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A regularity class for the Navier-Stokes equations in Lorentz spaces

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To Tosio Kato, in memoriam

Abstract. We extend Serrin's regularity class for weak solutions of the Navier-Stokes equations to a larger class replacing the Lebesgue spaces by Lorentz spaces.

1. Introduction and main result

Throughout this paper, Ω means the whole space $\mathbb{R}^3$, a bounded domain or an exterior domain in $\mathbb{R}^3$ with smooth (i.e. $C^\infty-$) boundary $\partial\Omega$. The time interval $[0, T)$ is given with $0 < T < \infty$. We assume that Ω is filled with a viscous incompressible fluid which can be described by the Navier-Stokes equations

$$u_t - \Delta u + u \cdot \nabla u + \nabla p = f, \tag{1.1}$$

$$\operatorname{div} u = 0. \tag{1.2}$$

Further we require the boundary condition

$$u|_{\partial\Omega} = 0 \tag{1.3}$$

if $\partial\Omega \neq \emptyset$, and the initial condition

$$u(0, \cdot) = u_0 \tag{1.4}$$

at time $t = 0$. Here $f = (f_1(t, x), f_2(t, x), f_3(t, x))$, $t \in [0, T)$, $x = (x_1, x_2, x_3) \in \Omega$, denotes the given external force. The initial value u_0 is given in the space $L^2_\sigma(\Omega)$, $u = (u_1(t, x), u_2(t, x), u_3(t, x))$ means the unknown velocity field of the fluid, and the scalar $p = p(t, x)$ means the unknown pressure. In this introduction we assume that f is a smooth function in the sense that $f \in C_0^\infty([0, T) \times \overline{\Omega})^3$, see Section 2 for notations.

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The physical background of the equation (1.1) is the validity of Newton's law. We write $u_t = du/dt$ for the time derivative and use the following notations:

$$\begin{aligned}
\Delta u &= D_1^2 u + D_2^2 u + D_3^2 u, \\
u \cdot \nabla u &= (u \cdot \nabla) u = u_1 D_1 u + u_2 D_2 u + u_3 D_3 u, \\
\nabla p &= (D_1 p, D_2 p, D_3 p)
\end{aligned}$$

with $D_j = \partial / \partial x_j$, $j = 1, 2, 3$, $x \in \Omega$.

The equation $\operatorname{div} u = \nabla \cdot u = D_1 u_1 + D_2 u_2 + D_3 u_3 = 0$ means that the fluid is incompressible.

Up to now it is an open problem whether the system (1.1)–(1.4) has a uniquely determined regular solution pair (u, p) in the classical sense. We only know certain partial results. For example, we know the existence of a classical solution pair in some initial interval $(0, T^\star)$, $0 < T^\star \le T$, depending on the "size" of the data f, u_0. Further we know the existence of a weak solution u in the whole interval $(0, T)$, see Definition 2.1 below. But we do not know whether such a weak solution is unique or regular. In this case, (1.1) is only satisfied in the sense of distributions together with some p. Concerning this problem we refer to [Hop51], [KiLa57], [FuKa64], [Lad69], [Sol77], [Tem79], [Hey80], [Miy81], [Mas84], [Kato84], [vWa85], [GiMi85], [Giga86], [Can95], [KoYa95], [Wie99], [Ama00].

In this situation it is our aim to improve such partial solvability results at least step by step. One aspect is to consider additional conditions under which a weak solution u is a regular smooth function. Each such condition determines a regularity class containing regular weak solutions. The other possibility is, using the structure of these classes, to try to find a counter example of a non-smooth weak solution.

The first important regularity class is due to Serrin [Ser63]: If a weak solution u satisfies the condition

$$u \in L^s(0, T; L^q) \quad \text{with} \quad 2 < s < \infty, \;\; 3 < q < \infty, \;\; \frac{3}{q} + \frac{2}{s} < 1, \tag{1.5}$$

then u is regular in $(0, T)$. This condition means that

$$\|u\|_{L^s(0,T;L^q)} = \left(\int_0^T \|u(t)\|_q^s \, dt \right)^{\frac{1}{s}} < \infty \tag{1.6}$$

with $\|u(t)\|_q = (\int_\Omega |u(t, x)|^q \, dx)^{\frac{1}{q}}$.

The next step was a "slight" improvement. It could be shown that $\frac{3}{q} + \frac{2}{s} < 1$ in (1.5) can be replaced by the weaker condition $\frac{3}{q} + \frac{2}{s} \le 1$, see [Soh83], [Giga86]. The case $\frac{3}{q} + \frac{2}{s} = 1$ yields the largest class. Thus we may restrict ourselves now to this case and obtain the following more general criterion: If a weak solution u satisfies the condition

$$u \in L^s(0, T; L^q) \text{ with } 2 < s < \infty, \ 3 < q < \infty, \ \frac{3}{q} + \frac{2}{s} = 1 \tag{1.7}$$

then u is regular in $(0, T)$.

There are several further regularity results. Da Veiga [BdV95] proved that the condition

$$u \in L^s(0, T; H^{1,q}) \text{ with } 1 < s < \infty, \ \frac{3}{2} < q < \infty, \ \frac{3}{q} + \frac{2}{s} = 2 \tag{1.8}$$

is sufficient for the regularity of a weak solution u. Da Veiga's class, defined by (1.8), is contained in Serrin's class only for special values s, q. We obtain a new class in several cases, e.g. if $s = 2, \ q = 3$, see [BdV95]. Here $H^{1,q}$ means the usual Sobolev space.

The case $q = 3, \ s = \infty$ is excluded in (1.7). In this critical case we only know that certain subspaces of $L^\infty(0, T; L^3)$ are regularity classes. The first result in this direction is due to von Wahl [vWa85]. He showed that the space $C(0, T; L^3)$ of continuous L^3-valued functions is a regularity class. In [KoSo97] it was shown that the space $BV(0, T; L^3)$ of L^3-valued functions of bounded variation is a regularity class. Further regularity classes within $L^\infty(0, T; L^3)$ are given in [KoSo97], [BdV97]. See [KoSo96] concerning a uniqueness result in this space.

Moreover, the regularity of a weak solution $u \in L^\infty(0, T; L^3)$ can be shown under an additional smallness condition on the norm of u in this space, see [Soh83], [Stru88].

The last result has been improved by Kozono [Kozo01]. He showed that the space L^3 in the last condition can be replaced by the (larger) Lorentz space $L^{3,\infty}$, see below. The use of this Lorentz space was an interesting aspect and a motivation to investigate more general Lorentz spaces in this context. See [BoMi95], [KoYa98], [Bers00] concerning other results in Lorentz spaces.

We mention some further results. In (1.7) it is possible to include the case $s = 2, \ q = \infty$. Moreover, Kozono and Tamiuchi [KoTa00] showed for the $\mathbb{R}^3$ that even the larger space $L^2(0, T; BMO) \supset L^2(0, T; L^\infty)$ is a regularity class.

Neustupa [Neu99] proved a result on the structure of singularities of weak solutions in $L^\infty(0, T; L^3)$. Neustupa, Novotny and Penel [NeNP99] proved a regularity criterion which requires (1.7) only for one of the three components of u, but with $\frac{3}{q} + \frac{2}{s} = 1$ replaced by the stronger condition $\frac{3}{q} + \frac{2}{s} = \frac{1}{2}$.

The main result of this paper is contained in the following theorems. Our aim is to extend Serrin's regularity criterion by introducing Lorentz spaces in both time and spatial direction. However, we need a restriction concerning the class of admitted weak solutions. We only consider weak solutions u which satisfy the energy inequality in the strong form (2.16), see Section 2. This restriction is not essential. We replace in Serrin's condition (1.7) the space L^s in time direction by any Lorentz space $L^{s,r}$, $s \le r \le \infty$, which is stricly larger than L^s if $s < r \le \infty$. Further we replace $L^q = L^q(\Omega)^3$ in spatial direction by the Lorentz space $L^{q,\infty} = L^{q,\infty}(\Omega)^3$ which is larger than L^q. This leads to Theorem 1.1 if $r < \infty$. In the more general case $r = \infty$ we have to suppose an additional smallness condition on the norm of u in this space; this leads to Theorem 1.2. Concerning Lorentz spaces we refer to [Lor50], [Hun64], [BuBe67], [BeLoe76], [Tri78], see the next section for definitions.

THEOREM 1.1. *Let* $u_0 \in L^2_\sigma(\Omega)$, $f \in C_0^\infty([0,T) \times \overline{\Omega})^3$, *and let*

$$u \in L^\infty(0,T; L^2_\sigma(\Omega)) \cap L^2(0,T; H^{1,2}_{0,\sigma}(\Omega)) \tag{1.9}$$

be a (weakly continuous) weak solution of the Navier-Stokes system (1.1)–(1.4) *with data* f, u_0, *satisfying the strong energy inequality* (2.16).

Suppose u *satisfies the condition*

$$u \in L^{s,r}(0,T; L^{q,\infty}) \text{ with } 3 < q < \infty, \ 2 < s \le r < \infty, \ \frac{3}{q} + \frac{2}{s} = 1. \tag{1.10}$$

Then u *is regular in the sense that* $u \in C^\infty((0,T) \times \overline{\Omega})^3$.

THEOREM 1.2. *Let* u_0, f, u *be as in* Theorem 1.1 *and let*

$$3 < q < \infty, \quad 2 < s < \infty \quad \text{with} \quad \frac{3}{q} + \frac{2}{s} = 1.$$

Then there exists a constant $\Gamma = \Gamma(\Omega, q, s) > 0$ *such that* u *is regular as above if*

$$u \in L^{s,\infty}(0,T; L^{q,\infty}) \tag{1.11}$$

and

$$\|u\|_{L^{s,\infty}(0,T;L^{q,\infty})} \le \Gamma. \tag{1.12}$$

In the proofs, see Sections 4 and 5, we will reduce the problem to Serrin's regularity criterion. Indeed, in both cases above we can show, there exist exponents

$$3 < q_1 < \infty, \quad 2 < s_1 < \infty \quad \text{with} \quad \frac{3}{q_1} + \frac{2}{s_1} = 1$$

such that

$$u \in L^{s_1}(\delta, T; L^{q_1}) \tag{1.13}$$

for all δ with $0 < \delta < T$, see (4.6).

In the case $r = s$ it holds $L^{s,s} = L^s$, and in the case $s < r < \infty$ we obtain larger spaces with continuous embeddings

$$L^q \subset L^{q,\infty}, \quad L^s \subset L^{s,r} \subset L^{s,\infty}, \quad s < r < \infty,$$

see (2.6), (2.11). Therefore, the regularity class defined by (1.10) is larger than Serrin's class.

We consider some examples of functions which are contained in the class (1.10) but not in Serrin's class (1.7).

Let u be a weak solution as in (1.9), let q, s be as in (1.10), and let $t_0 \in (0, T)$. Assume u satisfies the estimate

$$|u(t,x)| \leq C \left(\frac{1}{|t-t_0| \, |\ln |t-t_0|/T|} \right)^{\frac{1}{s}} |g(x)|, \quad t \in [0,T), \quad x \in \Omega \tag{1.14}$$

with some function $g \in L^{q,\infty}$ and some constant $C > 0$. Then it holds $u \in L^{s,r}(0,T; L^{q,\infty})$ with $s < r < \infty$, Theorem 1.1 yields the regularity of u but u need not be in Serrin's class $L^s(0,T; L^q)$.

Suppose that

$$|u(t,x)| \leq C \left(\frac{1}{|t-t_0|} \right)^{\frac{1}{s}} |g(x)|, \quad t \in [0,T), \quad x \in \Omega, \tag{1.15}$$

then $u \in L^{s,\infty}(0,T; L^{q,\infty})$ and the smallness condition (1.12) is satisfied if C is sufficiently small. Then Theorem 1.2 yields the regularity of u but u need not be in Serrin's class.

In a similar way we find examples with singularities in Ω. Let $x_0 \in \Omega$ and suppose that

$$|u(t,x)| \leq C \left(\frac{1}{|t-t_0| \, |\ln |t-t_0|/T|} \right)^{\frac{1}{s}} \left(\frac{1}{|x-x_0|} \right)^{\frac{3}{q}}. \tag{1.16}$$

Then $u \in L^{s,r}(0,T; L^{q,\infty})$ for $s < r < \infty$, Theorem 1.1 yields the regularity but u need not be in Serrin's class. In the more general case

$$|u(t,x)| \leq C \left(\frac{1}{|t-t_0|} \right)^{\frac{1}{s}} \left(\frac{1}{|x-x_0|} \right)^{\frac{3}{q}}, \tag{1.17}$$

we need an additional smallness assumption on $C > 0$. Then Theorem 1.2 yields the regularity of u but no other criterion is applicable.

On the other hand we see, if we try to find a counter example of a non-smooth weak solution, we have to investigate stronger singularities than those in (1.14)–(1.17).

The proof of the theorems above rests on a method which was used in principle already in [KoSo96]. We explain this method as follows.

If u is a weak solution satisfying the energy inequality (2.16), then for almost all $t \in [0, T)$ we are able to construct a classical (strong) solution $u^\star$ with initial value $u(t)$, defined in a certain existence interval $(t, t + T^\star)$, $T^\star > 0$. The energy inequality (2.16) enables us to identify u locally with $u^\star$. This yields the regularity of u in $(t, t + T^\star)$. To prove the regularity in the whole interval $(0, T)$ we need some information on the length $T^\star$ of the local existence intervals in order to cover $(0, T)$ by such local intervals.

The first result concerning local strong solutions is due to Kiselev-Ladyzhenskaya [KiLa57]. Further results are proved by Fujita-Kato [FuKa64], Solonnikov [Sol77], Heywood [Hey80], Miyakawa [Miy81], Kato [Kato84], Giga-Miyakawa [GiMi85], Cannone [Can95], Kozono-Yamazaki [KoYa98], Amann [Ama00], Kozono [Kozo01].

However, we have no result in the literature which enables us to estimate the length of the local existence intervals in an appropriate way. Therefore we develop such a result in Section 3 and give a complete proof. For this purpose we need the property of maximal regularity of the linear evolution equation

$$u_t + Au = f, \quad u(0) = u_0,$$

see [DoVe87], [PrSo90], [GiSo91], [Mon99] concerning this important property. Here $A = -P\Delta$ denotes the Stokes operator, see [Sol77], [Giga85], [Giga86], [BoSo87], [GiSo89], [GiSo91], and P denotes the Helmholtz decomposition, see [FuMo77], [Sol77], [SiSo92].

2. Notations and preliminaries

Let $1 < s \le \infty$, $0 < T < \infty$, and let X be any Banach space with norm $\|\cdot\|_X$. Then $L^s(0, T; X)$ means the usual Banach space of measurable X-valued (classes of) functions $t \mapsto v(t)$, $t \in [0, T)$, with norm

$$\|v\|_{L^s(0,T;X)} = \left(\int_0^T \|v(t)\|_X^s \, dt \right)^{\frac{1}{s}} < \infty. \tag{2.1}$$

Similarly we obtain the space $L^s(a, b; X)$ with $0 \le a < b \le T$. If $s = \infty$, an obvious modification of (2.1) yields the space $L^\infty(0, T; X)$. All spaces we consider here are real.

Let $1 < q < \infty$ and let $L^q = L^q(\Omega)^3$ be the usual Lebesgue space of vector fields $v = (v_1, v_2, v_3)$ with norm

$$\|v\|_q = \|v\|_{L^q} = \left(\int_\Omega |v(x)|^q \, dx \right)^{\frac{1}{q}}.$$

Then the norm of $L^s(0, T; L^q)$ will be denoted by

$$\|v\|_{L^s(0,T;L^q)} = \|v\|_{q,s} = \left(\int_0^T \left(\int_\Omega |v(t, x)|^q \, dx \right)^{\frac{s}{q}} dt \right)^{\frac{1}{s}}. \tag{2.2}$$

To define the Lorentz space $L^{q,r} = L^{q,r}(\Omega)^3$, here only for $q \le r \le \infty$, we consider on Ω a measurable function $v = (v_1, v_2, v_3)$ and define for $R \ge 0$ the Lebesgue measure

$$\mu(v, R) = m(\{x \in \Omega; |v(x)| > R\}) \tag{2.3}$$

of the set $\{x \in \Omega; |v(x)| > R\}$. Then $L^{q,r}$ is the space of all such v with

$$\|v\|_{L^{q,r}} = \left(\int_0^\infty R^r \mu(v, R)^{\frac{r}{q}} \frac{dR}{R}\right)^{\frac{1}{r}} < \infty \tag{2.4}$$

if $q \le r < \infty$, and with

$$\|v\|_{L^{q,\infty}} = \sup_{R \ge 0} (R\mu(v, R)^{\frac{1}{q}}) < \infty \tag{2.5}$$

if $r = \infty$. See [KoYa95, p. 757], [BuBe67, 3.3], [Tri78, 1.18,6], [BeLoe76, 1.3] concerning these spaces. We mention some important properties.

$L^{q,r}$ is a complete quasi-normed space with quasi-norm $\|\cdot\|_{L^{q,r}}$. This means that the triangle inequality only holds in the weaker form

$$\|v_1 + v_2\|_{L^{q,r}} \le k\,(\|v_1\|_{L^{q,r}} + \|v_2\|_{L^{q,r}})$$

with $k > 1$ instead with $k = 1$ for norms. However, in our case $1 < q < \infty$, there always exists a (more complicated) norm, equivalent to this quasi-norm, such that $L^{q,r}$ becomes a Banach space. Thus we may treat these spaces as Banach spaces, although we use (for simplicity) this quasi-norm.

We obtain $L^{q,q} = L^q$ in the sense that the quasi-norm $\|\cdot\|_{L^{q,q}}$ is equivalent to $\|\cdot\|_{L^q}$, and we get the continuous embeddings

$$L^q = L^{q,q} \subset L^{q,r} \subset L^{q,\infty}, \quad q < r < \infty. \tag{2.6}$$

Let $x_0 \in \Omega$. Then an elementary calculation shows that v defined by

$$v(x) = |x - x_0|^{-\frac{3}{q}} \tag{2.7}$$

is contained in $L^{q,\infty}$ but not in L^q. The space $L^{q,\infty}$ is also called the weak L^q-space and denoted by $L^{q,\infty} = L^q_w$.

In the same way we introduce the Lorentz space $L^{s,r}(0, T; X)$ in time direction for $s \le r \le \infty$. Let $v : t \mapsto v(t)$, $t \in [0, T)$, be a measurable X-valued function, and let

$$\mu(v, R) = m(\{t \in [0, T); \|v(t)\|_X > R\}), \quad R \ge 0, \tag{2.8}$$

denote the Lebesgue measure of $\{t \in [0, T); \|v(t)\|_X > R\}$.

Then $L^{s,r}(0, T; X)$ is the space of all such v with

$$\|v\|_{L^{s,r}(0,T;X)} = \left(\int_0^\infty R^r \mu(v, R)^{\frac{r}{s}} \frac{dR}{R}\right)^{\frac{1}{r}} < \infty \tag{2.9}$$

if $s \le r < \infty$, and with

$$\|v\|_{L^{s,\infty}(0,T;X)} \;=\; \sup_{R\ge 0}\,(R\mu(v,R)^{\frac{1}{s}}) \;<\infty \tag{2.10}$$

if $r = \infty$. As above we get $L^{s,s}(0,T;X) = L^s(0,T;X)$ and it hold the continuous embeddings

$$L^{s,s}(0,T;X) \;\subset\; L^{s,r}(0,T;X) \;\subset\; L^{s,\infty}(0,T;X) \tag{2.11}$$

for $s < r < \infty$.

In the same way as in (2.7) we see, if

$$\|v(t)\|_X \;\le\; C\,(|t-t_0|\;|\ln|t-t_0|/T|)^{-\frac{1}{s}}, \quad t\in[0,T), \tag{2.12}$$

then $v \in L^{s,r}(0,T;X)$ for $s < r \le \infty$, and if

$$\|v(t)\|_X \;\le\; C\,|t-t_0|^{-\frac{1}{s}}, \quad t\in[0,T), \tag{2.13}$$

then $v \in L^{s,\infty}(0,T;X)$; $C > 0$ means a constant. In both cases, v need not be in $L^s(0,T;X)$.

Next we have to explain several test function spaces. Let

$$\begin{array}{lll} C^\infty(\Omega)^3, & C_0^\infty(\Omega)^3, & C^\infty([0,T)\times\Omega)^3, \\ C_0^\infty([0,T)\times\Omega)^3, & C_0^\infty([0,T)\times\overline{\Omega})^3, & C^\infty((0,T)\times\overline{\Omega})^3 \end{array}$$

be the usual spaces of smooth functions $v = (v_1, v_2, v_3)$, $\overline{\Omega}$ means the closure of Ω. For example, $v \in C_0^\infty(\Omega)^3$ means that v is smooth with compact support contained in Ω, and $v \in C_0^\infty([0,T)\times\overline{\Omega})^3$ means that v is smooth with compact support contained in $[0,T)\times\overline{\Omega}$. In the last case the initial value

$$v(0) \;=\; v(0,\cdot) \;=\; v|_{t=0}$$

at $t = 0$ is well-defined.

Further we need the usual Sobolev spaces $H^{k,q} = H^{k,q}(\Omega)^3$ and $H_0^{k,q} = H_0^{k,q}(\Omega)^3$, $k \in \mathbb{N}$, $1 < q < \infty$, of functions $v = (v_1, v_2, v_3)$ with norm $\|v\|_{H^{k,q}}$. In particular we get

$$\|v\|_{H^{1,q}} \;=\; \|v\|_q + \|\nabla v\|_q$$

if $k = 1$, and

$$\|v\|_{H^{2,q}} \;=\; \|v\|_q + \|\nabla v\|_q + \|\nabla^2 v\|_q$$

if $k = 2$; see [Tri78].

Using the vector space

$$C_{0,\sigma}^{\infty} \;=\; C_{0,\sigma}^{\infty}(\Omega) \;=\; \{v \in C_0^{\infty}(\Omega)^3\;;\;\; \operatorname{div} v = 0\}$$

of solenoidal smooth functions, we define the Banach spaces

$$\begin{aligned} L_\sigma^q \;&=\; L_\sigma^q(\Omega) \;=\; \overline{C_{0,\sigma}^{\infty}}^{\|\cdot\|_q}, \\ H_{0,\sigma}^{1,q} \;&=\; H_{0,\sigma}^{1,q}(\Omega) \;=\; \overline{C_{0,\sigma}^{\infty}}^{\|\cdot\|_{H^{1,q}}} \end{aligned}$$

by taking the closure. Further we set

$$C_{0,\sigma}^{\infty}([0,T)\times\Omega) \;=\; \{v \in C_0^{\infty}([0,T)\times\Omega)^3\;;\; \operatorname{div} v = 0\}.$$

The Helmholtz projection P is a bounded operator from L^q onto L_σ^q, see [Sol77], [FuMo77], [SiSo92].

Let $1 < q < \infty,\; q' = \frac{q}{q-1}$. Then we set

$$\langle u, v\rangle_\Omega \;=\; \int_\Omega u\cdot v\, dx, \quad u\cdot v \;=\; u_1v_1 + u_2v_2 + u_3v_3$$

if $u \in L^q,\; v \in L^{q'}$, and correspondingly

$$\langle u, v\rangle_{\Omega,T} \;=\; \int_0^T \left(\int_\Omega u\cdot v\, dx\right) dt$$

in the time dependent case. If $q = 2$ we obtain the scalar products $\langle u, v\rangle_\Omega,\; \langle u, v\rangle_{\Omega,T}$, and similarly $\langle \nabla u, \nabla v\rangle_\Omega,\; \langle \nabla u, \nabla v\rangle_{\Omega,T}$.

The notion of a weak solution is formally obtained when we take in (1.1) the scalar product with a test function $v \in C_{0,\sigma}^{\infty}([0,T)\times\Omega)$ and use integration by parts. This yields the relation (2.15) below.

DEFINITION 2.1. Let $u_0 \in L_\sigma^2(\Omega),\; f \in L^2(0,T; L^2(\Omega)^3)$. Then a function

$$u \in L^\infty(0,T; L_\sigma^2(\Omega)) \;\cap\; L^2(0,T; H_{0,\sigma}^{1,2}(\Omega)) \tag{2.14}$$

is called a weak solution (with data u_0, f) of the Navier-Stokes system (1.1)–(1.4) if

$$\begin{aligned} &-\langle u, v_t\rangle_{\Omega,T} + \langle \nabla u, \nabla v\rangle_{\Omega,T} + \langle u\cdot\nabla u, v\rangle_{\Omega,T} \\ &\quad = \langle u_0, v(0)\rangle_\Omega + \langle f, v\rangle_{\Omega,T} \end{aligned} \tag{2.15}$$

holds for all $v \in C_{0,\sigma}^{\infty}([0,T)\times\Omega)$.

We know, there exists at least one weak solution u which satisfies additionally the energy inequality in the strong form

$$\frac{1}{2}\,\|u(t)\|_2^2 + \int_\tau^t \|\nabla u\|_2^2\, d\rho \;\le\; \frac{1}{2}\,\|u(\tau)\|_2^2 + \int_\tau^t \langle f, u\rangle d\rho \tag{2.16}$$

for $\tau = 0$, almost all $\tau \in (0, T)$, and all $t \in [\tau, T)$; see [Mas84, p. 626] for bounded domains, [GaMa86], [SoWW86], [MiSo88] for exterior domains, and [Ler34] for the whole space $\mathbb{R}^3$.

Note that this inequality is only problematic for unbounded domains, see [Mas84]. The case of exterior domains requires a complicated proof, and there is no proof up to now for general unbounded domains. If (2.16) only holds for $\tau = 0$, we obtain the "usual" energy inequality which is not problematic for arbitrary domains.

We mention some further facts. After a redefinition of $u(t)$ in a subset of $[0, T)$ of measure zero, u becomes weakly continuous as a function from $[0, T)$ to $L^2_\sigma(\Omega)$, see [Pro59], [Mas84, p. 625]. Therefore, we may assume in the following that each weak solution has this property. Then, in particular, $u(t) \in L^2_\sigma(\Omega)$ is well-defined for all $t \in [0, T)$.

Let u, v be two weak solutions of (1.1)–(1.4) with the same data u_0, f, let u satisfy the energy inequality (2.16) for $\tau = 0$, and assume that v satisfies the condition (1.7). Then Serrin's uniqueness criterion yields $u = v$; see [Ser63], [Mas84].

Let u be a (weakly continuous) weak solution with data u_0, f where f satisfies

$$f \in C_0^\infty([0, T) \times \overline{\Omega})^3. \tag{2.17}$$

Further suppose that u satisfies (1.7). Then Serrin's regularity criterion yields the regularity of u in the sense that

$$u \in C^\infty((0, T) \times \overline{\Omega})^3, \tag{2.18}$$

see [Ser63], [GaMa86], [Giga86], [Hey80]. Note that we cannot say anything on the regularity of u up to $t = 0$ without complicated compatibility conditions, see [Rau83].

A weak solution u as in Definition 2.1 is called a strong solution if Serrin's condition (1.7) is satisfied. Thus a strong solution is regular if f satisfies (2.17).

3. Existence result for strong solutions

The existence result below yields a strong solution in some interval $[0, T)$ where $T > 0$ is determined by a smallness condition, see (3.14) below. The proof rests on properties of the Stokes operator A, the fractional powers A^α, $A^{-\alpha}$, $0 \leq \alpha \leq 1$, and the semigroup operators e^{-tA}, $t \geq 0$. First we collect some of these properties, we refer to [Sol77], [Giga85], [Giga86], [BoSo87], [GiSo89], [GiSo91], [BoMi95], [KoYa95], [KoYa98], [Gal00], [Ama00]; see [KoYa95] for a summary.

Let $1 < q < \infty$. The Stokes operator A is defined by the domain $D(A) = L^q_\sigma \cap H_0^{1,q} \cap H^{2,q}$ and by $Au = -P\Delta u$, $u \in D(A)$, where P means the Helmholtz projection. Let $\widehat{A} = I + A$ with $D(\widehat{A}) = D(A)$ where I means the identity.

Then we know that the norms

$$\|v\|_q + \|Av\|_q, \quad \|\widehat{A}v\|_q, \quad \|v\|_{H^{2,q}} \tag{3.1}$$

are equivalent for $v \in D(A)$. Correspondingly, the norms

$$\|v\|_q + \|A^{\frac{1}{2}}v\|_q, \quad \|\widehat{A}^{\frac{1}{2}}v\|_q, \quad \|v\|_{H^{1,q}} \tag{3.2}$$

are equivalent for $v \in D(A^{\frac{1}{2}}) = D(\widehat{A}^{\frac{1}{2}})$.

Let $1 < q \leq r < \infty$, $0 < \alpha \leq 1$, $2\alpha + \frac{3}{r} = \frac{3}{q}$. Then we obtain the embedding estimate

$$\|v\|_r \leq C\, \|\widehat{A}^{\alpha}v\|_q, \quad v \in D(A^{\alpha}) = D(\widehat{A}^{\alpha}), \tag{3.3}$$

with some constant $C = C(\Omega, q, r) > 0$.

If $1 < q < \infty$, $0 \leq \alpha \leq 1$, $t > 0$, then it holds

$$\|\widehat{A}^{\alpha}e^{-t\widehat{A}}v\|_q \leq C\, t^{-\alpha}\|v\|_q, \quad v \in L^q_\sigma, \tag{3.4}$$

with some constant $C = C(\Omega, q, \alpha) > 0$.

If $1 < q \leq r < \infty$, $\alpha = \frac{1}{2}(\frac{3}{q} - \frac{3}{r})$, $t > 0$, then

$$\|e^{-t\widehat{A}}v\|_r \leq C\, t^{-\alpha}\|v\|_q, \quad v \in L^q_\sigma, \tag{3.5}$$

with $C = C(\Omega, q, r) > 0$.

If $1 < q < r < \infty$, then the norm $\|v\|_q = \|v\|_{L^q}$ in (3.5) can be replaced by the weak L^q-norm $\|v\|_{L^{q,\infty}}$, see [KoYa95, p. 763, (4)]. Then this inequality holds for all $v \in L^{q,\infty}_\sigma = L^{q,\infty}_\sigma(\Omega)$ where the last space is defined by

$$L^{q,\infty}_\sigma = \{w \in L^{q,\infty}(\Omega)^3;\ \operatorname{div} w = 0,\ N \cdot w|_{\partial\Omega} = 0\}, \tag{3.6}$$

see [BoMi95], [KoYa95; p. 757], [KoYa98; p. 754]. Here N means the exterior normal vector and $N \cdot w|_{\partial\Omega}$ the normal component of w at $\partial\Omega$. Note that $N \cdot w|_{\partial\Omega}$ is well-defined because of div $w = 0$, see [FuMo77], [SiSo92]. The last condition in (3.6) is omitted if $\Omega = \mathbb{R}^3$. Thus we obtain the following estimate:

If $1 < q < r < \infty$, $\alpha = \frac{1}{2}(\frac{3}{q} - \frac{3}{r})$, $t > 0$, then

$$\|e^{-t\widehat{A}}v\|_{L^r} \leq C\, t^{-\alpha}\|v\|_{L^{q,\infty}}, \quad v \in L^{q,\infty}_\sigma, \tag{3.7}$$

with $C = C(\Omega, q, r) > 0$. Using (3.6) we see in particular that (3.7) holds for all $v \in L^2_\sigma \cap L^{q,\infty} \subseteq L^{q,\infty}_\sigma$.

Let $1 < s < \infty$, $1 < q < \infty$, and consider the evolution equation

$$v_t + \widehat{A}v = f, \quad v(0) = 0, \quad 0 \leq t < T \tag{3.8}$$

with $f \in L^s(0, T; L^q_\sigma)$ and, for simplicity, with initial value zero. The solution v of this equation can be represented by the formula

$$v(t) = \int_0^t e^{-(t-\tau)\widehat{A}} f(\tau)\, d\tau, \quad 0 \le t < T, \tag{3.9}$$

and we obtain the important estimate

$$\|v_t\|_{L^s(0,T;L^q)} + \|\widehat{A}v\|_{L^s(0,T;L^q)} \le C\, \|f\|_{L^s(0,T;L^q)} \tag{3.10}$$

with $C = C(\Omega, q, s) > 0$, see [GiSo91], [PrSo90], [Mon99]. This estimate is problematic for general evolution equations, see [DoVe87]. Since v_t and $\widehat{A}v$ are contained in the same space as f, v possesses the maximal regularity.

Let $0 < \alpha < 1$. Applying $\widehat{A}^\alpha$ to (3.9) and using (3.4), we obtain with C as in (3.4) the estimate

$$\|\widehat{A}^\alpha v(t)\|_q \le C \int_0^t |t-\tau|^{-\alpha}\, \|f(\tau)\|_q\, d\tau, \quad 0 \le t < T. \tag{3.11}$$

Next we choose $r > s$ and $\alpha = 1 + \frac{1}{r} - \frac{1}{s}$ such that $1 - \alpha + \frac{1}{r} = \frac{1}{s}$. Then we use the Hardy-Littlewood estimate [Tri78, 1.18.8, Theorem 3] and obtain the inequality

$$\|\widehat{A}^\alpha v\|_{L^r(0,T;L^q)} \le C\, \|f\|_{L^s(0,T;L^q)} \tag{3.12}$$

with $C = C(\Omega, q, r, s) > 0$.

Using these properties we can prove the following solvability result:

THEOREM 3.1. *Let* $0 < T < \infty$,

$$3 < q < \infty, \quad 2 < s < \infty \quad \textit{with} \quad \frac{3}{q} + \frac{2}{s} = 1,$$

and let

$$u_0 \in L^2_\sigma(\Omega) \cap L^{q,\infty}(\Omega)^3, \quad f \in L^s(0, T; L^q(\Omega)^3 \cap L^2(\Omega)^3). \tag{3.13}$$

Then there exists a constant $K = K(\Omega, q, s) > 0$ *with the following property:*

If

$$\|f\|_{L^s(0,T;L^q)} + T^{\frac{1}{s}}\, \|u_0\|_{L^{q,\infty}} \le K\, e^{-T}, \tag{3.14}$$

then there is a uniquely determined weak solution

$$u \in L^\infty(0, T; L^2_\sigma(\Omega)) \cap L^2(0, T; H^{1,2}_{0,\sigma}(\Omega)) \tag{3.15}$$

of the Navier-Stokes system (1.1)–(1.4) *satisfying Serrin's condition*

$$u \in L^{s_1}(0, T; L^{q_1}) \quad \textit{with} \quad 3 < q_1 < \infty, \quad 2 < s_1 < \infty, \quad \frac{3}{q_1} + \frac{2}{s_1} = 1. \tag{3.16}$$

Thus u *is a strong solution and regular in the sense of* (2.18) *if* (2.17) *is satisfied.*

Proof. First we explain the method and consider for this purpose any weak solution u of (1.1)–(1.4) with (3.15). Then u satisfies the equation

$$u_t - \Delta u + u \cdot \nabla u + \nabla p = f$$

together with some p in the sense of distributions. Assuming that u is weakly continuous we get $u(0) = u_0$.

Multiplying the last equation with e^{-t} and setting

$$\widehat{u}(t) = e^{-t} u(t), \quad \widehat{p}(t) = e^{-t} p(t), \quad \widehat{f}(t) = e^{-t} f(t), \quad t \in [0, T), \tag{3.17}$$

we obtain the equation

$$\widehat{u}_t + (-\Delta + I)\widehat{u} + e^t \widehat{u} \cdot \nabla \widehat{u} + \nabla\, \widehat{p} = \widehat{f} \tag{3.18}$$

with $\operatorname{div} \widehat{u} = 0$, $\widehat{u}(0) = u_0$.

The next calculations will be justified later on. We apply the Helmholtz projection P, use the Stokes operator $A = -P\Delta$, $\widehat{A} = P(-\Delta + I) = A + I$, and obtain the equation

$$\widehat{u}_t + \widehat{A}u + e^t P \widehat{u} \cdot \nabla\, \widehat{u} \;=\; P\widehat{f}. \tag{3.19}$$

Introducing the matrix $\widehat{u}\,\widehat{u} \;=\; (\widehat{u}_j\, \widehat{u}_k)_{j,k=1}^3$ and setting

$$\operatorname{div}\,(\widehat{u}\,\widehat{u}) \;=\; \operatorname{div} \widehat{u}\,\widehat{u} \;=\; D_1(\widehat{u}_1 \widehat{u}) + D_2(\widehat{u}_2 \widehat{u}) + D_3(\widehat{u}_3 \widehat{u}), \tag{3.20}$$

we get with $\operatorname{div} \widehat{u} = 0$ the relations

$$\widehat{u} \cdot \nabla\, \widehat{u} \;=\; \widehat{u}_1 D_1 \widehat{u} \;+\; \widehat{u}_2 D_2 \widehat{u} \;+\; \widehat{u}_3 D_3 \widehat{u} \;=\; \operatorname{div}\,(\widehat{u}\,\widehat{u}), \tag{3.21}$$

$$\begin{aligned} \langle \widehat{u} \cdot \nabla\, \widehat{u}, \widehat{u} \rangle_\Omega \;&=\; \tfrac{1}{2} \langle \widehat{u}, \nabla |\widehat{u}|^2 \rangle_\Omega \\ &=\; -\tfrac{1}{2} \langle \operatorname{div} \widehat{u}, |\widehat{u}|^2 \rangle_\Omega \;=\; 0, \end{aligned} \tag{3.22}$$

and

$$\langle \widehat{u} \cdot \nabla\, \widehat{u}, \widehat{u} \rangle_\Omega \;=\; \langle \operatorname{div}\,(\widehat{u}\,\widehat{u}), \widehat{u} \rangle_\Omega \;=\; -\langle \widehat{u}\,\widehat{u}, \nabla\, \widehat{u} \rangle_\Omega. \tag{3.23}$$

Next we set

$$\begin{aligned} F(t) \;&=\; \int_0^t e^{-(t-\tau)\widehat{A}}\, P\widehat{f}(\tau)\, d\tau, \quad S(t) = e^{-t\widehat{A}} u_0, \\ W(t) \;&=\; F(t) + S(t), \quad U(t) = \widehat{u}(t) - W(t), \quad t \in [0, T), \end{aligned} \tag{3.24}$$

Then we get $S_t + \widehat{A}S = 0$, $F_t + \widehat{A}F = P\widehat{f}$, and the equation (3.19) can be written in the form

$$U_t + \widehat{A}U + e^t P \operatorname{div}\,((U + W)(U + W)) \;=\; 0 \tag{3.25}$$

with $U(0) = 0$. Applying formula (3.9) leads to the integral equation

$$U(t) = -\int_0^t e^{-(t-\tau)\widehat{A}} e^{\tau} P \operatorname{div} (U+W)(U+W)\, d\tau. \tag{3.26}$$

Now we define the nonlinear operator $\mathcal{F}$ by setting

$$(\mathcal{F}U)(t) = -\int_0^t e^{-(t-\tau)\widehat{A}} e^{\tau} P \operatorname{div} (U+W)(U+W)\, d\tau, \quad 0 \le t < T. \tag{3.27}$$

Then (3.26) can be written in the form

$$U = \mathcal{F}U. \tag{3.28}$$

In the next step we give this equation a precise meaning and construct a solution U in a certain Banach space by Banach's fixed point principle. Setting $\widehat{u} = U + W$, $u(t) = e^t \widehat{u}(t)$, we then obtain the desired solution u of our problem.

To carry out this procedure, we introduce exponents $q_1, s_1, q_2, s_2, q_3, s_3$ with the following properties:

$$q < q_1 < \infty, \quad 2 < s_1 < s, \quad \tfrac{3}{q_1} + \tfrac{2}{s_1} = 1,$$
$$q_2 = \tfrac{1}{2} q_1, \quad s_2 = \tfrac{1}{2} s_1, \quad \tfrac{3}{q_2} + \tfrac{2}{s_2} = 2,$$
$$\tfrac{1}{q_3} + \tfrac{1}{q_1} = \tfrac{1}{2}, \quad \tfrac{1}{s_3} + \tfrac{1}{s_1} = \tfrac{1}{2}, \quad q_3 > 2, \ s_3 > 2.$$

Further we define the Banach spaces D_{q_2,s_2} and $D_{2,2}$ in the following way. We use the notation (2.2) and write

$$|||U|||_{q_2,s_2} \equiv \|\widehat{A}^{-\frac{1}{2}} U_t\|_{q_2,s_2} + \|\widehat{A}^{\frac{1}{2}} U\|_{q_2,s_2} < \infty \tag{3.29}$$

if $U : t \mapsto U(t)$ has the following properties: $U : [0,T) \to L^{q_2}_\sigma$ is measurable, $\widehat{A}^{-\frac{1}{2}} U : [0,T) \to L^{q_2}_\sigma$ is strongly continuous, and it holds $\widehat{A}^{-\frac{1}{2}} U_t$, $\widehat{A}^{\frac{1}{2}} U \in L^{s_2}(0,T; L^{q_2}_\sigma)$.

Then D_{q_2,s_2} denotes the Banach space of all such U with norm $|||U|||_{q_2,s_2} < \infty$. Replacing q_2, s_2 by $2, 2$, we obtain in the same way the Banach space $D_{2,2}$ with norm

$$|||U|||_{2,2} \equiv \|\widehat{A}^{-\frac{1}{2}} U_t\|_{2,2} + \|\widehat{A}^{\frac{1}{2}} U\|_{2,2} < \infty. \tag{3.30}$$

Below we construct a solution $U \in D_{q_2,s_2}$ of (3.28) with $\widehat{A}^{-\frac{1}{2}} U(0) = 0$ if the condition (3.14) with some K is satisfied. Then we prove that $|||U|||_{2,2} < \infty$, that

$$\widehat{A}^{-\frac{1}{2}} U_t, \quad \widehat{A}^{\frac{1}{2}} U, \quad \widehat{A}^{-\frac{1}{2}} \widehat{u}_t, \quad \widehat{A}^{\frac{1}{2}} \widehat{u} \quad \in L^2(0,T; L^2_\sigma), \tag{3.31}$$

and that

$$u \in L^{s_1}(0,T; L^{q_1}) \tag{3.32}$$

with $\widehat{u}, u$ as above.

Then a calculation shows the validity of

$$\widehat{A}^{-\frac{1}{2}} U_t + \widehat{A}^{\frac{1}{2}} U + e^t \widehat{A}^{-\frac{1}{2}} P \text{ div } (U+W)(U+W) = 0$$

and of

$$\widehat{A}^{-\frac{1}{2}} \widehat{u}_t + \widehat{A}^{\frac{1}{2}} \widehat{u} + e^t \widehat{A}^{-\frac{1}{2}} P \text{ div } (\widehat{u}\, \widehat{u}) \; = \; \widehat{A}^{-\frac{1}{2}} P \widehat{f}. \tag{3.33}$$

It follows (3.18) and the relation (2.15) is satisfied for all $v \in C^\infty_{0,\sigma}([0,T) \times \Omega)$.

Taking in (3.18) the scalar product with $\widehat{u}$, integrating over $[0,t)$, $0 < t < T$, and using (3.21)–(3.23), (3.31), (3.32), (3.33), we obtain the validity of the energy equality

$$\frac{1}{2}\|u(t)\|_2^2 + \int_0^t \|\nabla u\|_2^2 \, d\tau \; = \; \frac{1}{2}\|u_0\|_2^2 + \int_0^t \langle f, u\rangle_\Omega \, d\tau \tag{3.34}$$

and it follows (2.14). This shows that u is a weak solution of (1.1)–(1.4). Because of (3.32), u is also a strong solution and the uniqueness follows from Serrin's criterion.

Thus it remains to prove the existence of a solution $U \in D_{q_2,s_2}$ of (3.28), satisfying $\widehat{A}^{-\frac{1}{2}} U(0) = 0$ and (3.30)–(3.32).

For this purpose we need several preparations.

Consider a matrix

$$M \; = \; (M_{j,k})^3_{j,k=1} \quad \in C^\infty_0(\Omega)^9,$$

and define

$$\text{div } M \; = \; (D_1 M_{1,k} + D_2 M_{2,k} + D_3 M_{3,k})^3_{k=1} \quad \in C^\infty_0(\Omega)^3$$

in the same way as in (3.20). Then we get

$$\begin{aligned} \langle \widehat{A}^{-\frac{1}{2}} P \text{ div } M, v\rangle_\Omega \; &= \; \langle \text{div } M, \widehat{A}^{-\frac{1}{2}} v\rangle_\Omega \\ &= \; -\langle M, \nabla \widehat{A}^{-\frac{1}{2}} v\rangle_\Omega \end{aligned}$$

for all $v \in C^\infty_{0,\sigma}(\Omega)$. With $1 < r < \infty$, $r' = \frac{r}{r-1}$, we conclude that

$$|\langle \widehat{A}^{-\frac{1}{2}} P \text{ div } M, v\rangle_\Omega| \; \leq \; \|M\|_r \, \|\nabla \widehat{A}^{-\frac{1}{2}} \, v\|_{r'}.$$

Using (3.2) we see that $\nabla \widehat{A}^{-\frac{1}{2}}$ is a bounded operator. Therefore we get the estimate

$$|\langle \widehat{A}^{-\frac{1}{2}} P \text{ div } M, v\rangle_\Omega| \; \leq \; C \, \|M\|_r \, \|v\|_{r'}$$

with $C = C(\Omega, r) > 0$, and this shows that the operator

$$\widehat{A}^{-\frac{1}{2}} P \text{ div } : M \mapsto \widehat{A}^{-\frac{1}{2}} P \text{ div } M, \quad M \in C^\infty_0(M)^9,$$

extends by closure to a bounded operator from $L^r(\Omega)^9$ to $L^r_\sigma(\Omega)$ with operator norm

$$\|\widehat{A}^{-\frac{1}{2}} P \operatorname{div}\| \;\le\; C. \tag{3.35}$$

Next we consider any $U \in D_{q_2,s_2}$ with $\widehat{A}^{-\frac{1}{2}} U(0) = 0$. Using (3.27) we obtain

$$\widehat{A}^{-\frac{1}{2}} (\mathcal{F}U)(t) \;=\; -\int_0^t e^{-(t-\tau)\widehat{A}}\, e^{\tau}\, \widehat{A}^{-\frac{1}{2}} P \operatorname{div} (U+W)(U+W)\, d\tau, \tag{3.36}$$

(3.9) yields

$$\widehat{A}^{-\frac{1}{2}} U(t) = \int_0^t e^{-(t-\tau)\widehat{A}}\, (\widehat{A}^{-\frac{1}{2}} U_\tau + \widehat{A}^{\frac{1}{2}} U)\, d\tau, \tag{3.37}$$

and together with (3.10) we get the estimate

$$|||U|||_{q_2,s_2} \;\le\; C \|\widehat{A}^{-\frac{1}{2}} U_t + \widehat{A}^{\frac{1}{2}} U\|_{q_2,s_2} \tag{3.38}$$

with $C = C(\Omega, q, s) > 0$.

Applying (3.10) to (3.36), using (3.35) and Hölder's inequality with $\frac{1}{q_2} = \frac{1}{q_1} + \frac{1}{q_1}$, $\frac{1}{s_2} = \frac{1}{s_1} + \frac{1}{s_1}$, we see that

$$\begin{aligned} |||\mathcal{F}U|||_{q_2,s_2} \;&\le\; C\, e^T\, \|U+W\|_{q_1,s_1}\, \|U+W\|_{q_1,s_1} \\ &\le\; C\, e^T\, (\|U\|_{q_1,s_1} + \|W\|_{q_1,s_1})^2 \end{aligned} \tag{3.39}$$

with $C = C(\Omega, q, s) > 0$.

Set $\beta = \frac{3}{4q_2}$. Then we get $2\beta + \frac{3}{q_1} = \frac{3}{q_2}$, $1 - (\frac{1}{2} + \beta) + \frac{1}{s_1} = \frac{1}{s_2}$, write (3.37) in the form

$$\widehat{A}^{\beta} U(t) = \int_0^t \widehat{A}^{\beta+\frac{1}{2}}\, e^{-(t-\tau)\widehat{A}}\, (\widehat{A}^{-\frac{1}{2}} U_\tau + \widehat{A}^{\frac{1}{2}} U)\, d\tau,$$

and apply first (3.3) and then (3.11), (3.12). This yields the estimate

$$\|U\|_{q_1,s_1} \;\le\; C_1\, \|\widehat{A}^{\beta} U\|_{q_2,s_1} \;\le\; C_2 |||U|||_{q_2,s_2} \tag{3.40}$$

with constants $C_1, C_2 > 0$.

Next we choose $0 < \gamma < \frac{1}{2}$, $\rho > s_1$ such that $2\gamma + \frac{3}{q_1} = \frac{3}{q}$, $\frac{1}{s_1} = \frac{1}{s} + \frac{1}{\rho}$, write $F(t)$ in the form

$$\widehat{A}^{\gamma} F(t) = \widehat{A} \int_0^t e^{-(t-\tau)\widehat{A}}\, \widehat{A}^{-(1-\gamma)} P \widehat{f}\, d\tau,$$

apply first (3.3), then (3.10), and then Hölder's inequality with $\frac{1}{s_1} = \frac{1}{s} + \frac{1}{\rho}$. This leads to the estimate

$$\begin{aligned} \|F\|_{q_1,s_1} &\le C_1 \|A^\gamma F\|_{q,s_1} \le C_2 \|\widehat{A}^{-(1-\gamma)} P \widehat{f}\|_{q,s_1} \\ &\le C_3 \|\widehat{f}\|_{q,s_1} = C_3 \left(\int_0^T (e^{-t} \|f(t)\|_q)^{s_1} \, dt \right)^{\frac{1}{s_1}} \\ &\le C_3 \left(\int_0^T e^{-t\rho} \, dt \right)^{\frac{1}{\rho}} \|f\|_{q,s} \le C_4 \|f\|_{q,s} \end{aligned}$$

with constants $C_1, C_2, C_3, C_4 > 0$ not depending on T.

Setting $\alpha = \frac{1}{2}(\frac{3}{q} - \frac{3}{q_1})$, and using (3.7) with $r = q_1$, $v = u_0 \in L^2_\sigma \cap L^{q,\infty}$, we obtain with $-\alpha s_1 + 1 = \frac{s_1}{s}$ the estimate

$$\begin{aligned} \|S\|_{q_1,s_1} &= \left(\int_0^T \|e^{-t\widehat{A}} u_0\|_{q_1}^{s_1} \, dt \right)^{\frac{1}{s_1}} \le C \left(\int_0^T t^{-\alpha s_1} \, dt \right)^{\frac{1}{s_1}} \|u_0\|_{L^{q,\infty}} \\ &= C (s/s_1)^{\frac{1}{s_1}} T^{\frac{1}{s}} \|u_0\|_{L^{q,\infty}} \end{aligned}$$

where $C > 0$ does not depend on T.

Using $W = F + S$ we see that

$$\|W\|_{q_1,s_1} \le C \, (\|f\|_{q,s} + T^{\frac{1}{s}} \|u_0\|_{L^{q,\infty}}) \tag{3.41}$$

where $C = C(\Omega, q, s) > 0$.

Setting $b = \|f\|_{q,s} + T^{\frac{1}{s}} \|u_0\|_{L^{q,\infty}}$, we get from (3.39), (3.40), (3.41) the estimate

$$|||\mathcal{F}U|||_{q_2,s_2} \le C \, e^T \, (|||U|||_{q_2,s_2} + b)^2 \tag{3.42}$$

with $C = C(\Omega, q, s) > 0$.

Next we set $a = Ce^T$ with C from (3.41) and obtain the inequality

$$|||\mathcal{F}U|||_{q_2,s_2} + b \le a \, (|||U|||_{q_2,s_2} + b)^2 + b \tag{3.43}$$

To prepare the fixed point argument for (3.28), we consider the equation

$$y = ay^2 + b, \quad y^2 - \frac{1}{a} y + \frac{b}{a} = 0, \quad y > 0; \tag{3.44}$$

see [Sol77, Lemma 10.2] concerning this argument. Then we fix the constant K in (3.14) and set

$$K = \frac{1}{5} C^{-1}$$

with C from (3.42). The assumption (3.14) leads to $4ab < 1$. The minimal root y_1 of the equation (3.44) is given by

$$0 < y_1 = 2b(1 + \sqrt{1 - 4ab})^{-1} < 2b.$$

Next we define the closed subset

$$D = \{U \in D_{q_2,s_2}\,;\ \widehat{A}^{-\frac{1}{2}}\, U(0) = 0,\ \ |||U|||_{q_2,s_2} + b \ \le\ y_1\} \subseteq D_{q_2,s_2}.$$

Since $y_1 = ay_1^2 + b > b$, we see that $D \neq \emptyset$.

Let $U \in D$. Then from (3.43) we get

$$|||\mathcal{F}U|||_{q_2,s_2} + b \ \le\ ay_1^2 + b \ =\ y_1$$

and therefore $\mathcal{F}U \in D$.

Let $U, V \in D$. Then the same calculation as above for (3.42) leads to the inequality

$$\begin{aligned} |||\mathcal{F}U - \mathcal{F}V|||_{q_2,s_2} \ &\le\ a(|||U - V|||_{q_2,s_2}\,(|||U|||_{q_2,s_2} + b) \\ &\qquad + |||U - V|||_{q_2,s_2}(|||V|||_{q_2,s_2} + b)) \\ &\le\ 2ay_1|||U - V|||_{q_2,s_2} \ \le\ 4ab|||U - V|||_{q_2,s_2}. \end{aligned}$$

Since $4ab < 1$, we can apply Banach's fixed point principle and obtain a solution $U \in D$ of the equation (3.28).

With $\widehat{u} = U + W,\ u(t) = e^t\,\widehat{u}(t),\ t \in [0, T)$, we conclude from (3.40), (3.41) that

$$\|\widehat{u}\|_{q_1,s_1} \ \le\ \|U\|_{q_1,s_1} + \|W\|_{q_1,s_1} \ < \infty,$$

which leads to (3.32).

Finally we have to prove the regularity properties (3.30) and (3.31).

Using (3.28) and (3.36), we see that

$$\begin{aligned} \widehat{A}^{-\frac{1}{2}}U_t + \widehat{A}^{\frac{1}{2}}U \ &=\ -e^t\widehat{A}^{-\frac{1}{2}}P \text{ div } (U + W)(U + W) \\ &=\ -e^t\widehat{A}^{-\frac{1}{2}}P \text{ div } (\widehat{u}\,U) - e^t\widehat{A}^{-\frac{1}{2}}P \text{ div } (\widehat{u}\,W). \end{aligned} \tag{3.45}$$

Applying (3.9) we derive from (3.45) the integral equation

$$\begin{aligned} \widehat{A}^{-\frac{1}{2}}U(t) \ =\ -\int_0^t e^{-(t-\tau)\widehat{A}}\,e^{\tau}(\widehat{A}^{-\frac{1}{2}}P \text{ div } (\widehat{u}\,U) \\ + \widehat{A}^{-\frac{1}{2}}P \text{ div } (\widehat{u}\,W))\,d\tau. \end{aligned}$$

Assume for a moment that we already know that $|||U|||_{2,2} < \infty$. Then we apply the same method as used to derive inequality (3.39). This leads to the estimate

$$|||U|||_{2,2} \ \le\ C\,e^T\,\|\widehat{u}\|_{q_1,s_1}(\|U\|_{q_3,s_3} + \|W\|_{q_3,s_3})$$

with some $C > 0$. The same argument as used for (3.40) yields the estimate

$$\|U\|_{q_3,s_3} \;\leq\; C\,|||U|||_{2,2}$$

with some $C > 0$. This proves the inequality

$$|||U|||_{2,2} \;\leq\; Ce^T\,\|\widehat{u}\|_{q_1,s_1}\,|||U|||_{2,2} \;+\; Ce^T\,\|\widehat{u}\|_{q_1,s_1}\,\|W\|_{q_3,s_3} \tag{3.46}$$

with $C = C(\Omega, q, s) > 0$.

Next we observe that

$$\|W\|_{q_3,s_3} \;\leq\; \|F\|_{q_3,s_3} \;+\; \|S\|_{q_3,s_3} \;<\infty.$$

To estimate $\|F\|_{q_3,s_3}$ we use the same method as above for $\|F\|_{q_1,s_1}$. This yields

$$\begin{aligned}\|F\|_{q_3,s_3} \;&\leq\; C_1\,\|\widehat{A}^{-\frac{1}{2}}(F_t + \widehat{A}F)\|_{2,2} \;=\; C_1\|\widehat{A}^{-\frac{1}{2}}P\widehat{f}\|_{2,2}\\ &\leq\; C_2\,\|f\|_{2,2} \;<\; \infty\end{aligned} \tag{3.47}$$

with constants $C_1, C_2 > 0$. Note that $f \in L^2(0,T;L^2)$ because of (3.13), $s > 2$, $T < \infty$.

To show that $\|S\|_{q_3,s_3} < \infty$, we consider some $T' > T$ and a smooth function $t \mapsto \varphi(t)$, $t \in [0, T']$, satisfying $0 \leq \varphi \leq 1$, $\varphi(t) = 1$ for $0 \leq t \leq T$, $\varphi(T') = 0$. Then we set $\widehat{S}(t) = \varphi(T'-t)S(T'-t)$, get $\widehat{S}(0) = 0$, $\widehat{S}(T') = u_0$, and a calculation yields that

$$\|S\|_{q_3,s_3} \;\leq\; \left(\int_0^{T'} \varphi(t)^{s_3}\,\|S(t)\|_{q_3}^{s_3}\,dt\right)^{\frac{1}{s_3}} \;=\; \left(\int_0^{T'} \|\widehat{S}(\tau)\|_{q_3}^{s_3}\,d\tau\right)^{\frac{1}{s_3}}. \tag{3.48}$$

Further we obtain

$$\begin{aligned}\int_0^{T'} \|\widehat{A}^{\frac{1}{2}}S(t)\|_2^2\,dt \;&=\; \int_0^{T'} \langle \widehat{A}^{\frac{1}{2}}e^{-t\widehat{A}}u_0, \widehat{A}^{\frac{1}{2}}e^{-t\widehat{A}}u_0\rangle_\Omega\,dt\\ &=\; \int_0^{T'} \langle \widehat{A}e^{-2t\widehat{A}}u_0, u_0\rangle_\Omega\,dt\\ &=\; -\tfrac{1}{2}\int_0^{T'} \frac{d}{dt}\langle e^{-2t\widehat{A}}u_0, u_0\rangle_\Omega\,dt\\ &=\; \tfrac{1}{2}(\langle u_0, u_0\rangle_\Omega - \langle e^{-2T'\widehat{A}}u_0, u_0\rangle_\Omega)\\ &\leq\; C\,\|u_0\|_2^2\end{aligned}$$

with some $C > 0$.

Writing

$$\widehat{S}(t) \;=\; \int_0^t \widehat{A}^{\frac{1}{2}}e^{-(t-\tau)\widehat{A}}\,(\widehat{A}^{-\frac{1}{2}}\widehat{S}_\tau \;+\widehat{A}^{\frac{1}{2}}\widehat{S})\,d\tau$$

see (3.37), and using the same method as in (3.47), we obtain

$$\begin{aligned}\left(\int_0^{T'} \|\widehat{S}(t)\|_{q_3}^{s_3}\, dt\right)^{\frac{1}{s_3}} &\le C_1 \left(\left(\int_0^{T'} \|\widehat{A}^{-\frac{1}{2}}\widehat{S}_t\|_2^2\, dt\right)^{\frac{1}{2}} + \left(\int_0^{T'} \|\widehat{A}^{\frac{1}{2}}\widehat{S}\|_2^2\, dt\right)^{\frac{1}{2}}\right)\\ &\le C_2\, \|u_0\|_{L^2}\end{aligned}$$

with constants $C_1, C_2 > 0$. Using (3.48) we see that $\|S\|_{q_3,s_3} < \infty$ and therefore that $\|W\|_{q_3,s_3} < \infty$.

Consider now the inequality (3.46). We may assume, without loss of generality, that

$$Ce^T\, \|\widehat{u}\|_{q_1,s_1} = Ce^T \left(\int_0^T \|\widehat{u}\|_{q_1}^{s_1}\, dt\right)^{\frac{1}{s_1}} < 1. \tag{3.49}$$

with C from (3.46). Otherwise we choose $0 < T^\star \le T$ such that (3.49) holds with T replaced by $T^\star$. Then we can repeat this argument with $u(T^\star)$ instead of u_0, and so on.

Assuming (3.49) we get from (3.46) the inequality

$$(1 - Ce^T\|\widehat{u}\|_{q_1,s_1})\, |\|U|\|_{2,2} \le Ce^T\|\widehat{u}\|_{q_1,s_1}\, \|W\|_{q_3,s_3}. \tag{3.50}$$

This estimate has been developed under the assumption that $|\|U|\|_{2,2} < \infty$. Since this is not yet known, we use a procedure to approximate U by a sequence $(U_j)_{j=1}^{\infty}$ such that (3.50) holds with U replaced by U_j. For this purpose we consider (3.45) as a linear equation for U with fixed $\widehat{u}$, the term $-e^t\widehat{A}^{-\frac{1}{2}}\, P\,\mathrm{div}\,(\widehat{u}U)$ is treated as a perturbation. Then we let $j \to \infty$ and get the desired property $|\|U|\|_{2,2} < \infty$.

Next we show that

$$|\|W|\|_{2,2} \le |\|F|\|_{2,2} + |\|S|\|_{2,2} < \infty.$$

For this purpose we argue as in (3.38), (3.47) and obtain

$$|\|F|\|_{2,2} \le C_1\, \|\widehat{A}^{-\frac{1}{2}}F_t + \widehat{A}^{\frac{1}{2}}F\|_{2,2} \le C_2\|f\|_{2,2} < \infty \tag{3.51}$$

with constants $C_1, C_2 > 0$. Similarly we get with $S_t = -\widehat{A}S$ and the above estimate that

$$\begin{aligned}|\|S|\|_{2,2} &= \|\widehat{A}^{-\frac{1}{2}}S_t\|_{2,2} + \|\widehat{A}^{\frac{1}{2}}S\|_{2,2}\\ &= 2\,\|\widehat{A}^{\frac{1}{2}}S\|_{2,2}\\ &\le C\,\|u_0\|_2 < \infty\end{aligned}$$

with $C > 0$. Using $\widehat{u} = U + W$ we see that

$$\begin{aligned}|\|\widehat{u}|\|_{2,2} &= \|\widehat{A}^{-\frac{1}{2}}\widehat{u}_t\|_{2,2} + \|\widehat{A}^{\frac{1}{2}}\widehat{u}\|_{2,2}\\ &\le |\|U|\|_{2,2} + |\|W|\|_{2,2} < \infty.\end{aligned}$$

This yields (3.31), and the proof of Theorem 3.1 is complete. $\square$

4. Proof of Theorem 1.2

First we prove Theorem 1.2, Theorem 1.1 is essentially a corollary.

Consider u_0, f, u, q, s as in Theorem 1.2. We set

$$\Gamma \;=\; 2^{-(2+\frac{1}{s})} K e^{-1}$$

where $K = K(\Omega, q, s) > 0$ is the constant in Theorem 3.1. Then we will show that u is regular if (1.12) is satisfied. Thus we have to prove, using (2.9) with $X = L^{q,\infty}$, that

$$\|u\|_{L^{s,\infty}(0,T;L^{q,\infty})} \;=\; \sup_{R\geq 0} \,(R\mu(u,R)^{\frac{1}{s}}) \;\leq\; \Gamma \tag{4.1}$$

implies the regularity of u. Recall that

$$\mu(u,R) \;=\; m(\{t \in [0,T);\, \|u(t)\|_{L^{q,\infty}} \;>\; R\}), \quad R \;\geq\; 0,$$

where m denotes the Lebesgue measure. Thus we suppose (4.1).

We use the following arguments. First we choose some T_0 with $0 < T_0 \leq \frac{1}{2}$ such that

$$\|f\|_{L^s(t,t+2T_0;L^q)} \;=\; \left(\int_t^{t+2T_0} \|f(\tau)\|_{L^q}^s \, d\tau\right)^{\frac{1}{s}} \;\leq\; \frac{K}{2}\, e^{-1} \tag{4.2}$$

is satisfied for all $t \in [0,T)$ with $t + 2T_0 \leq T$. This is possible, choosing T_0 sufficiently small, since

$$\|f\|_{L^s(0,T;L^q)} \;=\; \left(\int_0^T \|f(\tau)\|_{L^q}^s \, d\tau\right) \;<\; \infty.$$

Further we will prove the following property: For each $t \in [0,T)$ and each $T_1 > 0$ with $t + 3T_1 \leq T$, $T_1 \leq T_0$, there exists some $t_1 \in [t, t+T_1)$, such that the energy inequality (2.16) holds with $\tau = t_1$, and

$$u(t_1) \in L^2_\sigma \cap L^{q,\infty}, \quad (2T_1)^{\frac{1}{s}} \|u(t_1)\|_{L^{q,\infty}} \;\leq\; \frac{K}{2}\, e^{-1}. \tag{4.3}$$

Consider any t, T_1, t_1 with this property. Then we conclude with (4.3) that

$$\|f\|_{L^s(t_1,t_1+2T_1;L^q)} + (2T_1)^{\frac{1}{s}} \|u(t_1)\|_{L^{q,\infty}} \;\leq K\, e^{-1} \;\;\leq\;\; K\, e^{-2T_1}.$$

Therefore we can use Theorem 3.1 and find a strong solution $u^\star$ with initial value $u(t_1)$, defined in the interval $[t_1, t_1 + 2T_1)$; $u^\star$ satisfies Serrin's condition

$$u^\star \;\in L^{s_1}(t_1, t_1 + 2T_1; L^{q_1})$$

with exponents $3 < q_1 < \infty,\; 2 < s_1 < \infty,\; \frac{3}{q_1} + \frac{2}{s_1} = 1$.

Using the energy inequality (2.16) with $\tau = t_1$, we conclude with Serrin's uniqueness criterion that $u = u^\star$ holds in $[t_1, t_1 + 2T_1)$. Since $t_1 \le t + T_1$, we conclude that

$$u \in L^{s_1}(t + T_1, t + 2T_1; L^{q_1}). \tag{4.4}$$

Serrin's regularity criterion now yields in particular that

$$u \in C^\infty((t + T_1, t + 2T_1) \times \overline{\Omega})^3. \tag{4.5}$$

Since (4.5) holds for each $t \in [0, T)$ and each $T_1 > 0$ with $t + 3T_1 \le T$, $T_1 \le T_0$, we see that

$$u \in C^\infty((0, T) \times \overline{\Omega})^3,$$

and this yields the result of Theorem 1.2.

Using (4.5), we conclude in the same way that Serrin's condition

$$u \in L^{s_1}(T_1, T; L^{q_1}) \tag{4.6}$$

is satisfied for each T_1 with $0 < T_1 < T$.

Thus it remains to prove the existence of t_1 in (4.3). For this purpose let $t \in [0, T)$, $T_1 > 0$ with $t + 3T_1, \le T$, $T_1 \le T_0$. Then we set

$$\mu_1(u, R) = m(\{\tau \in [t, t + T_1); \|u(\tau)\|_{L^{q,\infty}} > R\}) \tag{4.7}$$

for all $R \ge 0$ where m means the Lebesgue measure. Using the definition in (2.8), (2.10) we see that $\mu_1(u, R) \le \mu(u, R)$ for all $R \ge 0$. Hence we obtain

$$\|u\|_{L^{s,\infty}(t,t+T_1;L^{q,\infty})} = \sup_{R \ge 0} (R\mu_1(u, R)^{\frac{1}{s}}) \le \|u\|_{L^{s,\infty}(0,T;L^{q,\infty})} \le \Gamma. \tag{4.8}$$

Note that the function $R \mapsto \mu_1(u, R)$, $R \ge 0$, is right-continuous and non-increasing. This follows by an elementary consideration, see [BeLoe76], [BuBe67], [Tri78]. Further we see that $0 \le \mu_1(u, R) \le T_1$.

To construct t_1 we consider the following cases a), b), c):

a) Let $\mu_1(u, 0) < T_1$. Then we see that

$$m(\{\tau \in [t, t + T_1)\,;\ \|u(\tau)\|_{L^{q,\infty}} = 0\}) \ne 0. \tag{4.9}$$

Therefore we can choose $t_1 \in [t, t + T_1)$ with $\|u(t_1)\|_{L^{q,\infty}} = 0$ in such a way that (2.16) holds for $\tau = t_1$. We obtain

$$(2T_1)^{\frac{1}{s}} \|u(t_1)\|_{L^{q,\infty}} = 0 \le \frac{K}{2} e^{-1}.$$

Since u is weakly continuous as a function from $[0,T)$ to $L^2_\sigma(\Omega)$, we get $u(t) \in L^2_\sigma$ for all $t \in [0,T)$. Thus it holds (4.3).

b) Let $\mu_1(u,0) = T_1$, and let $\mu_1(u,R_0) < T_1$ where $R_0 \geq 0$ is defined by

$$R_0 \;=\; \sup\{R \geq 0\,;\; \mu_1(u,R) \;=\; T_1\}. \tag{4.10}$$

We see that $R_0 > 0$ and that the function $R \mapsto \mu_1(u,R)$ is not continuous at $R = R_0$. This shows that

$$m(\{\tau \in [t,t+T_1);\; \|u(\tau)\|_{L^{q,\infty}} = R_0\}) \;\neq\; 0. \tag{4.11}$$

Now we choose $t_1 \in [t,t+T_1)$ with $\|u(t_1)\|_{L^{q,\infty}} = R_0$. Using (4.8) we see that

$$R\mu_1(u,R)^{\frac{1}{s}} \;=\; RT_1^{\frac{1}{s}} \;\leq\; \Gamma, \quad 0 \;\leq\; R \;<\; R_0$$

and therefore we get

$$\begin{aligned}(2T_1)^{\frac{1}{s}}\,\|u(t_1)\|_{L^{q,\infty}} \;&=\; (2T_1)^{\frac{1}{s}}\,R_0\\ &=\; 2^{\frac{1}{s}}\,\sup\{T_1^{\frac{1}{s}}R\,;\;0\leq R\leq R_0\}\\ &\leq\; 2^{\frac{1}{s}}\Gamma \;=\; \frac{K}{4}\,e^{-1} \;\leq\; \frac{K}{2}\,e^{-1}\end{aligned} \tag{4.12}$$

Using (4.11) instead of (4.9), we see in the same way as in a) that t_1 can be chosen such that (2.16) holds for $\tau = t_1$. Thus t_1 has the desired property (4.3).

c) Let $\mu_1(u,0) = T_1$ and let $\mu_1(u,R_0) = T_1$ with R_0 as in (4.10). We see that the function $R \mapsto \mu_1(u,R)$ is continuous at R_0. Therefore, for each $\varepsilon > 0$ it holds

$$m(\{\tau \in [t,t+T_1)\,;\; R_0 \leq \|u(\tau)\|_{L^{q,\infty}} \;\leq\; R_0+\varepsilon\}) \;\neq\; 0. \tag{4.13}$$

Otherwise we would get $\mu_1(u,R) = T_1$ for $R_0 \leq R < R_0+\varepsilon$ which is a contradiction to (4.10).

Now we choose $t_1 \in [t,t+T_1)$ with $R_0 \leq \|u(t_1)\|_{L^{q,\infty}} \leq R_0+\varepsilon$. Then we get

$$(2T_1)^{\frac{1}{s}}\,R_0 \;\leq\; 2^{\frac{1}{s}}\Gamma \;\leq\; \frac{K}{4}\,e^{-1}$$

in the same way as in (4.12), but in this case we conclude that

$$\begin{aligned}(2T_1)^{\frac{1}{s}}\,\|u(t_1)\|_{L^{q,\infty}} \;&\leq\; (2T_1)^{\frac{1}{s}}\,(R_0+\varepsilon)\\ &\leq\; \frac{K}{4}\,e^{-1} + \varepsilon R_0^{-1}\frac{K}{4}\,e^{-1} \;\leq\; \frac{K}{2}\,e^{-1}\end{aligned}$$

if $0 < \varepsilon \leq R_0$. If $R_0 = 0$, we choose $\varepsilon > 0$ such that

$$(2\,T_1)^{\frac{1}{s}}\,\varepsilon \;\leq\; \frac{K}{2}\,e^{-1}.$$

Then (4.3) is satisfied. Using (4.13) we can reach as before that (2.16) holds for $\tau = t_1$.

Thus it holds (4.3) in all cases and the proof of the theorem is complete.

5. Proof of Theorem 1.1

Let u be a weak solution as in this theorem and suppose that (1.10) is satisfied with $\mu(u, R)$ as in (4.1).

Using the definition in (2.8), (2.9) with $\mu(u, R)$ from (4.1), we obtain with Hölder's inequality the estimate

$$
\begin{aligned}
\|u\|_{L^{s,r}(0,T;L^{q,\infty})} &= \left(\int_0^\infty R^r \mu(u, R)^{\frac{r}{s}} \frac{dR}{R}\right)^{\frac{1}{r}} \\
&\leq \left(\int_0^{R_0} R^r \mu(u, R)^{\frac{r}{s}} \frac{dR}{R}\right)^{\frac{1}{r}} + \left(\int_{R_0}^\infty R^r \mu(u, R)^{\frac{r}{s}} \frac{dR}{R}\right)^{\frac{1}{r}} \\
&\leq T^{\frac{1}{s}} \left(\int_0^{R_0} R^{r-1}\, dR\right)^{\frac{1}{r}} + \left(\int_{R_0}^\infty R^{r-1} \mu(u, R)^{\frac{r}{s}}\, dR\right)^{\frac{1}{r}} \\
&= T^{\frac{1}{s}} R_0 r^{-\frac{1}{r}} + \left(\int_{R_0}^\infty R^{r-1} \mu(u, R)^{\frac{r}{s}}\, dR\right)^{\frac{1}{r}}
\end{aligned}
$$

for all $0 < R_0 < \infty$.

Given $\varepsilon > 0$, we first choose some $R_0 > 0$ with

$$
\left(\int_{R_0}^\infty R^{r-1} \mu(u, R)^{\frac{r}{s}}\, dR\right)^{\frac{1}{r}} \leq \frac{\varepsilon}{2}. \tag{5.1}
$$

This is possible since the norm above is finite. Then we choose some T_0 with $0 < T_0 \leq T$ such that

$$
T_0^{\frac{1}{s}}\, r^{-\frac{1}{r}}\, R_0 \leq \frac{\varepsilon}{2}.
$$

Applying the above estimate with T replaced by T_0, we see that (5.1) remains valid. Thus we get

$$
\|u\|_{L^{s,r}(0,T_0;L^{q,\infty})} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \tag{5.2}
$$

Since the function $R \mapsto \mu(u, R)$ is non-increasing, we obtain

$$
\begin{aligned}
\|u\|_{L^{s,r}(0,T;L^{q,\infty})} &\geq \left(\int_0^{R_0} R^{r-1} \mu(u, R)^{\frac{r}{s}}\, dR\right)^{\frac{1}{r}} \\
&\geq \mu(u, R_0)^{\frac{1}{s}}\, r^{-\frac{1}{r}}\, R_0,
\end{aligned}
$$

and therefore

$$\begin{aligned}\|u\|_{L^{s,\infty}(0,T;L^{q,\infty})} &= \sup_{R_0\geq 0} (R_0\mu(u,R_0)^{\frac{1}{s}}) \\ &\leq r^{\frac{1}{r}} \|u\|_{L^{s,r}(0,T;L^{q,\infty})}.\end{aligned}$$

Next we apply the last estimate with T replaced by T_0, and use (5.2) with $\varepsilon = r^{-\frac{1}{r}}\Gamma$, where $\Gamma = \Gamma(\Omega, q, s)$ is the constant in (4.1). Then we get

$$\|u\|_{L^{s,\infty}(0,T_0;L^{q,\infty})} \leq \Gamma,$$

and we can apply Theorem 1.2 with T replaced by T_0. This yields the regularity of u in $(0, T_0)$. The estimate (5.2) remains valid if $[0, T_0)$ is replaced by any interval $[t, t + T_0)$ with $t \in [0, T)$, $t + T_0 \leq T$. Therefore, we obtain the regularity of u in the whole interval $(0, T)$. The proof of Theorem 1.1 is complete.

We see, Theorem 1.1 is a consequence of Theorem 1.2. In particular we see that u satisfies Serrin's condition (1.13) for all δ with $0 < \delta < T$.

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