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Ray Tracing in Elastic and Viscoelastic Media

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Abstract—We consider several extensions of ray tracing (uniform asymptotics, complex rays, space-time rays) interrelated by the fact that they must be used jointly in order to deal with both focusing and attenuation. Two representative models of acoustic wave propagation are considered: elasticity and viscoelasticity. Basic ideas behind canonical functions and Maslov integrals for uniformly asymptotic evaluation of the wave field from ray field parameters are discussed. Complex space-time ray tracing algorithms for dispersive and attenuating media are presented. Two models of attenuation in a viscoelastic medium are compared: (1) complex space-time ray methods for general attenuation/dispersion, (2) real ray methods for weak attenuation.

Key words: Asymptotic ray theory, viscoelasticity, attenuation.

1. Introduction

Asymptotic methods based on geometrical optics are important for computations of 3-D wave fields because they offer the advantages of locality (each signal is explicitly related to a local source and a sequence of transmission, reflection, wave conversion and scattering events) and moderate dependence of computational cost on space dimension. The advantage of "full-wave" methods of computing a wave field consists in the possibility of indiscriminate modelling of all the signals carried by the wave field. This becomes a disadvantage when the paths of the signals are required.

In the context of hereditary viscoelasticity and poroelasticity, the flexibility of asymptotic methods allows a more accurate physical characterization of the medium, in particular the relaxation mechanisms involved. On the other hand efficient implementations of "full-wave" methods in viscoelasticity (cf., CARCIONE, 1990; CARCIONE and QUIROGA-GOODE, 1995) place serious restrictions on the physical relaxation models, excluding diffusion relaxation (KELBERT and CHABAN, 1986), fractional frequency power dispersion laws (SZABO, 1994), fractional viscoelastic models (CAPUTO and MAINARDI, 1976; TORVIK and BAGLEY, 1983;

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KOELLER, 1986; SLONIMSKY, 1967), high-frequency waves in poroelastic media (MAINARDI *et al.*, 1977) and seismic waves in microinhomogeneous poroelastic media (GUREVICH and LOPATNIKOV, 1995; GELINSKY *et al.*, 1998; HANYGA and ROK, 1999). Standard finite-difference algorithms do not capture such interesting phenomena as wave fields which are smooth at wavefronts (BUCHEN and MAINARDI, 1975; LOKSHIN and ROK, 1978; NARAIN and JOSEPH, 1982; RENARDY, 1982; PRÜSS, 1993; HANYGA and SEREDYŃSKA, 1999b, although FD methods can be extended at some extra expense to thise case too (HANYGA, 1999b).

In this paper we review basic ray tracing techniques in elastic media and discuss their modifications for dispersive and attenuating media. For definiteness hereditary viscoelastic media are chosen to illustrate the treatment of dispersion and attenuation in asymptotic ray theory. Evanescent waves appear in caustic shadows and in media with attenuation. In ray asymptotics evanescent waves are represented by complex rays. Attenuation is accompanied by dispersion, which implies that propagation speeds and ray fields depend on frequency. The asymptotic limit $\omega \to \infty$ is thus inadequate for attenuating media. Space-time rays and time-domain formulation provide a solution.

Asymptotic methods fall into two categories:

(i) eikonal solvers;

(ii) ray tracing methods.

Until the advent of a few recent papers (SYMES, 1996; BENAMOU, 1996; BEVC, 1996), the first method yielded only an estimate of the first arrival time (VIDALE, 1988; VAN TRIER and SYMES, 1991; PODVIN and LECOMTE, 1991). That was highly unsatisfactory since the first arrival need not be the most interesting one. The amplitude computation was not addressed. More recent methods of determination of multiple-valued travel times by either successive downward continuation (SYMES, 1996; BEVC, 1996) or by successive tracing of rays originating in a solid angle at a point source (BENAMOU, 1996) have a limited range of applicability.

The second group of methods is based on various strategies of ray tracing. Such methods fall into two classes, corresponding to the two kinds of applications commonly encountered in practice. The first kind of applications involves computation of a synthetic seismogram for a linear array of receivers. This problem is addressed by a two-stage procedure based on two-point ray tracing (HANYGA, 1988; JULIAN and GUBBINS, 1977; PEREYRA *et al.*, 1980; PEREYRA, 1992). In the first stage the two-point kinematic ray tracing problem is solved in order to determine the initial data for the ray equations. In the second stage the paraxial system of equations (HANYGA, 1988) is integrated in order to determine the amplitude. In HANYGA (1988) the two-point ray tracing problem is reduced to a finite system of N nonlinear equations of N unknowns and can be solved by an iterative algorithm (Section 6.3).

Two-point boundary value problems have multiple solutions. An iterative two-point ray tracing algorithm converges to a solution that depends on the initial approximation and the outcome cannot always be predicted. This difficulty is partly overcome by transforming the two-point algorithm to a corresponding point-to-curve (P2C) algorithm (HANYGA, 1996c; HANYGA and PAJCHEL, 1995). The point-to-curve algorithm traces a curve in a space of parameters specifying the rays for a given one-parameter family of two-point problems, e.g., for a set of receivers lying on a curve. Multiple solutions of the two-point problems lie on different branches of this curve and the branches are connected at caustics. Additional connections between real ray branches are provided by complex ray branches in caustic shadows. Unlike ordinary two-point ray tracing algorithms, point-to-curve ray tracing encounters no difficulty when the receiver intersects a caustic. It should however be emphasized that there are other algorithms which trace rays to receivers lying exactly on a caustic (HANYGA, 1989b). Due to complex ray connections between real ray branches, the P2C algorithm automatically finds topologically disconnected real ray branches for a given receiver curve.

A different application is imaging/migration, where it is necessary to calculate asymptotic Green functions on a 3-D grid of points. This problem is addressed by wavefront ray tracing (VINJE *et al.*, 1993; SUN, 1993; HANYGA *et al.*, 1995; LAMBARÉ *et al.*, 1996). The appropriate strategy involves tracing a densely sampled field of rays (VINJE *et al.*, 1993; SUN, 1993) or bicharacteristics (LAMBARÉ *et al.*, 1996; HANYGA *et al.*, 1995) across a finite set of isochrons in order to generate a grid of data for interpolating travel times and amplitudes. Wavefront ray tracing generates grid data for all the *real* branches of the ray field. Wavefront ray tracing can be modified to generate multiply reflected and diffracted ray fields in media with discontinuity surfaces and diffracting edges (VINJE *et al.*, 1993; DRUZHININ *et al.*, 2000; HANYGA, 1996d).

In some of these algorithms care has been taken to deal with ray field singularities associated with singularities of travel times (HANYGA *et al.*, 1995). At caustics the travel time has an algebraic singularity (a cusp), while the ray amplitudes are unbounded. At a shadow boundary the amplitudes of diffracted rays are infinite. It is therefore necessary to cope with two problems: (1) reliable interpolation of singular functions in caustic regions (HANYGA *et al.*, 1995), (2) expressing the wave field locally in terms of well-behaved ray field variables. To this effect two different techniques have been developed.

In Asymptotic Diffraction Theory (HANYGA, 1997, 1996a,b, 1993, 1994, 1995; HANYGA and SEREDYŃSKA, 1991), the wave field at a point in the space is expressed in terms of a canonical function depending on the travel times and the reduced amplitudes of the rays joining this point to the source. Reduced amplitudes (without the ray spreading factor) are bounded at the caustics. The method is well adapted to two-point tracing and its developments, such as point-to-curve ray tracing. In shadows, where some rays are absent, the missing ray branches are replaced by their analytic continuations. In particular, in caustic shadows the rays corresponding to evanescent waves are complex.

The second approach involves wavefront ray tracing and wave field computation based on a time-domain Maslov theory (HANYGA *et al.*, 1995). The input data for evaluating Maslov integrals do not require complex ray tracing. This is advantageous for media which cannot be described in terms of analytic functions. For the purpose of dealing with shadow boundary layers it is necessary to resort to canonical functions combined with appropriate paraxial estimates (HANYGA, 1996d).

The situation changes when the medium is dispersive and attenuating. In order to deal with attenuation and the accompanying dispersion it is necessary to trace rays in a complex space (for a set of frequencies, since space rays are frequency dependent) or in a complex space-time. Algorithms for two-point and point-to-curve complex ray tracing have long been used in seismology (HANYGA, 1988; HANYGA and HELLE, 1990; HANYGA, 1996c). The addition of the temporal dimension increases the computational cost rather insignificantly.

The main increase in computational cost is associated with the fact that a point source in space becomes a trajectory in space-time emitting rays with varying frequency. In a non-dispersive medium the signal is merely delayed in time, scaled by an amplitude and possibly phase-shifted. In order to determine the signal at a fixed space point, a single ray—or a few rays—connecting this point to the source are sufficient to determine all the relevant parameters. In a dispersive medium each infinitesimal element of a signal is carried by a separate ray which depends on the emission time and the arrival time. This leads to the concept of space-time ray tracing.

Wavefront ray tracing is incompatible with the complex ray topology. A complex ray can be naturally defined as a solution of a two-point boundary value problem (HANYGA, 1988). Considered as a solution of an initial value problem (with complex take-off angles at a point source or originating at a complex source) the ray leaves the real space immediately and in general does not return to there. A complex ray is a two-dimensional manifold, parameterized by the real and imaginary parts $\mathrm{Re}\, T$ and $\mathrm{Im}\, T$ of a complex-values travel time T, and it reaches the receiver in the real space for an *a priori* unknown complex value T_R of T. Initial-value ray tracing must therefore be preceded by solving a two-point boundary value problem to determine the initial data *and* T_R. The initial-value ray tracing then proceeds along any contour in the complex plane joining the points 0 and T_R in the complex T-plane and avoiding branching points. In a medium with discontinuity surfaces this problem is further compounded by the necessity of tracing the complex contour through the points in the complex T-plane corresponding to the intersections of the complex ray with interfaces (Sec. 4.1). In contrast, real rays can be traced from the source through successive interface intersections to a target surface with no *a priori* information regarding the interface intersections.

Complex ray asymptotics has all the properties of real ray tracing, listed in the beginning of this section, except for locality, which is valid in a more restricted sense (KRAVTSOV and ORLOV, 1993). A complex ray is reflected, transmitted or diffracted at some point in the complex continuation of the model space, and there it feels the influence of the real model rather indirectly, through analytic continuation. More complicated situations are considered in SMITH (1995), SMITH and TEW (1995), and TEW (1992).

Complex rays have long been used in problems of electromagnetism (WANG and DESCHAMPS, 1974), magnetohydrodynamics, viscoelasticity and other kinds of media. In elasticity and acoustics they have found numerous applications in modelling evanescent waves asssociated with caustics (e.g., HANYGA and SEREDYŃSKA, 1991; HANYGA and HELLE, 1990; HANYGA, 1993) and "non-geometric" S^* waves (BABICH and KISELEV, 1989), in convex body diffraction (CHAPMAN *et al.*, 1998a), edge diffraction (HANYGA, 1993; CHAPMAN *et al.*, 1998b) and in modelling directional sources (FELSEN, 1982; NORRIS and HANSEN, 1995). A global study of complex ray fields can be found in WHITE and PEDERSEN (1981).

In seismology complex rays were introduced in BUCHEN (1974), HEARN and KREBES (1990) to account for attenuation in a viscoelastic medium. A rigorous asymptotic theory of wave propagation in viscoporoelastic media is presented in HANYGA (1999a).

Complex rays can be avoided by assuming that attenuation is weak in the sense of being asymptotically negligible, so that it does not affect the dispersion relations and intervenes only at the level of transport equations (Sec. 7). In this spirit LEWIS and GRANOFF (LEWIS, 1965; GRANOFF and LEWIS, 1967) constructed an asymptotic theory of electromagnetic waves in dielectrics accommodating dispersion and weak attenuation. An approximate real-ray asymptotic theory of waves in viscoelastic media developed by CAVIGLIA and MORRO (1993) does not capture dispersion and is limited to weak attenuation. THOMSON (1997) makes an assumption equivalent to our assumption of small viscous stress (Sec. 7) and considers space-time rays as an alternative to complex rays. Thomson points out that the small viscous stress assumption cannot be satisfactory in application to marine sediments or gas-saturated porous rocks.

On the other hand, real-ray asymptotics provides fairly accurate and explicit solutions for viscoelastic media with singular memory kernels (BUCHEN and MAINARDI, 1975; HANYGA and SEREDYŃSKA, 1999a,b, 1998). The singular part of the memory plays a more conspicuous role than the regular part. The effect of the singular part is accounted by a complex phase function, which is explicitly expressed in terms of a real-valued eikonal function representing wavefront delay and a number of real-valued functions representing the effects of smoothing, amplitude decay and signal delay with respect to the wavefront. The regular part of the memory gives extra amplitude decay and modifies higher-order amplitudes. In this case real asymptotics can be viewed as an approximation to the complex ray asymptotics (HANYGA, 1998).

Space-time rays have often been invoked to deal with moving media and parametric excitation (e.g., in BABICH, 1979; BABICH *et al*., 1985; BABICH and ULIN, 1981; KIRPICHNIKOVA, 1990; KIRPICHNIKOVA and POPOV, 1983).

2. Ray Tracing Methods in Elastic Media

2.1. Rays and Bicharacteristics in an Elastic Medium

In this and in a few following sections we consider the case of anisotropic elasticity:

$$(c_{ijkl}(\mathbf{x})u_{k,l})_{,j} = \rho \ddot{u}_i. \tag{1}$$

The eikonal $T(\mathbf{x})$ can be determined by solving a Hamilton-Jacobi equation:

$$H(\mathbf{x}, \nabla T(\mathbf{x})) = 0 \tag{2}$$

with the Hamiltonian

$$H = \det\left[\frac{1}{\rho} c_{ijkl} p_j p_l - \delta_{ik}\right]. \tag{3}$$

The rays $\mathbf{x}(\sigma)$ can be determined as the projections of the integral curves of the Hamilton equations

$$\begin{aligned} \frac{d\mathbf{x}}{d\sigma} &= \frac{\partial H}{\partial \mathbf{p}} \\ \frac{d\mathbf{p}}{d\sigma} &= -\frac{\partial H}{\partial \mathbf{x}} \end{aligned} \tag{4}$$

(bicharacteristics) on the $\mathbf{x}$-space. At a surface of discontinuity $f(\mathbf{x}) = 0$ of the density and stiffness coefficients, the bicharacteristics satisfy the boundary conditions

$$[\mathbf{p}(\sigma_1 + 0) - \mathbf{p}(\sigma_1 - 0)] \times \nabla f(\mathbf{x}(\sigma_1)) = 0 \tag{5}$$

where $f(\mathbf{x}(\sigma_1)) = 0$ (Snell's law). The eikonal is obtained by integrating an additional equation

$$\frac{dT}{d\sigma} = \mathbf{p} \cdot \frac{\partial H}{\partial \mathbf{p}} \tag{6}$$

along the bicharacteristics. By lifting rays to the phase space and dealing with bicharacteristics, we can untangle a multiple-valued ray field, which is important for interpolation in wavefront ray tracing.

The first step in ray tracing is solution of a boundary value problem. In particular, in two-point ray tracing the two ends of the ray (the "source" and the "receiver") are specified

$$\begin{aligned}\mathbf{x}(0) &= \mathbf{x}_0 \\ \mathbf{x}(\Sigma) &= \mathbf{x}_1\end{aligned} \tag{7}$$

while the value Σ of the parameter σ at the receiver end of the ray must be determined. Iterative algorithms for solving two-point and other BVPs of ray tracing are presented in JULIAN and GUBBINS (1977), HANYGA (1988) and PEREYRA *et al.* (1980). The algorithms include normal incidence, edge diffraction and complex rays.

The algorithms described in HANYGA (1988) are based on solving a nonlinear system of equations

$$\mathbf{F}(\mathbf{z}, \mathbf{P}) = 0 \tag{8}$$

where $\mathbf{F}: \mathbb{R}^N \to \mathbb{R}^N$, with $N = 2n$ in the case of a piecewise homogeneous 3-D medium and $N = 7n + 4$ in the general heterogeneous 3-D case (n denotes the number of interfaces intersected by the ray), with some input parameters $\mathbf{P}$ including the coordinates of the endpoints $\mathbf{x}_0$ and $\mathbf{x}_1$. The equations are solved by an extension of the Newton method. In the most general case the ray is split into several segments, approximating its intersections with model blocks. The unknowns $\mathbf{z}$ consist of the initial data and the range of the independent variable Σ for integrating equations (4) for each segment. The equations involve Snell's law, closing the gap between the successive segments and forcing both segment ends to lie at appropriate interfaces.

It is worth noting that this formulation eliminates the difficult problem of finding intersection points of rays with interfaces. For complex rays this problem is intractable (Sec. 4.1).

Two-point ray tracing fails on two counts if the travel times are multiple-valued. Firstly, the iterative algorithm is expected to fail at caustics because the Jacobian matrix in Newton's method becomes singular. In practice the solution can be obtained even for a receiver situated very close to the caustic because the initial estimate of the Jacobian is calculated for some approximate ray not yet reaching the caustic.[1] Secondly, the two-point ray tracing problem has multiple solutions and their number is not *a priori* known. Each solution is generated by a trial-and-error choice of an initial approximation.

[1] Ray tracing algorithms for receivers lying exactly on a caustic require a reformulation of the two-point BVP (HANYGA, 1989b).

2.2. Seismic Trace Evaluation by Geometric Theory of Diffraction (GTD)

The contribution of a single isolated ray to a seismic trace at a point **x** has the form

$$\mathrm{Re}[A(\mathbf{x})\, s\,(t - T(\mathbf{x}))] \tag{9}$$

where the amplitude $A(\mathbf{x})$ is given by the function

$$A = C\left|\frac{J_0\rho_0}{J\rho}\right|^{1/2} \tag{10}$$

with $C = \mathrm{const} \in \mathbb{C}$, $s(t)$ is an analytic signal, J is the ray spreading and the subscript 0 refers to the initial values of ρ, J. The ray spreading is found by integrating the linearized Hamiltonian equations (paraxial ray equations)

$$\begin{aligned} \frac{d\mathbf{Q}}{dt} &= H_{\mathbf{px}}\mathbf{Q} + H_{\mathbf{pp}}\mathbf{P} \\ \frac{d\mathbf{Q}}{dt} &= -H_{\mathbf{xx}}\mathbf{Q} - H_{\mathbf{xp}}\mathbf{P} \end{aligned} \tag{11}$$

for the Jacobian 3×2 matrices

$$\begin{aligned} Q_{i\alpha} &= \frac{\partial x_i}{\partial u_\alpha} \\ P_{i\alpha} &= \frac{\partial p_i}{\partial u_\alpha} \end{aligned} \tag{12}$$

of $(\mathbf{x}, \mathbf{p})$ with respect to two parameters u_1, u_2 of the initial manifold $\mathbf{x}_0(u_1, u_2)$, $\mathbf{p}_0(u_1, u_2)$, and applying the formula

$$J = \det[\mathbf{x}_\sigma, \mathbf{x}_{u_1}, \mathbf{x}_{u_2}]. \tag{13}$$

For reflected/transmitted rays the plane-wave reflection/transmission coefficients and corrections for the ray spreading jump at the interface appear as additional factors:

$$A = R\, A_{\mathrm{inc}}\left|\frac{J_0\rho_0}{J\rho}\right|^{1/2} \tag{14}$$

where J_0 is the initial ray spreading of the reflected/transmitted wave and A_{inc} denotes the amplitude of the incident ray. For edge-diffracted rays the formula is only slightly different:

$$A^D = DA_{\mathrm{inc}}|J_0\rho_0/J\rho|^{1/2} \tag{15}$$

$$J_0 := \lim_{r \to 0} J/r \tag{16}$$

where r denotes the distance of the point $\mathbf{x}$ from the diffraction point and the diffraction coefficient $D(\theta)$ depends on the initial angle θ on the diffraction cone (Fig. 1) (HANYGA, 1995).

2.3. Ray Field Singularities

The ray amplitudes are singular at caustics and shadow boundaries. Let $T_1(\mathbf{x})$, $T_2(\mathbf{x})$ denote the two connected branches of travel time near a simple caustic, while d denotes the distance of the point $\mathbf{x}$ from the caustic. The singularity of the amplitude is due to the vanishing of J: $J = O[|T_2 - T_1|^{1/3}] = O[d^{1/2}]$ (HANYGA, 1989a).

At a shadow boundary the amplitude of the primary rays is bounded, but the diffracted ray amplitude blows up because of the singularity of the diffraction ray amplitude coefficient (HANYGA, 1995): $D = O[(T_D - T_P)^{-1/2}] = O[\theta^{-1}]$, where $T_D(\mathbf{x})$ and $T_P(\mathbf{x})$ denote the travel times of the diffracted and primary rays at $\mathbf{x}$ and θ denotes the angle between the diffraction ray and the shadow boundary on diffraction cone, as shown in Figure 1.

Some uniformly valid asymptotic expressions for the seismic trace are presented in the following sections.

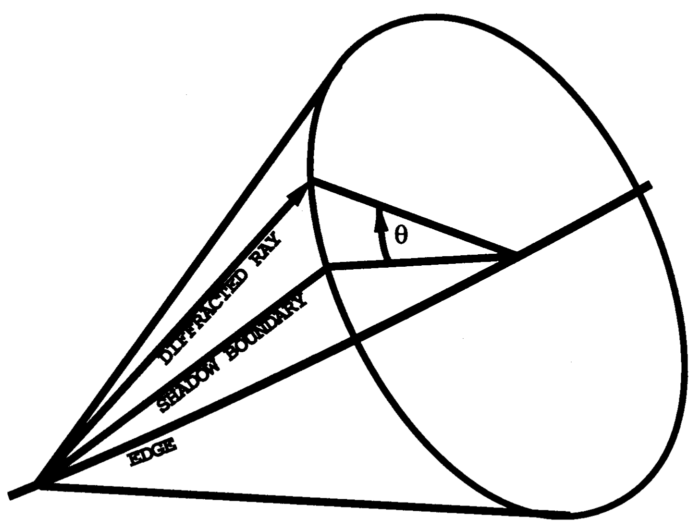

Figure 1
The angle θ on the diffraction cone.

3. Oscillatory Integrals, Canonical Functions and Maslov Theory

3.1. Uniform Asymptotics in Terms of Oscillatory Integrals

For an arbitrary wave field singularity there is a locally defined phase function $\phi(q, \mathbf{x})$ such that a uniformly valid asymptotic expansion of the wave field can be sought in the form

$$\hat{u}(\mathbf{x}, \omega) = \mathrm{Re}\left(\frac{2\pi\mathbf{i}}{\omega}\right)^{n/2} \int_{\mathscr{D}} e^{\mathbf{i}\omega\phi(q,\mathbf{x})}\left[a_0(q, \mathbf{x}) + \frac{1}{\mathbf{i}\omega} a_1(q, \mathbf{x}) + \cdots\right] dq \tag{17}$$

where $\mathscr{D}$ is a domain of $\mathbb{R}^n$, $n = 1, 2$ (HANYGA, 1997, 1996b). The Fourier transformation yields a time-domain series with the delta-spike signal in the leading term. For $\mathscr{D} = \mathbb{R}$

$$u(\mathbf{x}, t) = -\frac{1}{\sqrt{2\pi}} \int_{t > \phi} a_0(q, \mathbf{x}) \frac{1}{\sqrt{t - \phi(q, \mathbf{x})}} dq + \cdots \tag{18}$$

(HANYGA and SEREDYŃSKA, 1991). The higher-order terms involve indefinite integrals of the delta function.

Equation (17) can be transformed to a ray asymptotic expansion by applying the stationary phase method or the steepest descent method provided the stationary points or saddles are isolated and non-degenerate. Diffracted rays correspond to the contributions of boundary stationary points or saddle points (stationary points or saddle points of the restriction $\phi(\cdot, \mathbf{x})|\partial D$ of the phase function to the boundary ∂D of the region D). Complex rays correspond to complex saddles of an analytic phase function $\phi(\cdot, \mathbf{x})$. A singularity appears at a point $\mathbf{x}_0$ if for $\mathbf{x} \to \mathbf{x}_0$ at least two stationary points or saddles coalesce (Sec. 3.4).

The relation between ray fields and oscillatory integral expansions is summarized in the following theorem (HANYGA and SEREDYŃSKA, 1991):

Theorem 1.

- *For each ray field there is a (locally defined) phase function $\phi: \mathscr{D} \times \mathbb{R}^n \to \mathbb{R}$ such that the stationary points and boundary stationary points of ϕ correspond to primary and diffracted rays;*
- *expressions (17), (18) satisfy the appropriate PDEs in an asymptotic sense provided the ray theory expansions derived from (17) by the stationary phase approximation or by the steepest descent method satisfy the eikonal and transport equations.*

The singularities of GTD can be avoided by using the oscillatory integral expansions provided it is known how to calculate the phase function and the factors a_k for the given ray field. A method of constructing of the phase ϕ and the amplitudes a_k is provided for caustics by the Maslov theory (CHAPMAN and DRUMMOND, 1982; HANYGA, 1984b; THOMSON and CHAPMAN, 1985) (Sec. 3.2). Alternatively, an oscillatory integral can be reduced to a canonical function of an argument depending on the travel times and amplitudes of the rays joining the source to the point **x**. This is achieved by a generalization of the method of CHESTER *et al.* (1957) (Sec. 3.3).

3.2. Maslov Integrals

3.2.1. Recapitulation of local Maslov theory

In the case of a caustic cusp (travel-time triplication), the 3-D manifold Λ: $\mathbf{x}(\sigma, u_1, u_2)$, $\mathbf{p}(\sigma, u_1, u_2)$ spanned by the bicharacteristics in the phase space, has two folds. Due to the vertical orientation of Λ the folds are poorly parameterized by (x_1, x_2, x_3). Assuming that the caustics, which are the projections of the folds onto the (x_1, x_2, x_3) space, are transversal to the x_1 coordinate axis a neighborhood of the fold on Λ admits a local coordinate system (p_1, x_2, x_3) and the asymptotic contribution of the fold to the wave field should be expressed in terms of these coordinates. Let $\tau^{(1)}$ denote the following Legendre transform of the eikonal:

$$\tau^{(1)}(p_1, x_2, x_3) := T(X(p_1, x_2, x_3), x_2, x_3) - p_1 X(p_1, x_2, x_3) \tag{19}$$

where the function $X(p_1, x_2, x_3)$ is obtained by solving the equation

$$\frac{\partial T}{\partial x_1}(X, x_2, x_3) = p_1. \tag{20}$$

The phase function is now given by the formula

$$\phi(q, \mathbf{x}) := \tau^{(1)} + p_1 x_1 \tag{21}$$

and the slowly varying factor in the lowest-order integrand can be taken as

$$a_0 = c|J_1|^{-1/2} \tag{22}$$

where $J_1 = \partial(p_1, x_2, x_3)/\partial(\sigma, u_1, u_2) \neq 0$. In a neighborhood of an arbitrary point such that $J_1 \neq 0$ the function $\tau^{(1)}$ is regular.

Alternative Maslov integrals are provided by the transformations $x_2 \to p_2$, $x_3 \to p_3$, $(x_1, x_2) \to (p_1, p_2)$, etc.

3.2.2. Numerical implementation of time-domain Maslov theory in wavefront ray tracing

Wavefront ray tracing naturally yields multiple-valued solutions. Two kinds of difficulties remain. In the first place travel-time interpolation is unreliable in the vicinity of a caustic because the second-order derivatives of the eikonal are unbounded:

$$\frac{\partial^2 T}{\partial x_1^2} = O[d^{-1/2}] \tag{23}$$

where d denotes the distance from the caustic. Equation (23) follows from the estimate $T_2 - T_1 = O[d^{3/2}]$ (HANYGA, 1989a; Fig. 2). On the other hand, the

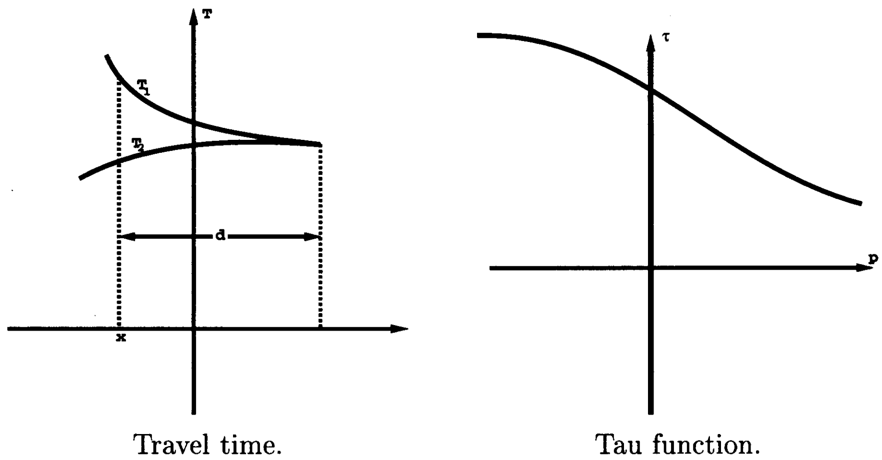

Figure 2
Travel time and τ near a caustic.

function $\tau^{(1)}(p_1, \ldots)$ has bounded derivatives. The amplitude A given by eq. (10) is unbounded but the slow factor a_0 in the Maslov integral is well-behaved.

A single Maslov integral does not quite solve the problem because the function a_0 has singularities at pseudo-caustics (HANYGA, 1984b; KENDALL and THOMPSON, 1993), defined by the equations

$$\frac{\partial^2 T}{\partial x_1^2} = 0, \quad \frac{\partial^2 \tau^{(1)}}{\partial p_1^2} = \pm \infty. \tag{24}$$

A theorem of symplectic geometry ensures that at least one type of Maslov integral or the GTD expression is locally well-defined. A globally valid expression is obtained by a weighted sum of Maslov integrals based on a partition of unity (HANYGA *et al.*, 1995).

3.3. Asymptotic Diffraction Theory

Asymptotic Diffraction Theory (ADT) (HANYGA, 1993, 1994, 1997, 1996b) is based on a local transformation of the oscillatory integrals to an expression depending only on those rays which pass through the point **x**, at which the wave field is being evaluated. The ADT expressions involve canonical functions which can be included in a mathematical function library. Time-domain functions are relatively simple. In the most interesting cases they are defined in terms of elementary functions and elliptic integrals.

The canonical functions for the most common singularities associated with focusing and edge diffraction have been derived by HANYGA (1993, 1994, 1995). The derivation begins with a local transformation of the integration variables carrying over the phase function into a polynomial, for example

$$\phi = \frac{1}{3} q^3 + y_1 q + y_0 \tag{25}$$

for a simple caustic (a smooth caustic surface without cusps and cuspidal edges) and

$$\phi = \frac{1}{2} q^2 + y_1 q + y_0, \quad q > 0 \tag{26}$$

for a simple shadow boundary (a shadow boundary diffeomorphic to a plane).

The ADT frequency-domain expression for a simple caustic is the well-known Kravtsov-Ludwig formula:

$$\hat{u}(\mathbf{x}, \omega) = [a_0(\mathbf{x}) \operatorname{Ai}(y_1) - \mathbf{i} b_0(\mathbf{x}) \omega^{-1/3} \operatorname{Ai}'(y_1)] \, \mathrm{e}^{\mathbf{i}\omega y_0} \tag{27}$$

where $y_0 = (T_1 + T_2)/2$, $y_1 = -[\frac{3}{4}(T_2 - T_1)]^{2/3}$, $a_0 = (1/\sqrt{2})(A_1 + A_2)(-y_1)^{1/4}$, $b_0 = \sqrt{2}(A_1 - A_2)(-y_1)^{-1/4}$. Theorem 1 implies that T_1, T_2 and A_1, A_2 are the travel times and the ray amplitudes of the two coalescent signals.

The time-domain formula for a simple shadow boundary is particularly simple:

$$u(\mathbf{x}, t) = A(\mathbf{x}) \frac{\partial G(\mathbf{x}, t)}{\partial t} \tag{28}$$

where

$$G = \begin{cases} 0 & \text{if } \iota = -1 \text{ and } t < T_\mathrm{P} \\ & \text{or } \iota = 1 \text{ and } t < T_\mathrm{D} \\ 1 & \text{if } \iota = -1 \\ & \text{and } T_\mathrm{P} < t < T_\mathrm{D} \\ \frac{1}{2} - \frac{\iota}{\pi} \sin^{-1} \sqrt{(T_\mathrm{D} - T_\mathrm{P})/(t - T_\mathrm{P})} & \text{if } t > T_\mathrm{D} \text{ and } \iota = \pm 1 \end{cases} \tag{29}$$

and

$$\iota := \begin{cases} -1 & \text{in the light} \\ +1 & \text{in the shadow} \end{cases} \tag{30}$$

(HANYGA, 1994, 1995). T_P, T_D denote the primary and diffracted travel time. The pulse shape is shown in Figure 3.

The amplitude of the diffracted arrival in eq. (28) is wrong. A correction term depending on the diffraction coefficient should be added (HANYGA, 1993). Unfor-

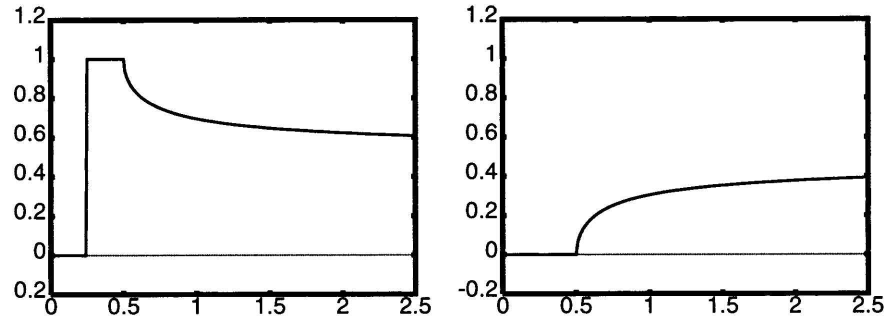

Figure 3
Edge-diffracted signals on the insonified (left) and shadow side (right) of the shadow boundary.

tunately diffraction coefficients are available for non-transparent diffracting objects and traction-free crack surfaces only.

A simple formula is also available for diffraction by a vertex at the intersection of several edges (Hanyga, 1995).

Equation (29) can be implemented in an algorithm based on two-point ray tracing. In the context of wavefront ray tracing it can be applied to compute the signals in the shadow boundary layer. In this case the excess travel time $T_{exc} := T_D - T_P$ can be estimated from paraxial data for the shadow boundary rays (Hanyga, 1996d; Druzhinin *et al.*, 2000).

Canonical functions for singularities involving shadow boundaries and caustics in time domain are presented in Hanyga and Seredyńska (1991) and Hanyga (1993).

3.4. Analytic Continuation and Complex Rays

The canonical expressions depend on a fixed number of travel times and ray amplitudes. In the shadows the missing travel times and amplitudes have to be replaced by the analytic continuation of their counterparts from the illuminated side.

Analytic continuation of a primary congruence in edge diffraction problems is quite straightforward.

Analytic continuation of a real ray congruence across the caustics results in complex rays. A direct analysis of the solutions of a BVP at a caustic and in the caustic shadow can be found in Hanyga (1989a). Complex rays correspond to the complex saddles of the oscillatory integral (17) with ϕ given by eq. (25). The behavior of the saddles at crossing a simple caustic $y_1 = 0$ is shown in Figure 4.

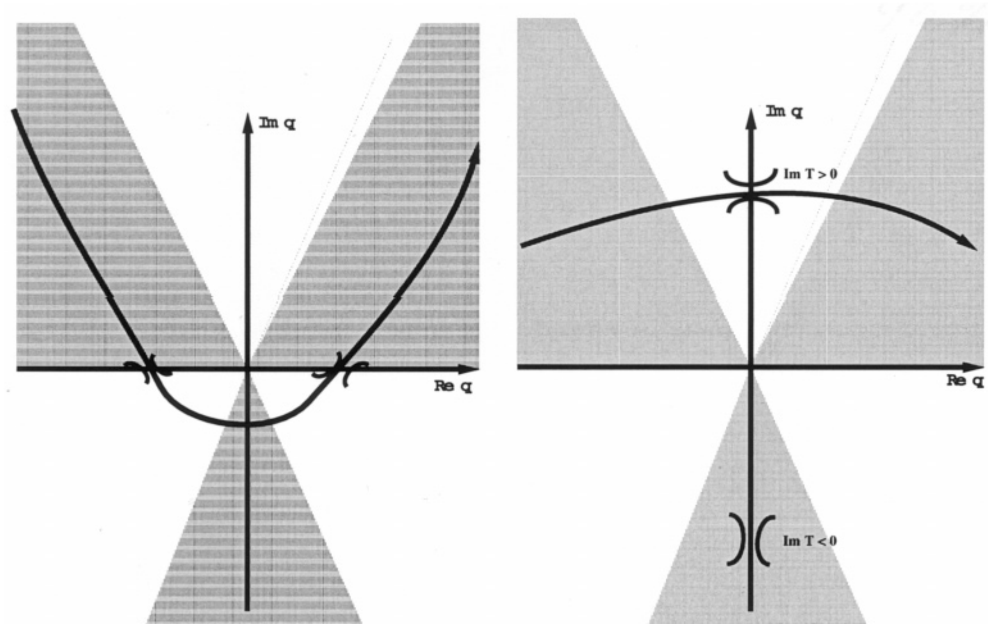

Figure 4
Complex saddle points of the oscillatory integral near a caustic (left: light; right: shadow).

In the caustic shadow $y_1 > 0$ complex rays and saddles appear in complex conjugate pairs. In particular $\mathrm{Im}\, T_2 = -\mathrm{Im}\, T_1$ and the saddle corresponding to an evanescent contribution ($\mathrm{Im}\, T_j > 0$) is picked by the complex contour. The contribution of the evanescent complex ray to the seismic trace is

$$u(\mathbf{x}, t) = \mathrm{Re}\left[\frac{1}{\mathbf{i}\pi} a_0(\mathbf{x}) \frac{1}{t - T(\mathbf{x})}\right]. \tag{31}$$

Expressing a_0 in terms of real-valued functions a_R, a_I,

$$a_0 = a_\mathrm{R} + \mathbf{i} a_\mathrm{I}$$

this yields

$$u(\mathbf{x}, t) = \frac{1}{\pi}\left[a_\mathrm{R} \frac{\mathrm{Im}\, T(\mathbf{x})}{[t - \mathrm{Re}\, T(\mathbf{x})]^2 + (\mathrm{Im}\, T(\mathbf{x}))^2} + a_\mathrm{I} \frac{t - \mathrm{Re}\, T(\mathbf{x})}{[t - \mathrm{Re}\, T(\mathbf{x})]^2 + (\mathrm{Im}\, T(\mathbf{x}))^2}\right]. \tag{32}$$

The signal shape function in eq. (32) is reminiscent of the delta function regularizer

$$\delta_\varepsilon(x) = \frac{1}{\pi} \frac{\varepsilon}{x^2 + \varepsilon^2}\,. \tag{33}$$

This suggests an interpretation of $\mathrm{Re}\, T$ as the physical travel time and $\mathrm{Im}\, T$ as a smoothing parameter (cf. also HANYGA and HELLE, 1990).

4. Complex Ray Tracing of Evanescent Wave Fields

4.1. Basic Facts about Complex Rays

In Section 3.4 we demonstrated the role of complex rays in asymptotic representation of evanescent wave fields in caustic shadows. In Section 5.4 complex rays will be discussed in the context of attenuation in viscoelastic media. In such common problems as total internal reflection and impedance boundary problems real rays excited at real parts of the boundary or at real initial data do not illuminate real space points, while complex rays originating at complex extension of the boundary or initial data provide the expected illumination (KELLER and KARAL, 1960; CHAPMAN *et al.*, 1998a).

Complex rays also arise when a point source radiating a beam is modeled by a complex point source (HEYMAN, 1989; NORRIS and HANSEN, 1995).

It is also worth noting that complex rays provide links between otherwise topologically disconnected parts of the real wave field (HANYGA, 1996c). A small perturbation of the model can replace a pair of real ray branches by two complex conjugate branches (HANYGA and PAJCHEL, 1995).

Two-point complex ray tracing is a straightforward extension of two-point real ray tracing. In the context of point-to-curve ray tracing complex rays become a tool for determining real rays because they provide links between topologically disconnected branches of real ray solutions (HANYGA, 1996c).

For practical purposes of complex ray tracing it is convenient to assume that the medium is described in terms of analytic functions. More specifically, material constants are represented by real analytic functions of the coordinates and the interfaces are given in terms of real analytic functions in the implicit form $f(\mathbf{x}) = 0$ or in the parametric form $\mathbf{x} = \mathbf{g}(v_1, v_2)$. For complex ray tracing these functions are replaced by their complex analytic continuations in proximity of the real space.

The assumption of analyticity can however be significantly relaxed. It is sufficient to assume that the functions defining the medium are $\mathscr{C}^\infty$-smooth and replace analytic continuation by quasi-analytic extension (MELLIN and SJÖSTRAND, 1974; TRÈVES, 1982). Quasi-analytic extensions are non-unique but they give rise to asymptotically equivalent results.

A complex ray $x_i = \xi_i(\sigma)$, $i = 1, 2, 3$, is defined in terms of complex analytic functions of a complex variable σ. It joins two points in the real space, the source and the receiver, but it leaves the source in a direction defined by complex-valued take-off angles. This fact can be demonstrated on a simple example borrowed from (HEARN and KREBES, 1990). Consider a medium consisting of several homogeneous plane layers of thickness h_j characterized by complex-valued phase propagation speeds v_j. For a given offset X the ray parameter p can be determined from the stationary phase condition

$$\partial \phi / \partial p = 0 \tag{34}$$

where the phase ϕ is given by

$$\phi = pX + \sum_j h_j \left(\frac{1}{v_j^2} - p^2 \right)^{1/2}. \tag{35}$$

From eq. (34) we get

$$p \sum_j h_j \left(\frac{1}{v_j^2} - p^2 \right)^{-1/2} = X \tag{36}$$

with X, $h_j \in \mathbb{R}$, $p \in \mathbb{C}$. Equation (36) is a ray tracing BVP in disguise. Assume that the ray direction is specified by a real vector $[n_1, n_3]^{\mathrm{T}}$ and let $a_j \in \mathbb{R}$ denote the value of n_1/n_3 in layer j. We then have $p/(v_j^{-2} - p^2)^{1/2} = a_j$, whence $v_j^{-2} = p^2(1 + a_j^2)/a_j^2$ and $\operatorname{Im} v_j^{-2}/\operatorname{Re} v_j^{-2} = \operatorname{Im} p^2/\operatorname{Re} p^2$. The last equality cannot be satisfied for arbitrary set of complex numbers v_j. Consequently, the vector $\mathbf{n}$ is complex if at least one speed v_j is complex.

Expression (9) is now defined in terms of the complex analytic continuations of the travel time $T(\mathbf{x})$ and amplitude $A(\mathbf{x})$ into the caustic shadow. The complex analytic continuation of the travel time can be numerically determined by applying analytic continuation to the two-point boundary value problem, essentially by allowing the variable $\mathbf{z}$ in eq. (8) to assume complex values (HANYGA, 1988). The solutions of the two-point problem with the receiver in the caustic shadow are complex rays. Even though the complex ray eventually reaches the real space at the receiver the travel time is complex valued. The amplitude is numerically determined by integrating the complex analytic continuation of eqs. (4). In the lowest order this can be reduced to the integration of the complex continuation of the paraxial ray equations (11).

A complex ray, considered as a real manifold in a six-dimensional space $\mathbb{R}^6$ of $\operatorname{Re} x_1$, $\operatorname{Im} x_1, \ldots, \operatorname{Im} x_3$, is a two-dimensional manifold, parameterized by the real variables $\operatorname{Re} \sigma$ and $\operatorname{Im} \sigma$. A complex ray satisfies eqs. (4) in the sense of complex analytic functions. It attains the receiver for $\sigma = \sigma_R$, where σ_R is a complex number. In contrast to real rays complex rays cannot be determined by numerical integration of eqs. (4) with given initial data. Numerical integration of complexified eqs. (4) requires the choice of a contour $\sigma(\tau)$, $\tau \in \mathbb{R}$, in the complex σ plane joining the origin $\sigma = 0$ to the point $\sigma = \sigma_R$, which must be *a priori* known.

Additional constraints for the integration contour arise for a complex ray representing a signal that is either transmitted at an interface or reflected at it. The interface is defined in the complex space $\mathbb{C}^3$ by the equation $f(\mathbf{x}) = 0$, where $f(\mathbf{x})$ is a complex analytic function, or, equivalently, by the two real equations $\operatorname{Re} f = 0$ and $\operatorname{Im} f = 0$. Consequently it has codimension two in $\mathbb{R}^6$ and its generic intersection with a complex ray corresponds to an isolated point σ_1 in the complex σ-plane. The contour $\sigma(\tau)$ for the numerical integration of eqs. (4) must pass through $\sigma = 0$, σ_1 and σ_R. Consequently the complex numbers σ_1, σ_R must be determined by solving a two-point boundary-value problem before numerical integration of eqs. (4).

The contour $\sigma(\tau)$ must also bypass the caustic points from an appropriate side, as discussed in the next section.

The second stage of ray tracing requires integration of transport equations (dynamic ray tracing, DRT). For a complex ray this reduces to integration of a system of ordinary differential equations along the complex contour $\sigma(\tau)$. The transport equations for complex rays can be obtained by straightforward analytic continuation of the transport equations for a real ray.

4.2. Complex Caustics and Phase Ambiguity

Complex caustics are discussed in KRAVTSOV (1967), HEYMAN and FELSEN (1983). The effect of attenuation is to shift the caustics to the complex space, which tends to remove singularities from the real space-time. Complex caustics, like real caustics, introduce a phase ambiguity which has to be addressed.

A complex ray (real dimension 2) intersects a complex caustic $I = 0$ (of real codimension 2) generically at an isolated point. The contour $\sigma(\tau)$, $\tau \in \mathbb{R}$, for DRT can bypass the caustic point leaving it either on the left side or on the right side. The phase of the complex ray spreading I is continuous along any path which does not contain caustic points. Indeed,

$$\frac{d \arg I}{d\tau} = \operatorname{Im} \operatorname{tr}\left[\frac{d\mathbf{W}}{d\tau} \mathbf{W}^{-1}\right] \tag{37}$$

and the right-hand side of eq. (37) is bounded along a path that does not contain caustic points. The two equivalence classes of paths lead to incompatible phase factors (Fig. 5). Only one of them is correct. A recourse to uniformly asymptotic oscillatory integral representation of the wave field followed by an application of the steepest descent method resolves the ambiguity.

Real rays can be embedded in complex rays in a natural way. A real ray specified in terms of real analytic functions $\mathbf{x}(\sigma)$, $\mathbf{p}(\sigma)$ can be extended to a complex ray by analytic continuation in the complex σ plane. The standard real ray tracing corresponds to complex ray tracing along the real axis of the σ-plane. Interface intersections, the receiver and points of contact of the ray with real caustics lie along the real axis. At a caustic point the ray spreading I changes sign and the value of the additional amplitude factor $(-1)^{-1/2} = \mp \mathbf{i}$ is at first undefined. Alternatively, the caustic point can be bypassed by a small detour in the complex σ-plane from either below or above, resulting in different values of $I^{-1/2}$.

For real rays phase ambiguity can be resolved by applying the stationary phase method to a uniformly asymptotic oscillatory integral representation of the wave field. The result can be expressed in terms of the Maslov (or KMAH) index, which can be calculated by an algebraic method (MASLOV and FEDORYUK, 1981; HANYGA, 1988) or by a geometric method (ARNOLD, 1967). The formula linking the phase factor to the Maslov index is based on the Maslov oscillatory integral representation of the wave field and cannot be applied to complex rays.

4.3. Stokes Surfaces and Shadow Boundaries of Complex Ray Congruences

Complex rays are not screened off by real obstacles. Indeed, the external surface of a real obstacle has codimension > 1 in the complexified space. A complex ray congruence does not "see" a real obstacle unless an intersection with the obstacle is part of its definition. Thus complex continuations of reflected/transmitted/diffracted rays satisfy appropriate Snell laws at the intersection with the obstacle but the complex continuation of the direct ray congruence does not.

Complex ray congruences have their own shadow boundaries associated with Stokes phenomena (CHAPMAN *et al.*, 1998a). The term "Stokes phenomena" refers to a sudden appearance and disappearance of a component of an asymptotic expansion of an analytic function (the component is itself an asymptotic series) as the complex argument of the function crosses a Stokes surface, which is a submanifold of codimension one. In the case of an analytic function of a single complex variable z the Stokes surface becomes a curve in the complex z-plane

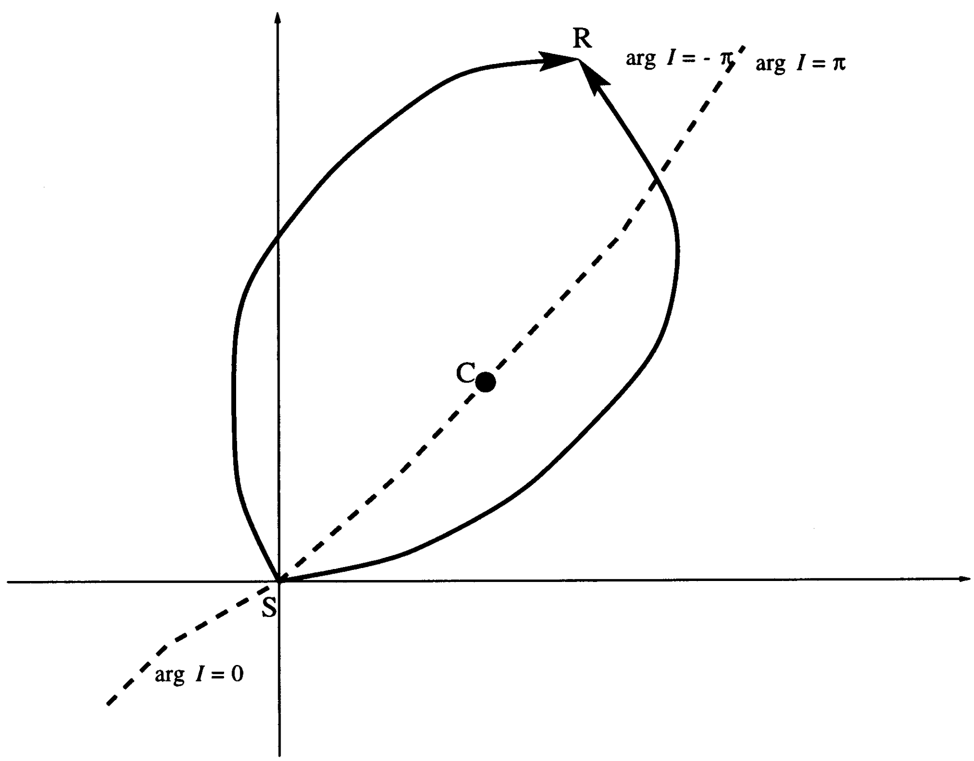

Figure 5
The σ-plane for a complex ray with a caustic point C defined by the equation $I = 0$. Two inequivalent paths and the curves arg $I = 0$, $\pm\pi$ are shown. The two paths from S to R result in different values of $I^{-1/2}$.

(Dingle, 1973; Meyer, 1989). Such a component of the asymptotic expansion can generally be associated with a complex ray and the Stokes surface plays the role of a shadow boundary of a branch of the multivalued complex ray field in the complex space.

Stokes surfaces are a proper subset of the set of surfaces (of codimension 1) defined by the equation $\mathrm{Re}[T_k(\mathbf{x}) - T_l(\mathbf{x})] = 0$ for $k \neq l$, where the indices k, l refer to different branches of the complex ray field. At the crossing of the Stokes surface $|\mathrm{Im}[T_k(\mathbf{x}) - T_l(\mathbf{x})]|$ has a local maximum so that one of the two rays k, l has a maximum dominance over the other one. Simultaneously, the Stokes surface is the shadow boundary of the subdominant ray branch. More accurate asymptotic analysis of the internal structure of the Stokes surface shows that the disappearance of the subdominant ray branch is a gradual but fast decrease in amplitude governed by a universal law (McLeod, 1992).

A caustic in the real space emanates Stokes surfaces which intersect the real space only at the caustic. In other cases Stokes surfaces intersect the real space outside the caustic along an ordinary shadow boundary, for example in edge diffraction problems, as shown in Chapman *et al.* (1998a). On the other hand, a complex caustic is generally a submanifold of codimension 2 in the complex space, defined by an equation $f(\mathbf{x}) = 0$, where $f(\mathbf{x})$ is a complex analytic function. A complex caustic does not divide the complex space into two parts illuminated by different sets of rays. It rather appears as a branching point emanating Stokes surfaces. The intersection of a complex caustic $\mathrm{Re}\, f(\mathbf{x}) = \mathrm{Im}\, f(\mathbf{x}) = 0$ with the real space can have codimension 1 or 2.

The first case is exemplified by the uniformly asymptotic solution (27). In the coordinates y_0, y_1 the complex caustic is given by the equations $\mathrm{Re}\, y_1 = \mathrm{Im}\, y_1 = 0$. For $\mathrm{Im}\, y_1 = 0$ two real rays contribute to the asymptotic expansion for $\mathrm{Re}\, y_1 < 0$ and one complex ray for $\mathrm{Re}\, y_1 > 0$. In the complex y_1-plane the two real rays continue as two complex rays in a neighborhood of the negative real axis. One ray disappears at the crossing of either of the two Stokes surfaces $\arg y_1 = \pm(2/3)\pi$. If the transition from the negative to positive real values of the variable y_1 proceeds along the real axis, then in the real space the caustic $y_1 = 0$ has codimension 1 and appears as a shadow boundary.

The other case is represented by edge diffraction in two dimensions. The edge, given by the equation $r = 0$ ($\mathrm{Re}\, r = \mathrm{Im}\, r = 0$ in the complex space) in terms of the polar coordinate r, is a diffracted ray caustic; it has codimension 2 in both the complex space and the real space.

Several case studies in Chapman *et al.* (1998a) demonstrate how a fictitious caustic, which either lies outside the domain of the wave field or is an envelope of a ray congruence that is not pertinent for the asymptotic wave field under consideration, emanates Stokes surfaces which are actual shadow boundaries for ray congruences appearing in the asymptotic expansion.

A Stokes surface does not merely divide the ray field into actual and fictitious rays. Stokes surfaces cut through individual rays, dividing them into active and inactive parts. A complex ray through a given real point $\mathbf{x}_R$ corresponds to a term in the asymptotic solution if $\mathbf{x}_R$ lies on the active part of the ray. This implies that complex ray tracing should be followed by an analysis of Stokes surfaces. This task involves the complete ray field and can in practice be carried out for small-size local canonical problems only.

A few methods described by CHAPMAN *et al.* (1998a) allow determination of Stokes surfaces. One of them fits well into our framework: replace the asymptotic solution by the oscillatory integral and consider the situations in which the complex contour appropriate for the solution begins/ceases to pick a saddle point. The corresponding coordinate of the receiver point lies on a Stokes surface.

A detailed in-depth analysis of the interplay between complex rays and Stokes surfaces can be found in CHAPMAN *et al.* (1998a).

5. Asymptotic Methods in Viscoelasticity

5.1. Introduction

In accordance with the Kramers-Kronig relations (AKI and RICHARDS, 1980), attenuation is always accompanied by dispersion. In ray asymptotics dispersion requires space-time rays while attenuation turns them into complex rays. We shall discuss the simplest mathematical model of attenuation and dispersion of viscoelastic waves. The basic conclusions of our considerations apply to other media with dispersion and attenuation, in particular to Biot's model of poroelasticity and its extension to viscoporoelasticity (HANYGA, 1999a).

The stress-strain relations are assumed in the hereditary form

$$\sigma_{ij}(t) = c^{(0)}_{ijkl}\, e_{kl}(t) + \int_0^\infty c'_{ijkl}(\theta) e_{kl}(t-\theta)\, d\theta. \tag{38}$$

Viscoelasticity based on strain-rate constitutive equations can be considered as an extreme case of the constitutive equation (38), with $c'_{ijkl}(\theta) \equiv C_{ijkl}\delta'(\theta)$.

We shall consider the asymptotics with respect to a large parameter defined by the ratio of an inhomogeneity scale (in time units) to a typical relaxation time. It is convenient to allow for inhomogeneity with respect to time, i.e., a non-stationary problem in a non-stationary medium. With the large parameter λ of dimension $[T^{-1}]$ the stress-strain relations assume the form

$$\sigma_{ij}(\mathbf{x}, t) = c^{(0)}_{ijkl}(\mathbf{x}, t)\, e_{kl}(\mathbf{x}, t) + \int_0^\infty c^{(1)}_{ijkl}(\tau, \mathbf{x}, t)\, e_{kl}(\mathbf{x}, t - \tau/\lambda)\, d\tau \tag{39}$$

where

$$c'_{ijkl}(\theta, \mathbf{x}, t) = \lambda c^{(1)}_{ijkl}(\lambda\theta, \mathbf{x}, t) \tag{40}$$

and

$$\lambda = 1/T_{\mathrm{R}} \gg \sup_{ijkl,\sigma} \left\{ \left| \frac{1}{c^{(\sigma)}_{ijkl}} \frac{\partial c^{(\sigma)}_{ijkl}}{\partial t} \right|, \left| \frac{v}{c^{(\sigma)}_{ijkl}} \frac{\partial c^{(\sigma)}_{ijkl}}{\partial x} \right| \right\}. \tag{41}$$

In eq. (41) T_{R} denotes the largest relaxation time, while v is the upper limit on the propagation velocity.

The basic problems can already be demonstrated in $1+1$-dimensional case.

5.2. One-dimensional Models

For the one-dimensional problems the stress-strain relation with a large parameter assumes the form

$$\sigma(t, x) = a(t, x)\, e(t, x) + \int_0^\infty \chi(\tau, t, x)\, e(t - \tau/\lambda, x)\, d\tau \tag{42}$$

with $\lambda = 1/T_\sigma$ where T_σ denotes the stress relaxation time. The requirement that mechanical energy should be dissipated in a cycle implies Graffi's inequality (Fabrizio and Morro, 1992):

$$\operatorname{Im} \hat{\chi} < 0 \tag{43}$$

where

$$\hat{\chi}(\Omega, t, x) := \int_0^\infty e^{\mathbf{i}\Omega\tau} \chi(\tau, t, x)\, d\tau. \tag{44}$$

For an oscillatory displacement field

$$u(t, x) = \left[U(t, x) + \frac{1}{\mathbf{i}\lambda} U_1 + O\left[\frac{1}{\lambda^2}\right]\right] \mathrm{e}^{\mathbf{i}\lambda S(t,x)} \tag{45}$$

the strain assumes the form

$$e(t, x) = \left[\mathbf{i}\lambda S_x U + U_x + O\left[\frac{1}{\lambda}\right]\right] \mathrm{e}^{\mathbf{i}\lambda S} \tag{46}$$

whence it is easily shown that

$$\sigma = \mathrm{e}^{\mathbf{i}\lambda S}\left[C(\mathbf{i}\lambda U S_x + U_x) - S_x \hat{\chi}_\Omega U_t - \hat{\chi}_\Omega S_{tx} U + \frac{1}{2} \hat{\chi}_{\Omega\Omega} S_x S_{tt} U + O[1/\lambda] \right] \tag{47}$$

with

$$C := a(t, x) + \hat{\chi}(-S_t, t, x). \tag{48}$$

Equation (47) can now be substituted into the momentum balance equation. Assuming that

$$\left|\frac{1}{k}\frac{\partial\Omega}{\partial x}\right| \ll \frac{1}{T_M} \tag{49}$$

where $k = \lambda K$, $K = S_x$, $\Omega = -S_t$, $\omega = \lambda\Omega$ and where $T_M \cong 10T_\sigma$ denotes the memory timespan, we can proceed formally collecting the $O[(\mathbf{i}\lambda)^2]$ and $O[\mathbf{i}\lambda]$ terms to obtain the dispersion relation

$$C(\Omega)K^2 - \rho\Omega^2 = 0 \tag{50}$$

and the transport equation:

$$-\left(\rho S_t + \frac{1}{2}S_x^2\,\hat{\chi}_\Omega\right)U_t + CS_xU_x - \frac{1}{2}(\rho S_{tt} - CS_{xx})U$$
$$-\frac{1}{2}\left[\left(\hat{\chi}_\Omega S_{tx}S_x - \frac{1}{2}\hat{\chi}_{\Omega\Omega}\,S_x^2\,S_{tt}\right) - \frac{dC}{dx}S_x\right]U = 0 \tag{51}$$

where

$$\frac{dC}{dx} := a_x + \frac{\partial\hat{\chi}}{\partial x} - \frac{\partial\hat{\chi}}{\partial\Omega}S_{xt}. \tag{52}$$

Equation (50) is the eikonal equation for the phase function $S(t, \mathbf{x})$, with the Hamiltonian

$$H = \frac{1}{2}[C(\Omega)K^2 - \rho\Omega^2]. \tag{53}$$

The Hamiltonian equations for (53) assume the form:

$$\begin{aligned}\frac{dt}{d\sigma} &= -\rho S_t - \frac{1}{2}\hat{\chi}_\Omega\,k^2\\ \frac{dx}{d\sigma} &= CS_x\end{aligned}\;. \tag{54}$$

Assume that $\rho = \text{const}$ in order to simplify the calculi and define

$$\mathbf{W} := \begin{bmatrix}\partial t/\partial\sigma, & \partial t/\partial u\\ \partial x/\partial\sigma, & \partial x/\partial u\end{bmatrix} \tag{55}$$

where $u = x_0$ for an initial-value problem and $u = t_0$ for a boundary-value problem (a point-source radiating in time). The space-time ray spreading $I := \det\mathbf{W}$ satisfies the equation

$$\frac{d \log I}{d\sigma} = \mathrm{tr}\left[\frac{d\mathbf{W}}{d\sigma}\,\mathbf{W}^{-1}\right] = CS_{xx} + \frac{dC}{dx}\,S_x - \rho S_{tt} - \hat{\chi}_\Omega\,S_x\,S_{tx} + \frac{1}{2}\,\hat{\chi}\,S_x\,S_{tt} - S_t\,\frac{\partial\rho}{\partial t} \quad (56)$$

whence the transport equation (51) can be explicitly integrated

$$U = U^{(0)}\left(\frac{I^{(0)}}{I}\right)^{1/2}. \quad (57)$$

For comparison with the more familiar space ray formula, we consider a solution of a boundary-value problem or a point source, in the multidimensional case. Additional parameters v specify the initial conditions for the ray:

$$\begin{aligned} t &= t_0 \\ x(0, v, t_0) &= x_0 \end{aligned}. \quad (58)$$

For the non-dispersive case ($\hat{\chi}_\Omega \equiv 0$), the time along a ray is given by the formula $t = t_0 + \rho(\omega_0/\lambda)\sigma$ and the spatio-temporal ray spreading I is related to the commonly used spatial ray spreading

$$J = \det[\partial x/\partial t, \quad \partial x/\partial v] \quad (59)$$

by the formula:

$$I = \det\begin{bmatrix} \rho\omega_0/\lambda, & 0, & 1 \\ \partial x/\partial\sigma, & \partial x/\partial v, & 0 \end{bmatrix} = \frac{\rho\omega_0}{\lambda}\,J \quad (60)$$

where x stands for $x_1, \ldots, x_d$ and v stands for $v_1, \ldots, v_{d-1}$. Graffi's inequality (Fabrizio and Morro, 1992)

$$\mathrm{Im}\,C < 0 \quad (61)$$

implies that $\mathrm{Re}\,k\;\mathrm{Im}\,k > 0$, i.e., the signal is attenuated in the direction of its propagation.

For a simple stress relaxation model

$$\chi(\tau) = b\;\mathrm{e}^{-\tau}, \quad \tau = \theta/T_\sigma. \quad (62)$$

Graffi's inequality amounts to $b < 0$. For the standard linear solid, defined by the relaxation law

$$\dot{\sigma} + T_\sigma\,\sigma = a(\dot{e} + T_e\,e) \quad (63)$$

$$\hat{\chi} = a\,\frac{1 + \omega^2 T_e T_\sigma + \mathbf{i}\omega(T_\sigma - T_e)}{1 + \omega^2 T_\sigma^2} \quad (64)$$

where T_σ, T_e denote the relaxation times, Graffi's inequality implies that

$$a(T_\sigma - T_e) < 0. \quad (65)$$

The following asymptotic expansions provide a generalization of wavefront expansions (COURANT, 1962):

$$u(t, x) \sim -\mathrm{Re}\left\{\frac{1}{\mathbf{i}\pi}\left[U\frac{1}{S} + U_1 \log S + \cdots\right]\right\}. \tag{66}$$

The first term of the expansion is

$$u = -\mathrm{Re}\left[\frac{1}{\mathbf{i}\pi}\frac{1}{S}U\right] = U_\mathrm{R}\frac{\mathrm{Im}\,S}{\pi|S|^2} - \frac{1}{\pi}U_\mathrm{I}\frac{\mathrm{Re}\,S}{|S|^2}$$

where $U = U_\mathrm{R} + \mathbf{i}U_\mathrm{I}$. The wavefront expansion (66) can be obtained from the expansion (45) by a Fourier transformation with respect to λ. In the limit of vanishing attenuation $\mathrm{Im}\,S \to 0+$ and

$$u \to U_\mathrm{R}\delta(S) - \frac{1}{\pi}U_\mathrm{I}\frac{1}{S}$$

as expected.

In a general problem involving a dispersive and non-stationary medium ($\partial H/\partial t = 0$) it is necessary to solve the ray-tracing BVP in space-time followed by DRT in space-time. For a stationary medium there is an alternative possibility of solving the ray-tracing BVP in space and integrating transport equations for a set of frequencies in the spectrum of the source signature. Ray tracing of harmonic wave fields $\omega = \omega_0, \omega_1, \ldots$ and a subsequent DFT synthesis is possible. For this purpose the phase function can be substituted by $S = T(x; \omega_k) - \omega_k t$ in the bicharacteristic equations.

5.3. 3-D Anisotropic Viscoelasticity

The basic conclusions of the previous section carry over to the 3-D case. We shall briefly discuss the polarizations of viscoelastic waves. In terms of the coefficients

$$C_{ijkl} = c^{(0)}_{ijkl}(\mathbf{x}, t) + \widehat{c'_{ijkl}}(\omega, \mathbf{x}, t) \tag{67}$$

the dispersion relation assumes the form

$$\det[C_{ijkl}(\mathbf{x}, t, \omega)k_j k_l - \rho(\mathbf{x})\omega^2\delta_{kl}] \equiv : H(\mathbf{x}, t, \mathbf{k}, \omega) = 0 \tag{68}$$

with $k_i = \partial S/\partial x_i$, $\omega = -\partial S/\partial t$. Graffi's inequality implies that

$$\mathrm{Im}\,C_{ijkl} \text{ is a negative definite } 6 \times 6 \text{ matrix.} \tag{69}$$

For isotropic media Graffi's inequality implies that $\mathrm{Re}\,\mathbf{k}\cdot\mathrm{Im}\,\mathbf{k} > 0$.

The bicharacteristics and the phase function are now given by the equations

$$\frac{d\mathbf{x}}{d\sigma} = \frac{\partial H}{\partial \mathbf{k}}, \qquad \frac{d\mathbf{k}}{d\sigma} = -\frac{\partial H}{\partial \mathbf{x}}$$
$$\frac{dt}{d\sigma} = -\frac{\partial H}{\partial \omega}, \quad \frac{d\omega}{d\sigma} = \frac{\partial H}{\partial t} \tag{70}$$

and

$$\frac{dS}{d\sigma} = \mathbf{k} \cdot \frac{\partial H}{\partial \mathbf{k}} + \omega \frac{\partial H}{\partial \omega} \equiv \omega \frac{\partial C_{ijkl}}{\partial \omega} k_j k_l A'_{ik} \tag{71}$$

where $\mathbf{A}'$ denotes the adjugate of the matrix

$$A_{ik} := C_{ijkl} k_j k_l - \rho \omega^2 \delta_{ik}. \tag{72}$$

For a complex unit vector

$$\mathbf{n} = \begin{bmatrix} \sin \vartheta \quad \cos \varphi \\ \sin \vartheta \quad \sin \varphi \\ \cos \vartheta \end{bmatrix} \tag{73}$$

($\vartheta, \varphi \in \mathbb{C}$), defining a generalized wavefront normal in the complex space, the polarization vectors $\mathbf{r}$ are defined by the equation

$$C_{ijkl}(\omega) n_j n_l r_k = \rho \omega^2 v^2 r_i \tag{74}$$

$C_{ijkl}(\omega)\, n_j n_l$ is a complex symmetric matrix. Existence of three independent polarization vectors can be established by perturbation from the non-lossy case, for sufficiently small Im C_{ijkl}. It is possible to show that the imaginary part of the matrix on the left-hand side is small provided the attenuation is sufficiently small. Indeed, Im $\mathbf{n} = O[\text{Im } C_{ijkl}]$ and Im v^2, Im $\mathbf{r} = O[\text{Im } C_{ijkl}\, n_j n_l]$.

5.4. Complex Rays, Inhomogeneous Waves and Dissipation

Complex rays represent local inhomogeneous plane waves. The relation between inhomogeneous plane waves and dissipation has often been studied (cf. a recent paper by BOULANGER and HAYES, 1990). We recall the basic relation in full generality, as appropriate for comparison with complex ray theory. The relation between the kinetic energy density

$$K = \frac{1}{2} \rho \dot{u}_i \dot{u}_i \tag{75}$$

and the energy flux $f_j = -\sigma_{ij} \dot{u}_i$ is given by the energy balance equation

$$\nabla \cdot \mathbf{f} + \frac{\partial K}{\partial t} + D = 0 \tag{76}$$

where D represents the dissipation rate

$$D = \sigma_{ij}\, \dot{e}_{ij}. \tag{77}$$

The stored energy of a dispersive medium is not uniquely defined, consequently it does not appear in equation (76). For a time-periodic wave field the operator

$$\langle \phi \rangle := \frac{\omega}{2\pi} \int_0^{2\pi/\omega} \phi(t)\, dt \tag{78}$$

denotes time averaging. The time-averaged energy flux density, kinetic energy density and dissipation rate associated with an inhomogeneous plane wave

$$u_l = \mathrm{Re}[A_l\, e^{\mathbf{i}(\mathbf{k}\cdot\mathbf{x} - \omega t)}] \tag{79}$$

are given by

$$\langle f_j \rangle = -\langle \sigma_{ij}\, \dot{u}_i \rangle = -\frac{1}{2}\omega\ \mathrm{Re}[C_{ijkl}\, k_l\, \bar{A}_i\, A_k]\, \mathrm{e}^{-2\,\mathrm{Im}\,\mathbf{k}\cdot\mathbf{x}} \tag{80}$$

$$\langle K \rangle = \left\langle \frac{1}{2}\rho \dot{u}_i\, \dot{u}_i \right\rangle = \frac{1}{4}\rho\omega^2 |\mathbf{A}|^2\, \mathrm{e}^{-2\,\mathrm{Im}\,\mathbf{k}\cdot\mathbf{x}} \tag{81}$$

$$\langle D \rangle = \frac{\omega}{4}\, \mathrm{Im}[C_{ijkl}\, k_j\, k_l\, A_i\, \bar{A}_k]\, \mathrm{e}^{-2\,\mathrm{Im}\,\mathbf{k}\cdot\mathbf{x}} \tag{82}$$

where

$$\sigma_{ij} = -\mathrm{Im}[C_{ijkl}(\omega) A_k\, k_l] \tag{83}$$

and

$$C_{ijkl} := c^{(0)}_{ijkl} + \widehat{c'_{ijkl}}(\omega). \tag{84}$$

Equations (74) imply the identity

$$C_{ijkl}\, k_j\, k_l\, A_k = \rho\omega^2 A_i. \tag{85}$$

From (82) and (85) we derive the relation

$$\langle f_j \rangle\ \mathrm{Im}\, k_j = \frac{1}{2}\langle D \rangle. \tag{86}$$

The second law of thermodynamics implies that energy is dissipated in closed loading cycles. In the context of wave propagation this suggests the inequality $\langle D \rangle \geq 0$ for every periodic motion of the medium (HANYGA and SEREDYŃSKA, 1999c). For an asymptotic ray expansion $k_j = \partial S/\partial x_j$ and

$$\frac{d\ \mathrm{Im}\, S}{d\tau} \geq 0 \tag{87}$$

along the integral curves of the energy flux:

$$\frac{d\mathbf{x}}{d\tau} = \langle \mathbf{f} \rangle \tag{88}$$

(Hanyga, 1999a).

6. *Ray Tracing in Complex Space-time*

6.1. Introduction

Complex ray tracing requires knowledge of complex travel times from the source to the intersections with successive interfaces and to the receiver. This information can be obtained by iterative solution of the BVP.

On account of dispersion it is necessary to apply either frequency-dependent space rays or space-time rays. The first method is restricted to stationary media (the elastic stiffness constants and the memory functions do not depend on the current time, the interfaces and diffractors do not move in space). In the case the Hamiltonian $H(\mathbf{k}, \omega, \mathbf{x}, t)$ does not depend on t and

$$\frac{d\omega}{d\sigma} = -\frac{\partial H}{\partial t} = 0 \tag{89}$$

so the frequency is constant along the ray. Each spectral component of the wave field propagates along its own ray field. For a narrow-band field it is reasonable to trace a few spectral components and calculate their superposition at the receiver.

Space-time rays provide a more economical method of ray tracing of short-duration signals. They are also applicable to non-stationary media (cf., Kravtsov *et al.*, 1974; Kirpichnikova and Popov, 1983; Kirpichnikova, 1990).

6.2. Snell's Law for Moving Interfaces and BVPs for Ray Tracing

A moving interface is a space-time surface. It can be specified in an implicit form:

$$f(\mathbf{x}, t) = 0 \tag{90}$$

or in a parametric representation:

$$\begin{aligned} \mathbf{x} &= \mathbf{g}(u_1, u_2, u_3) \\ t &= h(u_1, u_2, u_3) \end{aligned}. \tag{91}$$

Complex ray tracing requires that the functions f, $\mathbf{g}$ are analytic and admit a local analytic continuation to the complex argument space.

Phase matching of the asymptotic solution leads to the following equations for the incident and outgoing wavenumber and frequency:

$$\Delta\mathbf{k}\cdot\mathbf{g}_{u_\alpha} - \Delta\omega\, h_{u_\alpha} = 0, \quad \alpha = 1, 2, 3 \tag{92}$$

with $\Delta\mathbf{k} := \mathbf{k} - \mathbf{k}'$, $\Delta\omega := \omega - \omega'$, generalizing Snell's law to moving interfaces.

6.3. Two-point Ray Tracing in Space-time

Equation (70) defines the mapping: $(\mathbf{x}(0), t(0), \mathbf{k}(0), \omega(0), \Sigma) \in \mathbb{C}^9 \to (\mathbf{x}(\Sigma), t(\Sigma), \mathbf{k}(\Sigma), \omega(\Sigma)) \in \mathbb{C}^8$. We therefore consider the nine variables $\mathbf{x}(0)$, $t(0)$, $\mathbf{k}(0)$, $\omega(0)$, Σ specifying the segment as unknowns. In view of the constraint

$$\mathbf{x}^{(1)}(0) = \mathbf{x}_0, \quad t^{(1)}(0) = t_0 \tag{93}$$

the first segment is specified in terms of the five variables $\mathbf{k}^{(1)}(0)$, $\omega^{(1)}(0)$, $\Sigma^{(1)}$. For each segment except for the first and the last one the following nine equations have to be satisfied:

$$\begin{aligned} & f(\mathbf{x}^{(n+1)}(0), t^{(n+1)}(0)) = 0 \quad \text{(segments join at an interface)} \\ & H(\mathbf{k}^{(n+1)}(0), \omega^{(n+1)}(0), \mathbf{x}^{(n+1)}(0), t^{(n+1)}(0)) = 0 \quad \text{(dispersion relation)} \\ & \Delta\mathbf{k}\cdot\mathbf{g}_{u_\alpha} - \Delta\omega\, h_{u_\alpha} = 0, \quad \alpha = 1, 2, 3 \quad \text{(Snell's law)} \\ & \left.\begin{aligned} \mathbf{x}^{(n)}(\Sigma) &= \mathbf{x}^{(n+1)}(0) \\ t^{(n)}(\Sigma) &= t^{(n+1)}(0) \end{aligned}\right\} \quad \text{(continuity).} \end{aligned} \tag{94}$$

For the first segment the first two lines drop out while for the last one the continuity equations are replaced by the objective of reaching the receiver:

$$\begin{aligned} \mathbf{x}^{(N)}(\Sigma) - \mathbf{x}_R &= 0, \\ t^{(N)}(\Sigma) - t_{\text{curr}} &= 0 \end{aligned} \tag{95}$$

where t_{curr} is the recording time at the receiver.

Equations (94–95) coincide with the equations derived in HANYGA (1988) except for the additional temporal coordinate. An extension of eqs. (94–95) to problems involving edge diffraction by moving edges can be made by adapting the analogous equations in HANYGA (1988).

In order to trace complex space-time rays it is necessary to work with complex analytic continuations of the functions f, $\mathbf{g}$, h, defined for complex arguments. Consequently the algorithms described in HANYGA (1988) can be applied to these equations for real and complex rays. The total number of equations $9N + 5$, where N denotes the number of interfaces intersected by the ray, equals the number of unknowns. This can be compared with the $7N + 4$ equations for 3-D space ray tracing as described in HANYGA (1988).

Two-point ray tracing and, more generally, solution of BVPs for space-time rays, can be implemented by a straightforward generalization of the iterative ray tracing algorithms presented in HANYGA (1988). The algorithm described in Section 6.2 allows for non-stationary media in order to maintain the symmetry between time and spatial coordinates. The algorithms are based on a representation of the ray in terms of a set of ray segments. Each ray segment is specified by its initial data and interval of integration. A set of nonlinear equations $\mathbf{F}(\mathbf{z}) = \mathbf{0}$ expressing continuity and Snell's law, is solved by an iterative algorithm.

Complex ray tracing algorithms can be generated by complex analytic continuation of the function $\mathbf{F}(\mathbf{z})$, as shown by HANYGA (1988). Complex ray tracing can also be determined by point-to-curve ray tracing (HANYGA, 1996c).

The cost of time sampling is more significant. For N_s sources, N_r receivers and M_s, M_r time samples at the sources and receivers it is necessary to trace $N_s N_r M_s M_r$ space-time rays. This implies an extra factor $M_s M_r$ with respect to the non-dispersive case. For a short-duration signal it need not be a large number provided the following procedure is adopted:

- Pick a source signal with a central frequency Ω, finite duration ΔT and emission time T_{em}.
- Calculate the travel time $T(\Omega)$ to the receiver R.
- Trace space-time rays in the intervals $[T_{\mathrm{em}} - \Delta T, T_{\mathrm{em}} + \Delta T]$ at the source and $[T_{\mathrm{em}} + T(\Omega) - \Delta T, T_{\mathrm{em}} + T(\Omega) + \Delta T]$ at the receiver.

The point-to-curve algorithms can be derived by the rules discussed by HANYGA (1996c).

Space-time Dynamic Ray Tracing closely follows the familiar DRT for space rays. Extension to complex rays is discussed in detail by HANYGA (1988). Integration proceeds along an arbitrary contour in the complex σ-plane, passing through the values of σ corresponding to the interface intersections and the receiver and leaving the caustic points on the appropriate side.

6.4. Space-time Ray Field Singularities

Ray field singularities acquire new aspects in the asymptotic theory developed here. In the case of modelling attenuation by complex rays the caustics are in general shifted to the complex space. Space-time rays exhibit new kinds of focusing that are not related to space caustics. Finally, the time domain canonical functions associated with space-time ray singularities are identical with the frequency domain canonical functions of the non-dispersive case. The last result is due to the fact that we are dealing with large-parameter asymptotic expansions rather than wavefront expansions.

Space-time focusing involves new effects which are distinct from space caustics. We shall discuss two selected examples.

In a stationary dispersive medium, a focusing effect resembling the gradient catastrophe (COURANT, 1962) is common. Gradient catastrophe in a nonlinear

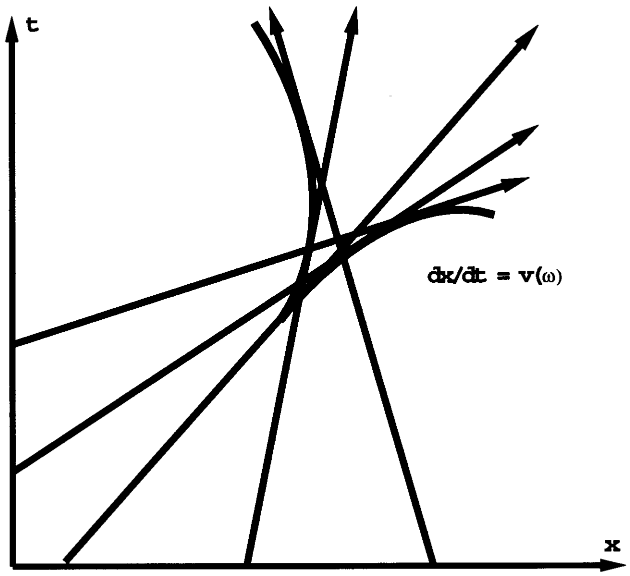

Figure 6
Linear dispersive medium: inside the cusp the solution is a superposition of three frequencies, cf. Figure 7.

hyberbolic system of equations is possible because the propagation speed depends on the field itself. In the simplest case a scalar field assumes a constant value u along each characteristic and the slope of the characteristic in the space-time depends on u. Gradient catastrophe arises when some characteristics overtake those in front of them (Fig. 6). In the theory of continuous media the particle velocity is assumed univalued, which imposes the necessity of introducing a shock wave discontinuity (Fig. 7) (HANYGA, 1984a). The corresponding effect in a stationary dispersive medium results from the dependence of the propagation speed on the frequency. In a stationary medium the frequency is constant along the space-time ray. For simplicity we consider a $1+1$ dimensional space time.

Let the group velocity be given by a function $v_g(\omega)$ and consider an intitial-value problem. In a first step we shall neglect attenuation and concentrate on the space-time aspect of the problem for a while. Figure 6 shows the space-time cusp arising when $v_g > 0$ but $(dv_g/d\omega)(d\omega/dx) < 0$.

A semi-explicit solution of the initial-value problem is given by the equations

$$\omega = \omega_0(x - v_g(\omega)t) \tag{96}$$

$$u(0,x) = A(0,x)\,\mathrm{e}^{\mathrm{i}\lambda S(0,x)} \tag{97}$$

$$S_t(0,x) = \omega_0(x). \tag{98}$$

We assume

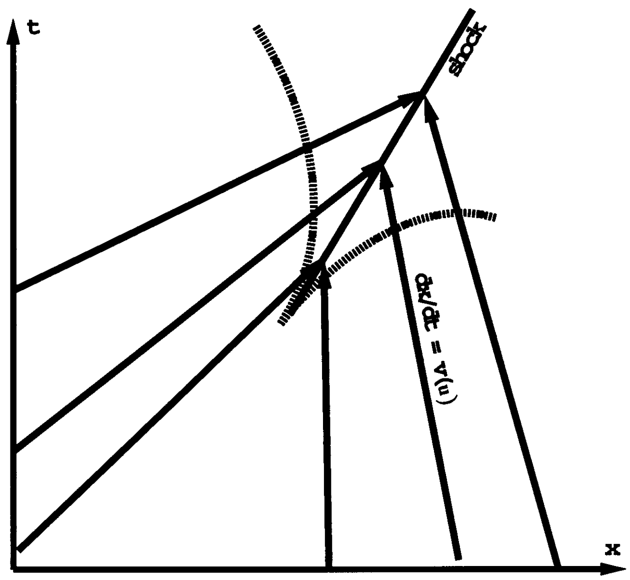

Figure 7
Gradient catastrophe in a nonlinear medium: multiple-valued solution is replaced by a discontinuous one by introducing a shock wave.

$$dk/d\omega > 0, \quad \frac{\partial^2 S}{\partial x^2}(0, x) > 0. \tag{99}$$

The caustic cusp (Fig. 6) is defined by the vanishing of the Jacobian

$$\frac{\partial(x, t)}{\partial(x_0, t)} \equiv 1 + v_g' \omega_0' t = 0. \tag{100}$$

A similar singularity for a non-stationary homogeneous medium is discussed in KRAVTSOV *et al.* (1974). It is obtained by considering a BVP at $x = 0$ with $S_x(t, 0) = k_0(x)$.

Another well-known space-time caustic is the trajectory of the Airy phase in a waveguide (Fig. 8):

$$u(\mathbf{x}, t) \propto \mathrm{Ai}\left(-\left[\frac{3}{4}(S_2 - S_1)\right]^{2/3} \right) e^{\mathbf{i}(S_1 + S_2)/2}. \tag{101}$$

Two waves with different carrier frequencies but equal group velocities travel along the path $x = v_3 t$. They coalesce along the trajectories $x = v_2 t$ and disappear in the shadow (e.g., along $x = v_1 t$).

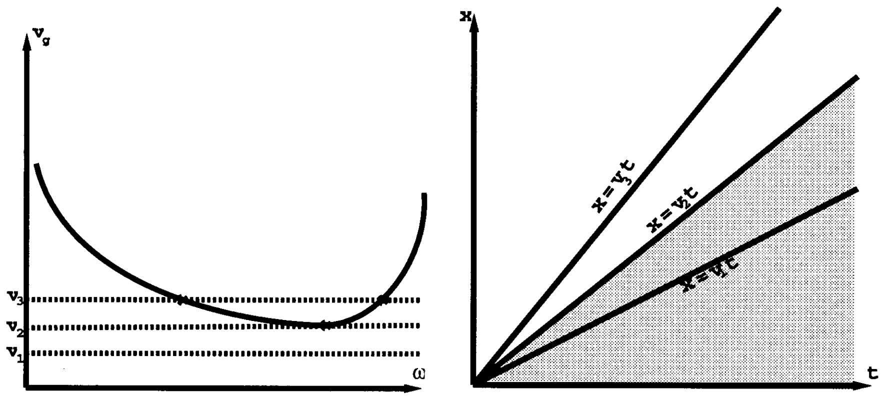

Figure 8
The Airy phase.

7. Small Viscous Stress

Attenuation does not affect ray kinematics provided the viscous part of the stress is vanishing small (of order $O[\lambda^{-1}]$) with respect to the elastic stress. In this case the rays are real and attenuation manifests itself in an exponential amplitude decay. The appropriate assumption is

$$c'_{ijkl}(\theta, \mathbf{x}, t) = \Lambda c^{(1)}_{ijkl}(\lambda\theta, \mathbf{x}, t) \tag{102}$$

where Λ is a constant frequency and $\Lambda/\lambda \ll 1$.

We consider the 1-D case only. In order to derive the transport equations from the transport equations derived in Section 5.2 it is enough to replace $\hat{\chi}$ by $(\Lambda/\lambda)\hat{\chi}$ and reshuffle various terms according to the power of λ^{-1} involved. The stress is given by the following asymptotic formula

$$\sigma = e^{\mathbf{i}\lambda S}[a(\mathbf{i}\lambda US_x + U_x) + \mathbf{i}\Lambda\hat{\chi}US_x + O[1/\lambda]]. \tag{103}$$

The dispersion relation assumes the form

$$aK^2 - \rho\Omega^2 = 0. \tag{104}$$

The coefficient of $\mathbf{i}\lambda$ leads to the transport equation

$$-2\rho S_t U_t + 2aS_x U_x - (\rho S_{tt} - aS_{xx})U + \mathbf{i}\Lambda\,\hat{\chi}\,S_x^2\,U + \frac{dC}{dx}S_x\,U = 0. \tag{105}$$

The first two terms represent the derivative $dU/d\sigma$ along a real ray

$$\begin{aligned} \frac{dt}{d\sigma} &= -\rho S_t \\ \frac{dx}{d\sigma} &= a S_x \end{aligned} . \tag{106}$$

The zero-order amplitude is now given by the formula

$$U = U^{(0)} \left(\frac{I^{(0)}}{I} \right)^{1/2} \mathrm{e}^{-\mathbf{i}(\Lambda/2) \int_0^\sigma \mathrm{Re}\, \hat{\chi} S_x^2 \, d\sigma} \, \mathrm{e}^{(\Lambda/2) \int_0^\sigma \mathrm{Im}\, \hat{\chi} S_x^2 \, d\sigma} \tag{107}$$

where I denotes the ray spreading of a real space-time ray field and the attenuation rate is determined by the exponent:

$$(\Lambda/2\lambda^2) \int \mathrm{Im}\, \hat{\chi} K^2 \, d\sigma = \frac{1}{2} \int K Q^{-1} \frac{dx}{d\sigma} \, d\sigma. \tag{108}$$

The quality factor Q is given by the formula

$$Q^{-1} = \frac{\Lambda}{\lambda^2} \frac{\mathrm{Im}\, \hat{\chi}}{a} = O[\lambda^{-2}]. \tag{109}$$

We have thus obtained a mathematical model of one-dimensional VE wave propagation without dispersion. The attenuation contributes an exponential decay factor and does not affect the ray field. This model is close to phenomenological seismic attenuation theory but unsatisfactory on physical grounds. In three dimensions the Q factor cannot be expressed in terms of the constitutive parameters. In the 3-D case the Q factor would have to be formulated along one of the following lines:

(i) the linear part of the Taylor expansion of the eikonal has the form $(k_j^{\mathrm{Re}} + \mathbf{i} a_j) x_j$, where

$$\mathbf{a} = \mathbf{A} \mathbf{k}^{\mathrm{Re}} \tag{110}$$

and $\mathbf{A}$ is a matrix-valued function of $(\omega, \mathbf{k}, \mathbf{x})$;

(ii) $\mathbf{k} = s(\mathbf{n}, \omega)\mathbf{n}$, where $\mathbf{n}$ is a real unit vector and $s = s^{\mathrm{Re}}(1 + \mathbf{i}/2Q)$.

In either case it is assumed that $\mathbf{A}$, $\mathbf{a} = O[\varepsilon]$. The last assumption can be reduced to an assumption pertinent to the constitutive parameters of the medium (Hanyga, 1999a).

For case (i), we note that the only relation between $\mathbf{a}$ and the real part $\mathbf{k}^{\mathrm{Re}}$ of the wavenumber vector $\mathbf{k}$ is provided by the dispersion relation. For a fixed $\mathbf{k}^{\mathrm{Re}}$ the dispersion relation allows the attenuation vector $\mathbf{a}$ to vary on a submanifold of codimension 2 in the wavenumber vector space (a circle in the isotropic case). The actual value of $\mathbf{a}$ at any point of the ray depends on the data for the boundary problem satisfied by the ray.

For case (ii), the dispersion relation can be expressed in the form $D(\mathbf{k}, \omega) = 0$ and the complex slowness $s(\mathbf{n}, \omega)$ can be determined from the equation $D(s\mathbf{n}, \omega) = 0$ for real $\mathbf{n}$. The quality factor can then be determined from the relation

$s = s^{\mathrm{Re}}(1 + \mathbf{i}/2Q)$. The corresponding ray is real (after a transformation of the parameter σ). The theory is consistent with the boundary value problem for the ray tracing if it can be proved that $\mathrm{Im}\, \mathbf{n} = O[\varepsilon^2]$ while the $O[\varepsilon]$ terms appear only in the term $1/(2Q)$. It is however easy to see that $\mathrm{Im}\, \mathbf{n} = O[\varepsilon]$ in the plane-layered case considered in Section 5.4.

A perturbative construction of complex ray theory with an *ad hoc* introduced quality factor is presented in ZHU and CHUN (1994). The authors concentrate on the local approximation of the complex ray and fail to address the boundary value problem.

8. Conclusions

Initial-value ray tracing and its derivatives such as wavefront ray tracing are possible in non-attenuating media (and in media with small attenuation). Otherwise rays are generally complex and two-point ray tracing becomes unavoidable.

In a viscoelastic medium

- dispersion implies either space-time rays or frequency-dependent space rays;
- attenuation shifts rays to the complex space;
- the attenuation model is not local (attenuation does not develop incrementally along a real path).

Ray methods based on BVP solving have natural counterparts in lossy dispersive media. Ray methods based on solving IVPs—such as wavefront ray tracing—appear difficult to implement. A direct solution of the eikonal equation is more appropriate.

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