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Determining the self-rotation number following a Naimark–Sacker bifurcation in the periodically forced Taylor–Couette flow

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Abstract. Systems which admit waves via Hopf bifurcations and even systems that do not undergo a Hopf bifurcation but which support weakly damped waves may, when parametrically excited, respond quasiperiodically. The bifurcations are from a limit cycle (the time-periodic basic flow) to a torus, i.e. Naimark–Sacker bifurcations. Floquet analysis detects such bifurcations, but does not unambiguously determine the second frequency following such a bifurcation. Here we present a technique to unambiguously determine the frequencies of such quasiperiodic flows using only results from Floquet theory and the uniqueness of the self-rotation number (the generalization of the rotation number for continuous systems). The robustness of the technique is illustrated in a parametrically excited Taylor–Couette flow, even in cases where the bifurcating solutions are subject to catastrophic jumps in their spatial/temporal structure.

Keywords. Floquet theory, self-rotation number, parametric excitation, quasiperiodic flow, Taylor–Couette flow.

1. Introduction

The usual discussion of parametric excitations is in terms of systems where the governing equations are reduced to systems of Mathieu-type equations (Mathieu [16]). These reductions are strictly only possible in systems whose natural frequencies are fixed by external constraints. A typical example is Faraday waves (Faraday [5]), surface waves due to a harmonic oscillation of a container of fluid in the direction parallel to gravity. In ideal fluids of infinite extent and subjected to small amplitude oscillations, this excitation of the free surface is described by the Mathieu equation

$$\ddot{\eta} + \left(\Omega^2 - a \sin 2\omega t\right) \eta = 0, \tag{1}$$

where Ω is the natural frequency of surface waves in the unmodulated system, a and ω are the amplitude and frequency of the vertical oscillations of the container, and η is the vertical displacement of the free surface from its flat, mean position. This equation of Mathieu ([16]) has been studied extensively (see for example Jordan & Smith [10]). A simple mechanical system where it arises is in characterizing the motion of a simple pendulum subjected to a vertical oscillation of its pivot. This equation has provided the starting point for the study of parametric resonance. It is significant to note that the first reported observation of parametric resonance was by Faraday ([5]) in a hydrodynamic system and has since led to many important implications in many branches of engineering and physics. Examples of parametric resonance include the response of mechanical and elastic systems to time-varying loads. Parametric resonance due to even very small vibrational loading can stabilize an unstable system, or destabilize a stable system, depending on particular characteristics of the system.

For systems that are governed by Mathieu-type equations (including linear damping terms), their response to parametric excitations can be expected to be either synchronous with the applied periodic forcing, or to have a subharmonic response (Davis & Rosenblat [4]). This means that when the trivial solution, i.e. the fixed point $\eta = 0$ in (1), loses stability, the bifurcating solution is either T-periodic (synchronous) or $2T$-periodic (subharmonic), where $T = 2\pi/\omega$ is the period of the applied forcing.

Hydrodynamic systems in which parametric resonance has been identified and studied are typically characterized by their ability to support waves in the absence of external modulations (e.g. see Miles & Henderson [17]), and these are waves in the classic sense, i.e. surface waves, gravity waves, Rossby waves, etc. Many such hydrodynamic systems have been studied in certain distinguished limits (e.g. Benjamin & Ursell [2]; Gershuni & Zhukhovitskii [6]; Kelly [13]; Gresho & Sani [7]; Craik & Allen [3]), where the governing equations reduce to either a Hill's or (damped) Mathieu's equation. Not all hydrodynamic systems of interest reduce to these simple forms, but they still may be susceptible to parametric excitation. In general, the governing equations for the departures from the unforced state reduce to a form

$$\dot{\mathbf{x}} = (\mathbf{A} + \mathbf{B}\sin\omega t)\,\mathbf{x} + \mathbf{f}(\mathbf{x}). \tag{2}$$

In general, (2) cannot be reduced to a system of (damped) Mathieu equations, and the response to parametric excitation can be more complicated than either synchronous or subharmonic.

An important difference between the stability of systems governed by Mathieu's equation and general systems is that in the former case the base state is a fixed point independent of the amplitude and frequency of the external forcing, whereas in more general systems, the base state is a periodic orbit that depends on the forcing parameters, usually with the same frequency as that of the forcing. In the classical Faraday experiment, for example, the basic state of the forced system is a rigid body motion and is at rest in the frame of reference of the container; the base state in this reference frame is unaware of the forcing. In general, there is no such reference frame when only some of the boundary conditions change due to the parametric excitation. So, in general, the basic state is also a function of the amplitude and frequency of the forcing, and the base state is periodic with the period of the forcing.

In Hu & Kelly [9] and Marques & Lopez [15], the stability of such a hydrodynamic system was investigated using Floquet theory. The system in question is the flow between two co-axial cylinders, the outer one being stationary and the inner one rotating at some fixed rate (the usual Taylor–Couette flow) and subjected to a harmonic oscillation in the axial direction. This system has also been investigated experimentally by Weisberg, Kevrekidis & Smits [20]. The stability of the corresponding time-periodic basic state is reduced to the determination of the growth rates of the solutions of a linear system of the form:

$$G\dot{\mathbf{x}} = H(t)\mathbf{x} = (A + B\sin\omega t + C\cos\omega t)\mathbf{x}. \tag{3}$$

The entries in the matrices G and H are given in the appendix of Marques & Lopez. H is periodic, of period $2\pi/\omega$ where ω is the frequency of the axial oscillations of the inner cylinder, and G is time-independent and positive definite. The system is governed by a number of nondimensional parameters. Dimensionally, the inner cylinder oscillates in the axial direction with velocity $U\sin\Omega t$ and rotates at constant angular velocity Ω_i. Its radius is r_i and the radius of the outer stationary cylinder is r_o. The annular gap between the cylinders is $d = r_o - r_i$. These parameters are combined to give the following nondimensional governing parameters:

$$\begin{aligned} \text{the radius ratio} \quad & e = r_i/r_o, \\ \text{the Couette flow Reynolds number} \quad & Re_i = dr_i\Omega_i/\nu, \\ \text{the axial Reynolds number} \quad & Re_a = dU/\nu, \\ \text{the nondimensional frequency} \quad & \omega = d^2\Omega/\nu, \end{aligned}$$

where ν is the kinematic viscosity of the fluid. The axial and azimuthal wave numbers of the bifurcating solutions are k and n respectively.

The stability of the basic state is determined by applying classical Floquet theory (e.g. Joseph [11], Guckenheimer & Holmes [8]) and numerical integration to (3). The fundamental matrix of (3) is the solution of the system

$$G\dot{X} = H(t)X, \qquad X(0) = I, \tag{4}$$

where I is the identity matrix. Integrating over a complete period $T = 2\pi/\omega$, one obtains the monodromy matrix of the system $X(T)$, whose eigenvalues γ_j, $j = 1, \dots, 4M$, called Floquet multipliers, control the growth rate of the perturbations (M is the size of the discretized system).

From a dynamical systems point of view, integration over one period is equivalent to considering the Poincaré map over a complete period. Therefore, we move from the analysis of a periodic ODE, to the analysis of an autonomous map. The base state of (2) is a fixed point of the map. The eigenvalues of the monodromy matrix are the eigenvalues of the linearized Poincaré map in the neighborhood of the fixed point. If all the eigenvalues have moduli less than one, all the perturbations of the basic state go to zero, and the basic state is asymptotically stable (an attractor). The basic flow losses stability when at least one eigenvalue of the monodromy matrix crosses the unit circle. There are three different generic cases to be considered. If the critical eigenvalue crosses at $+1$ (a fold bifurcation), the bifurcated state is a fixed point of the map, corresponding to a periodic orbit of the original ODE with the same frequency as that of the forcing. The bifurcation is said to be synchronous and no new frequency is introduced. If the critical eigenvalue crosses at -1, then we have a period doubling bifurcation where the fixed point becomes a period-2 fixed point of the map, corresponding to a periodic orbit of the original ODE with a frequency half of the forcing frequency, the so-called subharmonic case. For the Mathieu equation (with or without damping) these are the only possibilities when the basic state bifurcates (see Davis & Rosenblat [4]; Jordan & Smith [10]).

The third generic case corresponds to a loss of stability due to a pair of complex-conjugate eigenvalues crossing the unit circle not at ± 1. Then, an attracting invariant circle emerges from the fixed point of the map. It is a Hopf bifurcation for maps, called a Naimark–Sacker bifurcation (see Arnold [1], Kuznetsov [14] for details). The periodic orbit of the original ODE is now surrounded by an invariant torus. On this torus, the solution of the system has two frequencies. One of the frequencies is the forcing frequency (the frequency of the basic state, ω), which survives the bifurcation. The other bifurcating frequency, denoted ω_s, is associated with the phase (angle of crossing) of the complex-conjugate critical eigenvalues of moduli one, $\gamma_{1,2} = e^{\pm i\phi}$, $\phi = 2\pi\omega_s/\omega$. General hydrodynamic systems of the form (2) can experience such a bifurcation. It is this case that is of primary interest here.

Notice that for the angle ϕ in $\gamma_{1,2} = e^{\pm i\phi}$, its absolute value is unique only $\mathrm{mod}(2\pi)$. Therefore, the definition of the bifurcating frequency as $\omega_s = \omega\phi/2\pi$ is ambiguous. However, this ambiguity can be removed for continuous systems. Near the bifurcation, the Poincaré map $\mathbf{P}$ is a diffeomorphism of the invariant bifurcating circle. For such a diffeomorphism the *rotation number* is defined as the average angle by which the map rotates the invariant circle; the definition involves a limit for $n \to \infty$ of the iterates $\mathbf{P}^n$. The rotation number is unique $\mathrm{mod}(2\pi)$, removing the previous sign ambiguity. When $\mathbf{P}$ is the period-1 map of a continuous system such as (2), then the remaining ambiguity associated with the $\mathrm{mod}(2\pi)$ can also be removed by following the continuous system during a whole period and *continuously* monitoring the angle rotated. This unambiguously defined angle, a generalization of the rotation number for continuous systems, is called the *self-rotation number* ϕ_{sr}. All the pertinent definitions and proofs can be found in Peckham [18]. The bifurcating eigenvalues at criticality are $\gamma_{1,2} = e^{\pm i\phi_{sr}}$. We finally *define* the bifurcating frequency as $\omega_s = \omega\phi_{sr}/2\pi$. In the following, we will refer to the self-rotation number simply as ϕ.

In Mathieu's equation (1), the natural frequency, ω_s, is known *a priori* and is independent of the forcing. In a generic ODE, and in our PDE problem, the second frequency following the Hopf bifurcation, ω_s, is not known *a priori*, and in general depends on the forcing in a nonlinear and possibly discontinuous fashion. In order to determine the self-rotation number one must determine the imaginary parts of the Floquet exponents unambiguously, as these give ω_s. The Floquet analysis does not determine the imaginary parts of the Floquet exponents unambiguously and so when a quasiperiodic state results, the question arises as to how to determine its frequencies.

2. Correcting the phase and extracting the frequency

From Floquet theory, a linear problem with periodic coefficients like (4) has a set of fundamental solutions at criticality of the form

$$\mathbf{x}(t) = \mathbf{x}_p(t)e^{\pm i\omega_s t}, \tag{5}$$

with $\mathbf{x}_p(t)$ periodic, i.e. $\mathbf{x}_p(t+T) = \mathbf{x}_p(t)$, where $T = 2\pi/\omega$ is the period of the applied forcing. Therefore, the critical Floquet multipliers are $\gamma_{1,2} = e^{\pm i\omega_s T}$. However, the Floquet analysis does not give the self-rotation number $\phi = \omega_s T$, but rather an angle $\tilde{\phi} \in [0, \pi]$. The relationship between $\tilde{\phi}$ and ϕ is

$$\phi = 2l\pi \pm \tilde{\phi}, \tag{6}$$

where the sign and the integer multiple l are undetermined. So the question arises as to how to unambiguously determine ω_s. One could, of course, apply the definition of the self-rotation number in a brute force fashion. In that case, one must compute the solution of the system for very large times t; in fact, one would need the limit $t \to \infty$. Instead, we have developed a method to determine the self-rotation number from computations over one period of the base state at various points in parameter space that uses the continuity of the eigenvalues of the system and homotopy considerations.

Let ϕ be the self-rotation number, $\omega_s = \phi/T$, and $\tilde{\omega} = \tilde{\phi}/T$. We first establish the relationship between the self-rotation number, ϕ, and the phase given by the Floquet analysis, $\tilde{\phi}$. When ϕ lies in the interval $[2l\pi, (2l+1)\pi]$, the Floquet analysis gives $\tilde{\phi} = \phi - 2l\pi$; and when $\phi \in [(2l+1)\pi, (2l+2)\pi]$, the Floquet analysis gives $\tilde{\phi} = (2l+2)\pi - \phi$. Incorporating both cases into a single expression, then for $\phi \in [m\pi, (m+1)\pi]$,

$$\phi = \big(m + \frac{1}{2}\big)\pi + (-1)^m\big(\tilde{\phi} - \frac{\pi}{2}\big). \tag{7}$$

For an isolated point in parameter space, we do not know ϕ and m is undetermined. However, if we know the value of m for a particular state of the system, corresponding to a certain combination of parameter values, we can determine the value of m for any other state continuously connected with the known particular state. In fact, m remains constant as the parameters are varied continuously unless $\tilde{\phi}$ goes through zero or π. In these cases, from (7),

$$\tilde{\phi} \to 0 \Rightarrow m \to m - (-1)^m, \quad \text{and} \quad \tilde{\phi} \to \pi \Rightarrow m \to m + (-1)^m. \tag{8}$$

The deduction is as follows. When $\tilde{\phi} \to 0$ for a given value of m (i.e. $\phi \in [m\pi, (m+1)\pi]$), (7) gives $\phi \to (m + (1-(-1)^m)/2)\pi$. For m even, $\phi \to m\pi$, and therefore ϕ moves to the interval $[(m-1)\pi, m\pi]$, and hence m decreases by one. For m odd, $\phi \to (m+1)\pi$, and therefore ϕ moves to the interval $[(m+1)\pi, (m+2)\pi]$, and so m increases by one. An analogous argument applies when $\tilde{\phi} \to \pi$.

In order to apply (8) we need to know the value of m for some state of the system. The problem is that we do not know what m is for any isolated case. In general however, in the limit that the forcing amplitude goes to zero, the second bifurcation frequency ω_s asymptotes to the natural frequency of the unforced system, and in the limit of large forcing frequency, one obtains $\omega_s < \omega/2$ so that $\phi = 2\pi\omega_s/\omega < \pi$ and hence $m = 0$. For any particular problem, there may be other means of determining m in some part of parameter space by taking appropriate limits.

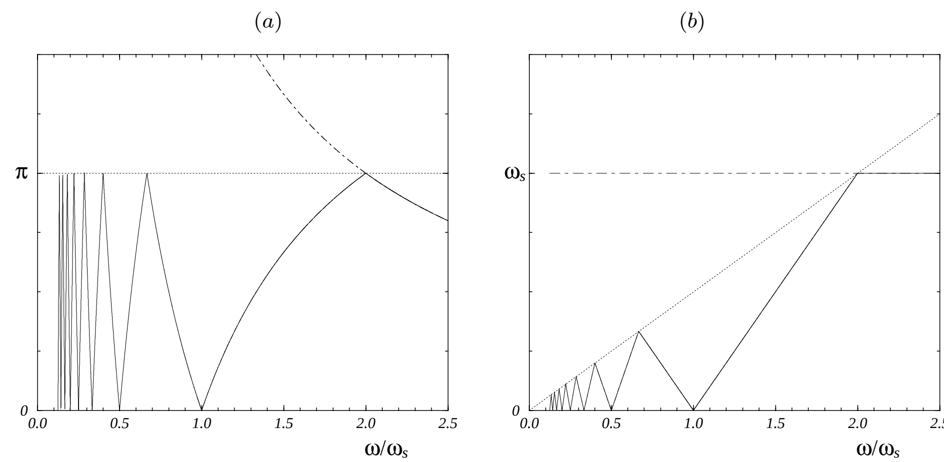

Figure 1.
(a) Phase of the bifurcating solution vs. ω/ω_s; dot–dash line: self-rotation number $\phi = 2\pi\omega_s/\omega$; solid line: phase from the analysis $\tilde{\phi}$; (b) frequency of the bifurcating solution vs. ω/ω_s; dot-dash line: true frequency ω_s; solid line: frequency from the analysis $\tilde{\omega}$; and dotted line: $\omega_s = 0.5\omega$.

As an example of this method of determining the frequency ω_s from $\tilde{\phi}$, we consider the idealized case in which the bifurcating frequency, ω_s, is independent of the forcing frequency ω. The self-rotation number and the bifurcating frequency are related by

$$\frac{\phi\omega}{2\pi} = \omega_s = \text{constant}, \tag{9}$$

so we have $\phi = (2\pi\omega_s)/\omega$, i.e. the self-rotation number is inversely proportional to the frequency of the basic flow (the forcing frequency ω). Figure 1a shows this relationship as a dot–dash line, along with the phase $\tilde{\phi}$ that results from the Floquet analysis as a solid line. The values of ω where $\tilde{\phi} = 0$ correspond to locations where the bifurcating frequency ω_s is an integer multiple of the forcing frequency ω. The values of ω where $\tilde{\phi} = \pi$ are not special and correspond to subharmonic responses, i.e. the bifurcating frequency is an odd multiple of $\omega/2$. Apart from these two classes of forcing frequencies, the system responds quasiperiodically. In the simple case that ω_s is independent of ω, it is straight forward to determine ω_s from $\tilde{\omega} = \tilde{\phi}/T$ using (8). This is shown in Figure 1(b).

In general, and in particular as the amplitude of the forcing is increased, ω_s will be a function of ω. The relationship $\tilde{\omega} = \tilde{\phi}/T$ still applies and (7) and (8) can still be used, but now we lack *a priori* knowledge of exactly where the synchronous and subharmonic points are, i.e. the value of m at any given ω. This is not a serious limitation for ω corresponding to small m, but as $\omega \to 0$ it very quickly becomes exceedingly difficult to determine the corresponding value of m.

There are further complications that arise when there are catastrophic jumps in the spatial structure of the solution as either the frequency or amplitude of the applied forcing is varied smoothly. We have assumed that the phases ϕ and $\tilde{\phi}$ are continuous functions of the parameters of the system, and in particular of ω. This is true for systems of finite dimension, but for infinite dimensional systems, the issue is more difficult (Kato [12]). Nevertheless, our analysis refers to numerically computed phases obtained from the discretization of the system, which is always of finite dimension. So we will consider that the phases are continuous functions of the parameters. Only an additional problem remains: for particular parameter values two different pairs of complex-conjugate eigenvalues can simultaneously cross the unit circle. In these cases, the most dangerous eigenvalue can change from one complex-conjugate pair to another in a neighborhood of the critical parameter values. Then, although the phases on both eigenvalue branches are continuous, the phase of the critical state is discontinuous because we must switch branches when following the most dangerous eigenvalue. This behavior may arise when more than one parameter is varied and higher codimension points are encountered where more than one mode becomes critical. In the following section we illustrate cases where this problem arises and how our technique may still be used to robustly and unambiguously determine the frequencies of these quasiperiodic flows.

3. Examples of quasiperiodic response in Taylor–Couette flow

In the Taylor–Couette flow with axial oscillations of the inner cylinder, the basic state consists of circular Couette flow with a superimposed annular Stokes flow. It is independent of the axial and azimuthal directions, and time-periodic with the period of the forcing; an analytic description of the basic flow is derived in Marques & Lopez [15]. Over an extensive range of parameter space, the primary bifurcation is to an axisymmetric state that is periodic in the axial direction and time, with the same temporal period as the forcing (Weisberg *et al.* [20]; Marques & Lopez [15]). Due to the symmetries of the system, the bifurcation is not the generic fold or saddle–node bifurcation, but a pitchfork for periodic orbits (Kuznetsov [14]). When the basic solution loses stability, two time-periodic solutions resembling Taylor vortices appear; the symmetry S changes one to the other.

The analysis of Marques & Lopez [15] however, showed that in narrow windows of parameter space, where interaction and competition between different axial modes occurs, the primary bifurcation is to a state that is periodic in both the axial and the azimuthal directions, and temporally has the forcing frequency as well as a new frequency ω_s, so that the dynamics are on a torus. These regions in parameter space are pockets of spatio-temporal complexity. Part of their figure 10 is reproduced here as figure 2, showing examples for the radius ratio $e = 0.905$ case. When $Re_a = 75$ there is a range of ω over which the azimuthal mode $n = 1$ is most dangerous and for $Re_a = 100$, the $n = 2$ mode is most dangerous. Normally, in the unforced Taylor–Couette flow, these azimuthal modes are interpreted as either single ($n = 1$) or double ($n = 2$) spirals, but here, they can manifest themselves as tilted, wobbling, and deforming Taylor cells, due to the interaction with the axial and temporal periodicities. Such tilted cells were noted in the experiments of Weisberg [19] within the same parameter range, but were not investigated in detail in that study. Hu & Kelly [9] only considered the axisymmetric modes ($n = 0$) for this flow, but did consider non-axisymmetric modes in the Taylor–Couette flow with an imposed time-periodic axial pressure gradient. In the range of parameters they considered, the axisymmetric mode was most dangerous.

We shall begin by analyzing the $Re_a = 75$ case where the amplitude of the periodic forcing is large enough that over a range of the forcing frequency the bifurcation is to a torus and the resulting second frequency, ω_s, varies with the forcing frequency ω (as well as with the forcing amplitude Re_a). From figure 2*a*, we see that the axisymmetric mode $n = 0$ is the most dangerous (i.e. for a given Re_a, ω, and n, the lowest value of Re_i over the range of axial wave numbers k at which a pair of Floquet multipliers first cross the unit circle), except in the range $5.6 < \omega < 9.8$, where the azimuthal mode $n = 1$ is most dangerous. The higher azimuthal modes ($n \geq 2$) have larger critical Re_i for any given ω at this forcing amplitude ($Re_a = 75$), and so would not normally be observed in any physical realization of the flow.

A limit in which it is clear how to extract ω_s from $\tilde{\phi}$ is in the limit of very

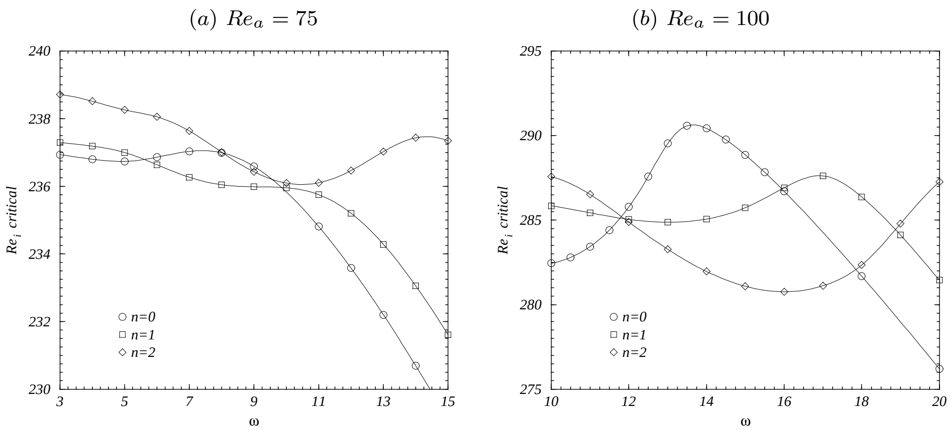

Figure 2.
Critical Re_i versus ω in for (a) $Re_a = 75$ and (b) $Re_a = 100$, and various azimuthal modes n as indicated.

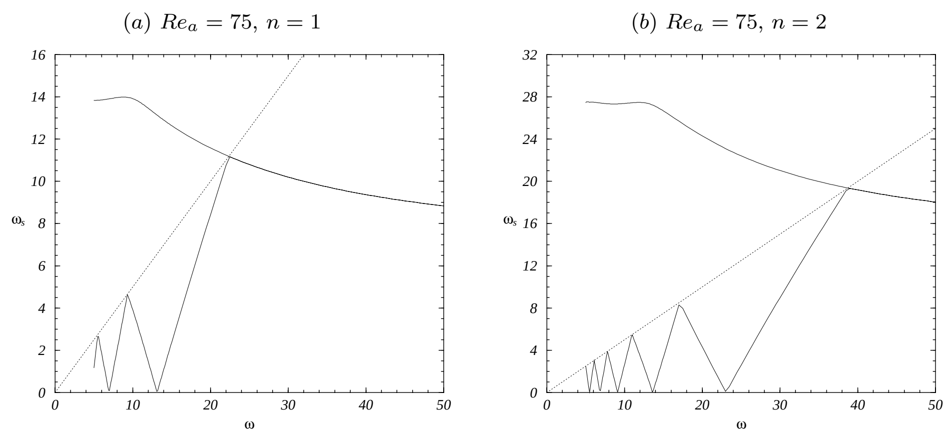

Figure 3.
Frequency of the bifurcating solution $\omega_s(\omega)$ (together with $\tilde{\omega}$ under the dotted line) when $Re_a = 75$ for azimuthal modes (a) $n = 1$ and (b) $n = 2$.

weak forcing ($Re_a \to 0$), as in this limit ω_s is independent of the forcing (Re_a and ω). By dividing $\tilde{\phi}$ from the Floquet analysis of a weakly forced system by the forcing frequency ω, and adjusting the sign and adding the multiples of 2π so that it matches the natural frequency of the unforced ($Re_a = 0$) system, one can then determine ω_s. This is the technique employed by Hu & Kelly [9]. However, it is not applicable as Re_a becomes larger, and in the present example with $Re_a = 75$ it is ambiguous. At high Re_a, ω_s may be some multiple of 2π different from the natural frequency of the unforced flow. Another limit in which it is possible to determine ω_s unambiguously from $\tilde{\phi}$ is $\omega \to \infty$. For ω large enough, the effect of the external forcing on the flow goes to zero, because it is confined to the Stokes boundary layer of thickness $\sqrt{2/\omega}$ (Marques & Lopez [15]). Therefore ω_s remains constant, and from (6), $\phi = 2\pi\omega_s/\omega \to 0$. Then, $\phi \in [0, \pi]$ and $m = 0$. In figure 3a, we plot both $\tilde{\omega}$ and ω_s obtained by determining the self-rotation number as in §2. It should be compared with figure 1b, which corresponds to the ideal cases where ω_s = constant.

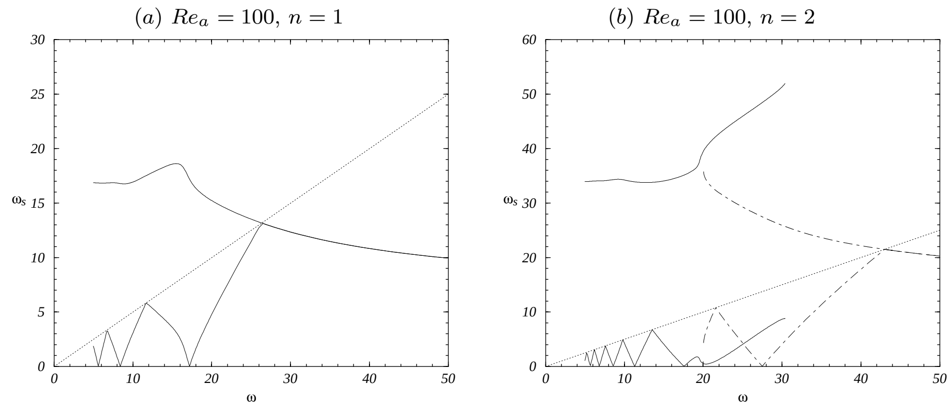

Figure 4.
Frequency of the bifurcating solution $\omega_s(\omega)$ (together with $\tilde{\omega}$ under the dotted line) when $Re_a = 100$ for azimuthal modes (a) $n = 1$ and (b) $n = 2$.

For larger Re, the dependence of ω_s on ω becomes increasingly more nonlinear. For the $n = 1$ case at $Re_a = 100$, the quasiperiodic response when ω is in the neighborhood of 15 is particularly nonlinear (see figure 4a). However, the locus of primary Hopf bifurcation points is continuous with varying ω, and using the techniques of §2, we are still able to determine the self-rotation number, and hence the second bifurcating frequency ω_s. Note that the critical Re_i and k change dramatically with both Re_a and ω (see figure 5). From figure 2b, for $Re_a = 100$, the $n = 0$ mode is the most dangerous except in the range $11.91 < \omega < 17.84$, where the $n = 2$ mode is the most dangerous, and for a very small range $11.74 < \omega < 11.91$, $n = 1$ dominates. For lower Re_a, the window of non-axisymmetric response is shared more evenly between the $n = 1$ and $n = 2$ modes, and as Re_a is reduced further $n = 1$ dominates, as described above.

The determination of the frequency ω_s in the $n = 2$ case is much more complicated. Here, the locus of primary Hopf bifurcation points is not continuous in ω for fixed Re_a. Instead, we find a range $20.045 < \omega < 30.380$ over which the stability curves Re_i vs. k have two minima (figure 6), each corresponding to distinct branches (i.e. loci of local minima in Re_i for variable ω and fixed Re_a) of bifurcating solutions. There is a large difference in the axial wavelengths associated with these two branches. Over the range of ω where the two $n = 2$ solutions co-exist,

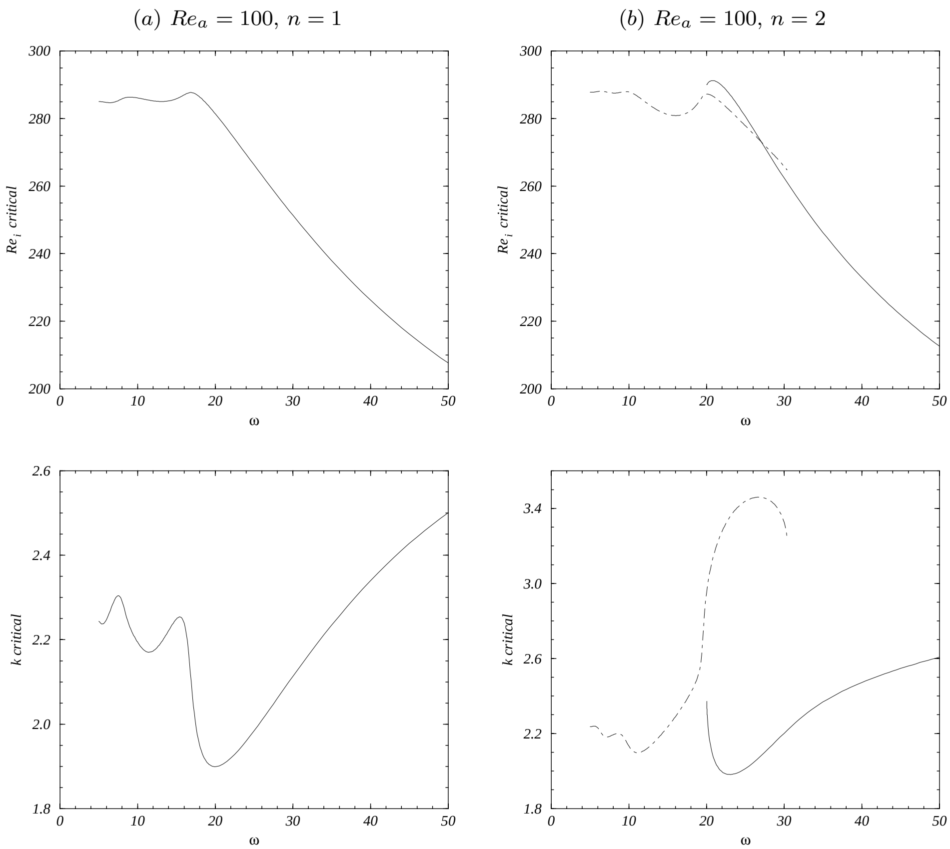

Figure 5.
Critical Re_i and k versus ω for $Re_a = 100$ and azimuthal modes (a) $n = 1$ and (b) $n = 2$.

the $n = 0$ solution is the most dangerous and hence the $n = 2$ solutions would not be physically realized. The two branches, where they exist, are continuous, branch 1 for $\omega > 20.045$ and branch 2 for $\omega < 30.380$. From figure 2b we see that over the range $11.8 < \omega < 17.8$, branch 2 is physically observable, and it would be of great interest to be able to predict the frequencies associated with this quasiperiodic flow.

The determination of ω_s on branch 1 is straightforward. Since the branch extends beyond $\omega > 2\omega_s$, we can directly apply the technique from §2, starting from a suitably large ω where $m = 0$ and detect the synchronous ($\tilde{\phi} = 0$) and subharmonic ($\tilde{\phi} = \pi$) points as ω is reduced in order to determine m. Such a straightforward application is not possible for branch 2 as, for fixed $Re_a = 100$, it ceases to exist for some $\omega < 2\omega_s$, so we do not have a simple method to determine m. However, branch 2 is continuous, as illustrated by the curves of critical Re_i

and k in figure 5b, and so it is reasonable to expect (Kato [12]) that ω_s will also be continuous on branch 2.

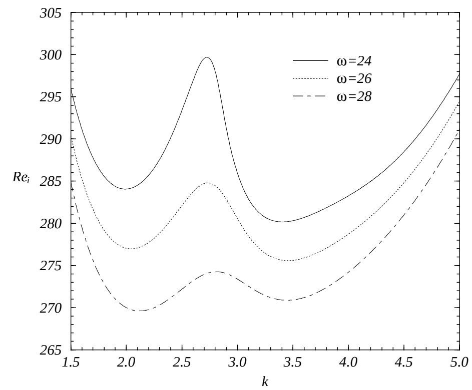

Figure 6.
Stability boundaries in (Re_i, k) space for $Re_a = 100$, azimuthal mode $n = 2$, and ω as indicated.

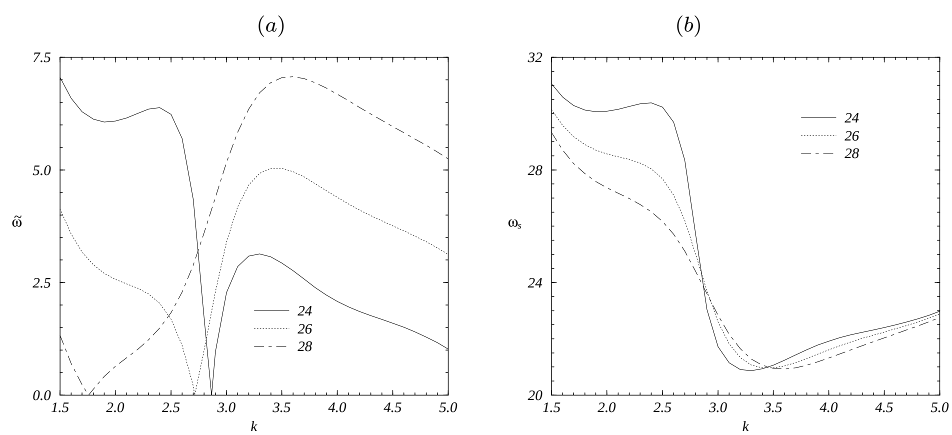

Figure 7.
(a) Frequency from the Floquet analysis $\tilde{\omega}$ and (b) the corresponding ω_s, versus axial wavenumber k, for $Re_a = 100$, $n = 2$, and forcing frequency ω as indicated

We shall use the continuity of eigenvalues from finite dimension dynamical systems theory (Kato [12]) in order to determine ω_s corresponding to branch 2, which does not extend to $\omega \to \infty$. The point is that the eigenvalues (in this case, ω_s) are not only continuous functions of the forcing frequency ω, but of *all* parameters governing the flow, and in particular of the axial wavenumber k. The locus of points where the eigenvalues cross the unit disk in the multi-dimensional parameter space governing the system is a continuous manifold, and this manifold may have folds so that a particular cut through the fold with all parameters fixed except for one of them, may have a discontinuity in the eigenvalue as a function of that parameter. However, with a suitable variation of the other parameters, a continuous connection between any two points on the manifold can be established. So, when we talk about branches, we mean particular cuts through this manifold on which the eigenvalues are continuous functions of the varying parameter.

In order to determine ω_s for branch 2, as illustrated in figure 4*b*, we have selected an ω where both branches co-exist, and where we know the value of m corresponding to branch 1 (e.g. $\omega = 28$, corresponding to the dot-dash line in figure fig6). We start from the minimum on the Re_i versus k curve corresponding to branch 1 ($k \approx 2.2$), where from figure 4*b* we know that $m = 1$. Then we vary k at the fixed $\omega = 28$ value, keeping track of when $\tilde{\phi} \to 0$ or π to increment m, until we reach the other minimum, corresponding to the branch 2 solution ($k \approx 3.4$). Once this is done, we have the value of m on a particular point on branch 2, and by continuity with varying ω, the ω_s on the entire branch 2 can be determined in the manner described earlier. This determination is illustrated in figure 4*b*. Examples at other fixed ω values for varying k are shown in figure 7. These distinct determinations with varying k are consistent with the results shown in figure fig4*b*, and give an additional check of the continuity-based technique.

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