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Long-time stabilization of solutions to a phase-field model with memory

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Abstract. We prove that any global bounded solution of a phase field model with memory terms tends to a single equilibrium state for large times. Because of the memory effects, the energy is not a Lyapunov function for the problem and the set of equilibria may contain a nontrivial continuum of stationary states. The method we develop is applicable to a more general class of equations containing memory terms.

1. Introduction

The time evolution of the phase variable $\chi(t, x)$ and the temperature $\vartheta(t, x)$ in the phase-field model proposed by Caginalp [5] is governed by the system of differential equations:

$$\partial_t \chi - \Delta\chi + W'(\chi) = \lambda'(\chi)\vartheta, \psi \tag{1.1}$$

$$\partial_t(\vartheta + \lambda(\chi)) + \operatorname{div} \mathbf{q} = 0 \tag{1.2}$$

where W and λ are given functions and $\mathbf{q}$ denotes the heat flux. Here we shall assume that $\mathbf{q}$ is determined by the linearized Coleman - Gurtin [6] constitutive relation:

$$\mathbf{q} = -k_I \nabla\vartheta - k * \nabla\vartheta\psi \tag{1.3}$$

where the constant $k_I > 0$ is the instantaneous heat conductivity, k is a suitable dissipative kernel, and the symbol $*$ denotes the time convolution:

$$k * v(t) = \int_0^\infty k(s)v(t-s)\,\mathrm{d}s.$$

The material occupies a bounded regular domain $\Omega \subset R^3$ and the system (1.1)–(1.3) is complemented by the homogeneous Neumann boundary condition for the phase variable

$$\nabla\chi.\mathbf{n}|_{\partial\Omega} = 0, \text{ with } \mathbf{n} \text{ the outer normal vector,} \tag{1.4}$$

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while ϑ obeys the homogeneous Dirichlet condition

$$\vartheta|_{\partial\Omega} = 0. \tag{1.5}$$

Systems of the same or comparable type have been recently studied by many authors (see Aizicovici and Barbu [1], Colli and Laurençot [7], Giorgi et al. [12], [13], etc). In particular, the long-time behavior of solutions seems to be well understood and the equilibrium (stationary) solutions of the problem

$$-\Delta\chi_\infty + W'(\chi_\infty) = 0,\ \nabla\chi_\infty.\mathbf{n}|_{\partial\Omega} = 0 \text{ and } \vartheta_\infty \equiv 0 \tag{1.6}$$

have been identified as the only candidates to belong to the ω-limit set of each individual trajectory (cf. [1, Theorems 3.3, 3.4] or [7, Theorem 2.6]). If the problem (1.6) admits only a finite number of solutions, then any solution $\chi(t)$, $\vartheta(t)$ converges as $t \to \infty$ to a single stationary state. Positive results in this direction in the one-dimensional case were obtained, e.g., by Aizicovici and Barbu [1], Elliott and Zheng [9] or Grinfeld and Novick-Cohen [14]. However, the structure of the set of stationary solutions for a general domain may be quite complicated, in particular, the set in question may contain a continuum of nonradial solutions if Ω is a ball or an annulus. If this is the case, it seems highly nontrivial to decide whether or not the solutions converge to a single stationary state. In fact, it is well-known that this might not be the case even for finite-dimensional dynamical systems (cf. Aulbach [3]), and similar examples for semilinear parabolic equations were derived by Poláčik and Rybakowski [19].

In 1983, Simon [21] developed a method to study the long-time behaviour of gradient-like dynamical systems based on deep results from the theory of analytic functions of several variables due to Lojasiewicz [18]. Roughly speaking, an analytic function behaves like a polynomial (of a sufficiently high degree) in a neighbourhood of any point where its gradient vanishes (critical points). More specifically, the following assertion holds (see [18, Theorem 4, page 88]):

PROPOSITION 1.1. *Let $G : U(a) \to C$ be a real analytic function defined on an open neighbourhood $U(a)$ of a point $a \in R^n$.*

Then there exist $\theta \in (0, \frac{1}{2})$ and $\delta > 0$ such that

$$|\nabla G(z)| \geq |G(z) - G(a)|^{1-\theta} \text{ for all } z \in R^n,\ |z - a| < \delta.$$

Simon succeded in proving a generalized version of the above theorem applicable to analytic functionals on Banach spaces. Later on, Jendoubi [17], and Haraux and Jendoubi [15] simplified considerably Simon's original approach making it accessible for application to a broad class of semilinear problems with variational structure. Related results in this direction were also obtained by Feireisl and Takáč [11], Hoffmann and Rybka [16] etc.

Last but not least, the same method has been successfully modified to deal with degenerate parabolic equations of porous media type (see [10]).

In some cases, Simon's approach can be used to deal with problems with only a partial variational structure. A typical example could be the system (1.1)–(1.3) with the memory term omitted in (1.3) (i.e., for $k = 0$). Indeed the "elliptic" part of (1.1) is the variational derivative of the free energy functional with respect to χ while (1.2) is not. On the other hand, since the temperature always tends to zero when time is large, it is possible to modify Simon's method to prove convergence of the phase variable χ to a single stationary state under fairly general conditions imposed on λ and W (see [2, Theorem 2.1]). It is the aim of the present paper to show that similar results can be obtained when the memory effects are taken into account in (1.3). Specifically, our main result is the following:

THEOREM 1.2. *Let* $\Omega \subset R^3$ *be a bounded domain of class* $C^{2+\mu}$, $\mu > 0$. *Suppose, moreover, that the nonlinearities* λ, W *satisfy the following hypotheses:*

$$\text{The function } \lambda \text{ is of class } C^{1+\mu}(R),\ \lambda(0) = 0,\ |\lambda'(z)| \leq \Lambda \text{ for all } z \in R; \tag{1.7}$$

The "free energy" function W *is twice continuously differentiable on* R,

$$\begin{aligned} W'(z)z &> 0 \text{ for all } z,\ |z| > 1,\ W'(z)\,\mathrm{sgn}(z) \\ &\geq \mu|z| - Q,\ \mu > 0,\ Q \geq 0 \text{ for all } z \in R, \end{aligned} \tag{1.8}$$

and

$$W \text{ is real analytic on the open interval } (-1, 1). \tag{1.9}$$

In addition, we assume that the instantaneous heat conductivity $k_I > 0$ *is strictly positive and the kernel* k *satisfies:*

$$\begin{aligned} &k \in L^1(0, \infty),\ k \text{ is convex on } (0, \infty), \\ &\quad \mathrm{d}k'(s) + \delta k'(s)\,\mathrm{d}s \geq 0 \text{ for a certain } \delta > 0. \end{aligned} \tag{1.10}$$

Then for any globally defined (classical) solution χ, ϑ *of the problem* (1.1)–(1.5), *there exists* χ_∞ *- a solution of the stationary problem* (1.6) *such that*

$$\chi(t) \to \chi_\infty,\ \vartheta(t) \to 0 \text{ in } C(\overline{\Omega}) \text{ as } t \to \infty.$$

REMARK 1.3. Here a globally defined classical solution means that χ_t, ϑ_t, $D_x^2\chi$, $D_x^2\vartheta$ are continuous on $(0, \infty) \times \Omega$, $\nabla\chi$, $\nabla\vartheta$ are continuous up to the boundary $\partial\Omega$, (1.1)–(1.3) hold on $(0,\infty)\times\Omega$, and the boundary conditions (1.4), (1.5) are satisfied for all $t \in (0,\infty)$. Moreover, the past values of ϑ are given for $t \in (-\infty, 0]$, and ϑ, $\nabla\vartheta$, and $D_x^2\vartheta$ are bounded uniformly for $t \in (-\infty, 0]$, $x \in \Omega$.

REMARK 1.4. A typical example of a kernel k satisfying (1.10) is $k(s) = s^{-\alpha}e^{-\beta s}$, $0 \le \alpha < 1$, $\beta > 0$.

Note that the class of functions λ, W covers all the cases considered in the literature; in particular, the polynomial nonlinearities investigated by Brochet et al. [4], and Elliott and Zheng [9] are included. Moreover, in contrast to Simon's original paper [21] as well as its subsequent adaptations by Jendoubi [17] or Hoffmann and Rybka [16], we require analyticity of W only on the interval containing all zeros of W'. This allows for a much broader class of nonlinearities to which our result applies. Finally note that λ needs not be analytic at all.

The assumption that χ, ϑ is a classical solution of the problem is not restrictive. It will be clear from the estimates presented in Section 2 that any weak solution emanating from smooth initial data will be globally defined and regular on the interval $(0,\infty)$.

2. A priori estimates. Asymptotic compactness

We present some a priori estimates of solutions of the problem (1.1)–(1.5) based on more or less standard arguments. Comparable results can be found in the existing literature (cf. e.g., Dafermos [8], Giorgi et al. [12], etc.) As a consequence, we obtain useful information on the structure of the ω-limit sets related to globally defined solutions. Before we start, let us review some properties of the kernel k that follow from the hypothesis (1.10).

LEMMA 2.1. *Let k satisfy* (1.10).
Then

$$\lim_{s\to 0+} s\,k(s) = \lim_{s\to 0+} s^2 k'(s) = 0, \tag{2.1}$$

$$\lim_{s\to\infty} s\,k(s) = \lim_{s\to\infty} s^2 k'(s) = 0, \tag{2.2}$$

$$s\,k'(s) \in L^1(0,1),\ \int_0^1 s^2 \mathrm{d}k'(s) < \infty, \tag{2.3}$$

and

$$s^2 k'(s) \in L^1(1,\infty),\ \int_1^\infty s^2 \mathrm{d}k'(s) < \infty. \tag{2.4}$$

Proof. The relations (2.1) and (2.3) are proved by Prüss [20, Proposition 3.6]. Moreover, since $k \in L^1(0,\infty)$ is nonnegative and nonincreasing, we have

$$\lim_{s\to\infty} sk(s) = 0.$$

Using

$$\int_1^\infty (-k)'(s)s \,\mathrm{d}s = [-k(s)s]_{s=1}^{s=\infty} + \int_1^\infty k(s)\,\mathrm{d}s,$$

we see that $(-k)'(s)s \in L^1(0,\infty)$, $(-k)'(s)s \geq 0$. Moreover, by virtue of (1.10), we have

$$\begin{aligned}
&(-k')(z_1+)z_1 - (-k')(z_2-)z_2 = z_1[(-k')(z_1+) - (-k')(z_2-)]\\
&\quad + (-k')(z_2-)(z_1 - z_2) = z_1 \int_{(z_1,z_2)} \mathrm{d}k'(z) + (-k')(z_2-)(z_1 - z_2)\\
&\quad \geq \delta z_1 \int_{z_1}^{z_2} (-k')(z)\,\mathrm{d}z + (-k')(z_2-)(z_1 - z_2)\\
&\quad \geq (-k')(z_2-)(z_2 - z_1)(\delta z_1 - 1) \text{ for all } 0 < z_1 < z_2.
\end{aligned}$$

Consequently, $s(-k')(s)$ is nonincreasing for large values of s and we have

$$\lim_{s\to\infty} s^2 k'(s) = 0$$

which completes the proof of (2.2).

Finally,

$$\int_{(\alpha,\beta)} s^2\,\mathrm{d}k'(s) = [s^2 k'(s)]_{s=\alpha+}^{s=\beta-} - 2[sk(s)]_{s=\alpha}^{s=\beta} + 2\int_\alpha^\beta k(s)\,\mathrm{d}s$$

which, together with (2.1), (2.2), proves the second relation in (2.4); the first one then follows immediately by (1.10). □

Now, we are ready to deduce the energy equality. Multiplying the equation (1.1) by χ_t, (1.2) by ϑ, and integrating by parts, we obtain:

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \left[\frac{1}{2}(|\nabla\chi|^2 + |\vartheta|^2) + W(\chi)\right]\mathrm{d}x + \|\chi_t\|^2_{L^2(\Omega)}\\
&\quad + k_I\|\nabla\vartheta\|^2_{L^2(\Omega)} + \int_\Omega (k * \nabla\vartheta)\nabla\vartheta\,\mathrm{d}x = 0.
\end{aligned} \tag{2.5}$$

Following [8], [12] we introduce the quantity

$$\eta(t,s,x) = \int_{t-s}^{t} \vartheta(z,x)\,\mathrm{d}z,\ \ s \geq 0.$$

Accordingly, making use of Lemma 2.1, we can write

$$k * \nabla\vartheta = \int_0^\infty k(s)\frac{\partial}{\partial s}\nabla\eta(t,s)\,\mathrm{d}s = -\int_0^\infty k'(s)\nabla\eta(t,s)\,\mathrm{d}s.$$

Thus

$$\int_\Omega (k * \nabla\vartheta)\nabla\vartheta\,\mathrm{d}x = \frac{1}{2}\left[\frac{d}{dt}\int_0^\infty (-k')(s)\|\nabla\eta(t,s)\|^2_{L^2(\Omega)}\,\mathrm{d}s + \int_0^\infty (-k)'(s)\frac{\partial}{\partial s}\|\nabla\eta(t,s)\|^2_{L^2(\Omega)}\,\mathrm{d}s\right];$$

whence, by virtue of Lemma 2.1,

$$\int_\Omega (k * \nabla\vartheta)\nabla\vartheta\,\mathrm{d}x = \frac{1}{2}\left[\frac{d}{dt}\int_0^\infty (-k')(s)\|\nabla\eta(t,s)\|^2_{L^2(\Omega)}\,\mathrm{d}s + \int_0^\infty \|\nabla\eta(t,s)\|^2_{L^2(\Omega)}\,\mathrm{d}k'(s)\right].$$

Consequently, the relation (2.5) takes the form

$$\begin{aligned}&\frac{d}{dt}\left[\int_\Omega \frac{1}{2}(|\nabla\chi(t)|^2 + |\vartheta(t)|^2) + W(\chi(t))\mathrm{d}x + \frac{1}{2}\int_0^\infty (-k')(s)\|\nabla\eta(t,s)\|^2_{L^2(\Omega)}\mathrm{d}s\right]\\ &+ \|\chi_t(t)\|^2_{L^2(\Omega)} + k_I\|\nabla\vartheta(t)\|^2_{L^2(\Omega)}\\ &+ \frac{1}{2}\int_0^\infty \|\nabla\eta(t,s)\|^2_{L^2(\Omega)}\,\mathrm{d}k'(s) = 0.\end{aligned} \tag{2.6}$$

We thereby arrive at:

LEMMA 2.2. *Under the hypotheses of Theorem* 1.2, *there exists* E_0 *depending only on the quantities*

$$\sup_{t\in(-\infty,0]} \|\nabla\vartheta(t)\|_{L^2(\Omega)},\ \|\nabla\chi(0)\|_{L^2(\Omega)},\ \|\chi(0)\|_{L^\infty(\Omega)}$$

such that

$$\sup_{t>0}\|\vartheta(t)\|_{L^2(\Omega)} + \sup_{t>0}\|\nabla\chi(t)\|_{L^2(\Omega)} \le E_0, \tag{2.7}$$

$$\int_0^\infty \|\chi_t(t)\|^2_{L^2(\Omega)} + \|\vartheta(t)\|^2_{W_0^{1,2}(\Omega)}\,\mathrm{d}t < E_0. \tag{2.8}$$

Similarly, we can multiply (1.2) by $-\Delta\vartheta$ and integrate by parts to deduce

$$\begin{aligned}&\frac{d}{dt}\|\nabla\vartheta\|^2_{L^2(\Omega)} + 2k_I\|\Delta\vartheta\|^2_{L^2(\Omega)} + 2\int_\Omega (k * \Delta\vartheta)\Delta\vartheta \,\mathrm{d}x \\ &= \int_\Omega \lambda'(\chi)\chi_t\Delta\vartheta \,\mathrm{d}x.\end{aligned} \tag{2.9}$$

Now, using (2.8) and treating the convolution term in the same way as when passing from (2.5) to (2.6), we obtain:

LEMMA 2.3. *Under the hypotheses of Theorem* 1.2, *there exists* E_1 *depending only on the quantities*

$$\sup_{t\in(-\infty,0]} \|\vartheta(t)\|_{W^{2,2}(\Omega)},\ \|\nabla\chi(0)\|_{L^2(\Omega)},\ \|\chi(0)\|_{L^\infty(\Omega)}$$

such that

$$\sup_{t>0}\|\vartheta(t)\|_{W_0^{1,2}(\Omega)} + \int_0^\infty \|\Delta\vartheta(t)\|^2_{L^2(\Omega)} \,\mathrm{d}t < E_1. \tag{2.10}$$

Moreover,

$$\begin{aligned}&\frac{d}{dt}\left[\|\nabla\vartheta\|^2_{L^2(\Omega)} + \int_0^\infty (-k')(s)\|\Delta\eta(t,s)\|^2_{L^2(\Omega)}\, ds\right] + \\ &c_1\left[\|\nabla\vartheta\|^2_{L^2(\Omega)} + \int_0^\infty (-k')(s)\|\Delta\eta(t,s)\|^2_{L^2(\Omega)}\, ds\right] \le c_2\Lambda\|\chi_t\|^2_{L^2(\Omega)}\end{aligned} \tag{2.11}$$

for certain positive constants c_1, c_2.

As a straighforward consequence of (2.8) and (2.11), we get

$$\vartheta(t) \to 0 \text{ in } W_0^{1,2}(\Omega) \text{ as } t \to \infty. \tag{2.12}$$

With the hypotheses (1.7), (1.8) at hand, we can use the comparison principle to deduce

$$\chi(t) \le S(t) \text{ for all } t \ge 0$$

where S solves the linear equation

$$S_t - \Delta S + \mu S = \Lambda|\vartheta| + Q,\ \nabla S.\mathbf{n} = 0,\ S(0) = \sup_{x\in\Omega}|\chi(0,x)|.$$

Now it is easy to see, by virtue of (2.12) and standard parabolic regularity estimates, that

$$\limsup_{t\to\infty} \sup_{x\in\Omega} |S(t,x)| \le \overline{S}$$

where $\overline{S}$ depends only on μ, Q, and Λ. This yields an upper bound on the solution χ which is independent of the initial data. Similarly, a lower bound can be obtained. Moreover, the trajectory $\{\chi(t)\}_{t>T}$ is precompact in $C(\overline{\Omega}) \cap W^{1,2}(\Omega)$ and, by virtue of (2.8), (2.12) the ω-limit set

$$\omega(\chi) = \left\{ \chi_\infty \mid \chi_\infty = \lim_{t_n\to\infty} \chi(t_n) \text{ in } C(\overline{\Omega}) \cap W^{1,2}(\Omega) \text{ for a certain } t_n \to \infty \right\}$$

is non-empty, compact in $C(\overline{\Omega}) \cap W^{1,2}(\Omega)$ and contained in the set of solutions of the problem (1.6). Finally, using a bootstrap argument, one can obtain uniform estimates on higher order derivatives of χ, ϑ, in particular, one can show that the convergence in (2.12) takes place in $C(\overline{\Omega})$.

Summing up the previous results we obtain the following auxiliary assertion:

PROPOSITION 2.4. *Under the hypotheses of Theorem* 1.2 *we have*

$$\vartheta(t) \to 0 \textit{ strongly in } C(\overline{\Omega}) \textit{ as } t \to \infty. \tag{2.13}$$

Moreover, any sequence of times $t_n \to \infty$ contains a subsequence (not relabeled) such that

$$\chi(t_n) \to \chi_\infty \textit{ strongly in } C(\overline{\Omega}) \cap W^{1,2}(\Omega) \tag{2.14}$$

where χ_∞ is a solution of the stationary problem (1.6).

3. The ω-limit sets

As we have seen in Proposition 2.4, the ω-limit set $\omega(\chi)$ of any global trajectory is contained in the set of stationary solutions. Accordingly, we review some information on the structure of this set.

LEMMA 3.1. *Assume W' is locally Lipschitz continuous and satisfies* (1.8). *Let χ_∞ be a (classical) solution of the stationary problem* (1.6).

Then either

$$\chi_\infty \equiv -1, \textit{ or } \chi_\infty \equiv 1,$$

or

$$-1 < \inf_{x\in\overline{\Omega}} \chi_\infty(x) \le \sup_{x\in\overline{\Omega}} \chi_\infty(x) < 1. \tag{3.1}$$

Proof. By virtue of (1.8) and the maximum principle, any solution of (1.6) satisfies

$$-1 \le \inf_{x\in\overline{\Omega}} \chi_\infty(x) \le \sup_{x\in\overline{\Omega}} \chi_\infty(x) \le 1.$$

Consequently, the function $w = 1 - \chi_\infty$ satisfies

$$-\Delta w + z(x)w \ge 0,\ \ w \ge 0,\ \ \nabla w.\mathbf{n}|_{\partial\Omega} = 0$$

for a certain bounded z. By virtue of the strong maximum principle, either $w \equiv 0$ or

$$\inf_{x\in\overline{\Omega}} w(x) > 0$$

which is the right-most inequality in (3.1). Now the rest of the proof is completely analogous, by applying the same argument to $w = \chi_\infty + 1$. □

The next assertion is an easy consequence of Lemma 3.1 and the topological properties of the ω-limit set $\omega(\chi)$.

LEMMA 3.2. *Under the hypotheses of Theorem* 1.2*, the ω-limit set $\omega(\chi)$ is either a singleton or there exists $r \in [0, 1)$ such that*

$$-1 < -r \le \inf_{x\in\overline{\Omega}} \chi_\infty(x) \le \sup_{x\in\overline{\Omega}} \chi_\infty \le r < 1 \textit{ for all } \chi_\infty \in \omega(\chi). \tag{3.2}$$

Proof. Assume that $\omega(\chi)$ is not a singleton and (3.2) does not hold. Then, since $\omega(\chi)$ is connected, there exists a sequence of functions $\chi_\infty^n \in \omega(\chi)$ such that

$$\sup_{x\in\overline{\Omega}} \chi_\infty^n = r_n < 1 \text{ and } r_n \to 1 \tag{3.3}$$

or

$$\inf_{x\in\overline{\Omega}} \chi_\infty^n = -r_n > -1,\ \text{ and } r_n \to -1.$$

We concentrate on the former situation showing that it leads to a contradiction; the latter case can be treated in a similar way.

It is easy to show that the set of solutions of (1.6) is compact in $C(\overline{\Omega})$. Consequently, we may assume

$$\chi_\infty^n \to \chi_\infty \text{ uniformly on } \overline{\Omega}$$

where, since $\omega(\chi)$ is closed, $\chi_\infty \in \omega(\chi)$, and, by virtue of (3.3),

$$\sup_{x\in\overline{\Omega}} \chi_\infty(x) = 1.$$

In view of Lemma 3.1, we have $\chi_\infty \equiv 1 \in \omega(\chi)$.

Now, since $1 \in \omega(\chi)$, for any n there exists a time t_n such that

$$\chi_\infty^n \le \chi(t_n);$$

whence, by the comparison principle,

$$\chi(t) \ge \chi_\infty^n \text{ for all } t \ge t_n. \tag{3.4}$$

Since $\chi_\infty^n \to 1$ uniformly on $\overline{\Omega}$, the relation (3.4) implies $\chi(t) \to 1$ as $t \to \infty$, i.e., $\omega(\chi)$ is a singleton - a contradiction. □

4. A generalized version of the Lojasiewicz theorem

In this section we collect some preparatory material for the proof of Theorem 1.2. To begin with, it is important to observe that the conclusion of Theorem 1.2 holds if $\omega(\chi)$ is a singleton. Consequently, from now on, we shall assume that $\omega(\chi)$ contains at least two different functions. In particular, by virtue of Lemma 3.2, $\omega(\chi)$ is contained in the interval $[-r, r]$ with $r < 1$. Accordingly, we are allowed to suppose, without loss of generality that W' has been modified outside of the interval $[-1, 1]$ in such a way that

$$W'(z) = \mu z + \gamma(z) \text{ where } \mu > 0,$$

$$\begin{aligned} &\gamma \text{ is real analytic on the interval } (-1, 1),\\ &\quad \text{and } |\gamma'(z)|, |\gamma(z)| \text{ are uniformly bounded on } R. \end{aligned} \tag{4.1}$$

The main idea of the present paper is the same as that of SIMON [21]; specifically, we derive an infinite dimensional analogue of the Lojasiewicz theorem (Proposition 1.1).

We introduce an elliptic operator $\mathcal{A}$,

$$\mathcal{A}z = -\Delta z + \mu z + \gamma(z),\ \nabla z.\mathbf{n}|_{\partial\Omega} = 0$$

where γ is the function from (4.1). The following result is standard:

LEMMA 4.1. *Let γ satisfy* (4.1)*. Then the operator $\mathcal{A}$ is continuously Fréchet differentiable on the spaces:*

$$\mathcal{A} : W_n^{2,p}(\Omega) = \{v \in W^{2,p}(\Omega) \mid \nabla v.\mathbf{n}|_{\partial\Omega} = 0\} \mapsto L^p(\Omega),\ \ p > 3$$

and

$$\mathcal{A} : W^{1,2}(\Omega) \mapsto [W^{1,2}]^*(\Omega).$$

Its derivative $\mathcal{H} = \mathcal{A}'$ *has the representation*

$$\mathcal{H}(z)\eta = -\Delta\eta + \mu\eta + \gamma'(z)\eta,$$

with $\mathcal{H}(z) \in \mathcal{L}(W_n^{2,p}(\Omega), L^p(\Omega))$ *in the first case, and*

$$\langle \mathcal{H}(z)\eta, \psi\rangle = \int_\Omega \nabla\eta\nabla\psi + \mu\eta\psi + \gamma'(z)\eta\psi \, dx \text{ for all } \psi \in W^{1,2}(\Omega),$$

with $\mathcal{H}(z) \in \mathcal{L}(W^{1,2}(\Omega), [W^{1,2}]^*(\Omega))$ *in the second case.*

Now, we report the following auxiliary result:

LEMMA 4.2. *Assume* $z \in W_n^{2,p}(\Omega)$, $p > 3$ *is such that*

$$-1 < -r < z(x) < r < 1 \text{ for all } x \in \Omega. \tag{4.2}$$

Then there exists a neighbourhood $U(z)$ *of* z *in* $W_n^{2,p}(\Omega)$ *such that*

$$\mathcal{A}|_{U(z)} \mapsto L^p(\Omega)$$

is analytic.

REMARK 4.3. We consider here the standard definition of analyticity (see e.g. Zeidler [22, Vol. I, Definition 8.8]:

An operator $\mathcal{A}$ acting between two Banach spaces X, Y is analytic in a neighbourhood of a point $z \in X$ if it may be expressed as

$$\mathcal{A}(z+h) - \mathcal{A}(z) = \sum_{n=1}^{\infty} T_n(z)[h, \dots, h] \text{ in } Y \text{ for any } h \in X,$$
$$\|h\|_X < \varepsilon, \ \varepsilon > 0 \text{ small enough,}$$

where $T_n(z)$ is a symmetric n-linear form on X with values in Y, and

$$\sum_{n=1}^{\infty} \|T_n(z)\|_{\mathcal{L}^n(X,Y)} \|h\|_X^n < \infty \text{ for all } \|h\|_X < \varepsilon.$$

The proof Lemma 4.2 can be done in exactly the same way as that of [2, Lemma 4.2] and we omit it.

Now, we are in a position to state the main result of this section. Let us define a functional

$$I(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 + \mu v^2 + 2\Gamma(v) \, dx \tag{4.3}$$

where $\Gamma(v) = \int_0^v \gamma(s)\, ds$. By virtue of Lemma 4.1, the functional I satisfies

$$I \in C^2(W^{1,2}(\Omega)) \text{ and } I'(z) = \mathcal{A}(z) \in [W^{1,2}]^*(\Omega).$$

The following assertion represents a version of Proposition 1.1 for analytic functionals on a Banach space:

PROPOSITION 4.4. *Let γ satisfy* (4.1)*; in particular, γ is real analytic on* $(-1, 1)$. *Let $z \in W_n^{2,p}(\Omega)$, $p > 3$ be a function satisfying*

$$-1 < -r < z(x) < r < 1 \text{ for all } x \in \Omega.$$

Then for any $P > 0$ there exist constants $\theta \in (0, \frac{1}{2})$ and $\xi(P),\ \varepsilon(P) > 0$ such that the inequality

$$|I(v) - I(z)|^{1-\theta} \leq \xi \| - \Delta v + \mu v + \gamma(v)\|_{[W^{1,2}]^*(\Omega)} \tag{4.4}$$

holds for any $v \in W^{1,2}(\Omega)$ satisfying

$$\|v - z\|_{L^2(\Omega)} < \varepsilon,\ |I(v) - I(z)| < P. \tag{4.5}$$

REMARK 4.5. Simon [21] obtained a similar result for F real analytic on R^1 and in the framework of classical solutions. Jendoubi [17] proved the same result for F real analytic on R^1 and the norm of the space $[W^{1,2}]^*$ replaced by that of the space L^2 in (4.4) and with the norm of $W^{2,p}$ in (4.5) (cf. also Hoffmann and Rybka [16]). The present version can be considered as a localized version of those results.

REMARK 4.6. Throughout the text, we are using the relation

$$W^{1,2}(\Omega) \subset L^2(\Omega) \approx [L^2(\Omega)]^* \subset [W^{1,2}]^*(\Omega)$$

to identify functions from $W^{1,2}$ as functionals in $[W^{1,2}]^*$ where L^2 is identified with its dual by the standard Riesz isometry.

The Proof of Proposition 4.4, based on Lemma 4.2, is identical with [10, Section 6, Proposition 6.1] and we omit it.

5. Proof of Theorem 1.2

With the results of the preceding section at hand, we can complete the proof of Theorem 1.2. We start with the following observation proved in [10, Lemma 7.1]:

LEMMA 5.1. *Let* $Z \geq 0$ *be a measurable function on* $(0, \infty)$ *such that*

$$Z \in L^2(0,\infty), \ \|Z\|_{L^2(0,\infty)} \leq Y$$

and there exist $\alpha \in (1, 2)$, $\xi > 0$ *and an open set* $\mathcal{M} \subset (0, \infty)$ *such that*

$$\left(\int_t^\infty Z^2(s)\, ds\right)^\alpha \leq \xi\ Z^2(t) \text{ for a.a. } t \in \mathcal{M}.$$

Then $Z \in L^1(\mathcal{M})$ *and there exists a constant* $c = c(\xi, \alpha, Y)$ *independent of* $\mathcal{M}$ *such that*

$$\int_{\mathcal{M}} Z(s)\, ds \leq c.$$

Now, we shall make use of the energy equality (2.6). Denoting by E the "total energy",

$$\begin{aligned} E(t) \ &= \ \frac{1}{2}\int_\Omega |\nabla\chi(t)|^2 + 2W(\chi(t))\, \mathrm{d}x + \frac{1}{2}\|\vartheta\|^2_{L^2(\Omega)} \\ &\quad + \frac{1}{2}\int_0^\infty (-k')(s)\|\nabla\eta(t,s)\|^2_{L^2(\Omega)}\, \mathrm{d}s, \end{aligned}$$

we have

$$E(t) \to E_\infty \text{ as } t \to \infty.$$

Moreover, by virtue of (2.12), one has

$$\|\vartheta(t)\|_{L^2(\Omega)} \to 0 \text{ as } t \to \infty, \tag{5.1}$$

and

$$\begin{aligned} &\int_0^\infty (-k)'(s)\|\nabla\eta(t,s)\|^2_{L^2(\Omega)}\, ds \leq \\ &\int_0^\tau (-k)'(s)s \sup_{t-\tau\leq z\leq t} \|\nabla\vartheta(z)\|^2_{L^2(\Omega)}\, \mathrm{d}s + \int_\tau^\infty (-k)'(s)s \sup_{z\in R} \|\nabla\vartheta(z)\|^2_{L^2(\Omega)}\, \mathrm{d}s \end{aligned}$$

If $\tau > 0$ is large enough, the second term on the right-hand side of the above inequality is small since $s(-k)'(s)$ is integrable. On the other hand, the first term tends to zero for large t for any fixed τ. Consequently, the right-hand side of the above inequality tends to zero when $t \to \infty$; whence

$$\int_0^\infty (-k)'(s)\|\nabla\eta(t,s)\|^2_{L^2(\Omega)}\, ds \to 0 \text{ as } t \to \infty. \tag{5.2}$$

Going back to (2.6) we see that

$$E(t) \to E_\infty = \frac{1}{2}\int_\Omega |\nabla\chi_\infty|^2 + 2W(\chi_\infty)\,\mathrm{d}x \text{ for any } \chi_\infty \in \omega(\chi).$$

In particular, the energy of all solutions $\chi_\infty \in \omega(\chi)$ equals the same constant E_∞, and, by virtue of (4.1),

$$\int_\Omega \frac{1}{2}|\nabla\chi(t)|^2 + W(\chi(t))\,dx = I(\chi(t)) \to E_\infty = I(\chi_\infty) \text{ as } t \to \infty \tag{5.3}$$

for arbitrary $\chi_\infty \in \omega(\chi)$, where I is the functional defined in (4.3).

Integrating (2.6) with respect to t and making use of (1.10), one obtains

$$\begin{aligned}
&\int_t^\infty \|\chi_t\|^2_{L^2(\Omega)} + k_I\|\nabla\vartheta\|^2_{L^2(\Omega)}\,ds + \\
&\quad\frac{\delta}{2}\int_t^\infty\int_0^\infty (-k)'(s)\|\nabla\eta(t,s)\|^2_{L^2(\Omega)}\,\mathrm{d}s \\
&\quad\le I(\chi(t)) - E_\infty + \frac{1}{2}\|\vartheta(t)\|^2_{L^2(\Omega)} \\
&\quad+ \frac{1}{2}\int_0^\infty (-k)'(s)\|\nabla\eta(t,s)\|^2_{L^2(\Omega)}\,\mathrm{d}s
\end{aligned} \tag{5.4}$$

Now, assume $\chi_\infty \in \omega(\chi)$ and $\omega(\chi)$ is not a singleton since otherwise there is nothing to prove. In accordance with Lemma 3.2, χ_∞ satisfies the hypotheses of Proposition 4.4. We take

$$\mathcal{M} = \{t \in (0,\infty) \mid \|\chi(t) - \chi_\infty\|_{L^2(\Omega)} < \varepsilon\}$$

where $\varepsilon > 0$ is the same as in Proposition 4.4. Since $E_\infty = I(\chi_\infty)$ we can use Proposition 4.4 to obtain

$$I(\chi(t)) - E_\infty \le c_4\|\partial_t\chi(t) - \lambda'(\chi(t))\vartheta(t)\|^{\frac{1}{1-\theta}}_{[W^{1,2}]^*(\Omega)},\ \theta \in \left(0, \frac{1}{2}\right).$$

which, combined with (5.4) and the Poincaré inequality, yields the following conclusion:

$$\begin{aligned}
&\int_t^\infty \|\chi_t\|^2_{L^2(\Omega)} + k_I\|\nabla\vartheta\|^2_{L^2(\Omega)}\,ds + \\
&\quad\frac{\delta}{2}\int_t^\infty\int_0^\infty (-k)'(s)\|\nabla\eta(t,s)\|^2_{L^2(\Omega)}\,\mathrm{d}s \\
&\quad\le c_5(\|\chi_t(t)\|^2_{L^2(\Omega)} + \|\vartheta(t)\|^2_{L^2(\Omega)})^{\frac{1}{2-2\theta}} + \frac{1}{2}\|\vartheta\|^2_{L^2(\Omega)} \\
&\quad+ \frac{1}{2}\int_0^\infty (-k')(s)\|\nabla\eta(t,s)\|^2_{L^2(\Omega)}\,\mathrm{d}s
\end{aligned}$$

provided $t \in \mathcal{M}$.

Making use of (5.1), (5.2), we can take

$$Z(t) = \left[\|\chi_t(t)\|^2_{L^2(\Omega)} + \|\nabla\vartheta(t)\|^2_{L^2(\Omega)} + \int_0^\infty (-k)'(s)\|\nabla\eta(t,s)\|^2_{L^2(\Omega)}\, \mathrm{d}s \right]^{\frac{1}{2}}$$

in Lemma 5.1 to conclude that

$$\int_{\mathcal{M}} \|\chi_t(t)\|_{L^2(\Omega)}\, dt < \infty$$

In particular, we have

$$\|\chi(t_1) - \chi(t_2)\|_{L^2(\Omega)} < \varepsilon/3$$

provided t_1, t_2 are large enough and the whole interval (t_1, t_2) lies in $\mathcal{M}$. Consequently, there exists $\tau > 0$ such that $(\tau, \infty) \subset \mathcal{M}$ which implies the convergence of $\chi(t)$ in, say, $L^2(\Omega)$ as $t \to \infty$.

This concludes the proof of Theorem 1.2.

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