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The asymptotic expansion of Gordeyev's integral

R. B. Paris

Abstract. We obtain asymptotic expansions for the integral

$$G_\nu(\omega, \lambda) = \omega \int_0^\infty \exp[i\omega t - \lambda(1-\cos t) - \frac{1}{2}\nu t^2]\, dt,$$

for large values of ω and λ and $\nu \to 0+$. For positive real parameters, the real part of the integral is associated with an exponentially small expansion in which the leading term involves a Jacobian theta function as an approximant. The asymptotic expansions are compared with numerically computed values of $G_\nu(\omega, \lambda)$.

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1. Introduction

In the study of the propagation of electrostatic waves in a hot magnetised Maxwellian plasma one encounters the function defined by

$$G_\nu(\omega, \lambda) = \omega \int_0^\infty \exp[i\omega t - \lambda(1-\cos t) - \frac{1}{2}\nu t^2]\, dt, \quad \mathrm{Re}(\nu) > 0, \tag{1.1}$$

known as Gordeyev's integral [4], in the expression for the dielectric function in the dispersion relation. The real part of the parameter ω represents the wave frequency normalised to the ion cyclotron frequency and the real parts of the parameters λ and ν are respectively the squares of the perpendicular and parallel components (with respect to the magnetic field) of the wave vector normalised to the ion Larmor radius.

In problems of physical application the behaviour of (1.1) is required for large positive values of the real parts of ω and λ (where ω or λ, or both, may possess small imaginary parts) and when the parameter ν is positive but very small (corresponding to propagation at large angles to the magnetic field). In such cases the representation in (1.1) becomes inconvenient for computational purposes, particularly for values of ω in the neighbourhood of a harmonic (represented by integer values of ω) on account of the oscillatory nature of the integrand. An alternative representation of $G_\nu(\omega,\lambda)$ can be obtained by expanding the factor $\exp(\lambda\cos t)$ in terms of modified Bessel functions $I_n(\lambda)$ to find [10, p. 176]

$$G_\nu(\omega,\lambda) = \frac{-i\omega}{\sqrt{2\nu}}\, e^{-\lambda} \sum_{n=-\infty}^{\infty} I_n(\lambda)\, Z\left(\frac{\omega-n}{\sqrt{2\nu}}\right), \tag{1.2}$$

where $Z(x)$ denotes the plasma dispersion function

$$Z(x) = i\sqrt{\pi}e^{-x^2}\mathrm{erfc}(-ix). \tag{1.3}$$

Although this form eliminates the problems associated with a highly oscillatory integrand, a large number of terms of the expansion must be retained when λ is large.

A different form of $G_\nu(\omega,\lambda)$ has been given by Johnston in [5] as an integral over a finite interval. By expanding the factor $\exp(\frac{1}{2}\nu t^2)$ in (1.1) as a Fourier integral followed by division of the range of integration into intervals of length 2π, he obtained after some manipulation the representation

$$G_\nu(\omega,\lambda) = \int_0^{2\pi} \exp[i\omega t - \lambda(1-\cos t) - \frac{1}{2}\nu t^2] K(t)\, dt, \tag{1.4}$$

where

$$K(t) = \{1 - e^{2\pi i(\omega + i\nu t)}\}^{-1} - \frac{(2\nu)^{-\frac{1}{2}}}{4\pi i} \sum_{n=-\infty}^{\infty} \frac{Z'(\zeta_n(t))}{\zeta_n(t)}$$

and $\zeta_n(t) = (\omega + i\nu t - n)/\sqrt{2\nu}$. The form (1.4), with suitable approximations for $K(t)$, was used by Brambilla in [2] to obtain asymptotic estimates for the real part of $G_\nu(\omega,\lambda)$ (the part of main interest in the above physical application) for $\omega = O(\lambda^{\frac{1}{2}})$ as $\lambda\to\infty$ with $\nu\to 0+$. However, his results were nonuniform in the frequency parameter ω, as separate approximations were given for ω close to and away from the neighbourhood of a harmonic. An expansion for $G_\nu(\omega,\lambda)$ valid in the same range of ω values has also been given in [6]. In the special case $\nu = 0$, the analytic continuation of $G_0(\omega,\lambda)$ to real values of ω has been studied in detail in [9].

In this paper we shall similarly reduce (1.1) to a finite range of integration but with an integrand which involves a factor closely related to the Jacobian theta functions [8]. In this manner, using the method of steepest descent, we obtain the asymptotic expansions for large ω and λ when ν is finite (and in particular as $\nu\to 0+$) that are *uniformly* valid in ω through a harmonic.

2. Modification of the integral representation

We shall mainly be concerned with positive real values of the parameters ω, λ and ν, although in §5 we do consider the extension of the expansion for $G_\nu(\omega,\lambda)$ to complex vales of ω and λ. The integral for $G_\nu(\omega,\lambda)$ in (1.1) can be expressed as an integral over a finite interval by dividing the range of integration into intervals of length 2π to find

$$G_\nu(\omega,\lambda) = \omega\int_0^{2\pi} e^{-\lambda f(t)} g(t)\{1+F(t)\}\,dt, \tag{2.1}$$

where

$$F(t) = \sum_{n=1}^{\infty} q^{n^2} \exp[2\pi i n(\Delta\omega + i\nu t)], \qquad q = e^{-2\pi^2\nu} \tag{2.2}$$

and $\omega = N + \Delta\omega$, with N a non-negative integer such that $|\Delta\omega| \leq \frac{1}{2}$. In (2.1) we have introduced the abbreviations

$$f(t) = 1 - \cos t - it\sinh\alpha, \qquad \sinh\alpha = \omega/\lambda \tag{2.3}$$

and $g(t) = \exp(-\frac{1}{2}\nu t^2)$. The function $F(t)$ in (2.2) is uniformly convergent in any bounded domain of ω and t provided $\mathrm{Re}(\nu) > 0$ (so that $|q| < 1$) and is closely related to the Jacobian theta functions [13, p. 462 *et seq.*]. It is readily shown that $F(t)$ satisfies the quasi-periodicity condition

$$F(t+2\pi) = e^{-2\pi i\Psi(t)}F(t) - 1, \qquad \Psi(t) = \Delta\omega + i\nu(\pi + t). \tag{2.4}$$

Use of (2.4) then enables us to separate the integral (2.1) into two parts

$$G_\nu(\omega,\lambda) = \omega\int_0^{\pi} e^{-\lambda f(t)} g(t)\,dt + \omega\int_{-\pi}^{\pi} e^{-\lambda f(t)} g(t)F(t)\,dt. \tag{2.5}$$

Since $F(t)$ does not depend upon the large variables N and λ it is a slowly varying function of t. With ν assumed finite, the integrand in (2.1) then possesses saddle points at the zeros of $f'(t) = 0$; that is, at the points given by $\sin t = i\sinh\alpha$. When $\lambda > 0$, there is a saddle point (which we shall call the principal saddle) in the domain of interest at P_0 given by $t_0 = i\alpha$, with the path of steepest descent through the saddle passing to infinity at $\pm\pi + \infty i$. Consider the path $\mathcal{C}_1$ starting from the origin in the t plane and passing to infinity at $\pi + \infty i$ and a second path $\mathcal{C}_2$ with endpoints at $\pm\pi + \infty i$, as shown in Fig. 1(a). Then, provided $\lambda > 0$, the first integral in (2.5) over $[0,\pi]$ may be deformed to pass over the path $\mathcal{C}_1$ together with the path CD parallel to the imaginary axis, while the second integral over

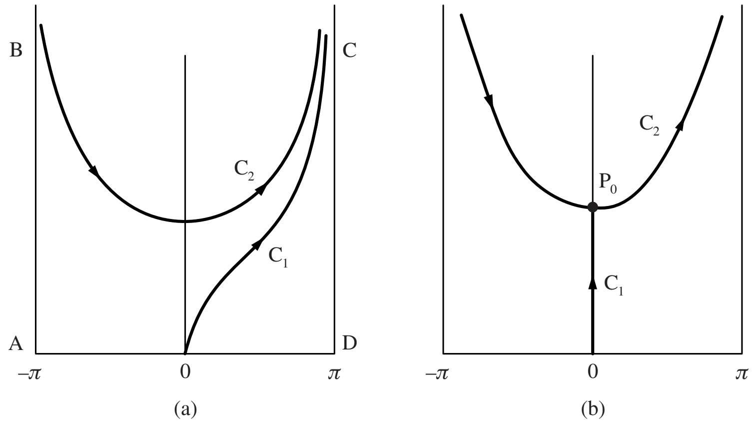

Figure 1.
(a) The paths of integration $\mathcal{C}_1$ and $\mathcal{C}_2$ and (b) the paths of steepest descent through the principal saddle P_0 when $\omega > 0$, $\lambda > 0$.

$[-\pi, \pi]$ can be deformed along the path $\mathcal{C}_2$ together with the paths AB and CD. We then find that

$$G_\nu(\omega, \lambda) = I_1 + I_2, \tag{2.6}$$

where

$$I_1 = \omega \int_{\mathcal{C}_1} e^{-\lambda f(t)} g(t)\, dt, \qquad I_2 = \omega \int_{\mathcal{C}_2} e^{-\lambda f(t)} g(t) F(t)\, dt, \tag{2.7}$$

since the contribution from the sides AB and CD, given by

$$\int_{-\pi}^{-\pi+\infty i} e^{-\lambda f(t)} g(t) F(t)\, dt - \int_{\pi}^{\pi+\infty i} e^{-\lambda f(t)} g(t)\{1 + F(t)\} dt$$

$$= \int_{-\pi}^{-\pi+\infty i} e^{-\lambda f(t)} g(t)\{F(t) - e^{2\pi i \Psi(t)}(1 + F(t + 2\pi))\} dt,$$

vanishes on account of (2.4).

3. The asymptotic expansion when $\omega/\lambda = O(1)$

We are now in a position to evaluate $G_\nu(\omega, \lambda)$ asymptotically from (2.6) by the method of steepest descent for large positive values of $\omega = N + \Delta\omega$ and λ, where it will be assumed that $\Delta\omega$, ν and ω/λ are finite. The path of steepest descent through the saddle P_0 is reconcilable with the path $\mathcal{C}_2$, while $\mathcal{C}_1$ can be deformed into the path with $\mathrm{Im} f(t) = 0$ which corresponds to the part of the imaginary axis between OP_0 and the path $\mathcal{C}_2$ lying in $\mathrm{Re}(t) > 0$; see Fig. 1(b). A straightforward application of the method of steepest descent [3, p. 119; 7, p. 127; 14, p. 90] then yields

$$I_1 \sim iJ_\alpha + \frac{\omega}{\sqrt{2\cosh\alpha}} e^{-\lambda f(i\alpha)} g(i\alpha) \sum_{k=0}^{\infty} \Gamma(\frac{1}{2}k + \frac{1}{2}) \frac{A_k}{\lambda^{\frac{1}{2}k+\frac{1}{2}}} \tag{3.1}$$

$$I_2 \sim \omega \left(\frac{2}{\cosh\alpha}\right)^{\frac{1}{2}} e^{-\lambda f(i\alpha)} g(i\alpha) \sum_{k=0}^{\infty} \Gamma(k + \frac{1}{2}) \frac{B_{2k}}{\lambda^{k+\frac{1}{2}}}, \tag{3.2}$$

where $f(i\alpha) = 1 - \cosh\alpha + \alpha\sinh\alpha$ and $g(i\alpha) = \exp(\frac{1}{2}\nu\alpha^2)$. The coefficients B_k are given by

$$B_0 = \gamma_0, \quad B_1 = (2\mathrm{sech}\alpha)^{\frac{1}{2}} \left\{\frac{1}{3} i\gamma_0 \tanh\alpha + \gamma_1\right\},$$

$$B_2 = \mathrm{sech}\alpha \left\{\frac{1}{12}\gamma_0(3 - 5\tanh^2\alpha) + i\gamma_1\tanh\alpha + \gamma_2\right\},$$

$$B_3 = (2\mathrm{sech}\alpha)^{\frac{3}{2}}$$
$$\left\{\frac{2}{135} i\gamma_0 \tanh\alpha\,(9 - 10\tanh^2\alpha) + \frac{1}{6}\gamma_1(1 - 2\tanh^2\alpha) + \frac{1}{3} i\gamma_2\tanh\alpha + \frac{1}{6}\gamma_3\right\},$$

$$B_4 = \mathrm{sech}^2\alpha \left\{\frac{1}{32}\gamma_0[\frac{77}{27}(5\tanh^2\alpha - 6)\tanh^2\alpha + 3] + \frac{1}{36} i\gamma_1\tanh\alpha\,(29 - 35\tanh^2\alpha)\right.$$
$$\left. + \frac{5}{36}\gamma_2(3 - 7\tanh^2\alpha) + \frac{5}{9} i\gamma_3\tanh\alpha + \frac{1}{6}\gamma_4\right\}, \ldots ,$$

where

$$\gamma_j = e^{\frac{1}{2}\nu t_0^2} (d/dt)^j \,[e^{-\frac{1}{2}\nu t^2} F(t)]_{t=t_0}, \quad (j = 0, 1, 2, \ldots)$$

and, from (2.2),

$$F(t_0) = F(i\alpha) = \sum_{n=1}^{\infty} q^{n^2} e^{2\pi i n\Omega}, \qquad \Omega = \Delta\omega - \alpha\nu. \tag{3.3}$$

The coefficients A_k are obtained from B_k by substitution of $F(t) \equiv 1$ in the quantities γ_j.

The integral J_α in (3.1) corresponds to the contribution to I_1 from the path OP_0. This is given by

$$J_\alpha = \omega \int_0^\alpha e^{-\lambda f(iu)} g(iu)\, du = \omega \int_0^{w(\alpha)} e^{-\lambda w} \frac{g(iu)}{w'(u)}\, dw, \tag{3.4}$$

since $w(u) \equiv f(iu) = u \sinh \alpha + 1 - \cosh u$ is monotonically decreasing on $0 \le u < \alpha$. For finite α (i.e., for ω/λ finite as $\lambda \to \infty$), the algebraic expansion associated with J_α can be obtained by Laplace's method; see [7, p. 81]. Introduction of the power series

$$\sinh \alpha \, \frac{e^{\frac{1}{2}\nu u^2}}{w'(u)} = \sum_{k=0}^{\infty} D_k w^k \qquad (|w| < w(\alpha)),$$

then leads to the expansion

$$J_\alpha \sim \sum_{k=0}^{\infty} D_k \frac{k!}{\lambda^k}, \quad \lambda \to \infty, \tag{3.5}$$

where the coefficients D_k are given by [3, p. 114]

$$D_0 = 1,\ D_1 = a,\ D_2 = \frac{1}{2} a(3a + \nu),$$

$$D_3 = \frac{1}{6} a^2 (15a + 6\nu + 1),$$

$$D_4 = \frac{1}{8} a^2 (35a^2 + 5(1 + 3\nu)a + \nu^2),$$

$$D_5 = \frac{1}{120} a^3 (945a^2 + 210(1 + 2\nu)a + 1 + 15\nu + 45\nu^2),$$

$$D_6 = \frac{1}{240} a^3 (3465a^3 + 525(1 + 3\nu)a^2 + 7(3 + 20\nu + 30\nu^2)a + 5\nu^3), \dots ,$$

and, for brevity, we have put $a = \operatorname{cosech}^2 \alpha$.

For positive values of the parameters, J_α is real and the coefficients A_k of even and odd order are respectively real and purely imaginary. Hence, the real and imaginary parts of $G_\nu(\omega, \lambda)$ in this case are

$$\mathrm{Re}\ G_\nu(\omega, \lambda) \sim \frac{\omega}{\sqrt{2\lambda \cosh \alpha}} e^{-\lambda f(i\alpha)} g(i\alpha) \sum_{k=0}^{\infty} \Gamma(k + \frac{1}{2}) \frac{C_{2k}}{\lambda^k} \tag{3.6}$$

$$\mathrm{Im}\ G_\nu(\omega, \lambda) \sim J_\alpha + \frac{\omega}{\sqrt{2\lambda \cosh \alpha}} e^{-\lambda f(i\alpha)} g(i\alpha)$$
$$\cdot \sum_{k=0}^{\infty} \frac{\Gamma(k + \frac{1}{2})}{\lambda^k} \left\{ 2\,\mathrm{Im}(B_{2k}) + \frac{\Gamma(k+1)}{\Gamma(k + \frac{1}{2})} \frac{A_{2k+1}}{\sqrt{\lambda}} \right\}, \tag{3.7}$$

where $C_{2k} = A_{2k} + 2\mathrm{Re}(B_{2k})$.

The accurate determination of the exponentially small part of $\mathrm{Im}\ G_\nu(\omega, \lambda)$ is complicated by the fact that the evaluation of I_1 for positive ω and λ is associated

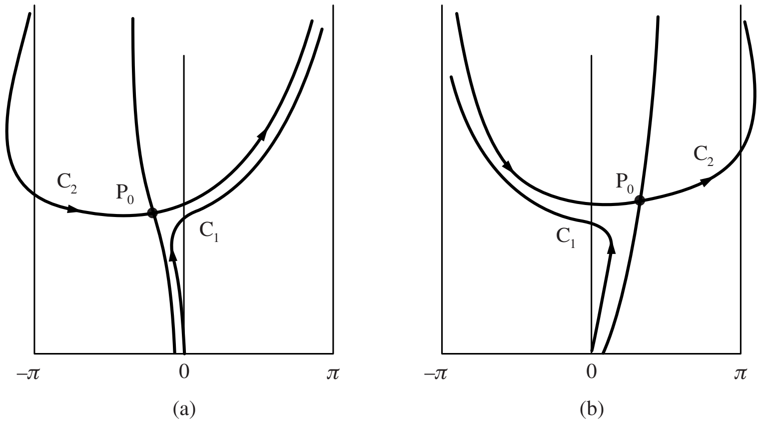

Figure 2.
The paths of steepest descent for $\mathrm{Re}(\omega)$, $\mathrm{Re}(\lambda) > 0$ with ω and λ possessing imaginary parts of O(1) when $\theta = -\arg(\omega/\lambda)$ is (a) negative and (b) positive.

with a Stokes phenomenon. This can be seen from (3.5) since, when the parameters are positive, all the coefficients in the algebraic expansion are of the same phase. In terms of the saddle point method, when ω and λ have imaginary parts of $O(1)$, the saddle point at P_0 will lie to the right or the left of the imaginary axis according as $\theta > 0$ or $\theta < 0$, respectively, where $-\theta = \arg(\omega/\lambda)$. When P_0 lies to the right, the path Im $f(t) = 0$ from the origin passes to infinity at $-\pi + \infty i$ so that the path $\mathcal{C}_1$ can be taken to coincide with this path together with the path of steepest descent through the saddle point. When P_0 lies to the left, the path Im $f(t) = 0$ passes to infinity at $\pi + \infty i$ and is reconcilable with $\mathcal{C}_1$; see Fig. 2. Thus, as θ varies with fixed $|\omega/\lambda|$ we find, upon optimal truncation of the expansion in (3.5) after K terms,

$$I_1 \sim \sum_{k=0}^{K-1} D_k \frac{k!}{\lambda^k} + \omega S(\theta) \left(\frac{2\pi}{\lambda \cosh \alpha}\right)^{\frac{1}{2}} e^{-\lambda f(i\alpha)} g(i\alpha), \tag{3.8}$$

where $S(\theta)$ is the Stokes multiplier which varies *smoothly* from 0 to 1 as θ varies from negative to positive values [1]. To leading order, the value of $S(\theta)$ on $\theta = 0$ is $\frac{1}{2}$, so that the second term on the right-hand side of (3.8) then corresponds to the leading term in the exponentially small expansion in (3.1). When the imaginary parts of ω and λ are small compared to their real parts, θ will be small and an accurate determination of the exponentially small contribution to I_1 will necessitate a detailed analysis of the Stokes phenomenon to take into account the smooth variation of $S(\theta)$ across the Stokes line $\theta = 0$; see §5 for further discussion on this point.

From (3.6) and (3.7), the leading asymptotic behaviour of $G_\nu(\omega, \lambda)$ (for real parameters) can therefore be expressed in the form

$$\mathrm{Re}\ G_\nu(\omega,\lambda) \sim \omega\sqrt{\frac{\pi}{2\lambda\cosh\alpha}}\exp[-\lambda(\alpha\sinh\alpha+1-\cosh\alpha)]\,e^{\frac{1}{2}\nu\alpha^2}\vartheta(\Omega,q) \quad (3.9)$$

$$\mathrm{Im}\ G_\nu(\omega,\lambda) \sim J_\alpha + \omega\sqrt{\frac{2\pi}{\lambda\cosh\alpha}}\exp[-\lambda(\alpha\sinh\alpha+1-\cosh\alpha)]\,e^{\frac{1}{2}\nu\alpha^2}\varphi(\Omega,q), \quad (3.10)$$

where, to display the dependence on Ω and q, we have put

$$\vartheta(\Omega,q) \equiv 1+2\mathrm{Re}\,F(i\alpha) = 1+2\sum_{n=1}^{\infty} q^{n^2}\cos 2\pi n\Omega$$

$$\varphi(\Omega,q) \equiv \mathrm{Im}\,F(i\alpha) = \sum_{n=1}^{\infty} q^{n^2}\sin\,2\pi n\Omega \quad (3.11)$$

with $F(i\alpha)$ and Ω given in (3.3); the function $\vartheta(\Omega,q)$ is the Jacobian theta function of the third kind [13, p. 463]. The real part is therefore exponentially small for $\alpha>0$ whereas the imaginary part is dominated by the algebraic expansion resulting from J_α in (3.5).

The approximations in (3.9) and (3.10) are not suitable in the limit $\nu\to 0+$ $(q\to 1-)$, since the sum $F(i\alpha)$ is then slowly convergent and accordingly becomes difficult to compute. To overcome this problem, we rewrite $F(i\alpha)$ by means of the Poisson-Jacobi transformation [11, §2.8; see also 8] in the form

$$F(i\alpha) = \frac{(2\nu)^{-\frac{1}{2}}}{2\pi i}\sum_{n=-\infty}^{\infty} Z\left(\frac{\Omega-n}{\sqrt{2\nu}}\right)-\frac{1}{2}, \quad (3.12)$$

where $Z(x)$ is defined in (1.3). Then the factors appearing in (3.9) and (3.10) become

$$\begin{aligned}\vartheta(\Omega,q) &= \frac{e^{-\Omega^2/2\nu}}{\sqrt{2\pi\nu}}\left\{1+2\sum_{n=1}^{\infty}e^{-(n^2/2\nu)}\cosh\frac{n\Omega}{\nu}\right\}\\ &\sim \frac{e^{-\Omega^2/2\nu}}{\sqrt{2\pi\nu}}\left\{1+2\exp(-\frac{1}{2\nu})\cosh\frac{\Omega}{\nu}\right\} \qquad (\nu\to 0+)\end{aligned} \quad (3.13)$$

and, upon use of the asymptotic behaviour of $Z(x)$ for large x,

$$\varphi(\Omega,q) \sim \frac{1}{2}\left\{\frac{ie^{-\Omega^2/2\nu}}{\sqrt{2\pi\nu}}\mathrm{erf}\left(\frac{-i\Omega}{\sqrt{2\nu}}\right)+\cot\pi\Omega-\frac{1}{\pi\Omega}\right\}. \qquad (\nu\to 0+) \quad (3.14)$$

We note that the approximations in (3.9), (3.10), (3.13) and (3.14) hold *uniformly* in $\Delta\omega$ through a harmonic ($\Delta\omega=\alpha\nu$).

4. The asymptotic expansion when $\omega = O(\lambda^{\frac{1}{2}})$

The expansions given in §3 describe the asymptotics of $G_\nu(\omega, \lambda)$ as ω and $\lambda \to \infty$ with $\omega = O(\lambda)$. For $\omega/\lambda = o(1)$, that is when $\alpha \to 0$ as $\lambda \to \infty$, the principal saddle P_0 moves down the imaginary axis towards the origin with the result that the exponential expansions in (3.6) and (3.7) [and also in (3.9) and (3.10)] remain valid in this limit. Thus, if we let $\xi_0 = \omega/\sqrt{2\lambda}$ and consider finite values of ξ_0 as $\lambda \to \infty$, we therefore obtain from (3.9) and (3.10) the leading approximations

$$\text{Re } G_\nu(\omega, \lambda) \sim \sqrt{\pi}\xi_0 e^{-\xi_0^2} \vartheta(\Delta\omega, q), \tag{4.1}$$

$$\text{Im } G_\nu(\omega, \lambda) \sim J_\alpha + 2\sqrt{\pi}\xi_0 e^{-\xi_0^2} \varphi(\Delta\omega, q). \tag{4.2}$$

In the limit $\nu \to 0+$, the factors $\vartheta(\Delta\omega, q)$ and $\varphi(\Delta\omega, q)$ can be approximated by (3.13) and (3.14) (with $|\Delta\omega| \leq \frac{1}{2}$). The algebraic expansion for J_α in (3.5), however, is not uniformly valid as $\alpha \to 0$, since it is clear from (3.4) that[1] $J_\alpha \to 0$. The breakdown in the expansion (3.5) can be seen to arise when the value of the exponential factor in the integrand in (3.4) evaluated at the saddle becomes O(1), i.e., when $\omega = O(\lambda^{\frac{1}{2}})$.

To determine the expansion of I_1 as $\alpha \to 0$, we take the path $\mathcal{C}_1$ in (2.7) to be the real axis between $[0, \pi]$ and the line $[\pi, \pi + \infty i)$ parallel to the imaginary axis. The contribution to I_1 from this latter part of the path is readily shown to be $O(\omega\lambda^{-\frac{1}{2}} e^{-2\lambda})$ as $\lambda \to +\infty$ so that, from (2.7),

$$I_1 = \omega \int_0^\pi e^{-\lambda f(t)} g(t)\, dt + O(\omega\lambda^{-\frac{1}{2}} e^{-2\lambda}). \tag{4.3}$$

Following [9], we use the expansion

$$e^{-\lambda(1-\cos t)} = e^{-\frac{1}{2}\lambda t^2} \sum_{n=0}^{\infty} \frac{(-)^n S_n(\lambda)}{(2n)!} t^{2n}, \qquad (|t| < \infty)$$

where the coefficients $S_n(\lambda)$ are given by[2]

$$S_0(\lambda) = 1, \quad S_1(\lambda) = 0, \quad S_2(\lambda) = S_3(\lambda) = 1, \quad S_4(\lambda) = \lambda + 35\lambda^2, \quad S_5(\lambda) = \lambda + 210\lambda^2, \ldots .$$

[1] More precisely, integration of (3.4) from the saddle P_0 along the path of steepest ascent to the origin shows that $J_\alpha \sim -i\sqrt{\pi}\xi_0 \exp(-\xi_0^2)\text{erf}(i\xi_0)$ as $\alpha \to 0$.

[2] Higher coefficients are defined by the recursion relation in [9, p.57]. For $n \to \infty$ and λ bounded away from zero, $S_n(\lambda) \sim (2n)!(\lambda/24)^m \delta_n/m!$, where $m = [n/2]$ and $\delta_n = 1$ (even n) and $\delta_n = m/30$ (odd n); see [9].

Substitution of the above series into the integral on the right-hand side of (4.3) then yields the convergent expansion[3]

$$\omega\int_0^\pi e^{-\lambda f(t)}g(t)\,dt = \sum_{n=0}^\infty \frac{S_n(\lambda)}{(2n)!}H_n, \tag{4.4}$$

where

$$\begin{aligned} H_n &= \omega\int_0^\pi t^{2n}\exp[i\omega t - \frac{1}{2}(\lambda+\nu)t^2]\,dt \\ &= \frac{i(-)^{n+1}\xi}{\{2(\lambda+\nu)\}^n}\{Z^{(2n)}(\xi)-R_n\}, \qquad \xi = \frac{\omega}{\sqrt{2(\lambda+\nu)}}, \end{aligned} \tag{4.5}$$

with

$$R_n = (2i)^{2n+1}\int_\Lambda^\infty \tau^{2n}e^{-\tau^2+2i\xi\tau}\,d\tau, \qquad \Lambda = \pi\left(\frac{\lambda+\nu}{2}\right)^{\frac{1}{2}}$$

and $Z^{(2n)}(\xi)$ denoting the $2n$th derivative of the plasma dispersion function in (1.3). Then, $|R_n| = 2^{2n}\Gamma(n+\frac{1}{2},\Lambda^2) = O(\lambda^{n-\frac{1}{2}}e^{-\frac{1}{2}\pi^2\lambda})$ for each integer value of n as $\lambda\to+\infty$, so that for finite values of ξ

$$H_n = \frac{i(-)^{n+1}\xi}{\{2(\lambda+\nu)\}^n}Z^{2n}(\xi)\left\{1+O(\lambda^{n-\frac{1}{2}}e^{-\frac{1}{2}\pi^2\lambda})\right\}, \quad \lambda\to\infty.$$

The Hadamard sum in (4.4) then furnishes the asymptotic expansion

$$\omega\int_0^\pi e^{-\lambda f(t)}g(t)\,dt \sim -i\xi\sum_{n=0}^\infty E_n(\xi)\tilde{S}_n(\lambda)\frac{\lambda^{[n/2]}}{(\lambda+\nu)^n}, \quad \lambda\to\infty, \tag{4.6}$$

where the square brackets denote the integer part, $\tilde{S}_n(\lambda)$ are the scaled coefficients $S_n(\lambda)$ given by

$$\tilde{S}_0(\lambda)=1,\quad \tilde{S}_1(\lambda)=0,\quad \tilde{S}_2(\lambda)=\tilde{S}_3(\lambda)=1,\quad \tilde{S}_4(\lambda)=1+\frac{1}{35\lambda},\quad \tilde{S}_5(\lambda)=1+\frac{1}{210\lambda},\ldots$$

and

$$\begin{aligned} E_0(\xi) &= Z(\xi), \quad E_1(\xi)=0, \\ E_2(\xi) &= \frac{1}{24}\{2\xi(-5+2\xi^2)+(3-12\xi^2+4\xi^4)Z(\xi)\}, \\ E_3(\xi) &= \frac{1}{720}\{2\xi(33-28\xi^2+4\xi^4)+(-15+90\xi^2-60\xi^4+8\xi^6)Z(\xi)\}, \end{aligned}$$

[3] For positive values of the parameters, $|H_n| < \omega\pi^{2n+1}/(2n+1)$ so that the sum in (4.4) is an absolutely convergent Hadamard sum (cf. [12, p. 204]).

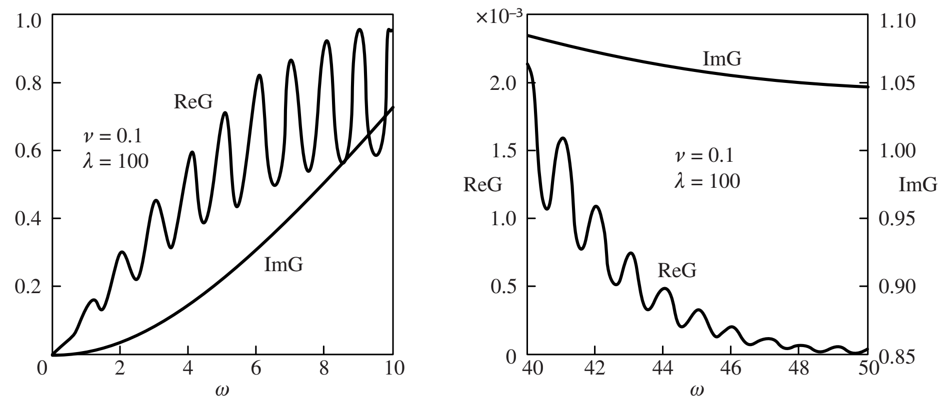

Figure 3.
Graphs of the real and imaginary parts of $G_\nu(\omega, 100)$ for different ranges of ω when $\nu = 0.1$.The oscillatory structure present in the imaginary part is not visible on the scale of these figures.

$$E_4(\xi) = \frac{1}{1152}\{2\xi(-279+370\xi^2-108\xi^4+8\xi^6)+(105-840\xi^2+840\xi^4-224\xi^6+16\xi^8)Z(\xi)\},$$

$$E_5(\xi) = \frac{1}{17280}\{2\xi(2895 - 5280\xi^2 + 2352\xi^4 - 353\xi^6 + 16\xi^8) + (-945 + 9450\xi^2 - 12600\xi^4 + 5040\xi^6 - 720\xi^8 + 32\xi^{10})Z(\xi)\}, \ldots ,$$

with ξ defined in (4.5). The expansion of $G_\nu(\omega, \lambda)$ for finite ξ (upon neglecting exponentially small terms) is then obtained from (2.6), where the expansion of I_2 is given in (3.2); this is equivalent to the result derived in [6, Eq. (6)]. For large values of ξ, it can be shown, by use of the asymptotic expansion of $Z(\xi)$, that (4.6) yields the algebraic expansion in (3.5).

5. Numerical results and complex values of ω, λ

In this section we compare the accuracy of the expansions developed in §§3, 4 with numerical results for $G_\nu(\omega, \lambda)$ computed using *Mathematica* from the sum in (1.2). In Fig. 3 we illustrate the behaviour of the real and imaginary parts of $G_\nu(\omega, \lambda)$ for two ranges of ω corresponding to $\omega = O(\lambda^{\frac{1}{2}})$ and $\omega = O(\lambda)$.

The expansions of the real and imaginary parts of $G_\nu(\omega, \lambda)$ in the case of positive parameter values are given by (3.6) (which holds for arbitrary values of ω) and (3.7). Fig. 4 shows the behaviour of the factors $\vartheta(\Omega, q)$ and $\varphi(\Omega, q)$ defined in (3.11) which appear in the leading terms in (3.9) and (3.10); see also (4.1) and (4.2). These factors contain the fine structure in ω: it can be seen that $\vartheta(\Omega, q)$ becomes strongly peaked as $\nu \to 0+$ in the neighbourhood of the harmonics

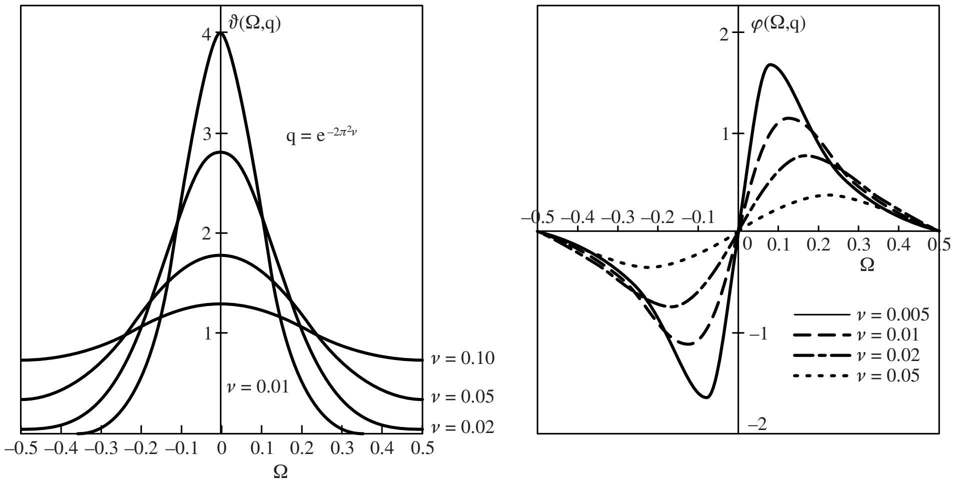

Figure 4.
The rapidly varying factors $\vartheta(\Omega, q)$ and $\varphi(\Omega, q)$ as a function of $\Omega = \Delta\omega - \alpha\nu$ for different values of ν.

$(\Delta\omega = \alpha\nu \simeq 0)$ with the width of the peak scaling roughly like $2\sqrt{2\nu}$, while $\varphi(\Omega, q)$ exhibits less pronounced resonance effects and vanishes at $\Omega = 0$ and $\pm\frac{1}{2}$.

Table 1 presents the results for Re $G_\nu(\omega, 100)$ for different ν and two ranges of ω. The asymptotic values are obtained using the first two terms in the expansion (3.6). In Table 2 we give corresponding values of Im $G_\nu(\omega, 100)$ obtained from (4.7) for the range $\omega = O(\lambda^{\frac{1}{2}})$ and from (3.10) and (3.5) for the range $\omega = O(\lambda)$. In both cases 6 terms were used in the computation of the respective algebraic expansions. It can be seen that these asymptotic approximations hold uniformly through a harmonic.

Fig. 5 summarises the results for the case $\nu \to 0+$ when $\omega = O(\lambda^{\frac{1}{2}})$. The approximate value of Re $G_\nu(\omega, \lambda)$ was obtained using (4.1) together with (3.13). To reveal the fine structure contained in Im $G_\nu(\omega, \lambda)$, the expansion for I_1 in (4.6) was subtracted off from the numerical value obtained from (1.2) and compared with the leading term in the expansion of I_2 (given by the second term on the right-hand side of (4.2)). Nonuniform approximations for the real part of $G_\nu(\omega, \lambda)$ in this case have been previously derived in [2] by use of the representation in (1.4). There, the approximation $\mathrm{Re}\, G_\nu(\omega, \lambda) \sim H(\Delta\omega)\xi_0 \exp[-\xi_0^2]$ was given, where in the neighbourhood of a harmonic $H(\Delta\omega) = \exp[-(\Delta\omega)^2/2\nu]/\sqrt{2\nu}$ (which agrees with (4.1)), while away from a harmonic $H(\Delta\omega) = \sqrt{\pi\nu^2}(\pi/\sin \pi\omega)^4$. For the case illustrated in Fig. 5, this latter approximation is found to be very poor, being in error by roughly two orders of magnitude when $\Delta\omega \simeq \frac{1}{2}$.

We now briefly discuss the extension of the expansions in §3 to complex values of ω and λ. We let $\beta = \arg \lambda$, $\gamma = \arg \omega$, $A = |\omega/\lambda|$ and restrict our attention to

Table 1. The computed and asymptotic values of Re $G_\nu(\omega, 100)$ for different ν when $\omega = O(\lambda^{\frac{1}{2}})$ and $\omega = O(\lambda)$.

	$\lambda = 100$	$\omega = N + \Delta\omega$		
	$\nu = 1.0$			
	$N = 10$		$N = 50$	
$\Delta\omega$	$\mathrm{Re}\, G_\nu(\omega, \lambda)$	Asymptotic	$\mathrm{Re} G_\nu(\omega, \lambda)$	Asymptotic
0.0	7.5953929×10^{-1}	7.5952920×10^{-1}	3.1492512×10^{-5}	3.1492102×10^{-5}
0.1	7.5951439×10^{-1}	7.5950429×10^{-1}	3.0078551×10^{-5}	3.0078159×10^{-5}
0.2	7.5933985×10^{-1}	7.5932976×10^{-1}	2.8725402×10^{-5}	2.8725028×10^{-5}
0.3	7.5901725×10^{-1}	7.5900717×10^{-1}	2.7430578×10^{-5}	2.7430221×10^{-5}
0.4	7.5854819×10^{-1}	7.5853811×10^{-1}	2.6191686×10^{-5}	2.6193428×10^{-5}
0.5	7.5793431×10^{-1}	7.5792424×10^{-1}	2.5006426×10^{-5}	2.5006101×10^{-5}
	$\nu = 0.1$			
0.0	9.7111401×10^{-1}	9.7110781×10^{-1}	3.6080839×10^{-5}	3.6080831×10^{-5}
0.1	9.3826929×10^{-1}	9.3826314×10^{-1}	3.4397747×10^{-5}	3.4397691×10^{-5}
0.2	8.3685821×10^{-1}	8.3685331×10^{-1}	3.0156276×10^{-5}	3.0156200×10^{-5}
0.3	7.0619053×10^{-1}	7.0618758×10^{-1}	2.4709080×10^{-5}	2.4709016×10^{-5}
0.4	5.9608918×10^{-1}	5.9608810×10^{-1}	1.9740274×10^{-5}	1.9740246×10^{-5}
0.5	5.4771793×10^{-1}	5.4771798×10^{-1}	1.6603339×10^{-5}	1.6603355×10^{-5}
	$\nu = 0.01$			
0.0	3.03009782	3.03007774	1.1241006×10^{-4}	1.1241003×10^{-4}
0.1	1.85622570	1.85621287	6.8316509×10^{-5}	6.8316390×10^{-5}
0.2	4.1828073×10^{-1}	4.1827797×10^{-1}	1.5273900×10^{-5}	1.5273856×10^{-5}
0.3	3.4671187×10^{-2}	3.4670990×10^{-2}	1.2562556×10^{-6}	1.2562510×10^{-6}
0.4	1.0571872×10^{-3}	1.0571827×10^{-3}	3.8012109×10^{-8}	3.8011958×10^{-8}
0.5	2.2528086×10^{-5}	2.2528088×10^{-5}	$6.8290175 \times 10^{-10}$	$6.8290240 \times 10^{-10}$

Table 2. The computed and asymptotic values of Im $G_\nu(\omega, 100)$ for $\nu = 1.0$ when $\omega = O(\lambda^{\frac{1}{2}})$ and $\omega = O(\lambda)$.

	$\lambda = 100$	$\omega = N + \Delta\omega$		
	$\nu = 1.0$			
	$N = 10$		$N = 50$	
$\Delta\omega$	$\mathrm{Im}\, G_\nu(\omega, \lambda)$	Asymptotic	$\mathrm{Im} G_\nu(\omega, \lambda)$	Asymptotic
0.0	0.72085642	0.72085647	1.04681856	1.04674228
0.2	0.74069539	0.74069543	1.04637664	1.04630456
0.4	0.76033872	0.76033877	1.04594153	1.04587342
0.6	0.77976723	0.77976730	1.04551307	1.04544874
0.8	0.79896249	0.79896257	1.04509114	1.04503036
1.0	0.81790681	0.81790689	1.04467557	1.04461816

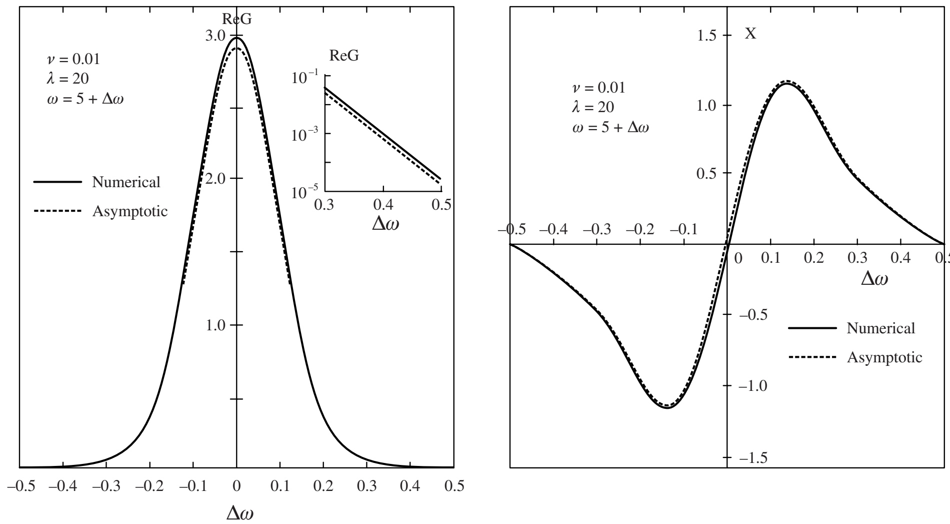

Figure 5.
The behaviour of $G_\nu(\omega, 20)$ for $\nu = 10^{-2}$ when $\omega = O(\lambda^{\frac{1}{2}})$: (a) the numerical value of $\operatorname{Re} G_\nu(\omega, 20)$ compared with the asymptotic approximation (4.1) and (b) the oscillatory structure in the expansion I_2. The figure shows the numerical value of $X \equiv \{\operatorname{Im} G_\nu(\omega, 20) - I_1\}/(2\sqrt{\pi}\xi_0 \exp(-\xi_0^2))$ compared with the asymptotic approximation $\varphi(\Delta\omega, q)$ [cf. (4.2)].

the range $|\beta| \leq \pi$, $|\gamma| \leq \frac{1}{2}\pi$ since, for $\nu > 0$, we have from (1.1)

$$G_\nu(-\omega, \lambda) = -G_\nu^*(\omega^*, \lambda^*),$$

where the asterisk denotes the complex conjugate. The principal saddle P_0 of $f(t)$ in (2.3) is located at $t_0 = i\operatorname{arcsinh}\{A \exp[i(\gamma - \beta)]\}$, with the other saddles $P_{\pm n}$ at $\pm\pi n + (-)^n t_0$ $(n = 1, 2, \ldots)$. As $\theta = -\arg(\omega/\lambda) = \beta - \gamma$ varies from 0 to π, the saddles P_0 and P_1 describe different loci[4] in the t plane according as $A \leq 1$ or $A > 1$, as illustrated in Fig. 6. This loci pattern is periodic and adjacent pairs of saddles P_{2n} and P_{2n+1} describe a similar behaviour. In addition to the above variation in the saddles, the paths of steepest descent of the function

$$e^{i\beta}(1 - \cos t - iAte^{-i\theta})$$

pass to infinity along paths parallel to the imaginary t axis with Re $(t) = \pm(2k + 1)\pi + \beta$ (when Im $(t) > 0$) and Re $(t) = \pm(2k+1)\pi - \beta$ (when Im $(t) < 0$), where k is a nonnegative integer.

This change in the position of the saddles and the lines of steepest descent as β and γ vary can result in the principal saddle P_0 connecting with one of the adjacent

[4] For $-\pi \leq \theta < 0$, the loci move in a symmetrical fashion in the opposite sense.

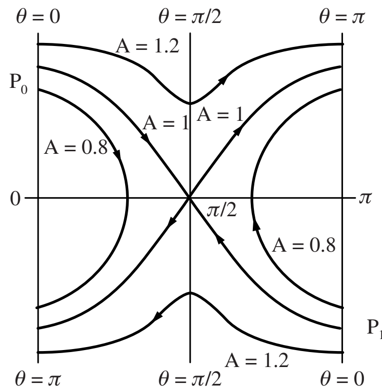

Figure 6.
The loci of the adjacent saddles P_0 and P_1 in the complex t plane as $\theta = \beta - \gamma$ increases from 0 to π. When $\theta = 0$, P_0 and P_1 are situated at $i\alpha$ and $\pi - i\alpha$, respectively, where $\alpha = \text{arcsinh}A$. When $A = 1$, $\theta = \frac{1}{2}\pi$ the saddles P_0, P_1 become coincident to form a double saddle.

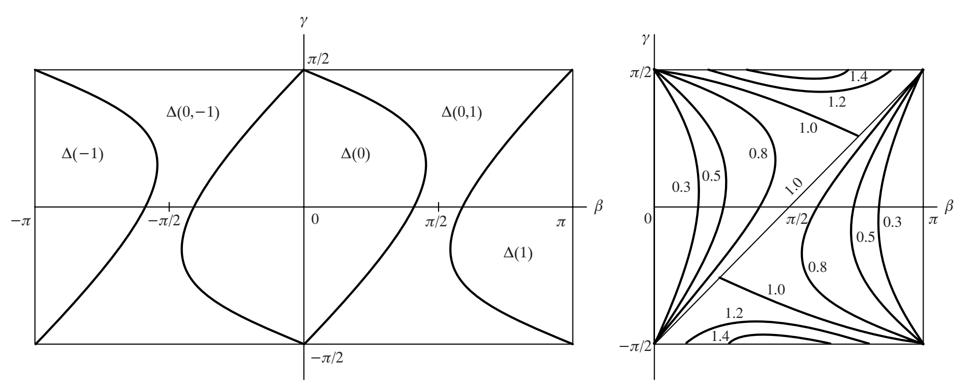

Figure 7.
(a) The boundaries of the domains $\Delta(0)$, $\Delta(\pm 1)$ and $\Delta(0, \pm 1)$ in the β, γ plane for $A = 0.8$. (b) The boundaries of $\Delta(0)$, $\Delta(0, 1)$ and $\Delta(1)$ for different values of $A = |\omega/\lambda|$. These domains possess odd symmetry about the lines $\beta = 0$ and $\beta = \pm\frac{1}{2}\pi$.

saddles $P_{\pm 1}$. A detailed study of the topology of the paths of steepest descent of (5.1) reveals that, for a given value of A, the β, γ plane is divided into domains $\Delta(m, n)$ determined by the saddles P_m, P_n which contribute to the asymptotics of the integrals I_1 and I_2 in (2.7); see Fig. 7(a). Thus, $\Delta(0)$ corresponds to the domain in which only the principal saddle P_0 contributes, while $\Delta(0, \pm 1)$ correspond to the domains in which both the saddles P_0 and $P_{\pm 1}$ contribute, and so on. These domains depend on the value of A; Fig. 7(b) shows the boundaries of the domains[5] $\Delta(0)$, $\Delta(1)$ and $\Delta(0,1)$ as a function of A on which a subdominant exponential expansion will appear (a Stokes phenomenon).

It is seen that for a finite value of $|\omega/\lambda|$, the domain $\Delta(0)$ encloses the origin (corresponding to positive values of ω and λ). For β and γ in the interior of $\Delta(0)$ only the principal saddle P_0 contributes to the expansion of I_1 and I_2, with the result that the expansions in §3 will hold for complex values of ω and λ in this domain. Outside $\Delta(0)$, the expansions must be modified to take account of an additional contributory saddle point. In particular, as arg λ increases from 0 to π with $\omega > 0$ (i.e., $0 \leq \beta \leq \pi$, $\gamma = 0$) the contributory saddle when $A > 1$ is P_0 throughout this range (and hence is not associated with a Stokes phenomenon), while when $A < 1$ the contributory saddle changes from P_0 to P_1, with an intermediate range in which both P_0 and P_1 contribute; in the special case $A = 1$, the change from P_0 to P_1 is via a double saddle when $\beta = \frac{1}{2}\pi$. In addition, the line $\gamma = \beta$ also corresponds to a Stokes phenomenon for the integral I_1, in which the contribution from the saddle P_0 is maximally subdominant with respect to the algebraic expansion. When $\theta < 0$ (i.e., $\gamma - \beta > 0$) in domain $\Delta(0)$, I_1 is given only by the algebraic expansion in (3.5), while when $\theta > 0$, I_1 contains, in addition, the contribution from the saddle P_0; compare Fig. 2.

From the foregoing discussion, it therefore follows that the expansions (3.1) and (3.2) remain valid when ω/λ is finite and either ω or λ (or both) have imaginary parts of O(1). As mentioned in §1, this is a situation which often arises in physical applications of (1.1).

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[5] Note that when $A > 1$, only the domains $\Delta(0)$ and $\Delta(0,1)$ are present in Fig. 7(b).

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R. B. Paris
Division of Mathematical Sciences
University of Abertay Dundee
Dundee DD1 1HG, UK
Fax: 01382 308877

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