger–Federer theorem is that the Kähler manifold may not contain any complex subvarieties while, by the general methods of geometric measure theory, one can represent any integral homology class by a volume minimizing integral current. Therefore, one has to ask for a converse of the Wirtinger–Federer theorem in suitably restricted settings, such as those about to be described.

If M is a Kähler manifold with positive Ricci curvature, then M is algebraic and $H^2(M,\mathbb{C})$ is purely of type $(1,1)$. Therefore it is reasonable to ask whether, in this case, an area minimizing surface is necessarily plus or minus holomorphic. This is still open but one does have the following important results:

Theorem 1.1 (Lawson–Simons, [LaS]). *The only closed stable minimal integral currents of $\mathbb{C}P^n$ with the Fubini–Study metric are the algebraic cycles.*

Theorem 1.2 (Siu–Yau, [SiuY]). *An area (energy) minimizing two-sphere in a Kähler manifold with positive holomorphic bisectional curvature is plus or minus holomorphic.*

A Riemannian manifold is called *special Kähler* if its holonomy group lies in $SU(n)$. This happens if and only if the Ricci curvature is zero and the metric is Kähler for some integrable complex structure (not necessarily unique). Any two dimensional cohomology class in a special Kähler manifold is of type $(1,1)$ with respect to one of the integrable complex structures compatible with the given metric. (This follows from the classification of special Kähler manifolds in [B].) The appropriate question now is: *is an area minimizing surface in a special Kähler manifold necessarily holomorphic with respect to one of the integrable complex structures compatible with the given metric?* For a long time it was felt that this question should have an affirmative answer. Evidence for this came from the following theorems of the second author.

Theorem 1.3 ([M1]). *An area minimizing surface in a flat four-torus is holomorphic with respect to a complex structure compatible with the given flat metric.*

Theorem 1.4 ([M2]). *Let $f\colon (\Sigma,\mu) \to (\mathbb{R}^n/\Lambda, eucl)$ be a conformal stable minimal immersion of a hyperelliptic Riemann surface (Σ,μ) into* *a flat torus* $(\mathbb{R}^n/\Lambda, eucl)$. *Then* $f(\Sigma)$ *lies in an even dimensional totally geodesic flat subtorus* T *of* $(\mathbb{R}^n/\Lambda, eucl)$ *and is holomorphic with respect to an integrable complex structure compatible with the flat metric on* T.

One of the main results of this paper is to establish the following theorem which shows that the hyperelliptic assumption in Theorem 1.4 is essential.

Theorem 1.5. *For any nonhyperelliptic Riemann surface* (Σ_γ, μ) *of genus* $\gamma \geqslant 4$, *there exists a conformal stable minimal immersion* $f\colon (\Sigma_\gamma, \mu) \to (\mathbb{R}^{2\gamma}/\Lambda, eucl)$ *into a flat torus, which is not holomorphic with respect to any complex structure compatible with the flat metric.*

An example of a hyperKähler four-manifold M which contains a stable minimal two-sphere which is not holomorphic with respect to any of the integrable complex structures compatible with the given metric on M can be found in ([MW]). (Micallef later realised that the minimal two-sphere in this example is, in fact, area minimizing.) This counterexample to the converse of the Wirtinger–Federer theorem is not totally satisfactory because M is noncompact.

The basic idea of the proof of Theorem 1.5 is to start with a holomorphic map of a Riemann surface (Σ_γ, μ) into a flat complex torus $(\mathbb{C}^n/\Lambda, eucl)$. We then deform the map and the metric on the torus, but not the conformal structure on Σ_γ, in such a way as to destroy holomorphicity but not minimality, stability and conformality.

More precisely, the proof has three key ingredients which are interesting in their own right.

(i) A useful characterization (Theorem 2.3) of holomorphic maps in terms of an algebraic property of the Weierstrass representation.

(ii) An algebraic geometric criterion for the deformation of a holomorphic immersion into a nonholomorphic minimal immersion (Theorem 2.4).

(iii) An algebraic geometric characterization of those holomorphic maps which have only translational Jacobi fields (Theorem 3.1). This guarantees the stability of the maps obtained by the deformation process in (ii).

We also generalize Theorem 1.5 to stable nonholomorphic minimal immersions of Σ_γ into $T^{2\gamma-2k}$, for $k = 1, 2$ or 3 and sufficiently large γ. We expect that such examples also exist for any $k \leqslant \gamma - 4$ but the algebraic geometric difficulties in our construction become considerable.

The proof of Theorem 1.5 makes it clear that stable minimal surfaces come in families. This is discussed towards the end of section 3 but we refer to [Ar1] for a thorough treatment of this topic. We complete this section by proving that stable nonholomorphic minimal surfaces exist in *any* flat torus of dimension $\geqslant 8$ (Theorem 3.6). However, we now lose track of topological and conformal properties of the surface other than the fact that it has to be nonhyperelliptic.

A similar strategy to the one described above has been used by the authors and G.P. Pirola ([ArMP]) to prove the existence of stable embedded nonholomorphic minimal surfaces of finite total curvature in euclidean spaces of sufficiently high dimension. In this case the technical difficulties are considerably harder since we have to deal with holomorphic differentials with assigned poles, and no real periods, on punctured Riemann surfaces.

At the end of section 3, we study surfaces which minimize area among maps from Σ_γ into $T^{2\gamma}$ which induce a fixed isomorphism $\rho\colon H_1(\Sigma_\gamma, \mathbb{Z}) \to H_1(T^{2\gamma}, \mathbb{Z})$. The techniques are now based on the variational methods employed by the first author in [Ar2]. We can only handle the case $\gamma = 4$. The cases $\gamma = 2$ and 3 were dealt with in [Ar2] where a classical result on principally polarised abelian surfaces and three-folds based on the Matsusaka–Ran criterion is proved by purely differential geometric methods. Roughly speaking, we prove that degeneration to a Riemann surface with nodes can happen during the area minimization process only if the metric is hermitian with respect to a complex structure which makes the torus split into a product of Jacobi varieties of Riemann surfaces of genus smaller than 4. As a consequence, we characterize (in Theorem 3.7) the flat metrics on T^8 which admit a nonholomorphic surface of genus 4 which minimizes area among surfaces which induce ρ. Some of the results we establish should still hold for $\gamma \geqslant 5$ but it does not seem worthwhile to put in the effort required to overcome the considerable technical difficulties.

We conclude the paper by remarking that the nonholomorphic area minimizing surfaces produced by Theorem 3.7 have a 6-fold cover which no longer minimizes area in its homology class. This contrasts with the two-sided hypersurface situation where stability implies the stability of any cover ([FS]).

2 Minimal and Holomorphic Immersions

It is best to start by fixing our notation.

Σ_γ	:=	a fixed compact oriented topological surface without boundary and of genus γ.
μ	:=	a conformal structure on Σ_γ.

K := the canonical bundle (i.e., the bundle of differential forms of type $(1,0)$) of (Σ_γ, μ).
T^n := torus of $\mathbb{R}$-dimension n, viewed as a Lie group but without any further geometric structure such as a metric, complex structure, etc.
Λ := a lattice of maximal rank in $\mathbb{R}^n$.
$eucl$:= the standard euclidean metric on $\mathbb{R}^n$.
g := a flat metric on T^n or $\mathbb{R}^n$. With respect to a parallel frame, g can be represented by a positive definite matrix.
J := a parallel (constant) complex structure on T^{2n} or $\mathbb{R}^{2n}$. J and g are compatible if J is an isometry w.r.t. g. In this case, we often refer to J as being orthogonal and g as being hermitian.
J_0 := $\begin{pmatrix} 0 & -Id_n \\ Id_n & 0 \end{pmatrix}$, the standard complex structure on $\mathbb{R}^{2n}$. We will denote $(\mathbb{R}^{2n}, J_0)$ by $\mathbb{C}^n$.

A flat torus will be denoted either by (T^n, g) or by $(\mathbb{R}^n/\Lambda, eucl)$. Of course, $(\mathbb{R}^n/\Lambda, eucl)$ is isometric to $(\mathbb{R}^n/\Lambda', eucl)$ (in which case they represent the same flat torus) iff $\Lambda' = O\Lambda$ for some $O \in O(n, \mathbb{R})$. A complex torus will be denoted by any of (T^{2n}, J), $\mathbb{C}^n/\Lambda$ and $(\mathbb{R}^{2n}/\Lambda, J_0)$. In the same way as for flat tori, $\mathbb{C}^n/\Lambda$ is biholomorphic to $\mathbb{C}^n/\Lambda'$ (in which case they represent the same complex torus) iff $\Lambda' = L\Lambda$ for some $L \in GL(n, \mathbb{C}) \subset GL(2n, \mathbb{R})$. A flat complex torus will be denoted by any of (T^{2n}, g, J), $(\mathbb{C}^n/\Lambda, eucl)$ and $(\mathbb{R}^{2n}/\Lambda, J_0, eucl)$. $(\mathbb{C}^n/\Lambda, eucl)$ and $(\mathbb{C}^n/\Lambda', eucl)$ are equivalent as Kähler tori iff $\Lambda' = U\Lambda$ for some $U \in U(n)$.

The basic tool in studying minimal surfaces in flat tori is given by the classical:

Theorem 2.1 (Generalized Weierstrass Representation). *If $f\colon (\Sigma_\gamma, \mu) \to (\mathbb{R}^n/\Lambda, eucl)$ is a conformal minimal immersion then, after a translation, f can be represented by*

$$f(p) = Re \begin{pmatrix} \int_{p_0}^{p} \eta_1 \\ \vdots \\ \int_{p_0}^{p} \eta_n \end{pmatrix} \mod \Lambda,$$

where $p_0 \in \Sigma_\gamma$ and the η_i's are holomorphic differentials on (Σ_γ, μ) satisfying

1. *$\sum_{i=1}^{n} \eta_i^2 = 0$, and*

2. *$\{Re\,(\int_\sigma \eta_1, \dots, \int_\sigma \eta_n)^t \mid \sigma \in H_1(\Sigma_\gamma, \mathbb{Z})\}$ is a sublattice of Λ.*

A useful way to rewrite the above theorem is the following: fix a basis $\{\omega_1, \dots, \omega_\gamma\}$ of the space of holomorphic differentials of (Σ_γ, μ) and write it as a column vector $\underline{\omega}$. Thus the row vector $\underline{\omega}^t = (\omega_1, \dots, \omega_\gamma)$. By Theorem 2.1 a conformal minimal immersion $f\colon (\Sigma_\gamma, \mu) \to (\mathbb{R}^n/\Lambda, eucl)$ is determined, up to translations, by n holomorphic differentials $\underline{\eta}^t = (\eta_1, \dots, \eta_n)$ which satisfy the quadratic condition (1). We therefore have $\underline{\eta} = M\underline{\omega}$, where M is a $n \times \gamma$ complex matrix and the conformality relation (1) becomes

$$\underline{\eta}^t \underline{\eta} = \underline{\omega}^t M^t M \underline{\omega} = 0\,. \tag{1}$$

The condition (2) on the periods can be translated into the following: let $\{\sigma_R,\ 1 \leqslant R \leqslant 2\gamma\}$ be a basis of $H_1(\Sigma_\gamma, \mathbb{Z})$ and define $\Omega_{jR} = \int_{\sigma_R} \omega_j$. Then the lattice generated by the columns of $Re(M\Omega)$ has to be a sublattice of Λ.

The set of matrices satisfying (1) and the above condition parametrizes the space of conformal minimal immersions of (Σ_γ, μ) into $(\mathbb{R}^n/\Lambda, eucl)$; we denote by f_M the immersion associated to the matrix M as above.

Given a compact Riemann surface there is a special class of conformal minimal immersions into flat tori of even dimension, namely the ones holomorphic w.r.t. a complex structure compatible with the metric. Let us recall that to every (Σ_γ, μ) it is possible to associate a complex torus, called the *jacobian* of the Riemann surface (denoted by $\mathcal{J}(\Sigma_\gamma, \mu)$ from now on), in the following way: take a basis $\{\omega_1, \dots, \omega_\gamma\}$ of the space of holomorphic differentials of (Σ_γ, μ), and consider

$$\Lambda_{\underline{\omega}} = \left\{ Re \left(\int_\sigma \omega_1, \dots, \int_\sigma \omega_\gamma, \int_\sigma i\omega_1, \dots, \int_\sigma i\omega_\gamma \right)^t \,\Big|\, \sigma \in H_1(\Sigma_\gamma, \mathbb{Z}) \right\}.$$

If $\underline{\eta} = L\underline{\omega}$ is a different basis, $L \in Gl(\gamma, \mathbb{C})$, it is clear that $(\mathbb{R}^{2\gamma}/\Lambda_{\underline{\omega}}, J_0)$ is biholomorphic via L to $(\mathbb{R}^{2\gamma}/\Lambda_{\underline{\eta}}, J_0)$. The jacobian $\mathcal{J}(\Sigma_\gamma, \mu)$ is the well defined point in the moduli space of complex tori represented by either of $(\mathbb{R}^{2\gamma}/\Lambda_{\underline{\omega}}, J_0)$ and $(\mathbb{R}^{2\gamma}/\Lambda_{\underline{\eta}}, J_0)$. Indeed we shall often write $\mathcal{J}(\Sigma_\gamma, \mu)$ for either of these complex tori.

By classical theorems due to Abel and Jacobi the map $j_{p_0}\colon (\Sigma_\gamma, \mu) \to \mathcal{J}(\Sigma_\gamma, \mu)$, defined by

$$j_{p_0}(p) = Re\left(\int_{p_0}^p \omega_1, \dots, \int_{p_0}^p \omega_\gamma, \int_{p_0}^p i\omega_1, \dots, \int_{p_0}^p i\omega_\gamma \right)^t \bmod \Lambda_{\underline{\omega}}$$

is a holomorphic embedding. It satisfies the following universal property among holomorphic maps, and is therefore called *the Abel–Jacobi map*:

Theorem 2.2 (Universal property of the Abel–Jacobi map). *If $f\colon (\Sigma_\gamma,\mu) \to (\mathbb{R}^{2n}/\Lambda, J)$ is a full holomorphic map of a Riemann surface (Σ_γ,μ) to a complex torus $(\mathbb{R}^{2n}/\Lambda, J)$, then f factors through $\mathcal{J}(\Sigma_\gamma,\mu)$, i.e. there exists a $\mathbb{C}$-linear map $A\colon \mathcal{J}(\Sigma_\gamma,\mu) \to (\mathbb{R}^{2n}/\Lambda, J)$ s.t. if p_0 is a fixed point in Σ_γ and $f_{p_0}(z) := f(z) - f(p_0)$ for all $z \in \Sigma_\gamma$, then*

$$\begin{array}{ccc} \Sigma_\gamma & \xrightarrow{f_{p_0}} & (\mathbb{R}^{2n}/\Lambda, J) \\ & \searrow_{j_{p_0}} & \uparrow_{A} \\ & & \mathcal{J}(\Sigma_\gamma,\mu) \end{array} \tag{2}$$

commutes. In particular $\mathcal{J}(\Sigma_\gamma,\mu)$ contains a codimension n complex subtorus, given by the kernel of A.

For *any* choice of basis $\underline{\omega}$, the map $j_{p_0}\colon (\Sigma_\gamma,\mu) \to (\mathbb{R}^{2\gamma}/\Lambda_{\underline{\omega}}, eucl)$ is a conformal minimal immersion. Observe that two different bases $\underline{\omega}$ and $\underline{\eta}$ of holomorphic differentials will lead to isometric flat tori if and only if $\underline{\eta} = L\underline{\omega}$ with L unitary. Thus there are many – in a sense which will be made more precise in section 3 – minimal immersions associated to the Abel–Jacobi map.

We now seek an effective way to recognize holomorphic maps in terms of the matrix M defined in (1) above. Suppose in fact that $f_M\colon (\Sigma_\gamma,\mu) \to (\mathbb{R}^{2n}/\Lambda, eucl)$ is an immersion holomorphic w.r.t. a compatible complex structure J. Then there exists $O \in \mathcal{O}(2n)$ s.t. $J = O^t J_0 O$.

By Theorem 2.2 there exists a complex linear map $L\colon (\mathbb{R}^{2\gamma}/\Lambda', J_0) \to (\mathbb{R}^{2n}/O\Lambda, J_0)$, where $(\mathbb{R}^{2\gamma}/\Lambda', J_0)$ is the jacobian of (Σ_γ,μ). For a fixed point $p_0 \in \Sigma_\gamma$, we have that $\tilde{f} := O \circ f_{p_0} = L \circ j_{p_0}$. On representing L by a complex $n \times \gamma$ matrix we obtain

$$\tilde{f}(p) = Re\int_{p_0}^{p} OM\underline{\omega} \;=\; Re\int_{p_0}^{p} (Id_n \;\; iId_n)^t L\underline{\omega} \mod O\Lambda\,.$$

We have proved the following:

Lemma 2.1. *Let $f\colon (\Sigma_\gamma,\mu) \to (\mathbb{R}^{2n}/\Lambda, eucl)$ be a full conformal minimal immersion into a flat torus, given by $f(p) = Re\int_{p_0}^{p} M\underline{\omega} \mod \Lambda$ and J a complex structure given by $O^t J_0 O$, with $O \in \mathcal{O}(2n)$. Then J is compatible with the euclidean metric and f is holomorphic w.r.t. the complex structure J if and only if there exists L, a complex $n \times \gamma$ matrix, s.t. $M = O(Id_n \;\; iId_n)^t L$.*

We can now establish the following characterization of holomorphic maps we have been seeking:

Theorem 2.3. *A full conformal minimal immersion $f\colon (\Sigma_\gamma,\mu) \to (\mathbb{R}^{2n}/\Lambda, eucl)$ given by $f(p) = Re\int_{p_0}^{p} M\underline{\omega} \mod \Lambda$ is holomorphic w.r.t.*

some complex structure compatible with the metric if and only if $M^tM = 0$.

Proof. By Lemma 2.1 we have to prove that $M^tM = 0$ implies $M = O(Id_n \;\; iId_n)^t L$ with L a complex $n \times \gamma$ matrix and $O \in \mathcal{O}(2n)$. Since f is full, we have $n \leqslant \gamma$, and then it is clearly sufficient to prove this claim for $n = \gamma$ and also for immersions that are not necessarily full because, under our assumptions, f defines a minimal immersion (now possibly not full) into a flat torus of dimension 2γ. We associate to M the $2\gamma \times 2\gamma$ real matrix $\tilde{M}$ given by $\tilde{M} = \left(\begin{smallmatrix} A & C \\ B & D \end{smallmatrix}\right)$ where $M^t = (A + iB \;\; C + iD)$. The condition $M^tM = 0$ is then equivalent to

$$\left.\begin{aligned} AA^t + CC^t - BB^t - DD^t = 0 \\ AB^t + BA^t + CD^t + DC^t = 0\,. \end{aligned}\right\} \tag{3}$$

We say that a $2\gamma \times 2\gamma$ real matrix $N = \left(\begin{smallmatrix} X & Y \\ Z & T \end{smallmatrix}\right)$ is in $\mathcal{M}(\gamma, \mathbb{C})$ (the set of $\gamma \times \gamma$ complex matrices) iff $Z = -Y$ and $X = T$, since such elements satisfy $NJ_0 = J_0N$. Given $N = \left(\begin{smallmatrix} X & Y \\ -Y & X \end{smallmatrix}\right)$ we denote by $N_{\mathbb{C}}$ the matrix $X - iY$.

Since

$$\tilde{M}\tilde{M}^t = \begin{pmatrix} AA^t + CC^t & AB^t + CD^t \\ BA^t + DC^t & BB^t + DD^t \end{pmatrix} \tag{4}$$

we have that $M^tM = 0$ implies $\tilde{M}\tilde{M}^t \in \mathcal{M}(\gamma, \mathbb{C})$. Moreover $\tilde{M}\tilde{M}^t$ is semipositive definite and therefore $MM^t = 0$ implies that $(\tilde{M}\tilde{M}^t)_{\mathbb{C}}$ is semipositive definite. Therefore $(\tilde{M}\tilde{M}^t)_{\mathbb{C}} = P_{\mathbb{C}}^2$, where $P_{\mathbb{C}}$ is hermitian semipositive definite. We then have $\tilde{M} = PO$, $O \in \mathcal{O}(2\gamma)$, and thus $M^t = P_{\mathbb{C}}(Id_\gamma \;\; iId_\gamma)O$ as we claimed. □

We now explain how to deform a holomorphic map through nonholomorphic minimal immersions:

Theorem 2.4. *Let* $f\colon (\Sigma_\gamma, \mu) \to (\mathbb{R}^{2n}/\Lambda, J_0)$ *be a holomorphic immersion given by* $f(p) = Re \int_{p_0}^{p} (\eta_1, \dots, \eta_n, i\eta_1, \dots, i\eta_n)^t \mod \Lambda$. *If the* η_i*'s satisfy a nontrivial quadratic relation, i.e. there exist* $a_{ij} \in \mathbb{C}$, *not all zero, s.t.* $a_{ij} = a_{ji}$ *and*

$$\sum_{i,j} a_{ij}\eta_i \cdot \eta_j = 0\,, \tag{5}$$

then there exists a family of conformal minimal immersions $f_s\colon (\Sigma_\gamma, \mu) \to (\mathbb{R}^{2n}/\Lambda_s, eucl)$ *s.t.* $f_0 = f$ *and* f_s *is holomorphic w.r.t. a compatible complex structure if and only if* $s = 0$.

Proof. Let us denote by A the matrix (a_{ij}) and by $\underline{\eta}^t$ the row vector $(\eta_1, \dots, \eta_n)$. For $s \in \mathbb{C}$, set $A_s = Id_n - sA$. For small $|s|$, A_s has rank n and therefore, by standard linear algebra we know that there exists an invertible $n \times n$ matrix L such that $L_s^t A_s L_s = Id_n$. (L_s can be chosen to depend smoothly on s.)

Let $M_s^t = (Id_n \;\; i(L_s^{-1})^t)$ and let $\Lambda_s = \{Re \int_\sigma M_s \underline{\eta} \mid \sigma \in H_1(\Sigma_\gamma, \mathbb{Z})\}$. Consider then the family of maps $f_s \colon (\Sigma_\gamma, \mu) \to (\mathbb{R}^{2n}/\Lambda_s, eucl)$ given by $f_s(p) = Re \int_{p_0}^p M_s \underline{\eta} \mod \Lambda_s$. Observe that $M_s^t M_s = sA$ and therefore, by Theorem 2.3, f_s is holomorphic w.r.t. some compatible complex structure for small s if and only if $s = 0$. On the other hand f_s is conformal for any s: indeed, $\underline{\eta}^t M_s^t M_s \underline{\eta} = s\underline{\eta}^t A \underline{\eta} = 0$ by (5). Thus, the maps f_s have all the required properties. □

COROLLARY 2.1. *Let $f \colon (\Sigma_\gamma, \mu) \to (\mathbb{C}^n/\Lambda, eucl)$ be a full holomorphic immersion given by $f(p) = Re\,(\int_{p_0}^p \eta_1, \ldots, \int_{p_0}^p \eta_n, \int_{p_0}^p i\eta_1, \ldots, \int_{p_0}^p i\eta_n)^t \mod \Lambda$. If*

$$\tfrac{1}{2}n(n+1) > 3\gamma - 3 \tag{6}$$

then there exists a family of conformal minimal immersions $f_s \colon (\Sigma_\gamma, \mu) \to (\mathbb{R}^{2n}/\Lambda_s, eucl)$ s.t. $f_0 = f$ and f_s is holomorphic w.r.t. a compatible complex structure if and only if $s = 0$.

Proof. Let V be the vector space generated by the η_i's. We will show that if n satisfies (6), then the assumptions of Theorem 2.4 are easily satisfied: indeed, a nontrivial quadratic relation among the η_i's corresponds precisely to a nontrivial element in the kernel of the cup-product map

$$Sym^2 \colon (V \otimes V)/\Lambda^2 V \to H^0\big((\Sigma_\gamma, \mu), 2K\big)\,.$$

Therefore, since $\dim(H^0((\Sigma_\gamma, \mu), 2K)) = 3\gamma - 3$ and $\dim((V \otimes V)/\Lambda^2 V) = n(n+1)/2$, condition (6) implies $\ker(Sym^2) \neq \{0\}$. □

Theorem 2.3 implies the holomorphicity of certain minimal immersions of surfaces of low genus into a flat torus. For example, we obtain the following result which was stated without proof in [M2], and which was also known to W. Meeks.

COROLLARY 2.2. *If $f \colon (\Sigma_3, \mu) \to (\mathbb{R}^6/\Lambda, eucl)$ is a conformal minimal immersion of a nonhyperelliptic Riemann surface of genus 3, then it is holomorphic w.r.t. some complex structure compatible with the metric.*

Proof. We have seen that f is given by

$$f(p) = Re\left(\int_{p_0}^p \eta_1, \ldots, \int_{p_0}^p \eta_6\right)^t \mod \Lambda\,,$$

where $\eta_1, \dots, \eta_6$ are $\mathbb{R}$-independent holomorphic differentials on (Σ_3, μ) satisfying

$$\sum_{i=1}^{6} \eta_i^2 = 0\,. \tag{7}$$

Choosing $\underline{\omega} = (\omega_1, \omega_2, \omega_3)^t$ a basis of $H^0((\Sigma_3, \mu), K)$ we get $\underline{\eta} = M\underline{\omega}$, where $\underline{\eta} = (\eta_1, \dots, \eta_6)^t$, $M \in \mathcal{M}(6 \times 3, \mathbb{C})$ and (7) becomes

$$\underline{\omega}^t M^t M \underline{\omega} = 0\,. \tag{8}$$

But Noether's theorem (see e.g. [ACGH]) shows that the canonical curve in $\mathbb{C}P^2$ of a nonhyperelliptic surface of genus 3 is not contained in any quadric and so (8) has to be the quadric of rank 0. Therefore $M^t M = 0$. □

Remark 2.1. The same argument as in the proof of Corollary 2.2 proves the following result.

Corollary 2.3. *Every conformal minimal immersion* $f\colon (\Sigma_2, \mu) \to (\mathbb{R}^4/\Lambda, eucl)$ *is holomorphic w.r.t. some complex structure compatible with the metric.*

Remark 2.2. The conclusion of Corollary 2.2 holds when (Σ_3, μ) is hyperelliptic provided that the map is stable. This was proved by Micallef in [M2].

Remark 2.3. The condition of stability in Remark 2.2 is essential. An example of a full, minimal immersion $(\Sigma_3, \mu) \to (\mathbb{R}^6/\Lambda, eucl)$ which is unstable can be constructed as follows: first observe that the canonical image of a hyperelliptic Riemann surface of genus 3 is contained in a nontrivial quadric. This follows directly from the fact (see [ACGH]) that such a (Σ_3, μ) is the Riemann surface of the algebraic function

$$w^2 = (z - a_1) \cdots (z - a_8)\,.$$

Therefore it has a basis of holomorphic differentials of the form

$$\omega_1 = \frac{dz}{w}\,, \quad \omega_2 = z\frac{dz}{w}\,, \quad \omega_3 = z^2\frac{dz}{w}\,,$$

where z, w are coordinates over $\mathbb{C}^2$. Therefore $\omega_2^2 = \omega_1\omega_3$. We now take M^t to be any 3×6 complex matrix with ReM and ImM of maximal rank and such that

$$M^t M = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & -1 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}. \tag{9}$$

The map

$$f(p) = Re \int_{p_0}^{p} M\underline{\omega} \mod \Lambda$$

is then a full conformal minimal immersion not holomorphic w.r.t. any compatible complex structure and therefore, by Remark 2.2, it is *unstable*. For example

$$M^t = \frac{1}{2}\begin{pmatrix} 1-i & 1+i & i & 1 & i & -1 \\ 0 & -2i & 2i & 0 & 2 & 0 \\ 1+i & 1-i & -i & -1 & i & 1 \end{pmatrix} \tag{10}$$

satisfies all the required properties.

3 Stable Minimal Surfaces via Deformations

In this section we prove the main theorems about the existence of stable minimal surfaces in flat tori which are not holomorphic w.r.t. any compatible complex structure. Our strategy is to start with a holomorphic map of a Riemann surface (Σ_γ, μ) into a flat complex torus $(\mathbb{C}^n/\Lambda, eucl)$. We then deform the map and the metric on the torus, but not the conformal structure on Σ_γ, in such a way as to destroy holomorphicity but not minimality, stability and conformality.

For this to work we need the Jacobi fields of the holomorphic map, seen as a minimal immersion, to come only from the translations on the torus. The following theorem gives an algebraic geometric condition which is equivalent to this property.

Theorem 3.1. *Let $f\colon (\Sigma_\gamma, \mu) \to (\mathbb{C}^n/\Lambda, eucl)$ be a holomorphic immersion. Let $\eta_1, \dots, \eta_n$ be the holomorphic differentials such that $f(p) = \int_{p_0}^{p} (\eta_1, \dots, \eta_n)^t \mod \Lambda$. Then the space of Jacobi fields of f as a minimal immersion is precisely the space of translations of the torus if and only if the linear map $(\alpha_1, \dots, \alpha_n) \mapsto \alpha_1 \cdot \eta_1 + \dots + \alpha_n \cdot \eta_n \colon H^0(\mathbb{C}^n \otimes K) \to H^0(2K)$ is surjective.*

Proof. By a theorem of Simons in [Si], the space of Jacobi fields is the space of holomorphic sections of the normal bundle $\nu\Sigma_\gamma$. To calculate this dimension we consider the following exact sequence:

$$0 \to T\Sigma_\gamma \to \mathbb{C}^n \to \nu\Sigma_\gamma \to 0$$

and the associated long exact sequence:

$$0 \to H^0(\mathbb{C}^n) \xrightarrow{\chi} H^0(\nu\Sigma_\gamma) \xrightarrow{\psi} H^1(T\Sigma_\gamma) \xrightarrow{\phi} H^1(\mathbb{C}^n) \to \cdots$$

Since this sequence is exact we have immediately $h^0(\nu\Sigma_\gamma) = \dim_{\mathbb{C}} H^0(\nu\Sigma_\gamma) \geqslant n$. Geometrically this is just the fact that $H^0(\mathbb{C}^n)$ corresponds to the translations in the torus which, of course, induce Jacobi fields on the surface.

By Serre–Kodaira duality $H^1(T\Sigma_\gamma) = (H^0(2K))^*$ and $H^1(\mathbb{C}^n) = (H^0(\mathbb{C}^n \otimes K))^*$. We observe that if ϕ is injective then $\psi = 0$ in which case $H^0(\nu\Sigma_\gamma) = \ker\psi = im\,\chi = \mathbb{C}^n$ which is what we wish to establish. Now ϕ is injective iff the dual map $\phi^*\colon H^0(\mathbb{C}^n\otimes K) \to H^0(2K)$ is surjective. All that remains is to calculate ϕ^* which we now do.

In the dual sequence

$$0 \to (\nu\Sigma_\gamma)^* \to (\mathbb{C}^n)^* \xrightarrow{df^*} (T\Sigma_\gamma)^* \to 0$$

df^* is just the pullback via f of 1-forms from $\mathbb{C}^n/\Lambda$ to Σ_γ. Therefore if $(\alpha_1, \dots, \alpha_n)$ is a n-tuple of holomorphic differentials, then $\phi^*(\alpha_1, \dots, \alpha_n) = \alpha_1 \cdot \eta_1 + \cdots + \alpha_n \cdot \eta_n$, where $\cdot$ denotes the symmetric product between holomorphic differentials. □

We can now prove one of the main results in this paper.

Theorem 3.2. *For any nonhyperelliptic Riemann surface* (Σ_γ, μ) *of genus* $\gamma \geqslant 4$, *there exists a conformal stable minimal immersion* $f\colon (\Sigma_\gamma, \mu) \to (\mathbb{R}^{2\gamma}/\Lambda, eucl)$ *into a flat torus, which is not holomorphic with respect to any complex structure compatible with the metric. (In fact, we shall produce a family of such minimal immersions.)*

Proof. Given a nonhyperelliptic Riemann surface (Σ_γ, μ) of genus $\gamma \geqslant 4$, we know, by Noether's Theorem (see [ACGH]), that its canonical image is contained in a quadric of rank k, $3 \leqslant k \leqslant \gamma$. Therefore we can apply Theorem 2.4 to the Abel–Jacobi map (for any choice of basis of holomorphic differentials) of such a Riemann surface, and get a family of conformal minimal immersions f_s which are not holomorphic w.r.t. any compatible complex structure for $s \neq 0$. (Of course, f_0 is the Abel–Jacobi map.)

We will now show that Theorem 3.1 guarantees the stability of f_s for small values of $|s|$. First, let us recall that f_0 is an embedding, and therefore, since the family f_s is smooth in s, it is clear that, for small $|s|$,

(i) f_s is also an embedding,

(ii) $E_s = \{(p, v) \in \Sigma_\gamma \times \mathbb{R}^{2\gamma} \mid v \in f_{s*}(T_p\Sigma_\gamma)\}$ is a smooth family of subbundles of the trivial bundle $\Sigma_\gamma \times \mathbb{R}^{2\gamma} = f_s^*(T(T^{2\gamma}))$,

(iii) the normal bundle $\nu(s)$ of f_s is the orthogonal complement of E_s in $\Sigma_\gamma \times \mathbb{R}^{2\gamma}$ with the Euclidean metric,

(iv) if $\Pi_s\colon C^\infty(\Sigma_\gamma, \mathbb{R}^{2\gamma}) \to \Gamma(\nu(s))$ is the obvious orthogonal projection and $i_0\colon \Gamma(\nu(0)) \to C^\infty(\Sigma_\gamma, \mathbb{R}^{2\gamma})$ is the obvious inclusion then, $P_s = \Pi_s \circ i_0\colon \Gamma(\nu(0)) \to \Gamma(\nu(s))$ is a bundle isomorphism,

(v) the metrics on the tangent bundle $T\Sigma_\gamma$ and the normal bundle $\nu(s)$ induced by f_s from the Euclidean metric on $\Sigma_\gamma \times \mathbb{R}^{2\gamma}$ vary smoothly with s.

By means of P_s we shall regard the second variation quadratic form $\partial^2\mathcal{A}_{f_s}$ as acting on $\Gamma(\nu(0))$. It is easy to verify that for small $|s|$,

$$\partial^2\mathcal{A}_{f_s}(\psi) \geqslant \tfrac{1}{2}\partial^2\mathcal{A}_{f_0}(\psi) + O(s)\int_{\Sigma_\gamma}|\psi|^2 dA_0 \tag{11}$$

where dA_s is the element of area for the metric on Σ_γ induced by f_s.

For each $v \in \mathbb{R}^{2\gamma}$, denote still by v the constant section $p \mapsto v$ of $\Sigma_\gamma \times \mathbb{R}^{2\gamma}$. Let $\Xi_s = \{\Pi_s(v) \mid v \in \mathbb{R}^{2\gamma}\}$ and denote by $\Xi_s^\perp$ the L^2-orthogonal complement of Ξ_s in $\Gamma(\nu(s))$.

Now by Noether's Theorem, the surjectivity hypothesis required in Theorem 3.1 is satisfied by the Abel–Jacobi map of any nonhyperelliptic Riemann surface of genus at least 4. Therefore, the space of Jacobi fields of f_0 is precisely Ξ_0. Furthermore, since f_0 is area minimizing (and, in particular, stable), there exists $\lambda > 0$ such that $\partial^2\mathcal{A}_{f_0}(\psi) \geqslant \lambda\int_{\Sigma_\gamma}|\psi|^2 dA_0$, for any $\psi \in \Xi_0^\perp$. It follows from (11) that, for $|s|$ small enough, $\partial^2\mathcal{A}_{f_s}(\psi) \geqslant \frac{\lambda}{4}\int_{\Sigma_\gamma}|\psi|^2 dA_0$ for all $\psi \in \Xi_0^\perp$ and therefore, index + nullity of f_s is at most 2γ. But the nullity of f_s is always at least 2γ (because of translations in the torus) and therefore the nullity equals 2γ and the index is zero, that is, f_s is stable for sufficiently small $|s|$. □

We want to apply the ideas used in the proof of the previous theorem also to the case of minimal immersions of surfaces of genus γ into flat tori of dimension less than 2γ. This requires us to produce a subspace V of $H^0((\Sigma_\gamma, \mu), K)$ of $\mathbb{C}$-dimension $n < \gamma$ with the property that any basis $\eta_1, \dots, \eta_n$ of V satisfies the following three conditions:

(i) the period condition 2 in Theorem 2.1 with Λ equal to the lattice of the jacobian of (Σ_γ, μ),
(ii) the surjectivity of the cup-product map in Theorem 3.2 and
(iii) the inequality (6) in Corollary 2.1.

To this aim, let us denote by F_1 the Hodge bundle over the moduli space $\mathcal{M}_\gamma$ of Riemann surfaces of genus γ, i.e. the bundle whose fibre over (Σ_γ, μ) is $H^0((\Sigma_\gamma, \mu), K)$, and by $Gr(k, F_1)$ the bundle over $\mathcal{M}_\gamma$ whose fibre is the grassmannian of k-dimensional subspaces of $H^0((\Sigma_\gamma, \mu), K)$. Points in $Gr(k, F_1)$ will be denoted by (μ, V) where V is a k-dimensional subspace of $H^0((\Sigma_\gamma, \mu), K)$.

Condition (ii) above will be fulfilled by appealing to the following

Theorem 3.3 (Gieseker [G]). *The points $(\mu, V) \in Gr(3, F_1)$ for which the cup-product map*

$$V \otimes H^0\big((\Sigma_\gamma, \mu), K\big) \to H^0\big((\Sigma_\gamma, \mu), 2K\big) \text{ is surjective} \tag{12}$$

form an open and dense subset $\mathcal{G}_3^\gamma$ of $Gr(3,F_1)$ whose projection onto $\mathcal{M}_\gamma$ is precisely the nonhyperelliptic locus.

Note that this theorem is not stated in this form in [G]. Nevertheless Theorem 1.1. in [G], combined with the well-known fact that property (12) is open under deformations of conformal structures, gives Theorem 3.3 as stated.

The next theorem will enable us to satisfy condition (i).

Theorem 3.4 (Colombo–Pirola, [CoP]). *For $k=1,2$ or 3 and $\gamma \geqslant 3$, there exists a dense subset $\mathcal{L}_k^\gamma$ of $Gr(\gamma-k,F_1)$ such that if $(\mu,V_k)\in\mathcal{L}_k^\gamma$ then, given any basis $\{\eta_1,\dots,\eta_{\gamma-k}\}$ of V_k, $\Lambda=\{(\int_\sigma\eta_1,\dots,\int_\sigma\eta_{\gamma-k})^t \mid \sigma\in H_1(\Sigma_\gamma,\mathbb{Z})\}$ is a sublattice of the lattice of the jacobian of (Σ_γ,μ) and the map*

$$p\mapsto\int_{p_0}^p(\eta_1,\dots,\eta_{\gamma-k})^t \bmod \Lambda$$

is a well defined holomorphic immersion into $\mathbb{C}^{\gamma-k}/\Lambda$.

Once again the above statement is not precisely the one in [CoP]. On the other hand, from the proofs of Theorems 1 and 3 in [CoP], we know that Theorem 3.4 holds if there exists $(\mu,V_k)\in Gr(\gamma-k,F_1)$ such that the cup-product map

$$V_k\otimes V_k^\perp\to H^0\big((\Sigma_\gamma,\mu),2K\big) \text{ is injective.} \tag{13}$$

For $k=1$ any (μ,V_1) satisfies (13), while for $k=2$ we can choose any V_2 which is base point free. For $k=3$ we can choose any (μ,V_k) such that V_k contains a 3-dimensional subspace W for which $(\mu,W)\in\mathcal{G}_3^\gamma$. Since $\phi\colon W\otimes H^0((\Sigma_\gamma,\mu),K)\to H^0((\Sigma_\gamma,\mu),2K)$ is surjective and $\dim H^0((\Sigma_\gamma,\mu),2K)=3\gamma-3$, we deduce that $\ker\phi=\Lambda^2(W)$ and therefore $\ker\phi\cap(V_k\otimes V_k^\perp)=\{0\}$ i.e. (13) holds.

Theorem 3.5. *For $k=1,2$ or 3 and $\gamma>\frac{5+2k+\sqrt{1+24k}}{2}$ (i.e. $\gamma\geqslant 7$ for $k=1$, $\gamma\geqslant 9$ for $k=2$, $\gamma\geqslant 10$ for $k=3$), there exists a dense subset $\mathcal{D}_k^\gamma$ of the moduli space of nonhyperelliptic Riemann surfaces of genus γ, such that if $\mu\in\mathcal{D}_k^\gamma$, then there exists a conformal stable minimal immersion $f\colon(\Sigma_\gamma,\mu)\to(\mathbb{R}^{2(\gamma-k)}/\Lambda, eucl)$ into a flat torus, which is not holomorphic w.r.t. any compatible complex structure.*

Proof. Let

$$\tilde{\mathcal{L}}_k^\gamma=\mathcal{L}_k^\gamma\cap\big\{(\mu,V)\in Gr(\gamma-k,F_1)\mid \exists\ W\subset V \text{ s.t. } (\mu,W)\in\mathcal{G}_3^\gamma\big\}$$

and let $\pi\colon\mathcal{L}_k^\gamma\to\mathcal{M}_\gamma$ denote the natural projection. $\tilde{\mathcal{L}}_k^\gamma$ is clearly not empty by Theorems 3.3 and 3.4; indeed, $\mathcal{D}_k^\gamma:=\pi(\tilde{\mathcal{L}}_k^\gamma)$ is dense in the nonhyperelliptic locus.

By Theorem 3.1, the holomorphic maps corresponding to $\tilde{\mathcal{L}}_k^\gamma$ via Theorem 3.4 have only translational Jacobi fields.

Moreover, if

$$\frac{(\gamma-k)(\gamma-k+1)}{2} > 3\gamma - 3 \tag{14}$$

then, by Corollary 2.1, we can deform these holomorphic maps and get a family of conformal minimal immersions which are not holomorphic w.r.t. any compatible complex structure. (Observe that (14) holds if and only if $\gamma > \frac{5+2k+\sqrt{1+24k}}{2}$.)

Finally, the same argument as in the proof of Theorem 3.2 shows that for small $|s|$ the deformed maps are stable. □

REMARK 3.1. (i) The simple dimensional count used in the paragraph after Theorem 3.4 shows that when $\gamma - k = 3$ the surjectivity of the map in (12) implies that the holomorphic differentials spanning V_3 cannot satisfy a nontrivial quadratic relation. Consequently, we cannot deform holomorphic maps to $(\mathbb{C}^3/\Lambda, eucl)$ which have only translational Jacobi fields to maps which are minimal but not holomorphic. In other words, we cannot produce stable nonholomorphic minimal immersions into flat tori of $\mathbb{R}$-dimension 6 by means of the strategy that was used to prove Theorems 3.2 and 3.5.

(ii) Another natural question is whether $\mathcal{D}_k^\gamma$ contains families of Riemann surfaces. From Colombo–Pirola's analysis one easily gets that $\mathcal{D}_3^\gamma$ does not contain any family, while $\mathcal{D}_1^\gamma$ and $\mathcal{D}_2^\gamma$ contain families of (real) dimensions $4\gamma - 4$ and $2\gamma - 2$ respectively.

It is evident from the proofs of Theorems 3.2 and 3.5 that stable nonholomorphic minimal surfaces come in families. We wish to describe these families more precisely. There are three key pieces of data associated to a conformal minimal immersion $f\colon (\Sigma_\gamma, \mu) \to (\mathbb{R}^n/\Lambda, eucl)$:

(i) the conformal structure μ,
(ii) the flat metric g on the torus (equivalently, the lattice Λ) and,
(iii) the induced action f_* from $H_1(\Sigma_\gamma, \mathbb{Z})$ to $H_1(T^n, \mathbb{Z})$.

In a continuous family of maps the action on homology does not change. Now if

$$f_0,\ f_1\colon (\Sigma_\gamma, \mu) \to (\mathbb{R}^n/\Lambda, eucl)$$

are two conformal minimal immersions of the same Riemann surface into the same flat torus such that $(f_0)_* = (f_1)_*$ then, in particular, they are homotopic harmonic maps and, by Hartman's Uniqueness Theorem, they can only differ by a translation. In the families produced so far, the conformal structure stays fixed and therefore, the metric of the torus must move.

There is an interaction between the metric and the induced action on the first homology which is particularly interesting when the action is an isomorphism (as is the case in Theorem 3.2). We digress briefly to describe this.

Consider a fixed isomorphism $\rho\colon H_1(\Sigma_\gamma,\mathbb{Z}) \to H_1(\mathbb{R}^{2\gamma}/\Lambda, eucl)$ (note that all maps inducing ρ represent the same 2-homology class though the converse is not true), and denote by Q_ρ the integral two-form on $T^{2\gamma}$ defined by

$$Q_\rho = \sum_{i=1}^{\gamma} \rho(\sigma_i)^* \wedge \rho(\tau_i)^* ,$$

where $\{\sigma_i, \tau_i\}$ is a symplectic basis of $H_1(\Sigma_\gamma,\mathbb{Z})$ with respect to the intersection form χ. If $f\colon \Sigma_\gamma \to T^{2\gamma}$ is any map which induces ρ on the first homology, then by construction, $f^*(Q_\rho)$ viewed as a skewsymmetric form on $H_1(\Sigma_\gamma,\mathbb{Z})$ is equal to χ.

Consider next a flat metric g on $T^{2\gamma}$. According to Calabi ([C]), there exists a unique complex structure $J_{\rho,g}$ on $T^{2\gamma}$ with respect to which g is hermitian and Q_ρ is of type $(1,1)$. The pair $(J_{\rho,g}, Q_\rho)$ endow $T^{2\gamma}$ with the structure of a principally polarized abelian variety. It is convenient at this point to recall the way various geometric structures on a torus are related.

Flat tori. The space of flat metrics $\mathcal{R}_n$ on T^n is easily identified with

$$O(n,\mathbb{R})\backslash Gl(n,\mathbb{R}) / Sl(n,\mathbb{Z})$$

where $A, A' \in Gl(n,\mathbb{R})$ are identified if $A' = OAX$ for some $O \in O(n)$ and $X \in Sl(n,\mathbb{Z})$. $\dim_{\mathbb{R}} \mathcal{R}_n = \frac{1}{2}n(n+1)$.

Kähler tori. The space of Kähler structures $\mathcal{K}_\gamma$ on $T^{2\gamma}$ is described by

$$U(\gamma)\backslash Gl(2\gamma,\mathbb{R})^+ / Sl(2\gamma,\mathbb{Z})$$

where $A, A' \in Gl(2\gamma,\mathbb{R})$ are identified if $A' = UAX$ for some $U \in U(\gamma)$ and $X \in Sl(2\gamma,\mathbb{Z})$. $\dim_{\mathbb{R}} \mathcal{K}_\gamma = 3\gamma^2$.

Symplectic tori. The space of symplectic structures $\mathcal{S}_\gamma$ on $T^{2\gamma}$ is described by

$$Sp(\gamma,\mathbb{R})\backslash Gl(2\gamma,\mathbb{R})^+ / Sl(2\gamma,\mathbb{Z})$$

where $A, A' \in Gl(2\gamma,\mathbb{R})$ are identified if $A' = SAX$ for some $S \in Sp(\gamma,\mathbb{R})$ and $X \in Sl(2\gamma,\mathbb{Z})$. $\mathcal{K}_\gamma$ is fibred over $\mathcal{S}_\gamma$, the fibres being those complex structures which are compatible with a fixed symplectic form ω. (Recall that a complex structure J is compatible with ω if there exists a metric g which is hermitian w.r.t. J such that ω is the Kähler form of (J,g).) Thus the space $\mathcal{A}_\omega$ of complex structures on $T^{2\gamma}$ which are compatible with ω can be identified with

$$U(\gamma)\backslash Sp(\gamma,\mathbb{R})/Sp(\gamma,\mathbb{Z})$$

$\dim_{\mathbb{R}} \mathcal{A}_\omega = \gamma^2 + \gamma$.

Principally polarised abelian varieties. Let Q be an unimodular integral skewsymmetric two-form on $T^{2\gamma}$. In particular, Q is a symplectic form. The space of abelian varieties principally polarised by Q is $\mathcal{A}_Q$.

Let $\mathcal{C}_\gamma$ denote the set of complex structures on $T^{2\gamma}$. (We warn the reader that, as explained by Kodaira and Spencer on page 413 in [KoS], a theorem of Siegel implies that $\mathcal{C}_\gamma$ is not even a Hausdorff space when $\gamma \geqslant 2$.) We have the obvious injective inclusion

$$i_A \colon \mathcal{A}_Q \to \mathcal{C}_\gamma$$

which simply forgets the polarisation Q. We also have the following fibrations:

$$\mathcal{Z}_\gamma \longrightarrow \mathcal{K}_\gamma \xrightarrow{\pi_R} \mathcal{R}_{2\gamma}$$

$$\mathcal{H}_\gamma \longrightarrow \mathcal{K}_\gamma \xrightarrow{\pi_C} \mathcal{C}_\gamma$$

where

$$\mathcal{H}_\gamma = \; U(\gamma)\backslash Gl(\gamma,\mathbb{C})/Sl(2\gamma,\mathbb{Z}) \cap Gl(\gamma,\mathbb{C})$$

is the space of flat metrics on $T^{2\gamma}$ which are hermitian w.r.t. a fixed complex structure J ($\dim_{\mathbb{R}} \mathcal{H}_\gamma = \gamma^2$), and

$$\mathcal{Z}_\gamma = \; U(\gamma)\backslash SO(2\gamma,\mathbb{R})/SO(2\gamma,\mathbb{Z})$$

is the space of complex structures on $T^{2\gamma}$ which are orthogonal w.r.t. a fixed flat metric g; it has real dimension equal to $\gamma^2 - \gamma$. We set:

$\mathcal{J}_\gamma := \; i_A(\widetilde{\mathcal{J}_\gamma})$ where

$\widetilde{\mathcal{J}_\gamma} := \; \{(\mathbb{C}^\gamma/\Lambda, Q_\rho) \in \mathcal{A}_{Q_\rho} \mid (\mathbb{C}^\gamma/\Lambda, Q_\rho)$ is the Jacobi variety of a Riemann surface of genus $\gamma\}$.

$\dim_{\mathbb{R}} \mathcal{J}_\gamma = 6\gamma - 6$ if $\gamma \geqslant 2$ and $\dim_{\mathbb{R}} \mathcal{J}_1 = 2$.

$\mathcal{J}_{\gamma,\,hy} := \; i_A(\{(\mathcal{J}(\Sigma_\gamma,\mu), Q_\rho) \in \widetilde{\mathcal{J}_\gamma} \mid (\Sigma_\gamma,\mu)$ is a hyperelliptic Riemann surface$\})$.

$\mathcal{J}_{\gamma,\,nhy} := \; \mathcal{J}_\gamma \setminus \mathcal{J}_{\gamma,\,hy}$.

$\mathcal{N}_\gamma := \; \{(\mathbb{R}^{2\gamma}/\Lambda, eucl) \in \mathcal{R}_{2\gamma} \mid (\mathbb{R}^{2\gamma}/\Lambda, J_0, Q_\rho) \in \widetilde{\mathcal{J}_\gamma}\} = \pi_R \circ (\pi_C)^{-1}(\mathcal{J}_\gamma)$ = space of flat tori arising from Abel–Jacobi maps.

$\mathcal{N}_{\gamma,\,hy}$ and $\mathcal{N}_{\gamma,\,nhy}$ are defined in the obvious way.

The rest of the paper will be devoted to studying the following problem:

Given a Riemann surface (Σ_γ,μ) and an isomorphism $\rho\colon H_1(\Sigma_\gamma,\mathbb{Z})\to H_1(T^{2\gamma},\mathbb{Z})$, describe the set

$$\mathcal{R}_{\mu,\rho} := \{g\in\mathcal{R}_{2\gamma} \mid \exists \text{ a stable conformal minimal immersion } f\colon(\Sigma_\gamma,\mu)\to (T^{2\gamma},g) \text{which is not holomorphic w.r.t. any complex structure compatible with } g \text{ and which induces } \rho\}\,.$$

We will also study $\mathcal{R}_\rho=\bigcup_{\mu\in\mathcal{M}_\gamma}\mathcal{R}_{\mu,\rho}$ where $\mathcal{M}_\gamma$ is the moduli space of Riemann surfaces of genus γ.

Given a flat torus $(T^{2\gamma},g)$ and an isomorphism $\rho\colon H_1(\Sigma_\gamma,\mathbb{Z})\to H_1(T^{2\gamma},\mathbb{Z})$, it would also be interesting to study the set

$$\mathcal{M}_{g,\rho} := \{(\Sigma_\gamma,\mu)\in\mathcal{M}_\gamma \mid \exists \text{ a stable conformal minimal immersion } f\colon (\Sigma_\gamma,\mu)\to(T^{2\gamma},g) \text{which is not holomorphic w.r.t. any complex structurecompatible with } g \text{ and which induces } \rho\}\,.$$

For instance, by analogy with the uniqueness of the Abel–Jacobi map among all holomorphic maps, it would be nice to prove that $\mathcal{M}_{g,\rho}$ contains at most one point. The first author has shown that such a result is, in general, false if the assumption of stability is dropped (Theorem 1.2 and Corollary 1.1 in [Ar1]).

We start our study of $\mathcal{R}_{\mu,\rho}$ by noting that Theorem 1.4 establishes that it is empty, irrespective of ρ, whenever (Σ_γ,μ) is hyperelliptic. If we also take Corollary 2.2 into account then we see that $\mathcal{R}_\rho$ is empty when $\gamma=2$ and 3. On the other hand, Theorem 3.2 shows that $\mathcal{R}_{\mu,\rho}$ is definitely nonempty whenever (Σ_γ,μ) is nonhyperelliptic and $\gamma\geqslant 4$. But how 'big' is it? The deformations in Theorem 3.2 move the metric out of $\mathcal{N}_{\gamma,\,nhy}$ inside $\mathcal{R}_{2\gamma}$. In order to calculate the dimension of $\mathcal{N}_{\gamma,\,nhy}$ we recall that $\mathcal{N}_\gamma$ is fibred over $\mathcal{J}_\gamma$, the fibres being $\mathcal{H}_\gamma$. Indeed, each fibre corresponds to all the minimal immersions associated to the Abel–Jacobi map of a fixed Riemann surface. We also see that $\dim\mathcal{N}_\gamma=\dim\mathcal{J}_\gamma+\dim\mathcal{H}_\gamma$. We further note that, for $\gamma\geqslant 3$, the nonhyperelliptic Riemann surfaces are open and dense in the space of all Riemann surfaces of the same genus γ and therefore, for $\gamma\geqslant 3$, $\dim\mathcal{N}_{\gamma,\,nhy}=\dim\mathcal{N}_\gamma$. We can now compare the dimensions of $\mathcal{N}_{\gamma,\,nhy}$ and $\mathcal{R}_{2\gamma}$:

(i) $\mathcal{N}_1=\mathcal{R}_2$ (this is a classical fact we shall not need) and,

(ii) for $\gamma\geqslant 2$, we have

$$\dim\mathcal{R}_{2\gamma}-\dim\mathcal{N}_\gamma=(2\gamma^2+\gamma)-(\gamma^2+6\gamma-6)=(\gamma-2)(\gamma-3)\,.$$

The above discussion leads to a fuller understanding of Theorem 3.2. It is not hard to show that, for $\gamma = 2$ and 3, $\mathcal{N}_\gamma$ is open and dense in $\mathcal{R}_{2\gamma}$. Therefore, we require $\gamma \geqslant 4$ to have any hope for the type of deformation argument used in Theorem 3.2 to work. We have already observed that the codimension of $\mathcal{N}_{\gamma,\,nhy}$ in $\mathcal{R}_{2\gamma}$ is equal to $(\gamma-2)(\gamma-3)$. But this is precisely the dimension of the space of quadrics satisfied by the canonical curve of a nonhyperelliptic Riemann surface. Hence, in light of Theorem 2.4, we expect $\mathcal{R}_{\mu,\rho}$ to contain a disc of $\mathbb{R}$-dimension $(\gamma-2)(\gamma-3)$ transverse to $\mathcal{N}_{\gamma,\,nhy}$ whenever (Σ_γ,μ) is nonhyperelliptic. This would imply the existence of a neighbourhood $\mathcal{V}_\gamma$ of $\mathcal{N}_{\gamma,\,nhy}$ in $\mathcal{R}_{2\gamma}$ such that $\mathcal{V}_\gamma \setminus \mathcal{N}_{\gamma,\,nhy}$ is contained in $\mathcal{R}_\rho$. We do not pursue this analysis here because such a result is implied by Theorem 1.5, part 2, in [Ar1] where the relation between the conformal structure of a minimal surface in a flat torus and the torus lattice is dicussed in great detail. However, we should point out that, rather surprisingly, the invertibility of the differential of the relevant map turns out to rely heavily yet again on the fact that the Jacobi fields for the Abel–Jacobi map of a nonhyperelliptic Riemann surface of genus at least 4 are all translational (c.f. Theorem 3.1 and Noether's Theorem).

It seems reasonable to expect that $\mathcal{R}_\rho$ contains the complement of the closure of $\mathcal{N}_\gamma$ in $\mathcal{R}_{2\gamma}$. We will prove this below (Theorem 3.7) for $\gamma = 4$. But first we prove

Theorem 3.6. *For any flat torus* $(\mathbb{R}^{2\gamma}/\Lambda, eucl)$, $\gamma \geqslant 4$, *there exists a stable conformal minimal immersion* $f\colon (\Sigma_\gamma,\mu) \to (\mathbb{R}^{2\gamma}/\Lambda, eucl)$ *of some nonhyperelliptic Riemann surface of genus* γ.

Proof. As mentioned above, the infinitesimal study of the period map in [Ar1] establishes that, for $\gamma \geqslant 4$, there exists an open subset $\mathcal{V}_\gamma$ of $\mathcal{N}_{\gamma,\,nhy}$ in $\mathcal{R}_{2\gamma}$ such that, whenever $\Lambda_0 \in \mathcal{V}_\gamma \setminus \mathcal{N}_{\gamma,\,nhy}$ there exists a stable conformal nonholomorphic minimal immersion f of a nonhyperelliptic Riemann surface of genus γ into $(\mathbb{R}^{2\gamma}/\Lambda_0, eucl)$ which induces an isomorphism between the first homology groups. Because of the density of $\mathbb{Q}$ in $\mathbb{R}$, it is easy to see that given any flat torus defined by a lattice Λ there exists a finite riemannian cover $I\colon (\mathbb{R}^{2\gamma}/\Lambda_0, eucl) \to (\mathbb{R}^{2\gamma}/\Lambda, eucl)$ for some $\Lambda_0 \in \mathcal{V}_\gamma \setminus \mathcal{N}_{\gamma,\,nhy}$. The map $I \circ f$ is then a conformal minimal immersion of a nonhyperelliptic Riemann surface of genus γ into the given flat torus with the following properties:

(i) *It is stable:* in fact if there exists a deformation of $I \circ f$ for which the second variation of area is negative, then the lift of this deformation to f would contradict its stability.

(ii) *It is not holomorphic w.r.t. any compatible complex structure:* in fact, if J is a complex structure on $(\mathbb{R}^{2\gamma}/\Lambda, eucl)$ compatible with the metric, let $\tilde{J}$ be the complex structure on $(\mathbb{R}^{2\gamma}/\Lambda_0, eucl)$ obtained by pulling back J via I. Observe that $\tilde{J}$ is compatible with the metric on $\mathbb{R}^{2\gamma}/\Lambda_0$ because I is a local isometry. Furthermore, I is a holomorphic map between the two tori equipped with these complex structures and, since it is locally invertible (not in a unique way, of course), $I \circ f$ is holomorphic w.r.t. J if and only if f is holomorphic w.r.t. $\tilde{J}$. The nonholomorphicity of f therefore implies the same for $I \circ f$. □

The above theorem indicates that loosing control on the topological requirements on the minimal map has the effect of making all flat tori indistinguishable when studying stable minimal maps. On the other hand the next results show that, by fixing the action on the first homology groups, also the complex geometry of abelian varieties comes into play. We therefore study the space $\mathcal{R}_\rho$ directly, that is, without viewing it as the union of $\mathcal{R}_{\mu,\rho}$ over all conformal structures μ.

Theorem 3.7. *Given an isomorphism $\rho\colon H_1(\Sigma_4,\mathbb{Z}) \to H_1(T^8,\mathbb{Z})$ and a flat metric g on T^8, suppose that $(T^8, Q_\rho, J_{\rho,g})$ is a principally polarized abelian variety which is neither a Jacobi variety, nor the product of lower dimensional Jacobi varieties. Then there exists a (possibly branched) nonholomorphic minimal immersion $f\colon \Sigma_4 \to (T^8, g)$ which induces ρ and which minimizes area among all maps which induce ρ. In particular, for $\gamma = 4$, $\mathcal{R}_\rho$ is open and dense in $\mathcal{R}_8$, the space of 8-dimensional flat tori.*

Remark 3.2. (i) The proof below will give no information on how the conformal structure induced on Σ by f depends on ρ and g.

(ii) It would be interesting to establish whether or not f is unique.

Proof. By the main result in [Ar2] we know that there exists a map $f\colon \tilde{\Sigma}_4 \to (T^8, g)$ of a Riemann surface of genus 4, possibly with nodes, such that on each part of $\tilde{\Sigma}_4$, f restricts to a stable minimal immersion. If $\tilde{\Sigma}_4$ has no nodes (in which case $\tilde{\Sigma}_4 = \Sigma_4$) and f is holomorphic w.r.t. a compatible complex structure J, then by Theorem 2.2 and because $f^*(Q_\rho)$ is the intersection form of the surface, (T^8, Q_ρ, J) is isomorphic as a principally polarized abelian variety to the jacobian of Σ_4. Now g is hermitian and Q_ρ is of type $(1,1)$ w.r.t. both J and $J_{\rho,g}$. Therefore, by a theorem of Calabi in [C] $J = J_{\rho,g}$ which results in $(T^8, Q_\rho, J_{\rho,g})$ being the jacobian of Σ_4, contrary to assumption. It follows that f cannot be holomorphic. It remains to prove that $\tilde{\Sigma}_4$ cannot have more than one part.

If $\tilde{\Sigma}_4$ does have nodes, then each of its parts, Σ_i, has genus $0 < \gamma_i < 4$. Therefore, by Theorem 1.4 and Corollary 2.2, $f_i = f_{|\Sigma_i} \colon \Sigma_i \to (T^{2\gamma_i}, g_i = g_{|T^{2\gamma_i}})$ has to be holomorphic w.r.t. a complex structure J_i on $T^{2\gamma_i}$ compatible with g_i. Let $Q_i = Q_{\rho|\Sigma_i}$ and $\chi_i =$ the intersection form of Σ_i. Then $f_i^*(Q_i) = \chi_i$ and again this, together with the universal property of the Abel–Jacobi map, implies that $(T^{2\gamma_i}, Q_i, J_i)$ are isomorphic as principally polarized abelian varieties to the jacobians of the Σ_i's. Hence, $(T^8, Q_\rho, \oplus_i J_i)$ is isomorphic as a principally polarized abelian variety to the product of the jacobians of the Σ_i's. We just need to show that $\oplus_i J_i = J_{\rho,g}$. This is done using Calabi's theorem as above once we establish that $\oplus_i J_i$ is a complex structure on T^8 compatible with g. But, as in the proof of Theorem 3.3 in [Ar2], this can be done by using Morgan's theorem in [Mo] (see [Ar2] for details).

Finally, let $\mathcal{X}$ be the space of abelian varieties that are principally polarized by Q_ρ and which are the product of p Jacobi varieties of Σ_i with $\sum_{i=1}^{p} \gamma_i = 4$. Then $\dim_{\mathbb{R}} \mathcal{X} = 6\sum_{i=1}^{p}(\gamma_i - 1) + 2k = 24 - 6p + 2k$ where k is the number of elliptic curves among the Σ_i's. Let $\mathcal{N}_{\mathcal{X}}$ denote the space of flat metrics on T^8 which are hermitian with respect to a complex structure J for which $(T^8, J, Q_\rho) \in \mathcal{X}$. Then $\dim_{\mathbb{R}} \mathcal{N}_{\mathcal{X}} = \dim_{\mathbb{R}} \mathcal{X} + \dim_{\mathbb{R}} \mathcal{H}_4 = 40 - 6p + 2k$. Of course, $p \geqslant 2$ if $k \geqslant 1$ and k is at most p. Therefore $40 - 6p + 2k \leqslant 34$ which is strictly less than 36, the dimension of $\mathcal{R}_8$. We showed above that $\mathcal{R}_\rho$ contains $\mathcal{R}_8 \setminus \mathcal{N}_{\mathcal{X}}$ and therefore, the complement of $\mathcal{R}_\rho$ in $\mathcal{R}_8$ has codimension at least 2. In particular, $\mathcal{R}_\rho$ is open and dense in $\mathcal{R}_8$. $\square$

Remark 3.3. If $(T^8, Q_\rho, J_{\rho,g})$ does split as a principally polarized abelian variety into a product of Jacobi varieties of Σ_i's then the area minimizing process must yield the Riemann surface with nodes whose parts are the Σ_i's. Thus the proof of Theorem 3.7 provides a criterion for degeneration to occur when minimizing area in certain homotopy classes in a flat 8-dimensional torus.

We expect $\mathcal{R}_\rho$ to be open and dense in $\mathcal{R}_{2\gamma}$ for all $\gamma \geqslant 4$. Unfortunately the proof above works only in the case when all the possible degenerations of the Riemann surface are holomorphic, and this might not be the case when $\gamma > 4$. It is unclear how to avoid this hypothesis when trying to prove that $\oplus_i J_i$ is compatible with g using Morgan's theorem. This probably means that the set of flat metrics for which degenerations occur is strictly larger than the set of metrics which are hermitian on nonsimple abelian varieties principally polarised by Q_ρ.

REMARK 3.4. Fisher–Colbrie and R. Schoen ([FS]) proved that the universal cover of a two-sided stable minimal hypersurface is also stable. The work of M. Ross and C. Schoen ([RS]) proved that the assumption on the hypersurface being two-sided is essential. Here, we note that a 6-fold cover of the nonholomorphic area minimizing surface produced in Theorem 3.7 no longer minimizes area in its homology class. This is because of the following proposition which is essentially contained in Remark 5.3 in [La].

PROPOSITION 3.1. *Let $(T^{2\gamma}, J, Q)$, $\gamma \geqslant 4$, be a principally polarized abelian variety. Let*

$$\rho\colon H_1(\Sigma_\gamma, \mathbb{Z}) \to H_1(T^{2\gamma}, \mathbb{Z})$$

be an isomorphism such that $Q_\rho = Q$. Let g be a flat metric on $T^{2\gamma}$ which is hermitian w.r.t. J. Suppose that $f\colon \Sigma_\gamma \to (T^{2\gamma}, g)$ is a (possibly branched) immersion such that

(i) $f_* = \rho$,

(ii) it minimizes area among all maps $u\colon \Sigma_\gamma \to (T^{2\gamma}, g)$ for which $u_* = \rho$ and,

(iii) it is not holomorphic w.r.t. any complex structure compatible with g.

Let $\pi\colon \Sigma_{\gamma'} \to \Sigma_\gamma$ be a covering map of degree $(\gamma-1)!$. Then $f \circ \pi$ is not area minimizing in the homology class $PD(Q^{\gamma-1}) = (\gamma-1)![f(\Sigma_\gamma)]$.

Proof. Simply observe that the class $PD(Q^{\gamma-1})$ can be represented by an effective holomorphic cycle, namely the theta divisor intersected with itself $\gamma - 1$ times (see, for example, [LB]). □

References

[ACGH] E. ARBARELLO, M. CORNALBA, P. GRIFFITHS, J. HARRIS, Geometry of Algebraic Curves, Grundlehren 267, Springer-Verlag, 1984.

[Ar1] C. AREZZO, On the Torelli theorem for minimal immersions, Int. J. of Math. 10 (1999), 697–706.

[Ar2] C. AREZZO, On minimal surfaces incompressible in homology and abelian varieties, J. London Maths. Soc. 60 (1999), 269–283.

[ArMP] C. AREZZO, M.J. MICALLEF, G.P. PIROLA, Stable nonholomorphic minimal surfaces of finite total curvature, Comm. Anal. Geometry, to appear.

[B] A. BEAUVILLE, Variétés kählériennes dont la première classe de Chern est nulle, J. Diff. Geom. 18 (1983), 755–782.

[C] E. CALABI, Isometric families of Kähler structures, Volume in honor of S.S. Chern, Berkeley, 1979.

[CoP] E. Colombo, G.P. Pirola, Some density results for curves with non-simple jacobians, Math. Ann. 288 (1990), 161–178.

[FS] D. Fisher–Colbrie, R. Schoen, The structure of complete stable minimal surfaces in 3-manifolds of non-negative scalar curvature, Comm. Pure and Appl. Math. 33 (1980), 199–211.

[G] D. Gieseker, A lattice version of the KdV equations, Acta Math. 168 (1992), 219–248.

[KoS] K. Kodaira, D.C. Spencer, On deformations of complex structures, II, Ann. of Math. 67 (1958), 403–466.

[LB] H. Lange, Ch. Birkenhake, Complex Abelian Varieties, Grundlehren 302, Springer-Verlag, 1992.

[La] B. Lawson, The stable homology of a flat torus, Math. Scand. 36 (1975), 49–73.

[LaS] B. Lawson, J. Simons, On stable currents and their applications to global problems in real and complex geometry, Ann. of Math. 98 (1973), 427–450.

[M1] M.J. Micallef, Stable minimal surfaces in euclidean space, J. Diff. Geom. 19 (1984), 57–84.

[M2] M.J. Micallef, Stable minimal surfaces in flat tori, Complex Differential Geometry and Nonlinear Differential Equations 49 (1984), 73–78.

[MW] M.J. Micallef, J. Wolfson, The second variation of area of minimal surfaces in four-manifolds, Math. Ann. 295 (1993), 245–267.

[Mo] F. Morgan, On the singular structure of two-dimensional area minimizing surfaces in $\mathbb{R}^n$, Math. Ann. 261 (1982), 101–110.

[RS] M. Ross, C. Schoen, Stable quotients of periodic minimal surfaces, Comm. Anal. Geom. 2 (1994), 451–459.

[S] R.M. Schoen, The role of harmonic mappings in rigidity and deformation problems, Proc. of the Osaka International Conference on Complex Geometry, (G. Komatsu, Y. Sakane, eds.) (1993), 179-200.

[Si] J. Simons, Minimal varieties in riemannian manifolds, Ann. of Math. 88 (1968), 62–105.

[SiuY] Y.T. Siu, S.T. Yau, Compact Kähler manifolds of positive bisectional curvature, Invent. Math. 59 (1980), 189–204.

Claudio Arezzo, Dipartimento di Matematica, Università di Parma, Via M. D'Azeglio 85, 43100 Parma, Italy

Mario J. Micallef, Mathematics Institute, University of Warwick, Coventry, CV4 7AL United Kingdom

Submitted: November 1998