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Partial regularity for minimizers of convex integrals with $L \log L$-growth

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Abstract. We prove $C^{1,\alpha}$-partial regularity of minimizers $u \in W^{1,1}_{loc}(\Omega; \mathbb{R}^N)$, with $\Omega \subset \mathbb{R}^n$, for a class of convex integral functionals with nearly linear growth whose model is

$$\int_\Omega \log(1 + |Du|)|Du| dx$$

In this way we extend to any dimension n a previous, analogous, result in [FS] valid only in the case $n \leq 4$.

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1 Introduction

The aim of this paper is to study regularity properties of minimizers of integral functionals of the calculus of variations,

$$\mathcal{F}(u) = \int_\Omega f(Du)dx, \tag{1.1}$$

with $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N, n \geq 2, N \geq 1$.

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The problem of finding sufficient conditions in order to guarantee regularity of minimizers has been intensively studied. We basically have, in the vectorial case, that if the integrand f is uniformly strictly quasiconvex and such that

$$0 \leq f(z) \leq L(1 + |z|^p), \quad p > 1, \tag{1.2}$$

then any local minimizer of $\mathcal{F}$ turns out to be partially regular, that is there exists an open subset $\Omega_0 \subset \Omega$ such that,

$$u \in C^{1,\alpha}(\Omega_0), \quad |\Omega - \Omega_0| = 0,$$

for some $0 < \alpha \leq 1$ (see [GM], [AF1] and [CFM] for precise statements). Some counterexamples show that in the vectorial case it is not possible to have in general regularity in the interior of Ω for local minimizers of $\mathcal{F}$, even when f is convex.
In the last years the problem of finding regularity properties of minimizers under growth conditions different from the one in (1.2) has been widely investigated (see [M2], [M3] for the scalar case $N = 1$). In particular, in the case of convex integrals a partial regularity result has been found in [AF3] for so called anisotropic functionals, whose model is

$$\int_\Omega \sum_{i \leq n} |D_i u|^{q_i} \; dx,$$

with $1 < q_1 < q_2 < \cdots < q_n$.
More recently, also motivated by problems arising in mathematical physics, more general types of growth have been considered. In particular the so called "nearly linear growth" has been studied, i.e.,

$$\lim_{|\xi| \to \infty} \frac{f(\xi)}{|\xi|} = +\infty, \quad \lim_{|\xi| \to \infty} \frac{f(\xi)}{|\xi|^q} = 0, \tag{1.3}$$

for any $q > 1$. We mention the paper [GIS] where higher integrability of minimizers of Dirichlet problems has been proved for a rather large class of convex functionals satisfying (1.3).
Here the model case is

$$\mathcal{F}(u) = \int_\Omega \log(1 + |Du|)|Du| \; dx. \tag{1.4}$$

In a recent paper (see [FS]) partial regularity of minimizers of $\mathcal{F}$ has been proved under the restrictive hypothesis that $n \leq 4$, leaving open the question for arbitrary dimensions. In this paper we prove partial regularity in any dimension (see Theorem 3.1) for local minimizers of $\mathcal{F}$ also providing suitable structure hypotheses on the energy density f to extend the result to more general cases, (see Section 4). The technique that we use combines some ideas from [EG], [CFM] and [FS]. In particular once we have the higher integrability of minimizers (proved in

[FS], [GIS]) we may use a blow-up procedure suited for functionals with polinomial growth as in (1.2), with p very near to 1 (in particular $p \to 1$ when $n \to +\infty$), finding a suitable decay estimate for the quantity

$$E(x,R) = \fint_{B(x,R)} |V(Du) - V((Du)_{x,R})|^2 dx,$$

where

$$V(Du) = (1+|Du|^2)^{\frac{p-2}{4}} Du.$$

Once proved this decay estimate (see Lemma 3.1), a standard iteration argument implies partial regularity for local minimizers.

2 Preliminaries and notation

In the following Ω will denote an open bounded domain in $\mathbb{R}^n$ and $B(x,R)$ will denote the open ball $\{y \in \mathbb{R}^n : |x-y| < R\}$. If F is an integrable function defined on $B(x,R)$ we will put

$$(F)_{x,R} = \fint_{B(x,R)} F(x)dx = \frac{1}{\omega_n R^n} \int_{B(x,R)} F(x)dx,$$

where ω_n is the Lebesgue measure of $B(0,1)$. We shall also adopt the convention of writing B_R and $(F)_R$ instead of $B(x,R)$ and $(F)_{x,R}$ respectively, when $x = 0$. Finally the letter c will freely denote a constant, not necessarily the same in any two occurencies, while only the relevant dependences will be highlighted.
We are going to deal with the following integral functional,

$$\mathcal{F}(u,\Omega) = \int_\Omega \log(1+|Du|)|Du|dx, \tag{2.1}$$

defined on $W^{1,1}(\Omega;\mathbb{R}^N)$. In order to make precise our variational problem we recall the following definition of minimizer, that in our case takes place:

Definition 2.1. A function $u \in W^{1,1}_{loc}(\Omega;\mathbb{R}^N)$ is a local minimizer of $\mathcal{F}$ iff

$$\log(1+|Du|)|Du| \in L^1_{loc}(\Omega)$$

and

$$\mathcal{F}(u, supp\varphi) \le \mathcal{F}(u+\varphi, supp\varphi),$$

for any $\varphi \in W^{1,1}_0(\Omega;\mathbb{R}^N)$.

Now we recall two results from [FS]. We start with a higher integrability result.

Theorem 2.1. *Let $u \in W^{1,1}_{loc}(\Omega;\mathbb{R}^N)$ be a local minimizers of $\mathcal{F}$, then*

$$\sqrt{1+|Du|} \in W^{1,2}_{loc}(\Omega).$$

Applying Sobolev embedding theorem we immediately have, by previous result, the following:

Corollary 2.1. *Let* $u \in W^{1,1}_{loc}(\Omega;\mathbb{R}^n)$ *be a local minimizer of* $\mathcal{F}$, *then*

$$|Du| \in L^p_{loc}(\Omega;\mathbb{R}^N),$$

with $p = \frac{n}{n-2}$ *if* $n > 2$ *and* $p < \infty$ *if* $n = 2$.

The second result we take from [FS] is the following Caccioppoli type inequality that also holds for local minimizers of $\mathcal{F}$,

Theorem 2.2. *Let* $u \in W^{1,1}_{loc}(\Omega;\mathbb{R}^N)$ *be a local minimizer of* $\mathcal{F}$. *Let* $B(x,R) \subset\subset \Omega$, $A \in \mathbb{R}^{nN}$ *and* $t \in (0,1)$ *then*

$$\int_{B(x,tR)} |D\sqrt{1+|Du|}\,|^2 dx \leq c(t) R^{-2} \int_{B(x,R)} \frac{\log(1+|Du|)}{|Du|} |Du - A|^2 dx.$$

Now we introduce a function that will be very useful below,

$$V_p(\xi) = (1+|\xi|^2)^{\frac{p-2}{4}} \xi, \tag{2.2}$$

for any $\xi \in \mathbb{R}^k$ and $p > 1$.
The following algebraic lemma collects some properties of V_p. For the proof we address the reader to [AF2], [CFM].

Lemma 2.1. *Let* $1 < p < 2$, *and* $V : \mathbb{R}^k \to \mathbb{R}^k$ *defined as in* (2.2), *then for any* $\xi, \eta \in \mathbb{R}^k, t > 0$

(i) $2^{\frac{p-2}{4}} \min\{|\xi|, |\xi|^{\frac{p}{2}}\} \leq |V(\xi)| \min\{|\xi|, |\xi|^{\frac{p}{2}}\}$
(ii) $|V(t\xi)| \leq \max\{t, t^{\frac{p}{2}}\} |V(\xi)|$
(iii) $|V(\xi+\eta)| \leq c(p)[|V(\xi)| + |V(\eta)|]$
(iv) $\frac{p}{2}|\xi - \eta| \leq \frac{|V(\xi)-V(\eta)|}{(1+|\xi|^2+|\eta|^2)^{\frac{p-2}{4}}} \leq c(k,p)|\xi-\eta|$
(v) $|V(\xi) - V(\eta)| \leq c(k,p)|V(\xi-\eta)|$
(vi) $|V(\xi-\eta)| \leq c(p,M)|V(\xi) - V(\eta)|$ *if* $|\eta| \leq M$ *and* $\xi \in \mathbb{R}^k$.

Beside those of V_p, we recall some properties of the growth function,

$$f(\xi) = \log(1+|\xi|)|\xi|,$$

with $\xi \in \mathbb{R}^{nN}$.
Easy computation show that

$$|Df(\xi)| \leq L(1+\log(1+|\xi|)) \tag{2.3}$$

$$\frac{|\lambda|^2}{\sqrt{2}(1+|\xi|^2)^{\frac{1}{2}}} \leq \langle D^2 f(\xi)\lambda, \lambda\rangle \leq 2\, \frac{\log(1+|\xi|)}{|\xi|}|\lambda|^2 \tag{2.4}$$

for any $\xi, \lambda \in \mathbb{R}^{nN}$.

Moreover

$$f \in C^2(\mathbb{R}^{nN}).$$

Finally we observe that for any $p > 1$ we have that

$$f(\xi) \le L|V_p(\xi)|^2$$

with L depending on p.
The following Proposition is concerned with the regularity of weak solutions of elliptic systems with constant coefficients. Althought we shall use it in the case of strongly elliptic systems we state it in the general case where the elliptic condition is a weaker one, namely the Legendre-Hadamard condition (see [CFM] for the proof).

Proposition 2.1. *Let $u \in W^{1,1}(\Omega; \mathbb{R}^N)$ such that*

$$\int_\Omega A^{ij}_{\alpha\beta} D_\alpha u^i D_\beta \phi^j \, dx = 0 \tag{2.5}$$

for any $\phi \in C^1_0(\Omega; \mathbb{R}^N)$, where $(A^{ij}_{\alpha\beta})$ is a constant matrix satisfying the strong Legendre-Hadamard condition,

$$A^{ij}_{\alpha\beta}\lambda^i\lambda^j\mu_\alpha\mu_\beta > \nu|\lambda|^2|\mu|^2 \quad \text{for any} \quad \lambda \in \mathbb{R}^N, \mu \in \mathbb{R}^n.$$

Then u is $C^\infty_{loc}(\Omega; \mathbb{R}^{nN})$ and for any $B_R(x_0) \subset \Omega$ and $0 < \rho < R$

$$\sup_{B_{\frac{R}{2}}} |Du| \le \frac{c}{R^n} \int_{B_R} |Du| dx, \tag{2.6}$$

where c depends only on n, N, p, ν and $\max |A^{ij}_{\alpha\beta}|$.

Remark. The main point in the previous result is that the weak solutions to the system are a priori allowed to lie only in $W^{1,1}(\Omega; \mathbb{R}^n)$ rather than in $W^{1,2}(\Omega; \mathbb{R}^n)$. So it also works for any solution in $W^{1,p}(\Omega; \mathbb{R}^n)$ with $1 \le p$ that is what we shall precisely need later.

We also state a Poincarè-type inequality on increasing spheres proved in [CFM],

Theorem 2.3. *If $1 < p < 2$, there exist $\frac{2}{p} < \alpha < 2$ and $\sigma > 0$ such that if $u \in W^{1,p}(B_{3R}(x_0); \mathbb{R}^N)$, then*

$$\left(\fint_{B(x_0,R)} \left| V\left(\frac{u - (u)_{x_0,R}}{R} \right) \right|^{2(1+\sigma)} dx \right)^{\frac{1}{2(1+\sigma)}} \le c \left(\fint_{B(x_0,3R)} |V(Du)|^\alpha dx \right)^{\frac{1}{\alpha}} \tag{2.7}$$

where $c \equiv c(n, p, N)$ is independent of R and u.

Finally, the following lemma is a straightforward consequence of (2.4) and mean value theorem.

Lemma 2.2. *Let* $A \in \mathbb{R}^{nN}, |A| \leq M, M > 0$ *and* $\lambda > 0$ *and define*

$$f_{A,\lambda}(\xi) = \lambda^{-2}[f(A + \lambda\xi) - f(A) - Df(A)\lambda\xi]$$

it follows that

$$|f_{A,\lambda}(\xi)| \leq L \frac{\log(1 + |\lambda\xi|)}{|\lambda\xi|} |\xi|^2$$

with $L \equiv L(M)$.

3 Blow-up and partial regularity

In this section we are going to prove the following main result of the paper:

Theorem 3.1. *Let* $u \in W^{1,1}_{loc}(\Omega)$ *be a local minimizer of* $\mathcal{F}$. *Then there exists an open set* $\Omega_0 \subset \Omega$ *such that*

$$|\Omega - \Omega_0| = 0 \quad u \in C^{1,\alpha}(\Omega_0; \mathbb{R}^N)$$

for any $0 < \alpha < 1$.

The proof of Theorem 3.1 rests upon a suitable decay estimate for the quantity (excess),

$$E_p(x, R) = \fint_{B(x,R)} |V_p(Du) - V_p((Du)_{x,R})|^2 dx \tag{3.1}$$

where $B_{x,R} \subset\subset \Omega$ and p is a fixed number such that

$$1 < p < \frac{n}{n-2}$$

if $n > 2$ and

$$1 < p < 2$$

if $n = 2$, while V_p is the function defined in (2.2).

The quantity E_p, roughly speaking, provides an integral measure of the oscillations of the gradient Du in a ball $B(x, R)$. The use of this particular quantity, with the above suitable choice of p, is the main technical trick of the proof. In fact we remark that the higher integrability stated in Corollary 2.1 allows to give sense to $E_p(x, R)$ when $p < \frac{n}{n-2}$ and therefore we may use a blow-up technique similar to the one used for functionals with p-growth, when $p < 2$. This fact is exploited in the proof of Lemma 3.1 below where a suitable, power like, decay estimate is provided for E_p. Finally let us mention that we will combine ideas from [EG], [CFM] and [FS] in order to overcome the technical difficulty that consist of

adapting a blow-up technique suited for functionals with p-growth to our case in which the growth is nearly linear.
For the proof of the following crucial lemma we will rely on the standard blow-up technique.

Lemma 3.1. *Let us fix $M > 0$, there exists a constant C_M such that for every $0 < \tau < \frac{1}{4}$ there exists an $\varepsilon \equiv \varepsilon(\tau, M)$ such that, if*

$$|(Du)_{x_0,R}| \le M \quad \textit{and} \quad E(x_0, R) < \varepsilon \tag{3.2}$$

then

$$E(x_0, \tau R) \le C_M \tau^2 E(x_0, R) \tag{3.3}$$

Proof. We preliminarly fix M and τ while C_M will be chosen later. We divide up the proof in five steps. In the following we shall omit the subscript p from V_p and E_p, that will be simply denoted by V and E, respectively.

Step 1. (Blow-up).

We argue by contradiction assuming that there exists a sequence of balls $B(x_m, R_m) \subset\subset \Omega$ such that

$$|(Du)_{x_m,R_m}| \le M \quad \lim_m E(x_m, R_m) = 0 \tag{3.4}$$

and

$$E(x_m, \tau R_m) > C_M \tau^2 E(x_m, R_m). \tag{3.5}$$

We put

$$a_m = (u)_{x_m,R_m} \quad A_m = (Du)_{x_m,R_m} \quad \lambda_m^2 = E(x_m, R_m) \tag{3.6}$$

and rescale the function u in each ball $B(x_m, R_m)$ in order to obtain a sequence of functions all defined in the same ball B_1,

$$v_m(y) = \frac{1}{\lambda_m R_m}[u(x_m + R_m y) - a_m - R_m A_m y] \tag{3.7}$$

Clearly we have

$$\begin{aligned} Dv_m(y) &= \frac{1}{\lambda_m}[Du(x_m + R_m y) - A_m] \\ (v_m)_{0,1} &= 0 \quad (Dv_m)_{0,1} = 0 \end{aligned}$$

Moreover from (ii) and (vi) of Lemma 2.1 we have that

$$\begin{aligned} \fint_{B(0,1)} |V(Dv_m(y))|^2 dy &= \fint_{B(x_m,R_m)} \left| V\left(\frac{Du(x) - A_m}{\lambda_m}\right)\right|^2 dx \\ &\le \frac{\tilde{c}(M)}{\lambda_m^2} \fint_{B(x_m,R_m)} |V(Du(x)) - V(A_m)|^2 dx = \tilde{c} \end{aligned}$$

Hence from (i) of Lemma 2.1 we may conclude that the sequence (Dv_m) is bounded in $L^p(B(0,1);\mathbb{R}^{nN})$,

$$\|Dv_m\|_{L^p(B_1)} \leq c \quad \text{for any} \quad m \in \mathbb{N}$$

and so up to (not relabelled) subsequences we may assume that

$$\begin{aligned} v_m &\longrightarrow v \quad \text{strongly in } L^p(B_1;\mathbb{R}^{nN}) \\ Dv_m &\rightharpoonup Dv \quad \text{weakly in } L^p(B_1;\mathbb{R}^{nN}) \\ \lambda_m Dv_m &\longrightarrow 0 \quad \text{strongly in } L^p(B_1;\mathbb{R}^{nN}) \\ A_m &\longrightarrow A \quad \text{in} \quad \mathbb{R}^{nN} \end{aligned} \tag{3.8}$$

Step 2. **(v solves a linear system.)**

From the Euler system for u, rescaled in each ball $B(x_m, R_m)$, we deduce that:

$$\int_{B_1} \langle Df(A_m + \lambda_m Dv_m), D\phi\rangle dx = 0$$

for any $\phi \in C_0^\infty(B_1;\mathbb{R}^{nN})$, and also

$$\frac{1}{\lambda_m}\int_{B_1} \langle Df(A_m + \lambda_m Dv_m) - Df(A_m), D\phi\rangle dx = 0. \tag{3.9}$$

Now we linearize about A_m thus obtaining

$$\begin{aligned} 0 &= \int_{B_1}\int_0^1 \langle D^2 f(A_m + s\lambda_m Dv_m)Dv_m, D\phi\rangle \; ds \; dx \\ &= \int_{B_1}\int_0^1 \langle (D^2 f(A_m + s\lambda_m Dv_m) - D^2 f(A_m))Dv_m, D\phi\rangle \; ds \; dx \\ &\quad + \int_{B_1} \langle D^2 f(A_m)Dv_m, D\phi\rangle \; dx = (I)_m + (II)_m \end{aligned} \tag{3.10}$$

From $(3.8)_2$ we obtain

$$\lim_m \, (II)_m = \int_{B_1} \langle D^2 f(A)Dv, D\phi\rangle \; dx \tag{3.11}$$

In order to deal with $(I)_m$ we fix $\delta > 0$ and by $(3.8)_3$ and Egorov's theorem we find $S \subset B_1$ such that

$$|B_1 - S| < \delta \quad \text{and} \quad \lambda_m Dv_m \longrightarrow 0 \text{ uniformly in } S \tag{3.12}$$

then we have that

$$\begin{aligned} |(I)_m| &\leq \int_S \int_0^1 |D^2 f(A_m + s\lambda_m Dv_m) - D^2 f(A_m)||Dv_m||D\phi| \; ds \; dx \\ &\quad + \int_{B_1 - S} \int_0^1 |D^2 f(A_m + s\lambda_m Dv_m) - D^2 f(A_m)||Dv_m||D\phi| \; ds \; dx \\ &= (III)_m + (IV)_m. \end{aligned}$$

From the boundedness of Dv_m in L^p it easily follows that

$$\lim_m \, (III)_m = 0 \tag{3.13}$$

while the remaining term is easily estimated as follows

$$\begin{aligned} |(IV)_m| &\leq c||D\phi||_{L^\infty} \int_{B_1 - S} \left(\frac{\log(1 + |A_m + \lambda_m Dv_m|)}{|A_m + \lambda_m Dv_m|} + 1 \right) |Dv_m| dx \\ &\leq c||D\phi||_{L^\infty} \int_{B_1 - S} |Dv_m| dx \\ &\leq c||D\phi||_{L^\infty(B_1)} ||Dv_m||_{L^p} |B_1 - S|^{\frac{p-1}{p}} \\ &\leq c\delta^{\frac{p-1}{p}} \end{aligned} \tag{3.14}$$

Connecting (3.10)–(3.14) and letting first $m \to \infty$ and then $\delta \to 0$ we finally obtain

$$\int_{B_1} \langle D^2 f(A) Dv, D\phi \rangle dx = 0$$

for any $\phi \in C_0^\infty(B_1; \mathbb{R}^N)$
By (2.4) the matrix $D^2 f(A)$ satisfies the following ellipticity condition

$$c^{-1}|\lambda|^2 \leq \langle D^2 f(A)\lambda, \lambda \rangle \leq c|\lambda|^2$$

for a suitable constant $c \equiv c(M) > 0$.
So by Proposition 2.1 it follows that $v \in C^\infty_{loc}(B_1; \mathbb{R}^N)$. Moreover from the theory of elliptic systems (see [G], Theorem 2.1, Chap. 3) and still by the estimate in Proposition 2.1 we get that if $0 < \tau < \frac{1}{2}$

$$\begin{aligned} \fint_{B_\tau} |Dv - (Dv)_\tau|^2 dy &\leq c(M)\tau^2 \fint_{B_{\frac{1}{2}}} |Dv - (Dv)_{\frac{1}{2}}|^2 dx \\ &\leq c(M)\tau^2 \sup_{B_{\frac{1}{2}}} |Dv|^2 \\ &\leq c(M)\tau^2 \left(\fint_{B_1} |Dv|^p dx \right)^{\frac{2}{p}} \\ &\leq c^*(M)\tau^2 \end{aligned} \tag{3.15}$$

Step 3. (Upper bound).

We introduce the following notation

$$f_m(\xi) = f_{A_m,\lambda_m}(\xi) = \frac{f(A_m + \lambda_m \xi) - f(A_m) - \lambda_m Df(A_m)\xi}{\lambda_m^2},$$

for any $\xi \in \mathbb{R}^{nN}$ and we define the following functionals

$$I_r^m(w) = \int_{B_r} f_m(Dw)\, dx,$$

for $w \in W^{1,1}_{loc}(B_1;\mathbb{R}^{nN})$ and $r < \frac{1}{3}$. With this notation the function v_m turns out to be a local minimizer of I_r^m for each m.

Here we want to prove that actually

$$\limsup_m [I_r^m(v_m) - I_r^m(v)] \leq 0 \tag{3.16}$$

for a.e. $r \in [0, \frac{1}{3}]$.
We choose $s < r$ and take $\eta \in C_0^\infty(B_r)$ such that $0 \leq \eta \leq 1$ and

$$\eta \equiv 1 \text{ on } B_s \quad |D\eta| \leq \frac{c}{r-s}$$

and we test the minimality of v_m with the test function $\phi_m = (v - v_m)\eta$. We obtain, using Lemma 2.2

$$\begin{aligned} I_r^m(v_m) - I_r^m(v) &\leq I_r^m(v_m + \phi_m) - I_r^m(v) \\ &= \int_{B_r - B_s} [f_m(Dv_m + D\phi_m) - f_m(Dv)]dx \leq \frac{c}{\lambda_m^2} \int_{B_r - B_s} [f(\lambda_m Dv) \\ &\quad + f(\lambda_m(v - v_m)D\eta + \lambda_m \eta Dv + \lambda_m(1-\eta)Dv_m)]dy \\ &\leq \frac{c}{\lambda_m^2} \int_{B_r - B_s} [f(\lambda_m Dv) + f(\lambda_m Dv_m) + f(\lambda_m(v - v_m)D\eta)]dy \end{aligned} \tag{3.17}$$

Now we define the following family of positive Radon measures on $B_{\frac{1}{3}}$

$$\mu^m(S) = \int_S \left[\frac{f(\lambda_m Dv_m)}{\lambda_m^2} + \frac{f(\lambda_m Dv)}{\lambda_m^2} \right] dx$$

whenever $S \subset B_{\frac{1}{3}}$. We observe that

$$\begin{aligned} \mu^m(B_{\frac{1}{3}}) &= \int_{B_{\frac{1}{3}}} \left[\frac{f(\lambda_m Dv_m)}{\lambda_m^2} + \frac{f(\lambda_m Dv)}{\lambda_m^2} \right] dx \\ &\leq c \left[\int_{B_{\frac{1}{3}}} \frac{|V(\lambda_m Dv_m)|^2}{\lambda_m^2} dx + 1 \right] \\ &\leq c(M) \end{aligned}$$

Where we used the smoothness of v and the definition of λ_m.

So we have that $\|\mu^m\|_{BV}$ is bounded uniformly with respect to m and so, up to (not relabelled) subsequence, we may suppose that

$$\mu^m \rightharpoonup \mu \text{ weakly in the sense of measures.}$$

Furthermore the set $\{t \in [0, \frac{1}{3}]\}$ such that

$$\mu(\partial B_t) \neq 0$$

is at most countable and so we may always assume that $r \in [0, \frac{1}{3}]$ is such that

$$\mu(\partial B_r) = 0 \tag{3.18}$$

Moreover, eventually passing to a (not relabelled) subsequence we may always suppose that

$$\lim_m [I_r^m(v_m) - I_r^m(v)]$$

exists.
Now we consider the last quantity appearing in (3.17). If $\frac{1}{2} = \theta + \frac{1-\theta}{2(1+\sigma)}$ with σ being as in Theorem 2.3, we may estimate, interpolating:

$$\begin{aligned}
\frac{1}{\lambda_m^2}\int_{B_r - B_s} f(\lambda_m(v - v_m)D\eta)dx &\leq \frac{c}{\lambda_m^2}\int_{B_r - B_s} |V(\lambda_m(v - v_m)D\eta)|^2 dx \\
&\leq \frac{c}{\lambda_m^2(r-s)^2}\int_{B_r - B_s} |V(\lambda_m(v - v_m))|^2 dx \\
&\leq \frac{c}{\lambda_m^2(r-s)^2}\left(\int_{B_r - B_s} |V(\lambda_m(v - v_m))| dx\right)^{2\theta} \\
&\quad \left(\int_{B_r - B_s} |V(\lambda_m(v - v_m))|^{2(1+\sigma)} dx\right)^{\frac{1-\theta}{1+\sigma}} \\
&\leq c\frac{\lambda_m^{2\theta}}{\lambda_m^2(r-s)^2}\left(\int_{B_r - B_s} |v - v_m| dx\right)^{2\theta} \\
&\quad \times \Bigg(\int_{B_r - B_s} |V(\lambda_m(v - v_m) - \lambda_m(v - v_m)_{0,\frac{1}{3}})|^{2(1+\sigma)} \\
&\quad + |V(\lambda_m(v - v_m)_{0,\frac{1}{3}})|^{2(1+\sigma)} dx\Bigg)^{\frac{1-\theta}{1+\sigma}} \\
&\leq c\frac{\lambda_m^{2\theta}}{\lambda_m^2(r-s)^2}\left(\int_{B_r - B_s} |v - v_m| dx\right)^{2\theta}\left[\left(\int_{B_1} |V(\lambda_m D v_m)|^2 dx\right)^{1-\theta}\right. \\
&\quad \left. + \lambda_m^{2(1-\theta)}\right] \leq \frac{c}{(r-s)^2}\left(\int_{B_1} |v - v_m| dx\right)^{2\theta}.
\end{aligned} \tag{3.19}$$

Where we used (ii) from Lemma 2.1, Poincaré inequality from Theorem 2.3 and once again the estimate

$$\fint_{B_1} \frac{|V(\lambda_m Dv_m)|^2}{\lambda_m^2} dx \leq c.$$

Collecting (3.17)–(3.19) we have,

$$I_r^m(v_m) - I_r^m(v) \leq c \left[\mu^m(B_r - B_s) + \frac{1}{(r-s)^2} \left(\int_{B_1} |v - v_m| dx \right)^{2\theta} \right],$$

and letting $m \to \infty$ we have

$$\lim_m [I_r^m(v_m) - I_r^m(v)] \leq \mu(B_r - B_s)$$

and the assertion in (3.16) (with lim sup replaced by lim) follows by letting $s \uparrow r$ in view of (3.18).

Step 4. (Lower bound).

Now we are going to prove that:

$$\limsup_m \frac{1}{\lambda_m^2} \int_{B_r} |V(\lambda_m(Dv_m - Dv))|^2 dx = 0 \tag{3.20}$$

for a.e. $r < \frac{1}{3}$.

We fix $0 < r < \frac{1}{3}$ such that (3.16) holds; if $w_m = v_m - v$ then we have:

$$\begin{aligned}
&I_r^m(v_m) - I_r^m(v) \\
&= \lambda_m^{-2} \int_{B_r} f(A_m + \lambda_m Dv_m) - f(A_m + \lambda_m Dv) - \lambda_m \langle Df(A_m), Dw_m \rangle dx \\
&= \lambda_m^{-1} \int_{B_r} \int_0^1 \langle Df(A_m + \lambda_m Dv + t\lambda_m Dw_m) - Df(A_m), Dw_m \rangle dt dx \\
&= \lambda_m^{-1} \int_{B_r} \int_0^1 \langle Df(A_m + \lambda_m Dv + t\lambda_m Dw_m) - Df(A_m + \lambda_m Dv), Dw_m \rangle dt dx \\
&\quad + \lambda_m^{-1} \int_{B_r} \langle Df(A_m + \lambda_m Dv) - Df(A_m), Dw_m \rangle dx = (I)_m + (II)_m
\end{aligned} \tag{3.21}$$

We further linearize and obtain

$$(II)_m = \int_{B_r} \int_0^1 \langle D^2 f(A_m + t\lambda_m Dv) Dv, Dw_m \rangle dt dx.$$

We have that v is smooth so that

$$D^2 f(A_m + t\lambda_m Dv) \longrightarrow D^2 f(A)$$

uniformly and by $(3.8)_2$ we get that

$$\lim_m (II)_m = 0 \tag{3.22}$$

In order to estimate $(I)_m$ from below we observe that, by (2.4)

$$\begin{aligned}(I)_m &= \int_{B_r}\int_0^1\int_0^1 \langle tD^2 f(A_m+\lambda_m Dv + st\lambda_m Dw_m)Dw_m, Dw_m\rangle ds\ dt\ dx \\ &\geq \int_{B_r}\int_0^1\int_0^1 t(1+|A_m+\lambda_m Dv+st\lambda_m Dw_m|^2)^{-\frac{1}{2}}|Dw_m|^2 ds\ dt\ dx.\end{aligned} \tag{3.23}$$

Now from (3.16), (3.21), (3.22) and (3.23) we have that

$$\limsup_m \int_{B_r}\int_0^1\int_0^1 t(1+|A_m+\lambda_m Dv+st\lambda_m Dw_m|^2)^{-\frac{1}{2}}|Dw_m|^2 ds\ dt\ dx = 0,$$

so that if we pick $L>0$ it follows that

$$\limsup_m \int_{\{x:\lambda_m|Dw_m|<L\}\cap B_r} |Dw_m|^2 dx = 0. \tag{3.24}$$

Now we want to prove that

$$\limsup_m \frac{1}{\lambda_m^2}\int_{\{x:\lambda_m|Dw_m|>L\}\cap B_r} |V(\lambda_m Dw_m)|^2 dx = 0 \tag{3.25}$$

provided L is chosen large enough (but independently of m).
In order to prove (3.25) we consider the following sequence of functions,

$$\varphi_m = \frac{1}{\lambda_m}[\sqrt{1+|\lambda_m Dv_m + A_m|} - \sqrt{1+|A_m|}].$$

Rescaling the formula appearing in the statement of Theorem 2.2 in the ball B_1, we may estimate

$$\begin{aligned}\int_{B_r}|D\varphi_m|^2 dx &\leq c(r)\int_{B_1}\frac{\log(1+|A_m+\lambda_m Dv_m|)}{|A_m+\lambda_m Dv_m|}|Dv_m|^2 dx \\ &\leq c(r,M)\int_{B_1}\frac{\log(1+|\lambda_m Dv_m|)}{|\lambda_m Dv_m|}|Dv_m|^2 dx \\ &\leq c(r,M)\int_{B_1}\frac{f(\lambda_m Dv_m)}{\lambda_m^2}dx \\ &\leq c(r,M)\int_{B_1}\frac{|V(\lambda_m Dv_m)|^2}{\lambda_m^2}dx \\ &\leq c(r,M)\end{aligned} \tag{3.26}$$

So using the definition of φ_m and Sobolev embedding theorem it follows that

$$\|\varphi_m\|_{L^{2q}(B_r)} \leq c(r, M).$$

where

$$q = \frac{n}{n-2}$$

if $n > 2$ and $p < q$ if $n = 2$.
On the other hand recalling that v is smooth, choosing $L \equiv L(M)$ large enough we also have that

$$\varphi_m \geq \frac{1}{2\lambda_m}\sqrt{\lambda_m|Dv_m|}$$

on $\{x : \lambda_m|Dw_m| > L\}$.
So we have, still with $L \equiv L(M)$ large enough,

$$\int_{\{x:\lambda_m|Dw_m|>L\}\cap B_r} |Dv_m|^q dx \leq c\lambda_m^q \int_{B_r} |\varphi_m|^{2q} dx \leq c(r, M)\lambda_m^q$$

and so, by Hölder inequality

$$\begin{aligned}
&\int_{\{x:\lambda_m|Dw_m|>L\}\cap B_r} \frac{|V(\lambda_m Dv_m)|^2}{\lambda_m^2} dx \\
&\leq c(M)\lambda_m^{p-2} \int_{\{x:|\lambda_m Dw_m|>L\}\cap B_r} |Dv_m|^p dx \\
&\leq c(M)\lambda_m^{p-2} \left(\int_{\{x:|\lambda_m Dw_m|>L\}\cap B_r} |Dv_m|^q dx\right)^{\frac{p}{q}} \leq c(L, M)\lambda_m^{2p-2} \longrightarrow 0, \quad (3.27)
\end{aligned}$$

by the fact that $p > 1$. On the other hand v is smooth and so

$$\begin{aligned}
&\frac{1}{\lambda_m^2}\int_{\{x:\lambda_m|Dw_m|>L\}\cap B_r} |V(\lambda_m Dv)|^2 dx \leq c(M)|\{x : \lambda_m|Dw_m| > L\} \cap B_r| \\
&\quad \leq \frac{c(M)}{L}\lambda_m \int_{B_R} |Dw_m| dx \leq c(M)\lambda_m \qquad (3.28)
\end{aligned}$$

Finally we have

$$\begin{aligned}
\frac{1}{\lambda_m^2}\int_{B_r} |V(\lambda_m Dw_m)|^2 dx \;&=\; \frac{1}{\lambda_m^2}\int_{\{x:\lambda_m|Dw_m|>L\}\cap B_r} |V(\lambda_m Dw_m)|^2 dx \\
&\quad + \frac{1}{\lambda_m^2}\int_{\{x:\lambda_m|Dw_m|<L\}\cap B_r} |V(\lambda_m Dw_m)|^2 dx
\end{aligned}$$

$$\leq \frac{1}{\lambda_m^2}\int_{\{x:\lambda_m|Dw_m|>L\}\cap B_r} |V(\lambda_m Dv)|^2 + |V(\lambda_m Dv_m)|^2 dx + c \fint_{\{x:\lambda_m|Dw_m|<L\}\cap B_r} |Dw_m|^2 dx \tag{3.29}$$

And by (3.26)–(3.28) all the quantities on the right hand side of (3.29) tend to 0 and (3.20) is proved.

Step 5. (Conclusion).

By step 4 we may suppose that, up to (not relabelled) subsequences:

$$\lim_m \frac{1}{\lambda_m^2}\int_{B_\tau} |V(\lambda_m(Dv_m - Dv))|^2 dx = 0$$

with $\tau < \frac{1}{4}$.
Now we fix $0 < \tau < \frac{1}{4}$ and using the previous formula, (iii), (iv) from Lemma 2.1 and (3.15) from Step 2, we get:

$$\begin{aligned}
\lim_m \frac{E(x_m, \tau R_m)}{\lambda_m^2} &= \lim_m \frac{1}{\lambda_m^2}\fint_{B(x_m,\tau R_m)} |V(Du) - V((Du)_{x_m,\tau R_m})|^2 dx \\
&\leq \lim_m \frac{c}{\lambda_m^2}\fint_{B(x_m,\tau R_m)} |V(Du - (Du)_{x_m,\tau R_m})|^2 dx \\
&= \lim_m \frac{c}{\lambda_m^2}\fint_{B_\tau} |V(\lambda_m(Dv_m - (Dv_m)_\tau))|^2 dx \\
&\leq \lim_m \frac{c}{\lambda_m^2}\fint_{B_\tau} [|V(\lambda_m(Dv_m - Dv))|^2 + |V(\lambda_m(Dv - (Dv)_\tau))|^2 \\
&\quad + |V(\lambda_m((Dv)_\tau - (Dv_m)_\tau))|^2] dx \\
&\leq C^*(M)[\tau^2 + \lim_m |(Du)_\tau - (Dv_m)_\tau|^2] = C^*(M)\tau^2,
\end{aligned}$$

since $Dv_m \rightharpoonup Dv$ in $L^p(B_1)$.

The contradiction (to (3.5)) now follows choosing in (3.5) $C_M = 2C^*(M)$ □

Proof of Theorem 3.1. Following the same arguments used in [FH], from decay estimate proved in Lemma 3.1 one can easily obtain that for any $M > 0$ there exist $0 < \tau < \frac{1}{4}$ and $\sigma > 0$ such that if

$$|(Du)_{x_0,R}| \leq M \quad \text{and } E_p(x_0, R) < \sigma \tag{3.30}$$

then for $k \in \mathbb{N}$

$$\begin{aligned} E_p(x_0, \tau^k R) &\leq C(M)\tau^{2k} E_p(x_0, R) \\ |(Du)_{x_0,\tau^k R}| &\leq 2M \end{aligned}$$

From the last estimate then one gets that if (3.30) hold, for any $0 < \rho < R$ we have that

$$|(Du)_{x_0,\rho}| \leq C(M) \quad \text{and } E_p(x_0,\rho) \leq C(M)\left(\frac{\rho}{R}\right)^2 E_p(x_0,R)$$

therefore by Lemma 2.1 we get that

$$\begin{aligned} &\fint_{B(x_0,\rho)} |Du - (Du)_{x_0,\rho}| dx \\ &\quad \leq \fint_{B(x_0,\rho)\cap\{x:|Du-(Du)_{x_0,\rho}|\leq 1\}} |Du - (Du)_{x_0,\rho}| dx \\ &\qquad + \int_{B(x,\rho)\cap\{x:|Du-(Du)_{x_0,\rho}|\geq 1\}} |Du - (Du)_{x_0,\rho}| dx \\ &\quad \leq c \fint_{B(x_0,\rho)} |V(Du - (Du)_{x_0,\rho})| dx + c\left(\int_{B(x,\rho)} |V(Du - (Du)_{x_0,R})|^2 dx\right)^{\frac{1}{p}} \\ &\quad \leq C(M)\left[\fint_{B(x_0,\rho)} |V(Du) - V((Du)_{x_0,\rho})| dx \right. \\ &\qquad \left. + \left(\int_{B(x_0,\rho)} |V(Du) - V((Du)_{x_0,\rho})|^2 dx\right)^{\frac{1}{p}}\right] \\ &\quad \leq C(M)[E_p^{\frac{1}{2}}(x_0,\rho) + E_p^{\frac{1}{p}}(x_0,\rho)] \leq C(M,R)\rho \end{aligned}$$

From this estimate it follows that if we set

$$\Omega_0 = \left\{x \in \Omega : \sup_{\rho>0} |(Du)_{x,\rho}| < \infty \text{ and } \lim_{\rho\to 0} E_p(x,\rho) = 0\right\}$$

then $u \in C^{1,\alpha}(\Omega_0)$ for any $0 < \alpha < 1$ and by the fact that $Du \in L^p_{loc}(\Omega)$ (Corollary 2.1) we also have that $|\Omega - \Omega_0| = 0$

4 Possible extensions

In this section we briefly comment on the hypotheses needed in order to extend the partial regularity result of Theorem 3.1 to a larger class of functionals with nearly linear growth. A crucial point in the proof of Theorem 3.1 is the possibility of adapting the blow-up argument of Lemma 3.1, previously used to prove partial regularity in the case of integrals with subquadratic growth (see [CFM]), to the case in which the integrand has nearly linear growth. This can be done using the higher integrability result of Corollary 2.1 and the Caccioppoli type inequality in Theorem 2.2. In this case (look at the proofs in [FS]) higher integrability of local minimizers as in Theorem 2.1 and Corollary 2.2 is a direct consequence of the (uniform) convexity of the function

$$\log(1+|\xi|)|\xi|$$

So in order to extend the result to more general functionals of the type

$$\int_\Omega f(Du)dx,$$

we make the following hypotheses on the energy density f,

$$\begin{gathered} f \in C^2(\mathbb{R}^{nN}) \\ \log(1+|\xi|)|\xi| \leq f(\xi) \leq L(\log(1+|\xi|)|\xi|+1) \\ \frac{c_1}{(1+|\xi|^2)^{\frac{1}{2}}}|\lambda|^2 \leq \langle D^2 f(\xi)\lambda,\lambda\rangle \leq c_2\frac{\log(1+|\xi|)}{|\xi|}|\lambda|^2 \end{gathered} \tag{4.1}$$

for any $\xi, \lambda \in \mathbb{R}^{nN}, c_2 > c_1 > 0, L > 0$.
In particular $(4.1)_3$ implies (strict and uniform) convexity. We remark that assuming (4.1) no other growth hypothesis is needed. In particular it is easy to see, using the same argument of [M1] (see also [M2] Lemma 2.1), that

$$|Df(\xi)| \leq L(\log(1+|\xi|)+1).$$

With the hypotheses stated above both the argument developed in [FS] to obtain Theorems 2.1 and 2.2 of this paper and the one developed here in Lemma 3.1 still work and partial regularity of local minimizers holds.
Finally we mention that very recently, some attention has been paid to functionals with nearly linear growth such as

$$\int_\Omega \log\log(\rho+|Du|)|Du|dx$$

or

$$\int_\Omega \log^{1+\delta}(1+|Du|)|Du|dx$$

with $\delta > 0$, and more generally

$$\int_\Omega G(|Du|)dx$$

where

$$G(t) = \int_0^t A(s)ds$$

with $A(s) > 0$ being an increasing function satisfying suitable hypotheses (see [GIS]). We think that methods developed here and in [FS] could be employed to obtain partial regularity theorems for some classes of such functionals, too.

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