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Blow-up of solutions of nonlinear heat equations in unbounded domains for slowly decaying initial data

Pierre Rouchon

Abstract. Consider the following equations: (E) $u_t - \Delta u = u^p$, (E') $u_t - \Delta u = u^p - \mu \mid \nabla u \mid^q$, (E'') $u_t - \Delta u = u^p + a.\nabla(u^q)$, in $\Omega \subset I\!R^d$. For any unbounded domain Ω, intermediate between a cone and a strip, we obtain a sufficient condition on the decay at infinity of initial data to have blow-up. This condition is related to the geometric nature of Ω. For instance, if Ω is the interior of a revolution surface of the form $\mid x'_d \mid < f(\mid x_d \mid)$, then the condition on the initial data is given by $\Phi(x) > Cf(\mid x \mid)^{-2/(p-1)}$ at infinity. Moreover, for a large class of domains Ω, we prove that those results are optimal (i.e. there exist global solutions with the same order of decay at infinity for their initial data).

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0. Introduction

This paper presents results on finite time blow-up for nonlinear heat equations in unbounded domains Ω in $I\!R^d$, $d \geq 2$. More precisely, we will give sufficient conditions for blow-up, involving the behaviour at infinity of the initial data and the geometric nature of Ω. We are first interested in the heat equation given by:

$$\begin{cases} u_t - \Delta u = |u|^{p-1}\, u \quad t > 0, \quad x \in \Omega, \\ u(0,x) = \Phi(x), \qquad \Phi(x) \geq 0, \quad x \in \Omega, \\ u = 0 \quad x \in \partial\Omega, \quad u(t,x) \to 0 \text{ when } |x| \to \infty, \end{cases} \tag{E}$$

where $p > 1$, is fixed. The first blow-up result of this kind was provided, to our knowledge, by T-Y Lee and W-M Ni in $\Omega = I\!R^d$ (see [6]). In a subsequent article (see [8]), Ph. Souplet and F. Weissler were interested in the case where Ω only contains a cone Ω', for equations of the form: $u_t - \Delta u = F(u, \nabla u)$. In the special case of (E) the result of [8] can be stated as follows:

Theorem A. *Let Ω be an unbounded domain of $I\!R^d$ containing a cone Ω'. There exists a constant $C = C(\Omega') > 0$ such that, if the initial data $\Phi \in C_0(\overline{\Omega})$ satisfies*

$$\liminf_{|x|\to\infty,\quad x\in\Omega'} \ |x|^{2/(p-1)} \ \Phi(x) \geq C,$$

then the solution of (E) blows up in finite time.

It is known that if Ω is a strip (see [4]), for Φ sufficiently small in $L^\infty(\Omega)$ norm the solution of (E) is global. What happens in the intermediate domains between a cone and a strip? The following work is a generalization (see Theorems 1, 2) of Theorem A for Ω containing an Ω' of this type. We also give a result of blow-up of the same type for the following equations, with the same conditions on the boundary of Ω and same intial data of (E):

$$u_t - \Delta u = u^p - \mu\,|\nabla u|^q \quad t > 0, \quad x \in \Omega, \tag{E'}$$

with p, $q > 1$, $\mu > 0$ fixed (Theorem 3);

$$u_t - \Delta u = u^p + a.\nabla(u^q) \quad t > 0, \quad x \in \Omega, \tag{E''}$$

with p, $q > 1$, $a \in I\!R^d$, $a \neq 0$ fixed (Theorem 4). This equation has been already studied by several authors, for instance in the case of $\Omega = I\!R^d$ by J. Aguirre and M. Escobedo (see [1]).

Lastly we prove that those results are optimal for the three equations (E) (Th. 5), (E') (Th. 6) and (E'') (Th. 7), for a large class of domains Ω' of paraboloïd type.

Finally, let us point out to the reader a recent article of N. Mizoguchi and E. Yanagida (see [7]), where the authors have studied the equation (E), but for Φ with changing sign. Their method of investigation is different from ours and their results are not exactly the same: for blow-up they cannot reach the critical exponent and for the optimality they have sometimes a gap which does not exist here. Moreover, since their approach heavily relies on energy arguments, it does not apply for "non variational" nonlinearities such as in (E') and (E'').

The principal tools of proof are the comparison principle and dilation arguments. The work is organized as follows. In Section 1 we present our results. Blow-up results are proved in Section 2. In this section we prove first Theorem 2 and then we give the necessary modifications to obtain Theorems 3 and 4. The Section 3 concerns optimality in the case of equation (E) (proof of Theorem 5). In Section 4 we also give the necessary modifications to obtain Theorems 6 and 7 and give elements of proof of optimality in the case of cones which has been already studied by several authors.

1. Main results

We will consider some domains $\Omega \supset \Omega'$, where Ω' is defined as follows:

$$\Omega' = \{(x_1, x_2, .., x_d) = (x'_d, x_d) \in I\!R^d / \ |x'_d| < f(x_d), \quad x_d > 0\}, \qquad (*)$$

where f is a given function. The next three theorems are results of blow-up. For optimality see Theorems 5-7. A first result concerns the simple case where Ω' is a paraboloïd:

Theorem 1. *Let Ω be a regular unbounded domain of $I\!R^d$ containing a paraboloïd Ω', defined by $(*)$ with:*

$$f(x_d) = x_d^{1/\beta}, \ \beta \in [1, \infty).$$

There exists a constant $C = C(\Omega') > 0$ such that, if the initial data $\Phi \in C_0(\overline{\Omega})$ verifies:

$$\liminf_{|x| \to \infty, \quad x \in \Omega'} \quad |x|^{2/(\beta(p-1))} \ \Phi(x) \geq C,$$

then the solution of (E) blows up in finite time.

Theorem 1 is, in fact, a special case of the following more general result:

Theorem 2. *Let Ω be a regular unbounded domain of $I\!R^d$ containing a set Ω' defined by $(*)$, where f verifies the following hypotheses:*

$$f : [a, \infty) \to (0, \infty), \quad a \geq 0, \ \textit{nondecreasing}, \qquad (1.1)$$

$$\exists k > 0, f(3s) \leq k f(s) \ \textit{for } s \textit{ large enough}, \qquad (1.2)$$

$$f(s) \leq s \ \textit{for } s \textit{ large enough}, \qquad (1.3)$$

and there exists a constant $C = C(\Omega') > 0$ such that, if the initial data $\Phi \in C_0(\overline{\Omega})$ verifies:

$$\liminf_{|x| \to \infty, \quad x \in \Omega'} \quad f(|x|)^{2/(p-1)} \Phi(x) \geq C, \qquad (1.4)$$

then the solution of (E) blows up in finite time.

Theorem 2 can be applied, for example, to the function $f(s) = \log s$ with $s \in [1, \infty)$. Notice here that the condition (1.2) is, in fact, not really restrictive, it just imposes a sufficiently regular growth of f.

Theorems 3 and 4 give a blow-up result for equations (E') and (E'').

Remark 1. For these problems, the natural functional frame is:

$$E = C_0^1(\overline{\Omega}) = \{f \in C^1(\overline{\Omega}), \ f(x) = 0 \text{ for } x \in \partial\Omega, \ f \to 0 \text{ and } \nabla f \to 0 \text{ as } |x| \to \infty\}.$$

Recall that there is local existence and uniqueness for these problems for all initial data $\Phi \in$ E.

Theorem 3. *Assume:*

$$\frac{2p}{p+1} \leq q < p, \tag{1.5}$$

and let Ω *be a regular unbounded domain of* $\mathbb{R}^d$ *containing a set* Ω' *defined by* $(*)$, *where* f *verifies the hypotheses (1.1), (1.2), (1.3). There exists a constant* $C = C(\Omega') > 0$ *such that, if the initial data* $\Phi \in C_0^1(\overline{\Omega})$ *verifies:*

$$\liminf_{|x|\to\infty,\quad x\in\Omega'} \quad f(|x|)^{2/(p-1)}\Phi(x) \geq C, \tag{1.6}$$

then the solution of (E') *blows up in finite time.*

Theorem 4. *Assume:*

$$q = \frac{p+1}{2}, \tag{1.7}$$

and let Ω *be a regular unbounded domain of* $\mathbb{R}^d$ *containing a set* Ω' *defined by* $(*)$, *where* f *verifies the hypotheses (1.1), (1.2), (1.3), there exists a constant* $C = C(\Omega') > 0$ *such that, if the initial data* $\Phi \in C_0^1(\overline{\Omega})$ *verifies (1.6), then the solution of* (E'') *blows up in finite time.*

The following theorems give the optimality in the case of Ω' being a paraboloïd. First, we consider the case of (E):

Theorem 5. *For any* $\beta \in (1, \infty)$, *there exists an unbounded domain* Ω *of* $\mathbb{R}^d$ *containing a paraboloïd* Ω' *defined by* $(*)$, *with* $f(x_d) = x_d^{1/\beta}$, *and there exists* $\Phi \geq 0$, $\Phi \in C_0(\overline{\Omega})$ *such that:*

$$\liminf_{|x|\to\infty,\quad x\in\Omega'} \quad |x|^{2/(\beta(p-1))}\ \Phi(x) > 0, \tag{1.8}$$

and

$$\text{the solution of } (E) \text{ is global and bounded.} \tag{1.9}$$

For (E') and (E''), as in the case of Theorems 3 and 4, we must here change the functional frame (see remark 1).

Theorem 6. *For any* $\beta \in (1, \infty)$, *there exists an unbounded domain* Ω *of* $\mathbb{R}^d$ *containing a paraboloïd* Ω' *defined by* $(*)$, *with* $f(x_d) = x_d^{1/\beta}$, *and there exists* $\Phi \geq 0$, $\Phi \in C_0^1(\overline{\Omega})$ *(see remark 1) such that:*

$$\liminf_{|x|\to\infty,\quad x\in\Omega'} \quad |x|^{2/(\beta(p-1))}\ \Phi(x) > 0, \tag{1.10}$$

and

$$\textit{the solution of } (E') \textit{ is global and bounded.} \tag{1.11}$$

Theorem 7. *Assume* (1.7). *For any* $\beta \in (1,\infty)$*, there exists an unbounded domain* Ω *of* $\mathbb{R}^d$ *containing a paraboloïd* Ω' *defined by* $(*)$*, with* $f(x_d) = x_d^{1/\beta}$*, such that for all* $a \in \mathbb{R}^d$ *with* $|a|$ *sufficiently small, there exists* $\Phi \geq 0$*,* $\Phi \in C_0^1(\overline{\Omega})$ *(see remark 1) such that* (1.10) *is verified and:*

$$\textit{the solution of } (E'') \textit{ is global and bounded.} \tag{1.12}$$

2. Proofs of blow-up results

We directly give here a proof of Theorem 2. Let $B_1 \subset \mathbb{R}^d$ be the unit ball. In order to simplify the proof, we choose v_0 subsolution of (E) in $[0,T)\times B_1$ which blows up at the finite time T, i.e.: $\|v_0(t)\|_\infty \to \infty$ when $t \to T$. Subsolutions of this kind clearly exist (see, e.g., [8]). From there we build a blowing up subsolution v of (E) on $\mathbb{R}^d$ by extending v_0 by 0 on $\mathbb{R}^d \backslash B_1$. Under assumption (1.4) on the initial data, we want to build a nonnegative and blowing up subsolution w of (E) of the form:

$$w(t,x) = \alpha^{2/(p-1)} \quad v(\alpha^2 t, \alpha\,|x-x_0|),$$

with

$$\operatorname{supp} w \subset \Omega'. \tag{2.1}$$

From assumption (1.4), there exists M such that:

$$\forall x \in \mathbb{R}^d, \quad |x| \geq M \Rightarrow \Phi(x) \geq \frac{C/2}{f(|x|)^{2/(p-1)}}, \tag{2.2}$$

and, from (1.3), we may also assume that:

$$\forall x \in \mathbb{R}^d, \quad |x| \geq M \Rightarrow 1 < f(|x|) \leq |x| \,. \tag{2.3}$$

Let us choose $x_0 = 2Me_d$. Since $\operatorname{supp} v(t,.) \subset \overline{B(0,1)}$, condition (2.1) is verified whenever:

$$\alpha\,|x-x_0| < 1 \Rightarrow |x'_d| < f(x_d),$$

i.e.

$$(x_d - 2M)^2 + |x'_d|^2 < 1/\alpha^2 \Rightarrow |x'_d| < f(x_d).$$

Thus, we just need to have:

$$1/\alpha^2 < (x_d - 2M)^2 + f^2(x_d), \; \forall x_d > 0.$$

Let us take $\alpha = 1/f(M)$. We have:

$$(x_d - 2M)^2 \geq f^2(M) - f^2(x_d).$$

(Indeed if $x_d \geq M$ then $f(x_d) \geq f(M)$, because f is nondecreasing by (1.1), and if $x_d < M$ then $(x_d - 2M)^2 + f^2(x_d) \geq M^2 \geq f^2(M)$, by (2.3).)

Having checked (2.1), we can now verify that w is a subsolution of (E):

(i) By construction, v is a subsolution hence $w_t - \Delta w = \alpha^{2p/(p-1)}(v_t - \Delta v) \leq \alpha^{2p/(p-1)} |v|^p = |w|^p$;

(ii) $\forall t \in [0, T/\alpha^2)$, $x \in \partial\Omega$, $w(t,x) = 0$ because supp $w \subset \Omega' \subset \Omega$,

(iii) Let us check that $w(0,x) \leq \Phi(x)$ in Ω. For every $x \in \Omega$ such that $w(0,x) \neq 0$, we have:

$$|x - x_0| \leq 1/\alpha,$$

that is,

$$|x - 2Me_d| \leq f(M),$$

hence

$$-f(M) \leq |x| - 2M \leq f(M),$$

so that:

$$M \leq 2M - f(M) \leq |x| \leq 2M + f(M) \leq 3M,$$

(see (2.3)). As, on the other hand, by (1.1), (1.2) we have:

$$f(|x|) \leq f(3M) \leq kf(M),$$

hence

$$\frac{1}{f(M)} \leq \frac{k}{f(|x|)}.$$

Therefore,

$$w(0,x) \leq \alpha^{2/(p-1)} \|v(0)\|_\infty = \left(\frac{1}{f(M)}\right)^{2/(p-1)} \|v(0)\|_\infty \leq \left(\frac{k}{f(|x|)}\right)^{2/(p-1)} \|v(0)\|_\infty .$$

Choosing $C(\Omega') = k^{2/(p-1)} \|v(0)\|_\infty$, it follows from (2.2) that:

$$w(0,x) \leq C(\Omega') \left(\frac{1}{f(|x|)}\right)^{2/(p-1)} \leq \Phi(x).$$

Let T^* be the maximal time of existence of u the solution of (E). According to the comparison principle we have:

$$u(t,x) \geq w(t,x), \quad 0 \leq t < \min(T^*, T/\alpha^2), \quad x \in \Omega.$$

We conclude that $T^* < \infty$. □

By [8, Thm 1], under the assumption $q < p$, we know that there exists a nonnegative subsolution $\bar{v}$ of (E') (resp. (E'')), such that $\bar{v}$ blows up in finite time, and that $\bar{v}$ remains supported in the unit ball of $\mathbb{R}^d$.

Using $\bar{v}$ instead of v_0 in the proof of Theorems 1 and 2, the proof is then almost unchanged, except for the verification of (i), that is: $Pw \equiv w_t - \Delta w - w^p + \mu \mid \nabla w|^q \leq 0$ (resp. $Pw \equiv w_t - \Delta w - w^p p - a.\nabla(w^q) \leq 0$, for (E'')).

To check (i) in the case of (E'), we note that:

$$Pw(t,x) = \alpha^{2p/(p-1)}\big(v_t - \Delta v - v^p + \alpha^{(q(p+1)-2p)/(p-1)}\mu\, |\nabla v|^q\,\big)(\alpha^2 t, \alpha\, |x - x_0|).$$

Since $\alpha = 1/f(M) < 1$ by (2.3), $\dfrac{2p}{(p+1)} \leq q < p$ by (1.5) and $\mu > 0$ by assumption, it follows that

$$Pw \leq \alpha^{2p/(p-1)}\big(v_t - \Delta v - v^p + \mu\, |\nabla v|^q\,\big)(\alpha^2 t, \alpha\, |x - x_0|) \leq 0,$$

(recall that v is a subsolution of (E')). □

To check (i) in the case of (E''), we now write:

$$\begin{aligned} Pw(t,x) &= \alpha^{2p/(p-1)}\big(v_t - \Delta v - v^p - \alpha^{2q-p-1} a.\nabla(v^q)\big)(\alpha^2 t, \alpha\, |x - x_0|), \\ &= \alpha^{2p/(p-1)}\big(v_t - \Delta v - v^p - a.\nabla(v^q)\big)(\alpha^2 t, \alpha\, |x - x_0|) \leq 0, \end{aligned}$$

since $q = \frac{p+1}{2}$. □

3. Proof of optimality for equation (E)

We always consider (E) defined on Ω. (In the special case of dimension 2, the following proof remains true when changing in $(*)$ $x'_d := x_2$, and $x_d := x_1$). The first order of matters is to define a suitable domain Ω. To do so, let us introduce spherical coordinates:

$$\begin{cases} x_d = r\cos\theta_1 \\ . \\ x_i = r\cos\theta_{d-i+1}\sin\theta_{d-i+2}...\sin\theta_1 \\ . \\ x_2 = r\sin\theta_1\sin\theta_2...\sin\theta_{d-2}\sin\theta_{d-1} \\ x_1 = r\sin\theta_1\sin\theta_2...\sin\theta_{d-2}\cos\theta_{d-1} \end{cases}$$

with $(r, \theta_1, .., \theta_{d-1}) \in [0, \infty) \times [0, \pi]^{d-2} \times [0, 2\pi)$. Let $r_0 > 0$ and $K \in (1, 2]$ to be fixed later. We then define:

$$\Omega = \left\{ (r, \theta_1, .., \theta_{d-1}) \in [0, \infty) \times [0, \pi]^{d-2} \times [0, 2\pi) \ / \theta_1^2 < \frac{K}{r^{2-2/\beta}}, \ \ 0 < r_0 \leq r, \ \theta_1 < \pi/2 \right\}. \tag{3.1}$$

Changing without loss of generality the definition $(*)$, we set:

$$\Omega' = \left\{ (x_1, x_2, .., x_d) = (x'_d, x_d) \in I\!R^d / \ |x'_d| < (x_d - 2r_0)^{1/\beta}, \quad x_d > 2r_0 \right\}. \tag{3.2}$$

We then have:

Lemma 3.1. *Let $K > 1$ and let Ω and Ω' be as defined in* (3.1) *and* (3.2)*. Then, for $r_0 \geq r_{0,1} > 0$ sufficiently large (depending on K), we have $\Omega' \subset \Omega$.*

Proof of Lemma 3.1. Ω' is written as:

$$r \sin \theta_1 \leq (r \cos \theta_1 - 2r_0)^{1/\beta}, \quad r \cos \theta_1 \geq 2r_0, \ \theta_1 \in [0, \pi/2). \tag{3.3}$$

There exist $r_{0,1}$ and $\theta_{1,max}$ such that:

$$x \in \Omega', \quad r \geq r_{0,1} \Rightarrow \frac{1}{\sqrt{K}} \theta_1 < \sin \theta_1 \quad \text{and} \quad 0 \leq \theta_1 \leq \theta_{1,max} < \pi/2. \tag{3.4}$$

(Recall that $K > 1$). Imposing that $r_0 \geq r_{0,1}$, we have by (3.4) and since $\beta > 1$:

$$\frac{r\theta_1}{\sqrt{K}} \leq r \sin \theta_1 < (r \cos \theta_1 - 2r_0)^{1/\beta} \leq r^{1/\beta}, \ \forall r \geq 2r_0, \ \theta_1 \leq \theta_{1,max} < \pi/2.$$

Then, Ω' is included in:

$$\theta_1^2 < K r^{2/\beta - 2}, \quad r \geq 2r_0, \ \theta_1 \leq \theta_{1,max} < \pi/2,$$

hence included in Ω. □

The next step is now to build a bounded positive stationnary supersolution of (E) on Ω. We fix a cut-off function $\eta \in C^\infty(I\!R)$ such that $0 \leq \eta \leq 1$, $\eta = 0$ on $[0, 1]$ and $\eta = 1$ on $(2, \infty)$.

Lemma 3.2. *Let*

$$v(t, x) = v(r, \theta_1) = r^{-\alpha} \left(\frac{K}{r^{2-2/\beta}} - \theta_1^2 \right), \qquad \textit{for } x \in \Omega,$$

with

$$\alpha + 2 - \frac{2}{\beta} = \frac{2}{\beta(p-1)}, \tag{3.5}$$

and let

$$\Phi(x) = v(0,x)\eta\left(\frac{x_d}{r_0}\right).$$

There exist $K \in (1,2]$ *and* $r_0 > 0$ *such that the following properties hold:*

$$\Phi \geq 0,\ \ \Phi \in C_0(\overline{\Omega}),$$

and

$$v \text{ is a supersolution of } (E), \quad v \geq 0.$$

Proof of Lemma 3.2. First, notice that v is clearly positive in Ω by (3.1). Rewrite v as:

$$v(r,\theta_1) = Kr^{-\gamma-2} - r^{-\alpha}\theta_1^2, \quad x \in \Omega, \text{ where } \gamma = \alpha - 2/\beta.$$

As v depends only on θ_1 and r, the Laplacian of v has a rather simple form (for more details see [3]):

$$\Delta v = v_{rr} + \frac{d-1}{r}\ \ v_r + \frac{1}{r^2} v_{\theta_1\theta_1} + \frac{d-2}{r^2}\ \ \cot\theta_1\ \ v_{\theta_1}.$$

Let us calculate those derivatives:

$$\begin{aligned}
v_r &= -K(\gamma+2)\ \ r^{-\gamma-3} + \alpha\ \ r^{-\alpha-1}\ \theta_1^2\\
v_{rr} &= K(\gamma+2)(\gamma+3)\ \ r^{-\gamma-4} - \alpha(\alpha+1)\ \ r^{-\alpha-2}\ \ \theta_1^2\\
v_{\theta_1} &= -2\theta_1 r^{-\alpha}\\
v_{\theta_1\theta_1} &= -2r^{-\alpha}
\end{aligned}$$

It follows that $-\Delta v$ is:

$$\begin{aligned}
-\Delta v &= K((d-1)(\gamma+2) - (\gamma+2)(\gamma+3))r^{-\gamma-4} + (\alpha(\alpha+1)\theta_1^2\\
&- \alpha(d-1)\theta_1^2 + 2(d-2)\theta_1\cot\theta_1 + 2)r^{-\alpha-2}\\
&= ((\alpha^2 + (2-d)\alpha)\theta_1^2 + 2(d-2)\theta_1\cot\theta_1 + 2)\ \ r^{-\alpha-2}\\
&+ K(\gamma+2)(d-\gamma-4)\ \ r^{-\gamma-4}\\
&= r^{-\alpha-2}\ \ ((\alpha^2 + (2-d)\alpha)\theta_1^2 + 2(d-2)\theta_1\cot\theta_1 + 2\\
&+ K(\gamma+2)(d-\gamma-4)\ \ r^{-\gamma+\alpha-2})
\end{aligned}$$

We want v to be a supersolution, i.e.:

$$-\Delta v \geq (Kr^{-\gamma-2} - r^{-\alpha}\theta_1^2)^p = v^p.$$

It is enough to have:

$$(\alpha^2+(2-d)\alpha)\theta_1^2+2(d-2)\theta_1\cot\theta_1+2+K(\gamma+2)(d-\gamma-4)\ \ r^{2/\beta-2} \geq K^p r^{(-\gamma-2)p+\alpha+2}. \tag{3.6}$$

Since $0 \le \theta_1 \le \pi/2$ and $d \ge 2$ we have $2(d-2)\theta_1 \cot\theta_1 \ge 0$. In order to have (3.6) we just need to have the following two conditions:

$$(-\gamma - 2)p + \alpha + 2 = 0 \tag{3.7}$$
$$K^p \le 2 + (2-d)\alpha\theta_1^2 + K(\gamma+2)(d-\gamma-4)r^{2/\beta-2} \tag{3.8}$$

We have (3.7) equivalent to:

$$\frac{2}{\beta(p-1)} = \gamma + 2,$$

that is (3.5). With this choice of γ, let us verify (3.8). There exists $r_{0,2} > 0$ such that:

$$x \in \Omega',\ r \ge r_{0,2} \Rightarrow |(2-d)\alpha|\ \theta_1^2 \le K/4 \text{ and } 2(\gamma+2)\ |d-\gamma-4|\ r^{2/\beta-2} \le K/4. \tag{3.9}$$

(Recall that $2/\beta - 2 < 0$ because $\beta > 1$). We now choose:

$$r_0 = \max(r_{0,1}, r_{0,2}).$$

By (3.9), in order to obtain (3.8) it is enough to have:

$$K^p < 2 - (K/2),\ \text{ for } x \in \Omega. \tag{3.10}$$

Let us choose now: $K = (12/11)^{1/p} \in (1,2]$, then (3.10) is verified and hence (3.8) too. With those choices of γ, K, r_0, we have:

$$Pv = v_t - \Delta v - v^p \ge 0.$$

On the other hand, since $v(0,x) \ge \Phi(x)$ and $v \ge 0$ by construction, we conclude that v is a supersolution of (E). Finally, it is clear that $\Phi = 0$ on $\partial\Omega$, and $\lim\limits_{|x|\to\infty, x\in\Omega} \Phi(x) = 0$ since $0 \le \Phi(x) \le K\ |x|^{\frac{-2}{\beta(p-1)}}$. □

Completion of the proof of Theorem 3: By Lemma 3.2, in view of the comparison principle, we have $0 \le u(t,x) \le v(x)$, $t \in [0, T^*)$, $x \in \Omega$, where T^* is the maximal time of existence of u, then it follows that:

$$\|u(t)\|_\infty \le C = \|v\|_\infty,\ \forall t \in [0, T^*). \tag{3.11}$$

We know that $T^* < \infty$ would imply $\lim\limits_{t\to T^*} \|u(t)\|_\infty = \infty$ which is impossible by (3.11). Hence u is bounded and global in time and (1.9) is verified.

To conclude the proof of Theorem 3, it just remains to check (1.8). We have:

$$\forall x \in \Omega',\ \Phi(x)\ |x|^{2/(\beta(p-1))} = v(r,\theta_1) r^{2/(\beta(p-1))}.$$

That is:

$$\Phi(x)\ |x|^{2/(\beta(p-1))} = (Kr^{-2/(\beta(p-1))} - r^{-\alpha}\theta_1^2)r^{2/(\beta(p-1))} = K - r^{2-2/\beta}\theta_1^2.$$

By (3.3) we know that:

$$\forall x \in \Omega',\ \sin^2\theta_1 < r^{2/\beta-2}\cos^{2/\beta}\theta_1;$$

and by (3.4):

$$\forall x \in \Omega', \quad K\sin^2\theta_1 \geq \theta_1^2.$$

It then follows that:

$$\forall x \in \Omega',\ \Phi(x)\ |x|^{2/(\beta(p-1))} \geq K - Kr^{2-2/\beta}\sin^2\theta_1 > K(1-\cos^{2/\beta}\theta_1).$$

Since $\theta_1 \leq \theta_{1,max} < \pi/2$, writing $A(\Omega) = K(1-\cos^{2/\beta}\theta_{1,max}) > 0$, we have:

$$\forall x \in \Omega'\ \Phi(x)\ |x|^{2/(\beta(p-1))} > A > 0.$$

Hence (1.8) is verified, which concludes the proof of Theorem 5. □

4. Proofs of optimality for equations (E') and (E") and additional remarks

In the case of (E'), (Theorem 6), the result is an immediate consequence of Theorem 5, since any solution of (E) is automatically a supersolution of (E'). We just need here to say that $\Phi \in C_0^1(\overline{\Omega})$ because v belongs, by construction, to this functional space. □

In the case of (E''), (Theorem 7), we use the same proof as in Section 3. To prove Theorem 7, we just need to verify that v is a supersolution. Hence, it just remains here to prove lemma 3.2 adapted to the case of (E'') (note that $\Phi \in C_0^1(\overline{\Omega})$ by construction). Hence we want:

$$-\Delta v \geq v^p + a.\nabla(v^q) = v^p + qv^{q-1}a.\nabla v, \tag{4.1}$$

where $a \in I\!R^d$ will be made precise below. Then $a.\nabla v$ is:

$$a.\nabla v = a_r v_r + \frac{1}{r}a_{\theta_1}v_{\theta_1} = -a_r K(\gamma+2)r^{-\gamma-3} + a_r\alpha\theta_1^2 r^{-\alpha-1} - 2a_{\theta_1}\theta_1 r^{-\alpha-1},$$

where a_r and a_{θ_1} denote the components of a on e_r and e_{θ_1}, and γ, α and K are given in Section 2. Select $r_{0,2}$ as in (3.9). In order to have (4.1), it is enough to have for $r \geq r_{0,2}$ and under (3.5):

$$2-K/2 \geq K^p + q\ |a|\ K^{q-1}r^{(-\gamma-2)(q-1)+\alpha+2}\big(K(\gamma+2)r^{-\gamma-3} + \alpha\theta_1^2 r^{-\alpha-1} + 2\theta_1 r^{-\alpha-1}\big). \tag{4.2}$$

At first, we study the powers of r in (4.2). Under (3.5), we wish to have:

$$(-\gamma-2)(q-1)+\alpha+2-\gamma-3\leq 0. \tag{4.3}$$

Knowing that $\alpha=\gamma+\dfrac{2}{\beta}$, $\gamma+2=\dfrac{2}{\beta(p-1)}$, and $q=\dfrac{p+1}{2}$, (4.3) becomes:

$$\frac{p+1}{2}\geq\frac{2p+\beta(1-p)}{2},$$

which is automatically verified, since $\beta>1$. Then, we have:

$$(-\gamma-2)(q-1)+\alpha+2-\alpha-1=1-\frac{1}{\beta}>0, \tag{4.4}$$

as $\beta>1$. Now, in a second time we are going to control the last two terms in (4.2), by using the properties of Ω. As we are in Ω, we know, by definition of Ω, that $\forall x\in\Omega$, $\theta_1^2<Kr^{2/\beta-2}$. It follows then that:

$$q\ |a|\ K^{q-1}r^{1-1/\beta}(\alpha\theta_1^2+2\theta_1)\leq 3q\ |a|\ K^{q-1}r^{1-1/\beta}\theta_1\leq 3q\ |a|\ K^{q-1/2}, \tag{4.5}$$

for $r>r_{0,3}=r_{0,3}(K,\alpha)$ sufficiently large. Now, compiling (3.4), (3.9) and (4.5), we choose:

$$r_0=\max(r_{0,1},r_{0,2},r_{0,3},1).$$

Then, under (3.5), (4.3) and as $r\geq r_0$, in order to have (4.2) it suffices that:

$$2\geq K^p+K/2+qK^q\ |a|\ (\gamma+2)+3q\ |a|\ K^{q-1/2}. \tag{4.6}$$

As $\gamma+2\leq\dfrac{2}{p-1}$, $q\leq p$ and $K>1$, to have (4.6) it suffices that:

$$2\geq K^p+\frac{K}{2}+\frac{p+1}{p-1}\ |a|\ K^p+\frac{3(p+1)}{2}\ |a|\ K^p. \tag{4.7}$$

Choosing now $a\in I\!R^d$ such that:

$$|a|\leq\max\left(\frac{1}{9(p+1)},\frac{p-1}{6(p+1)}\right),$$

in order to have (4.7) it is sufficient to have:

$$2\geq K^p+\frac{K^p}{3}+\frac{K}{2}. \tag{4.8}$$

As $K=(12/11)^{1/p}$, then (4.7) is verified, hence (4.2) too. Hence v is a supersolution of (E'') which belongs, by construction, to $C_0^1(\overline{\Omega})$. Same conclusions as in Section 3 hold here. This concludes the proof of Theorem 7. $\square$

Remarks: Optimality in the case of β =1

In the case of $\beta = 1$, it is known that our theorems of optimality remains true under additional hypotheses. That is the following proposition:

Proposition 4.1. *Assume:*

$$p > \frac{d}{d-2}, \quad d \geq 3. \tag{4.9}$$

There exists an unbounded domain Ω of $I\!R^d$ containing a cone Ω' defined by:

$$\Omega' = \left\{(x'_d, x_d) \in I\!R^d,\ |x'_d| < x_d - 2r_0, \quad x_d \geq 2r_0\right\},$$

such that for all $a \in I\!R^d$, $a \neq 0$, with $|a|$ sufficiently small, there exists $\Phi \geq 0$, $\Phi \in C_0^1(\overline{\Omega})$ such that:

$$\liminf_{|x|\to\infty,\quad x\in\Omega'} \ |x|^{2/(p-1)}\ \Phi(x) > 0, \tag{4.10}$$

and

the solution of (E'') is global and bounded. (4.11)

The same conclusion holds for equation (E), and for equation (E') with $\mu > 0$ and $q \geq 1$.

Remark 2. The condition (4.9) is not really surprising, at least for equation (E). Indeed in the case of a cone, there exists a critical exponent $p^*(\Omega')$ of Fujita's type, that is: if $1 < p \leq p^*(\Omega')$, then all positive solutions of (E) blow up in finite time. This exponent is less than $d/(d-2)$. For more details, see the articles of C. Bandle and H. A. Levine ([2]), and H. A. Levine and P. Meier (see [5]).

Remark 3. It follows easily from [2] and [5], that the Fujita's exponent for problem (E) in a paraboloïd Ω' is equal to 1.

Proof of proposition 4.1. Writing:

$$\Omega = \left\{(x'_d, x_d) \in I\!R^d,\ |x'_d| < Kx_d - r_0, \quad x_d \geq r_0\right\},$$

with $K > 1$ fixed, it follows immediately that $\Omega' \subset \Omega$. Let us fix two cut-off functions: let $\eta \in C^\infty(I\!R)$ be such that $0 \leq \eta \leq 1$, $\eta = 0$ on $[0,1]$ and $\eta = 1$ on $(2,\infty)$ and and let $\zeta \in C^\infty(I\!R^+)$ satisfy:

$$\zeta(\rho) = \begin{cases} 1, & \rho \leq x_d - 2r_0, \\ 0, & \rho \geq Kx_d - r_0, \end{cases}$$

and $0 \leq \zeta \leq 1$. First, we prove the following lemma:

Lemma 4.1. *Assume* (4.9) *and:*

$$|a| \leq \frac{p-1}{p+1}. \tag{4.12}$$

Let $v = K_1 r^{-2/(p-1)}$ *and let*

$$\Phi(x) = \eta\left(\frac{x_d}{r_0}\right)\zeta(\,|x'_d|\,)v(r).$$

There exists a $K_1 > 0$ *such that following properties hold:*

$$\Phi \geq 0,\ \ \Phi \in C_0^1(\overline{\Omega}),$$

and

$$v \text{ is a supersolution of } (E'').$$

Proof of Lemma 4.1. First note that $\Phi \geq 0$ and $\Phi \in C_0^1(\overline{\Omega})$ by construction because the function v verifies those properties. Hence we just need to prove that $v = K_1 r^{-\alpha}$ with $\alpha = 2/(p-1)$, is a supersolution of (E''). Hence we want:

$$-\Delta v \geq v^p + q(a.\nabla v)v^{q-1},$$

where q is given by (1.7). That is:

$$\alpha(d-2-\alpha)K_1 r^{-\alpha-2} \geq K_1^p r^{-\alpha p} - \alpha q a_r K_1^q r^{-\alpha q - 1}. \tag{4.13}$$

Knowing that $q < p$, in order to have (4.13) it suffices that:

$$\alpha(d-2-\alpha) \geq K_1^{p-1}(1 + q\alpha\,|a|\,r^{\alpha(1-q)+1}). \tag{4.14}$$

We have:

$$\alpha(1-q)+1 = \frac{2}{p-1}\frac{1-p}{2} + 1 = 0.$$

Under (1.7) and (4.12), in order to have (4.14) it is sufficient to have:

$$K_1^{p-1} = \frac{\alpha}{2}(d-2-\alpha),$$

that is in terms of p:

$$K_1^{p-1} = \frac{1}{p-1}\left(d-2-\frac{2}{p-1}\right) > 0, \tag{4.15}$$

by hypothesis (4.9). Choosing K_1 as in (4.15), (4.14) is verified and v is a supersolution of (E''). By construction too, we have $\forall x \in \Omega$, $0 \leq \Phi(x) \leq v(x)$ and $\forall x \in \partial\Omega$, $v \geq 0$. Hence v is a supersolution of (E''). □

Completion of the proof of Proposition 4.1. As v is a supersolution of (E''), which is positive, bounded and global in time, the comparison principle implies that $T^*(\Phi) = \infty$ (where $T^*(\Phi)$ is the maximal time of existence of u solution of (E'') with initial data Φ defined in Lemma 4.1). It follows then that u remains bounded by the same arguments as in Section 3. Hence (4.11) is verified.

To finish the proof, let us check (4.10):

$$\forall x \in \Omega', \quad \Phi(x)\, |x|^{2/(p-1)} = K_1 r^{-2/(p-1)} r^{2/(p-1)} = K_1 > 0,$$

by (4.9). Then (4.10) is verified which concludes the proof of Proposition 4.1. For equations (E) and (E') in the case $\beta = 1$, taking $v = K_2 r^{-\alpha}$, with $K_2^{p-1} = 2K_1^{p-1}$, as a supersolution, the proof is exactly the same. (Note that we have no condition on $|a|$). □

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Pierre Rouchon
LAGA, UMR CNRS 7539
Institut Galilée
Université Paris-Nord
93430 Villetaneuse
France
e-mail: rouchon@math.univ-paris13.fr

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